

New exact solutions of the mBBM equation

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Abstract. The enhanced modified simple equation method presented in this article is applied to construct the exact solutions of modified Benjamin -Bona -Mahoney equation. Some new exact solutions are derived by using this method. When some parameters are taken as special values, the solitary wave solutions can be got from the exact solutions. It is shown that the method introduced in this paper has general significance in searching for exact solutions to the nonlinear evolution equations.

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Key words: modified Benjamin -Bona -Mahoney equation, exact solutions, enhanced modified simple equation method

1 Introduction

Nonlinear evolution equations are widely used as models to describe complex physical phenomena in various fields of the sciences, especially in fluid mechanics, solid state physics, plasma physics, plasma waves and chemical physics [1]-[3]. The study of exact solutions of nonlinear evolution equations plays an important role in soliton theory. In the past two decades, Many effective methods have been established to obtain exact solutions of nonlinear evolution equations, such as Darboux transformations method [4], Bäcklund transformation method [5], Hirota's bilinear method [6], Painlevé expansions method [7], symmetry method [8], the tanh-method [9], the homogeneous balance method [10], the Jacobi-elliptic function method [11], the (G'/G) -expansion method [12], the modified simple equation method [13,14] and so on. The modified Benjamin-Bona-Mahoney(mBBM) equation is a significant model of medium-long wave unidirectional transmission among the weak nonlinear dispersion. Under the help of Mathematica, the exact solutions of mBBM equation is constructed by a method mixed with auxiliary equation method and the solutions of Nonlinear evolution equations in Ref.[15] and calculated

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by modified mapping method in Ref.[16]. As applications of the enhanced modified simple equation method in this paper, some new exact solutions can be constructed from the nonlinear evolution equation, namely, the modified Benjamin-Bona-Mahoney(mBBM) equation.

2 Description of the enhanced modified simple equation method

Suppose there have a nonlinear evolution equation in the form

$$F(u, u_t, u_x, u_{xx} \dots) = 0, \quad (1)$$

where F is a polynomial of u and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method:

Step1. Using the generalized wave transformation

$$u(x, t) = u(\xi), \quad \xi = p(t)x + q(t). \quad (2)$$

where $p(t)$ and $q(t)$ are differentiable function of t , from (1) and (2) we have the following ODE :

$$F[u, (\dot{p}(t)x + \dot{q}(t))u', p(t)u' \dots] = 0, \quad (3)$$

where $\dot{} \equiv d/dt, ' \equiv d/d\xi$.

Step2. Suppose that Eq.(3) has the formal solution

$$u(\xi) = \sum_{k=0}^N A_k(t) \left[\frac{\psi'(\xi)}{\psi(\xi)} \right]^k, \quad (4)$$

where $A_k(t)$ are functions of t , $A_k(t)$ and $\psi(\xi)$ are unknown functions to be determined later such that $A_N \neq 0$.

Step3. Determine the positive integer N in Eq.(4) by considering the homogeneous balance between the highest order derivatives and the nonlinear terms in Eq.(3).

Step4. Substitute Eq.(4) into Eq.(3), calculate all the necessary derivatives u', u'', \dots of the unknown function $u(\xi)$ and obtain the function $\psi(\xi)$. As a result, a polynomial of $\frac{\psi'(\xi)}{\psi(\xi)}$ and its derivatives can be got. In this polynomial, we gather all the terms of the same power of $\frac{\psi'(\xi)}{\psi(\xi)}$ and its derivatives. Equate with zero all the coefficients of this polynomial, this operation yields a system of equations which can be solved to find $A_k(t)$ and $\psi(\xi)$. Consequently, we can get the exact solutions of Eq.(1).

3 Applications

The mBBM equation is well known in Ref.[16] and has the form

$$u_t + \gamma u_x + \beta u^2 u_x + \alpha u_{xxt} = 0. \quad (5)$$

We use the generalized transformation (2) to reduce Eq.(5) to the following ODE:

$$[\dot{p}(t)x + \dot{q}(t) + \gamma p(t)]u' + \beta p(t)u^2 u' + 2\alpha p(t)\dot{p}(t)u'' + \alpha p^2(t)[\dot{p}(t)x + \dot{q}(t)]u''' = 0. \quad (6)$$

Integrating Eq.(6) with respect to ξ and equating integration constant to zero yields the following ODE:

$$3[\dot{p}(t)x + \dot{q}(t) + \gamma p(t)]u + \beta p(t)u^3 + 6\alpha p(t)\dot{p}(t)u' + 3\alpha p^2(t)[\dot{p}(t)x + \dot{q}(t)]u''' = 0. \quad (7)$$

Balancing with yields, consequently, Eq.(7) has the formal solution

$$u(\xi) = A_0(t) + A_1(t) \left[\frac{\psi'(\xi)}{\psi(\xi)} \right], \quad (8)$$

Where $A_0(t)$ and $A_1(t)$ are functions of t , to be determined such that $A_1(t) \neq 0$. It is easy to see that

$$u' = A_1(t) \left(\frac{\psi''}{\psi} - \frac{\psi'^2}{\psi^2} \right), \quad (9)$$

$$u'' = A_1(t) \left(\frac{\psi'''}{\psi} - 3 \frac{\psi' \psi''}{\psi^2} + 2 \frac{\psi'^3}{\psi^3} \right). \quad (10)$$

Substituting Eqs.(8)-(10) into Eq.(7) and equating all the coefficients of ψ^0 , $\psi^0 x$, ψ^{-1} , $\psi^{-1} x$, ψ^{-2} , $\psi^{-2} x$, ψ^{-3} and $\psi^{-3} x$ to zero, we respectively obtain

$$3\dot{q}(t)A_0(t) + 3\gamma p(t)A_0(t) + \beta p(t)A_0^3(t) = 0, \quad (11)$$

$$\dot{p}(t)A_0(t) = 0, \quad (12)$$

$$[\dot{q}(t)A_1(t) + \gamma p(t)A_1(t) + \beta p(t)A_0^2(t)A_1(t)]\psi' + 2\alpha p(t)\dot{p}(t)A_1(t)\psi'' + \alpha p^2(t)\dot{q}(t)A_1(t)\psi''' = 0, \quad (13)$$

$$\dot{p}(t)A_1(t)\psi' + \alpha p^2(t)\dot{p}(t)A_1(t)\psi''' = 0, \quad (14)$$

$$[\beta p(t)A_0(t)A_1^2(t) - 2\alpha p(t)\dot{p}(t)A_1(t)]\psi'^2 - 3\alpha p^2(t)\dot{q}(t)A_1(t)\psi'\psi'' = 0, \quad (15)$$

$$3\alpha p^2(t)\dot{p}(t)\psi'\psi'' = 0, \quad (16)$$

$$[\beta p(t)A_1^3(t) + 6\alpha p^2(t)\dot{q}(t)A_1(t)]\psi''' = 0, \quad (17)$$

$$2\alpha p^2(t)\dot{p}(t)A_1(t)\psi''' = 0. \quad (18)$$

Eqs.(11),(12),(14),(16),(17) and (18) give the results

$$p(t) = k, A_0(t) = 0, A_0(t) = \pm \sqrt{[-3\gamma k - 3\dot{q}(t)]/\beta k}, A_1(t) = \pm \sqrt{-6\alpha k \dot{q}(t)/\beta}, \quad (19)$$

where k is a constant of integration. Let us now discuss the following cases:

Case1. If $A_0(t) = 0$, Eq.(5) has no formal solution of (8).

Case2. If $A_0(t) = \pm \sqrt{[-3\gamma k - 3\dot{q}(t)]/\beta k}$,

Eqs.(13) and (15) reduce to

$$[\dot{q}(t) + \gamma k + \beta k A_0^2(t)]\psi' + \alpha k^2 \dot{q}(t)\psi''' = 0, \quad (20)$$

$$\beta A_0(t)A_1(t)\psi' - 3\alpha k \dot{q}(t)\psi'' = 0. \quad (21)$$

Eq.(21) gives

$$\psi' = \left[\frac{E_1(t)}{F_1(t)} \right] \psi'', \quad (22)$$

where $E_1(t) = 3\alpha k \dot{q}(t)$, $F_1(t) = \beta A_0(t)A_1(t)$.

Substituting Eq.(22) into Eq.(20) we conclude that

$$\frac{\psi'''}{\psi''} = \frac{E_1(t)G_1(t)}{F_1(t)H_1(t)}, \quad (23)$$

where $G_1(t) = 2\dot{q}(t) + 2\gamma k$, $H_1(t) = \alpha k^2 \dot{q}(t)$.

Integrating Eq.(23) with respect to ξ , yields

$$\psi'' = c_1(t) \exp \left[\frac{E_1(t)G_1(t)}{F_1(t)H_1(t)} \xi \right]. \quad (24)$$

Substituting Eq.(24) into Eq.(22) we conclude that

$$\psi' = \frac{c_1(t)E_1(t)}{F_1(t)} \exp \left[\frac{E_1(t)G_1(t)}{F_1(t)H_1(t)} \xi \right], \quad (25)$$

and then

$$\psi = c_2(t) + \frac{c_1(t)H_1(t)}{G_1(t)} \exp \left[\frac{E_1(t)G_1(t)}{F_1(t)H_1(t)} \xi \right], \quad (26)$$

where $c_1(t), c_2(t)$ and $q(t)$ are arbitrary functions of t . Now the exact solution of Eq.(5) has the form

$$u(x,t) = \pm \sqrt{\frac{-3\gamma k - 3\dot{q}(t)}{\beta k}} \pm \sqrt{\frac{-6\alpha k \dot{q}(t)}{\beta}} \frac{c_1(t)E_1(t)}{F_1(t)} \left\{ \frac{\exp \left[\frac{E_1(t)G_1(t)}{F_1(t)H_1(t)} (kx + q(t)) \right]}{c_2(t) + \frac{c_1(t)H_1(t)}{G_1(t)} \exp \left[\frac{E_1(t)G_1(t)}{F_1(t)H_1(t)} (kx + q(t)) \right]} \right\}. \quad (27)$$

If we set $c_2(t) = \pm 1, c_1(t) = \frac{G_1(t)}{H_1(t)}$ in Eq.(27), then we have the following solitary-like wave solutions:

$$u_1(x,t) = \pm \sqrt{\frac{-3\gamma k - 3\dot{q}(t)}{\beta k}} \pm \sqrt{\frac{-6\alpha k \dot{q}(t)}{\beta} \frac{E_1(t)G_1(t)}{2F_1(t)H_1(t)}} \left\{ 1 + \tanh \left[\frac{E_1(t)G_1(t)}{2F_1(t)H_1(t)} (kx + q(t)) \right] \right\}, \quad (28)$$

$$u_2(x,t) = \pm \sqrt{\frac{-3\gamma k - 3\dot{q}(t)}{\beta k}} \pm \sqrt{\frac{-6\alpha k \dot{q}(t)}{\beta} \frac{E_1(t)G_1(t)}{2F_1(t)H_1(t)}} \left\{ 1 + \coth \left[\frac{E_1(t)G_1(t)}{2F_1(t)H_1(t)} (kx + q(t)) \right] \right\}. \quad (29)$$

4 Conclusions

In this article, the enhanced modified simple equation method is applied to find the exact solutions of the mBBM equation. Also, the solitary-like wave solutions and the solitary wave solutions are derived from the exact solutions. When let $\gamma = 0, q(t) = \omega t, k = 1, \alpha = \frac{1}{2}$, then $u_2(x,t) = \pm \frac{\sqrt{-3\omega\beta}}{\beta} \pm \frac{\sqrt{-3\omega\beta}}{\beta} \pm \frac{\sqrt{-3\omega\beta}}{\beta} \coth(x + \omega t)$ which contains $u = \frac{\sqrt{6A\omega\beta}}{\beta \tanh(\xi)}$ where $\xi = x + \omega t, A = -\frac{1}{2}$, this solution is mentioned in Ref.[16]. It is a special case in this paper. In conclusion, it can be seen that the enhanced modified simple equation method is direct, effective and can be applied to many other nonlinear evolution equations.

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