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THE HOLE-FILLING METHOD AND THE UNIFORM MULTISCALE COMPUTATION OF THE ELASTIC EQUATIONS IN PERFORATED DOMAINS

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Abstract. In this paper, we discuss the boundary value problem for the linear elastic equations in a perforated domain Ω^ε. We fill all holes with a very compliant material, then we study the homogenization method and the multiscale analysis for the associated multiphase problem in a domain Ω without holes. We are interested in the asymptotic behavior of the solution for the multiphase problem as the material properties of one weak phase go to zero, which has a wide range of applications in shape optimization and in 3-D mesh generation. The main contribution obtained in this paper is to give a full mathematical justification for this limiting process in general senses. Finally, some numerical results are presented, which support strongly the theoretical results of this paper.

Key Words. homogenization, multiscale analysis, elastic equations, perforated domain, hole-filling method.

1. Introduction

In this paper, we consider the following boundary value problems of elastic equations in a perforated domain:

$$(1)\qquad \begin{cases} -\dfrac{\partial}{\partial x_j}(a_{ijkh}(\frac{x}{\varepsilon})\dfrac{\partial u_k^\varepsilon(x)}{\partial x_h}) = f_i(x), \quad i = 1, 2, \cdots, n, \text{ in } \quad \Omega^\varepsilon \\ \sigma_\varepsilon(u^\varepsilon) = 0, \quad \text{on} \quad S_\varepsilon \\ u^\varepsilon(x) = g_0(x), \quad \text{on} \quad \Gamma_1 \\ \sigma_\varepsilon(u^\varepsilon) = g_1(x), \quad \text{on} \quad \Gamma_2 \end{cases}$$

Following Oleinik's notation (see [22]), let $Q = \{\xi : 0 < \xi_j < 1, j = 1, \cdots, n\}$, and ω be an unbounded domain of R^n which satisfies the following conditions:

(B_1) ω is a smooth unbounded domain of R^n with a 1-periodic structure.

(B_2) The cell of periodicity $\omega \cap Q$ is a domain with a Lipschitz boundary.

(B_3) The set $Q\backslash\bar{\omega}$ and the intersection of $Q\backslash\bar{\omega}$ with the δ_0-neighborhood ($\delta_0 < \frac{1}{4}$) of ∂Q consist of a finite number of Lipschitz domains separated from each other and from the edges of the cube Q by a positive distance.

Suppose that Ω^ε is a domain which has the form: $\bar{\Omega}^\varepsilon = \bar{\Omega}_0^\varepsilon \cup (\bar{\Omega}\backslash\Omega_0)$, where Ω is a bounded Lipschitz convex domain of R^n without holes, $\bar{\Omega}_0 = \cup_{z\in T_\varepsilon}\varepsilon(z + \bar{Q}) \subset \Omega$, $\bar{\Omega}_0^\varepsilon = \bar{\Omega}_0 \cap \varepsilon\bar{\omega}$ is shown in Fig.1(a), T_ε is the subset of Z^n consisting of all z such that $\varepsilon(z + Q) \subset \Omega$. The domain $\Omega_1 = \Omega\backslash\bar{\Omega}_0$ denotes the boundary layer as shown in Fig.1(b). The boundary $\partial\Omega^\varepsilon$ of a perforated domain Ω^ε is composed of $\partial\Omega$ and

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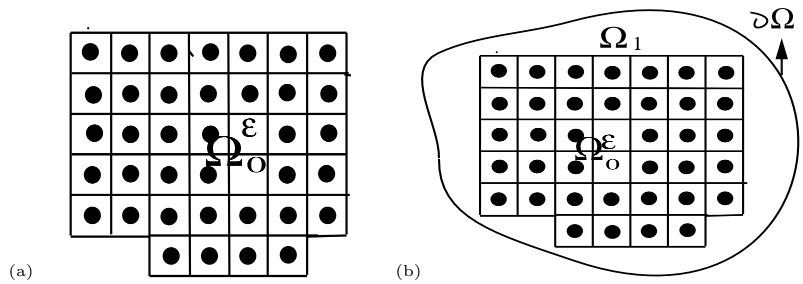


FIGURE 1. : (a) interior domain Ω_0^ε; (b) boundary layer.

the surfaces S_ε of cavities, where $\partial\Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_2, \Gamma_1 \cap \Gamma_2 = \emptyset$. Such a domain Ω^ε is called as a type-II domain (see, [22]).

In equations (1), $u^\varepsilon(x) = (u_1^\varepsilon(x), \cdots, u_n^\varepsilon(x))^T$ denotes a displacement function, $f(x) = (f_1(x), \cdots, f_n(x))^T$ is a body force, $g_0(x)$ is a given displacement function on the Dirichlet boundary Γ_1, $g_1(x)$ is a given surface stress on the Neumann boundary Γ_2, $\sigma_\varepsilon(u^\varepsilon) = (\sigma_{\varepsilon,1}(u^\varepsilon), \cdots, \sigma_{\varepsilon,n}(u^\varepsilon))$, $\sigma_{\varepsilon,i}(u^\varepsilon) \equiv \nu_j a_{ijhk}^\varepsilon \dfrac{\partial u_h^\varepsilon}{\partial x_k}$, $i = 1, \cdots, n$, where $\vec{n} = (\nu_1, \cdots, \nu_n)$ is the unit outer normal vector to $\partial\Omega^\varepsilon = \partial\Omega \cup S_\varepsilon$.

Suppose that

(A_1) Let $\xi = \varepsilon^{-1}x$, and the elements of a matrix $(a_{ijkh}(\xi))$ are 1-periodic functions in ξ.

(A_2) $a_{ijkh}(\xi) = a_{jikh}(\xi) = a_{khij}(\xi)$.

(A_3) $\gamma_0\eta_{ij}\eta_{ij} \le a_{ijkh}(\xi)\eta_{ij}\eta_{kh} \le \gamma_1\eta_{ij}\eta_{ij},\ \xi \in \omega,\ \gamma_0, \gamma_1 > 0,$ where (η_{ij}) is any real symmetric matrix.

(A_4) $a_{ijkh} \in L^\infty(\omega)$, $f \in L^2(\Omega^\varepsilon)$, $g_0 \in H^{\frac{1}{2}}(\Gamma_1)$, $g_1 \in L^2(\Gamma_2)$.

Remark 1.1. *Existence and uniqueness of the solution to problem (1) can be established on the basis of the assumptions* $(B_1) - (B_3)$*, and* $(A_1) - (A_4)$ *(see, e.g.* [22]*).*

The numerous studies of homogenization and its applications for problem (1) in perforated domains containing many small holes have been developed by so many contributions that it is impossible to quote them all. We refer the interested reader to these books and articles (see, e.g. [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 14, 16, 17, 18, 19, 20, 21, 22]).

When we solve the elastic equations with homogeneous Neumann boundary conditions on the surfaces of holes in a perforated domain, we usually fill these holes with an almost degenerated phase, which is also called the hole-filling method. Actually, engineers often use the method to predict the effective properties of perforated materials. From a physical point of view, when the material properties of the weak phase go to zero, this limit procedure is clear. But a full mathematical justification has not been seen in all the available literature. In this paper, we try to give a full mathematical justification for this limiting process in general cases. Furthermore, in order to compute the displacement and the stress field in a domain, we present a uniform multiscale method for solving the elastic equations (1) regardless of whether there are holes or not. The crucial step of the method is to define the cell functions which are different from those of classical homogenization method. On the other hand, from the viewpoint of numerical computation, the mesh generation in a 3-D perforated domain is somehow more difficult than that in a 3-D nonperforated domain. By means of the method of the paper, we only do the mesh generation in a 3-D nonperforated domain where all holes have been filled with an almost degenerated phase. It has a wide range applications in mechanical and engineering problems, in particular, in the shape optimal design of composite materials.

The remainder of this paper is organized as follows. In Section 2, we introduce the hole-filling method, and derive the rigorous proof of a convergence result for this method. In Section 3, we discuss the homogenization method and the multiscale asymptotic expansions for the original problem in a perforated domain and the associated multiphase problem in a domain without holes, respectively. We try to give the full mathematical justification for the hole-filling method. Finally, we show some numerical examples, which validate the theoretical results presented in the previous sections.

Denote uniformly by C the positive constant independent of ε, δ without distinction. For convenience, we use the Einstein summation convention on repeated indices.

2. The Hole-Filling Method and the Convergence Theorem

In this section, we first give the definition of the hole-filling method in a perforated domain, and then we present a full mathematical justification for this limiting process ,i.e. the weak material properties go to zero in the multiphase material.

Definition 2.1. Let Ω^ε be a type II perforated domain given in (1). If we fill all holes of Ω^ε with a very compliant material such that Ω^ε is changed into a nonperforated domain Ω, then it is called as the hole-filling method.

We now introduce some notation: $Q = \{\xi,\, 0 < \xi_j < 1,\, j = 1, \cdots, n\}$, the periodic cell $Q \cap \omega$, $\delta > 0$ is a sufficiently small constant, $\mathcal{V}_\delta = \{\xi \in (Q \backslash \omega),\, dist(\xi, \partial\omega) \le \delta\}$, $\mathcal{V}_0 = \{\xi \in (Q \setminus \omega),\, dist(\xi, \partial\omega) \ge \delta\}$, where $dist(A, B)$ denotes a distance between A and B(see Fig.2).

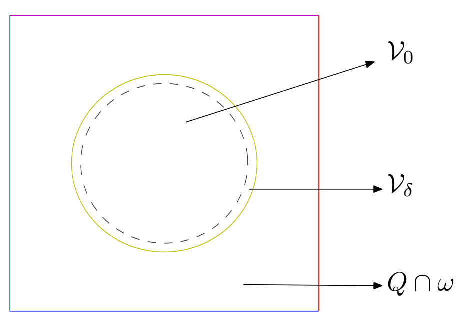


FIGURE 2. $\mathcal{V}_0$ and $\mathcal{V}_\delta$

Let $\xi = \varepsilon^{-1}x$, and define $a^*_{ijkh}(\frac{x}{\varepsilon}) = a^*_{ijkh}(\xi)$ in the following form:

$$a^*_{ijkh}(\xi) = \begin{cases} a_{ijkh}(\xi), & \xi \in Q \cap \omega \\ \phi^{\delta}_{ijkh}(\xi), & \xi \in \mathcal{V}_\delta \\ \delta^{\frac{1}{2}}\delta_{ik}\delta_{jh}, & \xi \in \mathcal{V}_0 \end{cases} \tag{2}$$

where $a_{ijkh}(\xi)$, $\xi \in Q \cap \omega$ is as given in (1), $\phi^{\delta}_{ijkh}(\xi) \in C^\infty(\mathcal{V}_\delta)$, $\delta^{1/2} \le \|\phi^{\delta}_{ijkh}\|_{L^\infty(\mathcal{V}_\delta)} \le M$, δ_{ij} is the Kronecker symbol.

If we assume that a tensor $(\phi^{\delta}_{ijkh}(\xi))$, $\xi \in (Q \setminus \omega)$ satisfies Conditions (A_1) and (A_2), then a tensor $(a^*_{ijkh}(\xi))$, $\xi \in Q$ has the similar properties. In this situation, we define $u^{\varepsilon,*}(x)$ is the solution of the following boundary value problems of elastic equations in a domain Ω without holes:

$$\text{(3)} \quad \begin{cases} -\dfrac{\partial}{\partial x_j}(a^*_{ijkh}(\frac{x}{\varepsilon})\dfrac{\partial u_k^{\varepsilon,*}(x)}{\partial x_h}) = \eta_\varepsilon(x) f_i(x), \quad i = 1, 2, \cdots, n, \text{ in } \quad \Omega \\ u^{\varepsilon,*}(x) = g_0(x), \quad \text{on} \quad \Gamma_1 \\ \sigma^*_\varepsilon(u^{\varepsilon,*}) = g_1(x), \quad \text{on} \quad \Gamma_2 \end{cases}$$

where

$$\text{(4)} \qquad \eta_\varepsilon(x) = \eta(\frac{x}{\varepsilon}) = \begin{cases} 1 & x \in \Omega^\varepsilon \\ 0 & x \in \Omega \setminus \bar{\Omega}^\varepsilon. \end{cases}$$

Since Ω^ε is a type-II perforated domain, i.e. $\overline{\Omega}^\varepsilon = \overline{\Omega}_0^\varepsilon \cup \Omega_1$, we thus have $\sigma^*_\varepsilon(u^{\varepsilon,*}) \equiv \sigma_\varepsilon(u^{\varepsilon,*})$, where $\sigma_\varepsilon(u)$ is as given in (1).

Our goal of this section is to give the estimate for $\|u^\varepsilon - u^{\varepsilon,*}\|_{H^1(\Omega^\varepsilon)}$. To this end, we first introduce Lemma 2.1 given by:

Lemma 2.1. *Suppose that Ω^ε is a type-II perforated domain as given in (1) and the boundary $S_\varepsilon \in C^1$, then there exists an extension operator such that*

$$P_\varepsilon : W^{1,\infty}(\Omega^\varepsilon) \longrightarrow W^{1,\infty}(\Omega)$$

and for any $v \in W^{1,\infty}(\Omega^\varepsilon)$,

$$P_\varepsilon v = v \quad a.e. \ in \ \Omega^\varepsilon,$$
$$\text{(5)} \qquad \|P_\varepsilon v\|_{W^{1,\infty}(\Omega)} \le C\|v\|_{W^{1,\infty}(\Omega^\varepsilon)}.$$

where C is a constant independent of ε.

Proof. In order to build an extension operator P_ε on Ω, it is sufficient to know how to build it on an arbitrary cell $\varepsilon(k+Q)$, $k \in Z^n$.

Each $x \in \Omega^\varepsilon$ can be written as

$$x = \varepsilon(k+\xi), \text{ where } \xi \in Q \cap \omega, k \in Z^n.$$

Set

$$\text{(6)} \qquad v_{\varepsilon,k}(\xi) = v(\varepsilon(k+\xi)) = v(x).$$

This function is defined on $Q \cap \omega$, and moreover, $v_{\varepsilon,k} \in W^{1,\infty}(Q \cap \omega)$. Following the lines of the proof of Theorem 1 of ([12],p.254), we can extend $v_{\varepsilon,k}$ to the whole of Q by $Pv_{\varepsilon,k}$. Due to definition (6), this clearly defines an extension on the cell $\varepsilon(k+Q)$. As the holes do not intersect the boundary of the cells, there is no trace problem when passing form a cell to any adjacent one, and we can define an extension to the whole of Ω by setting

$$(P_\varepsilon v)(x) = (Pv_{\varepsilon,k})(\frac{x - \varepsilon k}{\varepsilon}), \quad \forall x \in \Omega, \quad x \in \varepsilon(k+Q), \quad k \in Z^n.$$

Moreover,

$$\text{(7)} \qquad \begin{aligned} \|P_\varepsilon v\|_{W^{1,\infty}(\varepsilon(k+Q))} &= \|Pv_{\varepsilon,k}\|_{W^{1,\infty}(Q)} \\ &\le C\|v_{\varepsilon,k}\|_{W^{1,\infty}(Q\cap\omega)} = C\|v\|_{W^{1,\infty}(\varepsilon(k+Q\cap\omega))}. \end{aligned}$$

Since Ω^ε is of type II, due to (7), whether the essential supremum of v is take in the boundary layer or not, we can get

$$\|P_\varepsilon v\|_{W^{1,\infty}(\Omega)} \le C\|v\|_{W^{1,\infty}(\Omega^\varepsilon)}.$$

Next we give the convergence result involved in the hole-filling method.

Theorem 2.1. *Suppose that $\Omega^\varepsilon \subset R^n$ is a type-II perforated domain, the boundary $S_\varepsilon \in C^1$ and $\Omega \subset R^n$ is the associated Lipschitz convex domain without holes. Let*

u^ε be the weak solution of problem (1) in Ω^ε, and $u^{\varepsilon,*}$ be the weak solution of the corresponding multiphase problem (3) in Ω. If $u^\varepsilon \in W^{1,\infty}(\Omega^\varepsilon)$, then there holds

$$\|u^\varepsilon - u^{\varepsilon,*}\|_{H^1(\Omega^\varepsilon)} \leq C\delta^{\frac{1}{4}}\varepsilon^{-n+\frac{1}{2}}\|u^\varepsilon\|_{W^{1,\infty}(\Omega^\varepsilon)}. \tag{8}$$

Furthermore, we can choose a sufficiently small $\delta > 0$ such that

$$\|u^\varepsilon - u^{\varepsilon,*}\|_{H^1(\Omega^\varepsilon)} \leq C\varepsilon^{\frac{1}{2}} \tag{9}$$

where C is a constant independent of ε, $\delta > 0$. In particular, we are interested in the asymptotic behaviour of the solution $u^{\varepsilon,*}(x)$, as $\delta \to 0$.

Proof. Thanks to Lemma 2.1, there exists an extension operator

$$P_\varepsilon : W^{1,\infty}(\Omega^\varepsilon) \longrightarrow W^{1,\infty}(\Omega)$$

such that

$$P_\varepsilon u^\varepsilon = u^\varepsilon \quad \text{a.e. in } \Omega^\varepsilon,$$
$$\|P_\varepsilon u^\varepsilon\|_{W^{1,\infty}(\Omega)} \leq C\|u^\varepsilon\|_{W^{1,\infty}(\Omega^\varepsilon)}.$$

Let $\widetilde{u}^\varepsilon = P_\varepsilon u^\varepsilon$, and $\tilde{v}_i(x) = (u_i^{\varepsilon,*}(x) - \widetilde{u}_i^\varepsilon(x)) \in H^1(\Omega, \Gamma_1)$, where $H^1(\Omega^\varepsilon, \Gamma_1) = \{v \in H^1(\Omega^\varepsilon),\ v|_{\Gamma_1} = 0\}$. If we take $\tilde{v}_i = (u_i^{\varepsilon,*} - \widetilde{u}_i^\varepsilon)$ as test functions, then we obtain the variational forms for (1) and (3) given by:

$$\int_\Omega a^*_{ijkh}(\frac{x}{\varepsilon})\frac{\partial u_k^{\varepsilon,*}}{\partial x_h}\frac{\partial \tilde{v}_i}{\partial x_j}dx = \int_\Omega \eta_\varepsilon(x) f_i \tilde{v}_i dx + \int_{\Gamma_2} g_{1,i}(x)\tilde{v}_i d\Gamma, \tag{10}$$

$$\int_{\Omega^\varepsilon} a_{ijkh}(\frac{x}{\varepsilon})\frac{\partial u_k^{\varepsilon}}{\partial x_h}\frac{\partial \tilde{v}_i}{\partial x_j}dx = \int_{\Omega^\varepsilon} f_i \tilde{v}_i dx + \int_{\Gamma_2} g_{1,i}(x)\tilde{v}_i d\Gamma, \tag{11}$$

where $g_1(x) = (g_{1,1}(x), \cdots, g_{1,n}(x))^T$ is as given in (1).

Subtracting (10) from (11) gives

$$\begin{aligned}
0 \quad &\leq \int_{\Omega^\varepsilon} a_{ijkh}(\frac{x}{\varepsilon})\frac{\partial(u_k^{\varepsilon,*} - \widetilde{u}_k^\varepsilon)}{\partial x_h}\frac{\partial(u_i^{\varepsilon,*} - \widetilde{u}_i^\varepsilon)}{\partial x_j}dx \\
&= -\int_{\Omega\setminus\bar{\Omega}^\varepsilon} a^*_{ijkh}(\frac{x}{\varepsilon})\frac{\partial u_k^{\varepsilon,*}}{\partial x_h}\frac{\partial(u_i^{\varepsilon,*} - \widetilde{u}_i^\varepsilon)}{\partial x_j}dx \\
&= -\int_{\Omega\setminus\bar{\Omega}^\varepsilon} a^*_{ijkh}(\frac{x}{\varepsilon})\frac{\partial(u_k^{\varepsilon,*} - \widetilde{u}_k^\varepsilon)}{\partial x_h}\frac{\partial(u_i^{\varepsilon,*} - \widetilde{u}_i^\varepsilon)}{\partial x_j}dx \\
&\quad - \int_{\Omega\setminus\bar{\Omega}^\varepsilon} a^*_{ijkh}(\frac{x}{\varepsilon})\frac{\partial \widetilde{u}_k^\varepsilon}{\partial x_h}\frac{\partial(u_i^{\varepsilon,*} - \widetilde{u}_i^\varepsilon)}{\partial x_j}dx \\
&= I + I^\delta + I^0,
\end{aligned} \tag{12}$$

where

$$I = -\int_{\Omega\setminus\bar{\Omega}^\varepsilon} a^*_{ijkh}(\frac{x}{\varepsilon})\frac{\partial(u_k^{\varepsilon,*} - \widetilde{u}_k^\varepsilon)}{\partial x_h}\frac{\partial(u_i^{\varepsilon,*} - \widetilde{u}_i^\varepsilon)}{\partial x_j}dx \leq 0, \tag{13}$$

$$I^\delta = -\int_{V_\delta} a^*_{ijkh}(\frac{x}{\varepsilon})\frac{\partial \widetilde{u}_k^\varepsilon}{\partial x_h}\frac{\partial(u_i^{\varepsilon,*} - \widetilde{u}_i^\varepsilon)}{\partial x_j}dx, \tag{14}$$

$$I^0 = -\int_{V_0} a^*_{ijkh}(\frac{x}{\varepsilon})\frac{\partial \widetilde{u}_k^\varepsilon}{\partial x_h}\frac{\partial(u_i^{\varepsilon,*} - \widetilde{u}_i^\varepsilon)}{\partial x_j}dx, \tag{15}$$

and $V_\delta = \bigcup\limits_{z\in T_\varepsilon}(z + \mathcal{V}_\delta)$, $V_0 = \bigcup\limits_{z\in T_\varepsilon}(z + \mathcal{V}_0)$, $\mathcal{V}_\delta$, $\mathcal{V}_0$ are as given in (2).

Using Lemma 2.1, we obtain

$$\begin{aligned}|I_z^\delta| &= |\int_{(z+\mathcal{V}_\delta)} a^*_{ijkh}(\frac{x}{\varepsilon})\frac{\partial \widetilde{u}^\varepsilon_k}{\partial x_h}\frac{\partial(u_i^{\varepsilon,*}-\widetilde{u}_i^\varepsilon)}{\partial x_j}dx| \\ &\le C\|\nabla\widetilde{u}^\varepsilon\|_{L^2(z+\mathcal{V}_\delta)}\|\nabla(u^{\varepsilon,*}-\widetilde{u}^\varepsilon)\|_{L^2(z+\mathcal{V}_\delta)} \\ &\le C(\varepsilon\delta)^{\frac12}\|\nabla\widetilde{u}^\varepsilon\|_{L^\infty(z+\mathcal{V}_\delta)}\|\nabla(u^{\varepsilon,*}-\widetilde{u}^\varepsilon)\|_{L^2(z+\mathcal{V}_\delta)} \\ &\le C(\varepsilon\delta)^{\frac12}\|u^\varepsilon\|_{W^{1,\infty}(z+Q\cap\omega)}\|\nabla(u^{\varepsilon,*}-\widetilde{u}^\varepsilon)\|_{L^2(z+\mathcal{V}_\delta)},\end{aligned}$$

and consequently

$$\begin{aligned}|I^\delta| &= |\int_{\bigcup_{z\in T_\varepsilon}(z+\mathcal{V}_\delta)} a^*_{ijkh}(\frac{x}{\varepsilon})\frac{\partial \widetilde{u}^\varepsilon_k}{\partial x_h}\frac{\partial(u_i^{\varepsilon,*}-\widetilde{u}_i^\varepsilon)}{\partial x_j}dx| \\ &\le \sum_{z\in T_\varepsilon}|I_z^\delta| \le C(\varepsilon\delta)^{\frac12}\sum_{z\in T_\varepsilon}\|u^\varepsilon\|_{W^{1,\infty}(z+Q\cap\omega)}\|\nabla(u^{\varepsilon,*}-\widetilde{u}^\varepsilon)\|_{L^2(z+\mathcal{V}_\delta)} \\ &\le C(\varepsilon\delta)^{\frac12}\varepsilon^{-n}\|u^\varepsilon\|_{W^{1,\infty}(\Omega^\varepsilon)}\|\nabla(u^{\varepsilon,*}-\widetilde{u}^\varepsilon)\|_{L^2(V_\delta)},\end{aligned}$$

where $V_\delta = \bigcup_{z\in T_\varepsilon}(z+\mathcal{V}_\delta)$.

Similarly, we have

$$\begin{aligned}|I^0| &= |\int_{V_0} a^*_{ijkh}(\frac{x}{\varepsilon})\frac{\partial \widetilde{u}^\varepsilon_k}{\partial x_h}\frac{\partial(u_i^{\varepsilon,*}-\widetilde{u}_i^\varepsilon)}{\partial x_j}dx| \\ &\le C\delta^{\frac12}\|\nabla\widetilde{u}^\varepsilon\|_{L^2(V_0)}\|\nabla(u^{\varepsilon,*}-\widetilde{u}^\varepsilon)\|_{L^2(V_0)} \\ &\le C\delta^{\frac12}\|u^\varepsilon\|_{W^{1,\infty}(\Omega^\varepsilon)}\|\nabla(u^{\varepsilon,*}-\widetilde{u}^\varepsilon)\|_{L^2(V_0)},\end{aligned}$$

where $V_0 = \bigcup_{z\in T_\varepsilon}(z+\mathcal{V}_0)$.

It follows from the Korn's inequality and (12) that

$$\begin{aligned}&\delta^{\frac12}\|\nabla(u^{\varepsilon,*}-\widetilde{u}^\varepsilon)\|^2_{L^2(\Omega)} \\ &\le \int_\Omega a^*_{ijkh}(\frac{x}{\varepsilon})\frac{\partial(u_k^{\varepsilon,*}-\widetilde{u}_k^\varepsilon)}{\partial x_h}\frac{\partial(u_i^{\varepsilon,*}-\widetilde{u}_i^\varepsilon)}{\partial x_j}dx \le |I^\delta|+|I^0| \\ &\le C\delta^{\frac12}\varepsilon^{-n+\frac12}\|u^\varepsilon\|_{W^{1,\infty}(\Omega^\varepsilon)}\|\nabla(u^{\varepsilon,*}-\widetilde{u}^\varepsilon)\|_{L^2(\Omega)},\end{aligned}$$

i.e.

$$\|\nabla(u^{\varepsilon,*}-\widetilde{u}^\varepsilon)\|_{L^2(\Omega)} \le C\varepsilon^{-n+\frac12}\|u^\varepsilon\|_{W^{1,\infty}(\Omega^\varepsilon)}. \tag{16}$$

We thus obtain

$$|I^\delta| \le C\delta^{\frac12}\varepsilon^{-2n+1}\|u^\varepsilon\|^2_{W^{1,\infty}(\Omega^\varepsilon)},$$

$$|I^0| \le C\delta^{\frac12}\varepsilon^{-n+\frac12}\|u^\varepsilon\|^2_{W^{1,\infty}(\Omega^\varepsilon)}, \tag{17}$$

and

$$|I| \le |I^\delta|+|I^0| \le C\delta^{\frac12}\varepsilon^{-2n+1}\|u^\varepsilon\|^2_{W^{1,\infty}(\Omega^\varepsilon)}. \tag{18}$$

Thanks to the Korn's inequalities (see, e.g. [22]) and (12), we obtain

$$\begin{aligned}\gamma_0\|u^{\varepsilon,*}-\widetilde{u}^\varepsilon\|^2_{H^1(\Omega^\varepsilon)} &\le \int_{\Omega^\varepsilon} a_{ijkh}(\frac{x}{\varepsilon})\frac{\partial(u_k^{\varepsilon,*}-\widetilde{u}_k^\varepsilon)}{\partial x_h}\frac{\partial(u_i^{\varepsilon,*}-\widetilde{u}_i^\varepsilon)}{\partial x_j}dx \\ &\le C\delta^{\frac12}\varepsilon^{-2n+1}\|u^\varepsilon\|^2_{W^{1,\infty}(\Omega^\varepsilon)}\end{aligned}$$

i.e.

$$\|u^{\varepsilon,*}-\widetilde{u}^\varepsilon\|_{H^1(\Omega^\varepsilon)} \le C\delta^{\frac14}\varepsilon^{-n+\frac12}\|u^\varepsilon\|_{W^{1,\infty}(\Omega^\varepsilon)}. \tag{19}$$

Therefore we complete the proof of Theorem 2.1.

3. The Uniform Multiscale Method and the Associated Properties

In this section, we first introduce some known results of the homogenization method and the multiscale asymptotic expansions of the solution for problem (1) in a perforated domain Ω^ε without any justification(see, e.g [22]). Second, we give similarly some results of homogenization and the multiscale asymptotic expansions of the solution for the multiphase problem (3) in a Lipschitz convex domain Ω. Third, we discuss the relationship between two types of asymptotic methods, and obtain some convergence theorems when one material properties go to zero, i.e, as $\delta \to 0$. They are the main results obtained in this paper, and form the basis of the uniform multiscale numerical method presented in the next section.

Set formally:

$$\begin{aligned} U_1^\varepsilon(x) &= u^0(x) + \varepsilon N_{m,\alpha_1}(\xi)\tfrac{\partial u_m^0(x)}{\partial x_{\alpha_1}} \\ U_2^\varepsilon(x) &= u^0(x) + \varepsilon N_{m,\alpha_1}(\xi)\tfrac{\partial u_m^0(x)}{\partial x_{\alpha_1}} + \varepsilon^2 N_{m,\alpha_1\alpha_2}(\xi)\tfrac{\partial^2 u_m^0(x)}{\partial x_{\alpha_1}\partial x_{\alpha_2}} \end{aligned} \tag{20}$$

where the vector-valued functions

$$\begin{aligned} &U_s^\varepsilon(x) = (U_{s,1}^\varepsilon(x), \cdots, U_{s,n}^\varepsilon(x))^T, \quad s = 1,2; \quad u^0(x) = (u_1^0(x), \cdots, u_n^0(x))^T; \\ &N_{m,\alpha_1}(\xi) = (N_{1m,\alpha_1}(\xi), \cdots, N_{nm,\alpha_1}(\xi))^T; \\ &N_{m,\alpha_1\alpha_2}(\xi) = (N_{1m,\alpha_1\alpha_2}(\xi), \cdots, N_{nm,\alpha_1\alpha_2}(\xi))^T, \quad m, \alpha_i = 1,2,\cdots,n. \end{aligned} \tag{21}$$

The cell functions $N_{m,\alpha_1}(\xi)$ and $N_{m,\alpha_1\alpha_2}(\xi)$ on $Q \cap \omega$ are defined in turn

$$\begin{cases} -\dfrac{\partial}{\partial \xi_j}\Big(a_{ijkh}(\xi)\dfrac{\partial N_{km,\alpha_1}(\xi)}{\partial \xi_h}\Big) = \dfrac{\partial}{\partial \xi_j}\Big(a_{ijm\alpha_1}(\xi)\Big) \quad \text{in} \quad Q\cap\omega \\ a_{ijkh}(\xi)\dfrac{\partial N_{km,\alpha_1}}{\partial \xi_h}\nu_j = -a_{ijm\alpha_1}(\xi)\nu_j \quad \text{on} \quad Q\cap\partial\omega \\ N_{km,\alpha_1}(\xi) \quad \text{is 1-periodic in } \xi, \quad \int_{Q\cap\omega} N_{km,\alpha_1}(\xi)d\xi = 0 \end{cases} \tag{22}$$

$$\begin{cases} -\dfrac{\partial}{\partial \xi_j}\Big(a_{ijkh}(\xi)\dfrac{\partial N_{km,\alpha_1\alpha_2}(\xi)}{\partial \xi_h}\Big) = \dfrac{\partial}{\partial \xi_j}\Big(a_{ijk\alpha_2}(\xi)N_{km,\alpha_1}(\xi)\Big) \\ +a_{i\alpha_2kh}(\xi)\dfrac{\partial N_{km,\alpha_1}(\xi)}{\partial \xi_h} + a_{i\alpha_2m\alpha_1}(\xi) - \widehat{a}_{i\alpha_2m\alpha_1} \quad \text{in} \quad Q\cap\omega \\ a_{ijkh}(\xi)\dfrac{\partial N_{km,\alpha_1\alpha_2}}{\partial \xi_h}\nu_j = -a_{ijk\alpha_2}(\xi)N_{km,\alpha_1}\nu_j \quad \text{on} \quad Q\cap\partial\omega \\ N_{km,\alpha_1\alpha_2}(\xi) \quad \text{is 1-periodic in } \xi, \quad \int_{Q\cap\omega} N_{km,\alpha_1\alpha_2}(\xi)d\xi = 0 \end{cases} \tag{23}$$

where $\vec{n} = (\nu_1, \cdots, \nu_n)$ denotes the unit outer normal to $Q \cap \partial\omega$, and the homogenized coefficients are the following:

$$\widehat{a}_{ijkh} = \frac{1}{|Q\cap\omega|}\int_{Q\cap\omega}(a_{ijkh}(\xi) + a_{ijml}(\xi)\frac{\partial N_{mk,h}(\xi)}{\partial \xi_l})d\xi \tag{24}$$

where $|Q \cap \omega|$ is the Lebesgue measure of $Q \cap \omega$.

Let $u^0(x)$ be the unique weak solution of the following homogenized equations:

$$\begin{cases} -\dfrac{\partial}{\partial x_j}(\widehat{a}_{ijkh}\dfrac{\partial u_k^0(x)}{\partial x_h}) = f_i(x), \quad i = 1,2,\cdots,n, \quad \text{in} \quad \Omega \\ u^0(x) = g_0(x), \quad \text{on} \quad \Gamma_1 \\ |Q\cap\omega|\widehat{\sigma}(u^0) = g_1(x), \quad \text{on} \quad \Gamma_2 \end{cases} \tag{25}$$

where $\widehat{\sigma}(u^0) = (\widehat{\sigma}_1(u^0), \cdots, \widehat{\sigma}_n(u^0))$, $\widehat{\sigma}_i(u^0) \equiv \nu_j \hat{a}_{ijkh}\dfrac{\partial u_k^0(x)}{\partial x_h}$, $i = 1, \cdots, n$.

Remark 3.1. *It is emphasized that second-order correctors $N_{m,\alpha_1\alpha_2}(\xi)$ are very important to computing the displacement and the stress fields of composite structures. We refer the interested reader to Section 4.*

Lemma 3.1.[22] *Suppose that Ω^ε is a type-II perforated domain. Let $u^\varepsilon(x)$ be the unique weak solution of problem(1), and $U_1^\varepsilon(x)$, $U_2^\varepsilon(x)$ be as given in (20). Under the assumptions $(B_1)-(B_3)$, $(A_1)-(A_4)$, one obtains the following error estimates:*

$$\|u^\varepsilon - U_s^\varepsilon\|_{H^1(\Omega^\varepsilon)} \le C\varepsilon^{\frac{1}{2}}, \quad s = 1, 2 \tag{26}$$

where C is a constant independent of ε.

For the extended problem (3), we set formally:

$$\begin{aligned} U_1^{\varepsilon,*}(x) &= u^{0,*}(x) + \varepsilon N_{m,\alpha_1}^*(\xi)\frac{\partial u_m^{0,*}(x)}{\partial x_{\alpha_1}} \\ U_2^{\varepsilon,*}(x) &= u^{0,*}(x) + \varepsilon N_{m,\alpha_1}^*(\xi)\frac{\partial u_m^{0,*}(x)}{\partial x_{\alpha_1}} + \varepsilon^2 N_{m,\alpha_1\alpha_2}^*(\xi)\frac{\partial^2 u_m^{0,*}(x)}{\partial x_{\alpha_1}\partial x_{\alpha_2}} \end{aligned} \tag{27}$$

where the vector-valued functions

(28)

$$\begin{aligned} &U_s^{\varepsilon,*}(x) = (U_{s,1}^{\varepsilon,*}(x), \cdots, U_{s,n}^{\varepsilon,*}(x))^T, \quad s = 1, 2; \quad u^{0,*}(x) = (u_1^{0,*}(x), \cdots, u_n^{0,*}(x))^T; \\ &N_{m,\alpha_1}^*(\xi) = (N_{1m,\alpha_1}^*(\xi), \cdots, N_{nm,\alpha_1}^*(\xi))^T; \\ &N_{m,\alpha_1\alpha_2}^*(\xi) = (N_{1m,\alpha_1\alpha_2}^*(\xi), \cdots, N_{nm,\alpha_1\alpha_2}^*(\xi))^T. \end{aligned}$$

Similarly, we obtain the homogenized equations associated with problem (3) given by:

$$\begin{cases} -\dfrac{\partial}{\partial x_j}(\widehat{a}_{ijkh}^*\dfrac{\partial u_k^{0,*}(x)}{\partial x_h}) = (\frac{|Q\cap\omega|}{|Q|})f_i(x), \quad i = 1, 2, \cdots, n, \quad \text{in} \quad \Omega \\ u^{0,*}(x) = g_0(x), \quad \text{on} \quad \Gamma_1 \\ \hat{\sigma}^*(u^{0,*}) = g_1(x), \quad \text{on} \quad \Gamma_2 \end{cases} \tag{29}$$

where $\hat{\sigma}^*(u^{0,*}) = (\hat{\sigma}_1^*(u^{0,*}), \cdots, \widehat{\sigma}_n^*(u^{0,*}))$, $\hat{\sigma}_i^*(u^{0,*}) \equiv \nu_j\widehat{a}_{ijkh}^*\dfrac{\partial u_k^{0,*}(x)}{\partial x_h}$, $\vec{n} = (\nu_1, \cdots, \nu_n)$ is the unit outer normal vector to $\partial\Omega$, and

$$\widehat{a}_{ijkh}^* = \frac{1}{|Q|}\int_Q (a_{ijkh}^*(\xi) + a_{ijml}^*(\xi)\frac{\partial N_{mk,h}^*(\xi)}{\partial \xi_l})d\xi. \tag{30}$$

The cell functions $N_{m,\alpha_1}^*(\xi)$, $N_{m,\alpha_1\alpha_2}^*(\xi)$ in unit cell Q are given by:

$$\begin{cases} -\dfrac{\partial}{\partial \xi_j}\Big(a_{ijkh}^*(\xi)\dfrac{\partial N_{km,\alpha_1}^*(\xi)}{\partial \xi_h}\Big) = \dfrac{\partial}{\partial \xi_j}\Big(a_{ijm\alpha_1}^*(\xi)\Big) \quad \text{in} \quad Q \\ N_{km,\alpha_1}^*(\xi) \quad \text{is 1-periodic in } \xi, \quad \int_Q N_{km,\alpha_1}^*(\xi)d\xi = 0 \end{cases} \tag{31}$$

$$\begin{cases} -\dfrac{\partial}{\partial \xi_j}\Big(a_{ijkh}^*(\xi)\dfrac{\partial N_{km,\alpha_1\alpha_2}^*(\xi)}{\partial \xi_h}\Big) = \dfrac{\partial}{\partial \xi_j}\Big(a_{ijk\alpha_2}^*(\xi)N_{km,\alpha_1}^*(\xi)\Big) \\ +a_{i\alpha_2 kh}^*(\xi)\dfrac{\partial N_{km,\alpha_1}^*(\xi)}{\partial \xi_h} + a_{i\alpha_2 m\alpha_1}^*(\xi) - \dfrac{\eta(\xi)|Q|}{|Q\cap\omega|}\widehat{a}_{i\alpha_2 m\alpha_1}^* \quad \text{in} \quad Q \\ N_{km,\alpha_1\alpha_2}^*(\xi) \quad \text{is 1-periodic in } \xi, \quad \int_Q N_{km,\alpha_1\alpha_2}^*(\xi)d\xi = 0 \end{cases} \tag{32}$$

where $\xi = \varepsilon^{-1}x$, $\eta(\xi) = \eta(\frac{x}{\varepsilon})$ is as given in (4).

Remark 3.2. *The crucial idea of the hole-filling method presented in Section 2 is to consider the boundary value problems in a nonperforated domain Ω regardless of whether there are holes or not in an original problem. If Ω^ε is a type-II perforated domain containing many small holes, then we fill these holes with a very compliant material, see problem (3). Now we discuss the relationship between the homogenized coefficients of problem (1) and those of problem (3), and give a full mathematical justification.*

Theorem 3.1. *Suppose that ω is an unbounded domain of R^n which satisfies Conditions $(B_1)-(B_3)$, the boundaries of holes $\partial\omega \in C^1$, and a matrix (a^{ε}_{ijkh}) satisfies Conditions $(A_1)-(A_3)$, $a^{\varepsilon}_{ijkh} \in W^{1,\infty}(\omega)$. Let $\widehat{a}_{ijkh}$ and $\widehat{a}^*_{ijkh}$ denote the associated coefficients (24) and (30) for the homogenized equations, respectively. If the cell functions $N_{m,\alpha_1} \in W^{1,\infty}(Q\cap\omega)$, $\alpha_1 = 1,\cdots,n$, $n \geq 3$, then there holds*

$$|\widehat{a}^*_{ijkh} - |Q\cap\omega|\widehat{a}_{ijkh}| \leq C\delta^{\frac{1}{4}}. \tag{33}$$

where C is a constant independent of $\delta > 0$, $\delta > 0$ is a sufficiently small number given by (2) and $|Q\cap\omega|$ is the Lebesgue measure of $Q\cap\omega$.

Proof. Thanks to Theorem 4.3 of ([2],p.44), under the assumptions of $(A_1)-(A_3)$ and $a^{\varepsilon}_{ijkh} \in W^{1,\infty}(\omega)$, if $n=2$, then we can deduce that the cell functions $N_{m,\alpha_1} \in W^{1,\infty}(Q\cap\omega)$, $\alpha_1 = 1,\cdots,n$. But we need to add the assumption $N_{m,\alpha_1} \in W^{1,\infty}(Q\cap\omega)$, $\alpha_1 = 1,\cdots,n$ for $n \geq 3$.

Under the assumptions of Conditions $(B_1)-(B_3)$ and $\partial\omega \in C^1$, it follows from Lemma 2.1 that there exists an extension operator

$$P : W^{1,\infty}(Q\cap\omega) \longrightarrow W^{1,\infty}(Q)$$

such that

$$\begin{aligned} &PN_{m,\alpha_1} = N_{m,\alpha_1} \quad \text{a.e. in } Q\cap\omega, \\ &\|PN_{m,\alpha_1}\|_{W^{1,\infty}(Q)} \leq C\|N_{m,\alpha_1}\|_{W^{1,\infty}(Q\cap\omega)}. \end{aligned}$$

We now introduce some notation:

$$\begin{aligned} &M_Y(\psi) = \frac{1}{|Y|}\int_Y \psi(\xi)d\xi, \\ &W^1_{per}(Y) = \{\psi \in H^1(Y) : \psi \text{ is 1-periodic and } \int_Y \psi(\xi)d\xi = 0\}. \end{aligned}$$

In fact, here we take $Y = Q$ or $Y = Q\cap\omega$.

Let $\widetilde{N}_{m,\alpha_1} = PN_{m,\alpha_1}$, and

$$\begin{aligned} &v_1(\xi) = N^*_{mk,h} - \widetilde{N}_{mk,h} - M_Q(N^*_{mk,h} - \widetilde{N}_{mk,h}) \in W^1_{per}(Q), \\ &v_2(\xi) = N^*_{mk,h} - \widetilde{N}_{mk,h} - M_{Q\cap\omega}(N^*_{mk,h} - \widetilde{N}_{mk,h}) \in W^1_{per}(Q\cap\omega). \end{aligned}$$

We take $v_1(\xi)$ and $v_2(\xi)$ as test functions for (22) and (31), respectively, then we obtain

$$\begin{aligned} &\int_Q a^*_{ijkh}\frac{\partial N^*_{km,\alpha_1}}{\partial\xi_h}\frac{\partial(N^*_{im,\alpha_1} - \widetilde{N}_{im,\alpha_1})}{\partial\xi_j}d\xi \\ = \quad & -\int_Q a^*_{ijm\alpha_1}\frac{\partial(N^*_{im,\alpha_1} - \widetilde{N}_{im,\alpha_1})}{\partial\xi_j}d\xi \end{aligned} \tag{34}$$

and

$$\begin{aligned} &\int_{Q\cap\omega} a_{ijkh}\frac{\partial \widetilde{N}_{km,\alpha_1}}{\partial\xi_h}\frac{\partial(N^*_{im,\alpha_1} - \widetilde{N}_{im,\alpha_1})}{\partial\xi_j}d\xi \\ = \quad & -\int_{Q\cap\omega} a_{ijm\alpha_1}\frac{\partial(N^*_{im,\alpha_1} - \widetilde{N}_{im,\alpha_1})}{\partial\xi_j}d\xi. \end{aligned} \tag{35}$$

Subtracting (34) from (35), we have

$$
\begin{aligned}
0 \leq \quad & \int_{Q\cap\omega} a_{ijkh} \frac{\partial(N^*_{km,\alpha_1} - \widetilde{N}_{km,\alpha_1})}{\partial\xi_h} \frac{\partial(N^*_{im,\alpha_1} - \widetilde{N}_{im,\alpha_1})}{\partial\xi_j} d\xi \\
& = -\int_{Q\setminus\bar{\omega}} a^*_{ijkh} \frac{\partial N^*_{km,\alpha_1}}{\partial\xi_h} \frac{\partial(N^*_{im,\alpha_1} - \widetilde{N}_{im,\alpha_1})}{\partial\xi_j} d\xi \\
& \quad - \int_{Q\setminus\bar{\omega}} a^*_{ijm\alpha_1} \frac{\partial(N^*_{im,\alpha_1} - \widetilde{N}_{im,\alpha_1})}{\partial\xi_j} d\xi \\
& = -\int_{Q\setminus\bar{\omega}} a^*_{ijkh} \frac{\partial(N^*_{km,\alpha_1} - \widetilde{N}_{km,\alpha_1})}{\partial\xi_h} \frac{\partial(N^*_{im,\alpha_1} - \widetilde{N}_{im,\alpha_1})}{\partial\xi_j} d\xi \\
& \quad - \int_{Q\setminus\bar{\omega}} a^*_{ijkh} \frac{\partial\widetilde{N}_{km,\alpha_1}}{\partial\xi_h} \frac{\partial(N^*_{im,\alpha_1} - \widetilde{N}_{im,\alpha_1})}{\partial\xi_j} d\xi \\
& \quad - \int_{Q\setminus\bar{\omega}} a^*_{ijm\alpha_1} \frac{\partial(N^*_{im,\alpha_1} - \widetilde{N}_{im,\alpha_1})}{\partial\xi_j} d\xi \\
& = I_3 + I_1^\delta + I_1^0 + I_2^\delta + I_2^0,
\end{aligned} \tag{36}
$$

where

$$
\begin{aligned}
I_1^\delta &= -\int_{\mathcal{V}_\delta} a^*_{ijkh} \frac{\partial\widetilde{N}_{km,\alpha_1}}{\partial\xi_h} \frac{\partial(N^*_{im,\alpha_1} - \widetilde{N}_{im,\alpha_1})}{\partial\xi_j} d\xi, \\
I_1^0 &= -\int_{\mathcal{V}_0} a^*_{ijkh} \frac{\partial\widetilde{N}_{km,\alpha_1}}{\partial\xi_h} \frac{\partial(N^*_{im,\alpha_1} - \widetilde{N}_{im,\alpha_1})}{\partial\xi_j} d\xi, \\
I_2^\delta &= -\int_{\mathcal{V}_\delta} a^*_{ijm\alpha_1} \frac{\partial(N^*_{im,\alpha_1} - \widetilde{N}_{im,\alpha_1})}{\partial\xi_j} d\xi, \\
I_2^0 &= -\int_{\mathcal{V}_0} a^*_{ijm\alpha_1} \frac{\partial(N^*_{im,\alpha_1} - \widetilde{N}_{im,\alpha_1})}{\partial\xi_j} d\xi, \\
I_3 &= -\int_{Q\setminus\bar{\omega}} a^*_{ijkh} \frac{\partial(N^*_{km,\alpha_1} - \widetilde{N}_{km,\alpha_1})}{\partial\xi_h} \frac{\partial(N^*_{im,\alpha_1} - \widetilde{N}_{im,\alpha_1})}{\partial\xi_j} d\xi \leq 0.
\end{aligned}
$$

Note that $Q \setminus \omega = \mathcal{V}_\delta \cup \mathcal{V}_0$, and $\mathcal{V}_\delta = \{\xi \in (Q \setminus \omega),\, dist(\xi, \partial\omega) \leq \delta\}$, $\mathcal{V}_0 = \{\xi \in (Q \setminus \omega),\, dist(\xi, \partial\omega) \geq \delta\}$.

Using the properties of an extension operator P , we get

$$
\begin{aligned}
|I_1^\delta| = & \left| \int_{\mathcal{V}_\delta} a^*_{ijkh} \frac{\partial\widetilde{N}_{km,\alpha_1}}{\partial\xi_h} \frac{\partial(N^*_{im,\alpha_1} - \widetilde{N}_{im,\alpha_1})}{\partial\xi_j} d\xi \right| \\
& \leq C\|e(\widetilde{N}_{m,\alpha_1})\|_{L^2(\mathcal{V}_\delta)} \|e(N^*_{m,\alpha_1} - \widetilde{N}_{m,\alpha_1})\|_{L^2(\mathcal{V}_\delta)} \\
& \leq C\|\nabla\widetilde{N}_{m,\alpha_1}\|_{L^2(\mathcal{V}_\delta)} \|\nabla(N^*_{m,\alpha_1} - \widetilde{N}_{m,\alpha_1})\|_{L^2(\mathcal{V}_\delta)} \\
& \leq C\delta^{\frac{1}{2}}\|\nabla\widetilde{N}_{m,\alpha_1}\|_{L^\infty(\mathcal{V}_\delta)} \|\nabla(N^*_{m,\alpha_1} - \widetilde{N}_{m,\alpha_1})\|_{L^2(\mathcal{V}_\delta)} \\
& \leq C\delta^{\frac{1}{2}}\|N_{m,\alpha_1}\|_{W^{1,\infty}(Q\cap\omega)} \|\nabla(N^*_{m,\alpha_1} - \widetilde{N}_{m,\alpha_1})\|_{L^2(\mathcal{V}_\delta)}
\end{aligned}
$$

and

$$
\begin{aligned}
|I_1^0| &= |\int_{\mathcal{V}_0} a^*_{ijkh} \frac{\partial \widetilde{N}_{km,\alpha_1}}{\partial \xi_h} \frac{\partial (N^*_{im,\alpha_1} - \widetilde{N}_{im,\alpha_1})}{\partial \xi_j} d\xi| \\
&\le \delta^{\frac{1}{2}} \|e(\widetilde{N}_{m,\alpha_1})\|_{L^2(\mathcal{V}_0)} \|e(N^*_{m,\alpha_1} - \widetilde{N}_{m,\alpha_1})\|_{L^2(\mathcal{V}_0)} \\
&\le C\delta^{\frac{1}{2}} \|\nabla \widetilde{N}_{m,\alpha_1}\|_{L^2(\mathcal{V}_0)} \|\nabla (N^*_{m,\alpha_1} - \widetilde{N}_{m,\alpha_1})\|_{L^2(\mathcal{V}_0)} \\
&\le C\delta^{\frac{1}{2}} \|N_{m,\alpha_1}\|_{W^{1,\infty}(Q\cap\omega)} \|\nabla (N^*_{m,\alpha_1} - \widetilde{N}_{m,\alpha_1})\|_{L^2(\mathcal{V}_0)}.
\end{aligned}
$$

where $e(u) = (e_{ij}(u))$, $e_{ij}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$, $i, j = 1, \cdots, n$.

Similarly, we have

$$
\begin{aligned}
|I_2^\delta| &= |\int_{\mathcal{V}_\delta} a^*_{ijm\alpha_1} \frac{\partial (N^*_{im,\alpha_1} - \widetilde{N}_{im,\alpha_1})}{\partial \xi_j} d\xi| \\
&\le C \int_{\mathcal{V}_\delta} |e(N^*_{m,\alpha_1} - \widetilde{N}_{m,\alpha_1})| d\xi \\
&\le C\delta^{\frac{1}{2}} \|e(N^*_{m,\alpha_1} - \widetilde{N}_{m,\alpha_1})\|_{L^2(\mathcal{V}_\delta)} \\
&\le C\delta^{\frac{1}{2}} \|\nabla (N^*_{m,\alpha_1} - \widetilde{N}_{m,\alpha_1})\|_{L^2(\mathcal{V}_\delta)}
\end{aligned}
$$

and

$$
\begin{aligned}
|I_2^0| &= |\int_{\mathcal{V}_0} a^*_{ijm\alpha_1} \frac{\partial (N^*_{im,\alpha_1} - \widetilde{N}_{im,\alpha_1})}{\partial \xi_j} d\xi| \\
&\le C\delta^{\frac{1}{2}} \int_{\mathcal{V}_0} |e(N^*_{m,\alpha_1} - \widetilde{N}_{m,\alpha_1})| d\xi \\
&\le C\delta^{\frac{1}{2}} \|e(N^*_{m,\alpha_1} - \widetilde{N}_{m,\alpha_1})\|_{L^2(\mathcal{V}_0)} \\
&\le C\delta^{\frac{1}{2}} \|\nabla (N^*_{m,\alpha_1} - \widetilde{N}_{m,\alpha_1})\|_{L^2(\mathcal{V}_0)}.
\end{aligned}
$$

It follows from (36), and the Korn's inequality (see e.g. [22], pp.21) that

$$
\begin{aligned}
&\delta^{\frac{1}{2}} \|\nabla (N^*_{m,\alpha_1} - \widetilde{N}_{m,\alpha_1})\|^2_{L^2(Q)} \\
&\quad \le |\int_Q a^*_{ijkh} \frac{\partial (N^*_{km,\alpha_1} - \widetilde{N}_{km,\alpha_1})}{\partial \xi_h} \frac{\partial (N^*_{im,\alpha_1} - \widetilde{N}_{im,\alpha_1})}{\partial \xi_j} d\xi| \\
&\quad \le |I_1^\delta| + |I_1^0| + |I_2^\delta| + |I_2^0| \\
&\quad \le C\delta^{\frac{1}{2}} \|\nabla (N^*_{m,\alpha_1} - \widetilde{N}_{m,\alpha_1})\|_{L^2(Q\setminus\bar{\omega})} \\
&\quad \le C\delta^{\frac{1}{2}} \|\nabla (N^*_{m,\alpha_1} - \widetilde{N}_{m,\alpha_1})\|_{L^2(Q)},
\end{aligned}
$$

and consequently

$$
\|\nabla (N^*_{m,\alpha_1} - \widetilde{N}_{m,\alpha_1})\|_{L^2(Q)} \le C. \tag{37}
$$

In particular, it yields

$$
\|\nabla (N^*_{m,\alpha_1} - \widetilde{N}_{m,\alpha_1})\|_{L^2(Q\setminus\omega)} \le C. \tag{38}
$$

From (38), we derive

$$
I_1^\delta \le C\delta^{\frac{1}{2}}, \quad I_1^0 \le C\delta^{\frac{1}{2}}, \quad I_2^\delta \le C\delta^{\frac{1}{2}}, I_2^0 \le C\delta^{\frac{1}{2}}, \quad I_3 \le C\delta^{\frac{1}{2}}.
$$

From (36) , using the Korn's inequality again, we obtain

$$\begin{aligned}&\gamma_0\|\nabla(N^*_{m,\alpha_1}-\widetilde{N}_{m,\alpha_1})\|^2_{L^2(Q\cap\omega)}\\&\le\int_{Q\cap\omega}a_{ijkh}\frac{\partial(N^*_{km,\alpha_1}-\widetilde{N}_{km,\alpha_1})}{\partial\xi_h}\frac{\partial(N^*_{im,\alpha_1}-\widetilde{N}_{im,\alpha_1})}{\partial\xi_j}d\xi\\&\le|I_3|+|I_1^\delta|+|I_1^0|+|I_2^\delta|+|I_2^0|\le C\delta^{\frac12}\end{aligned}$$

i.e.

$$\|\nabla(N^*_{m,\alpha_1}-N_{m,\alpha_1})\|_{L^2(Q\cap\omega)}\le C\delta^{\frac14}. \tag{39}$$

We recall that

$$\begin{aligned}&|\widehat{a}^*_{ijkh}-|Q\cap\omega|\widehat{a}_{ijkh}|\\&=|\int_Q(a^*_{ijkh}+a^*_{ijml}\frac{\partial N^*_{mk,h}}{\partial\xi_l})d\xi-\int_{Q\cap\omega}(a_{ijkh}+a_{ijml}\frac{\partial N_{mk,h}}{\partial\xi_l})d\xi|\\&=|\int_{Q\setminus\bar\omega}a^*_{ijkh}d\xi+\int_{Q\cap\omega}a_{ijml}\frac{\partial(N^*_{mk,h}-N_{mk,h})}{\partial\xi_l}d\xi+\int_{Q\setminus\bar\omega}a^*_{ijml}\frac{\partial N^*_{mk,h}}{\partial\xi_l}d\xi|\\&=|\int_{Q\setminus\bar\omega}a^*_{ijkh}d\xi+\int_{Q\cap\omega}a_{ijml}\frac{\partial(N^*_{mk,h}-N_{mk,h})}{\partial\xi_l}d\xi\\&\quad+\int_{Q\setminus\bar\omega}a^*_{ijml}\frac{\partial(N^*_{mk,h}-\widetilde{N}_{mk,h})}{\partial\xi_l}d\xi+\int_{Q\setminus\bar\omega}a^*_{ijml}\frac{\partial\widetilde{N}_{mk,h}}{\partial\xi_l}d\xi|\\&=|E_1+E_2+I_2^\delta+I_2^0+E_3|,\end{aligned}$$

where

$$|E_1|=|\int_{Q\setminus\bar\omega}a^*_{ijkh}d\xi|\le|\int_{\mathcal{V}_\delta}a^*_{ijkh}d\xi|+|\int_{\mathcal{V}_0}a^*_{ijkh}d\xi|\le C\delta^{\frac12},$$

$$|E_2|=|\int_{Q\cap\omega}a_{ijml}\frac{\partial(N^*_{mk,h}-N_{mk,h})}{\partial\xi_l}d\xi|\le C\|\nabla(N^*_{k,h}-N_{k,h})\|_{L^2(Q\cap\omega)}\le C\delta^{\frac14},$$

$$|E_3|=|\int_{Q\setminus\bar\omega}a^*_{ijml}\frac{\partial\widetilde{N}_{mk,h}}{\partial\xi_l}d\xi|\le|\int_{\mathcal{V}_\delta}a^*_{ijml}\frac{\partial\widetilde{N}_{mk,h}}{\partial\xi_l}d\xi|+|\int_{\mathcal{V}_0}a^*_{ijml}\frac{\partial\widetilde{N}_{mk,h}}{\partial\xi_l}d\xi|\le C\delta^{\frac12},$$

and we thus obtain

$$|\widehat{a}^*_{ijkh}-|Q\cap\omega|\widehat{a}_{ijkh}|\le C\delta^{\frac14}.$$

Recalling (22) and (31),(23) and (32), we observe that their constraint conditions are inconsistent. Since $\int\limits_{Q\cap\omega}(N^*_{km,\alpha_1}(\xi)-N_{km,\alpha_1}(\xi))d\xi\neq0$, $\int\limits_{Q\cap\omega}(N^*_{km,\alpha_1\alpha_2}(\xi)-N_{km,\alpha_1\alpha_2}(\xi))d\xi\neq0$, it is difficult to give the error estimates for $\|N^*_{m,\alpha_1}-N_{m,\alpha_1}\|_{H^1(Q\cap\omega)}$ and $\|N^*_{m,\alpha_1\alpha_2}-N_{m,\alpha_1\alpha_2}\|_{H^1(Q\cap\omega)}$, although we have obtained the semi-norm estimate (39). To this end, we need to define the cell functions in the unit cell Q in the following forms:

$$\begin{cases}-\dfrac{\partial}{\partial\xi_j}\Big(a^*_{ijkh}(\xi)\dfrac{\partial\breve{N}^*_{km,\alpha_1}(\xi)}{\partial\xi_h}\Big)=\dfrac{\partial}{\partial\xi_j}\Big(a^*_{ijm\alpha_1}(\xi)\Big)\quad\text{in}\quad Q\\ \breve{N}^*_{km,\alpha_1}(\xi)\quad\text{is 1-periodic in }\xi,\quad\int_{Q\cap\omega}\breve{N}^*_{km,\alpha_1}(\xi)d\xi=0\end{cases} \tag{40}$$

$$
(41)\quad \begin{cases} -\dfrac{\partial}{\partial \xi_j}\Big(a^*_{ijkh}(\xi)\dfrac{\partial \breve{N}^*_{km,\alpha_1\alpha_2}(\xi)}{\partial \xi_h}\Big) = \dfrac{\partial}{\partial \xi_j}\Big(a^*_{ijk\alpha_2}(\xi)\breve{N}^*_{km,\alpha_1}(\xi)\Big) \\ +a^*_{i\alpha_2 kh}(\xi)\dfrac{\partial \breve{N}^*_{km,\alpha_1}(\xi)}{\partial \xi_h} + a^*_{i\alpha_2 m\alpha_1}(\xi) - \dfrac{\eta(\xi)|Q|}{|Q\cap\omega|}\breve{\hat{a}}^*_{i\alpha_2 m\alpha_1} \quad \text{in} \quad Q \\ \breve{N}^*_{km,\alpha_1\alpha_2}(\xi) \quad \text{is 1-periodic in } \xi, \quad \int_{Q\cap\omega}\breve{N}^*_{km,\alpha_1\alpha_2}(\xi)d\xi = 0 \end{cases}
$$

where $\eta(\xi) = \eta(\frac{x}{\varepsilon})$ is as given in (4), and

$$
(42)\qquad \breve{\hat{a}}^*_{ijkh} = \frac{1}{|Q|}\int_Q (a^*_{ijkh}(\xi) + a^*_{ijml}(\xi)\frac{\partial \breve{N}^*_{mk,h}(\xi)}{\partial \xi_l})d\xi.
$$

where $\breve{N}^*_{mk,h}(\xi)$, $m,k,h = 1,\cdots,n$ are as given in (40).

Following along the lines of the proof of Theorem 3.1, we can prove

$$
(43)\qquad \|\nabla(\breve{N}^*_{m,\alpha_1} - N_{m,\alpha_1})\|_{L^2(Q\cap\omega)} \le C\delta^{\frac{1}{4}}.
$$

Furthermore

$$
(44)\qquad |\breve{\hat{a}}^*_{ijkh} - |Q\cap\omega|\widehat{a}_{ijkh}| \le C\delta^{\frac{1}{4}}.
$$

Therefore we obtain the following homogenized equations associated with problem (3), which is equivalent to (29):

$$
(45)\qquad \begin{cases} -\dfrac{\partial}{\partial x_j}(\breve{\hat{a}}^*_{ijkh}\dfrac{\partial \breve{u}^{0,*}_k(x)}{\partial x_h}) = (\frac{|Q\cap\omega|}{|Q|})f_i(x), \quad i = 1,2,\cdots,n, \quad \text{in} \quad \Omega \\ \breve{u}^{0,*}(x) = g_0(x), \quad \text{on} \quad \Gamma_1 \\ \breve{\hat{\sigma}}^*(\breve{u}^{0,*}) = g_1(x), \quad \text{on} \quad \Gamma_2 \end{cases}
$$

where $\breve{\hat{\sigma}}^*(\breve{u}^{0,*}) = (\breve{\hat{\sigma}}^*_1(\breve{u}^{0,*}),\cdots,\breve{\hat{\sigma}}^*_n(\breve{u}^{0,*}))$, $\breve{\hat{\sigma}}^*_i(\breve{u}^{0,*}) \equiv \nu_j\breve{\hat{a}}^*_{ijkh}\dfrac{\partial \breve{u}^{0,*}_k(x)}{\partial x_h}$, $\vec{n} = (\nu_1,\cdots,\nu_n)$ is the unit outer normal vector to $\partial\Omega$, and $\breve{\hat{a}}^*_{ijkh} = \frac{1}{|Q|}\int_Q(a^*_{ijkh}(\xi)+a^*_{ijml}(\xi)\frac{\partial \breve{N}^*_{mk,h}(\xi)}{\partial \xi_l})d\xi$.

Similarly to (27), we define formally :

$$
(46)\qquad \begin{aligned} &\breve{U}^{\varepsilon,*}_1(x) = \breve{u}^{0,*}(x) + \varepsilon\breve{N}^*_{m,\alpha_1}(\xi)\tfrac{\partial \breve{u}^{0,*}_m(x)}{\partial x_{\alpha_1}} \\ &\breve{U}^{\varepsilon,*}_2(x) = \breve{u}^{0,*}(x) + \varepsilon\breve{N}^*_{m,\alpha_1}(\xi)\tfrac{\partial \breve{u}^{0,*}_m(x)}{\partial x_{\alpha_1}} + \varepsilon^2\breve{N}^*_{m,\alpha_1\alpha_2}(\xi)\tfrac{\partial^2 \breve{u}^{0,*}_m(x)}{\partial x_{\alpha_1}\partial x_{\alpha_2}} \end{aligned}
$$

Remark 3.3. *Observing (33) and (44), we know that the corresponding homogenized coefficients are the same regardless of whether we use (31) or (40). It is very important to the homogenization method in shape optimization. Because homogenization method provides an effective way of optimizing the domain topology without having to keep track of complex hole boundaries, we use (31) to compute the homogenized coefficients* $\widehat{a}^*_{ijkh}$ *in the most cases.*

It remains to give the error estimates for $\|\breve{N}^*_{m,\alpha_1} - N_{m,\alpha_1}\|_{H^1(Q\cap\omega)}$ and $\|\breve{N}^*_{m,\alpha_1\alpha_2} - N_{m,\alpha_1\alpha_2}\|_{H^1(Q\cap\omega)}$. To do so, we next introduce the lemma given by:

Lemma 3.2.([13], Lemma B.63, p.490) *Let* $1 \le p < +\infty$, *and* Y *be a bounded connected open set having the* $(1,p)-$ *extension property. Let* f *be a linear form on* $W^{1,p}(Y)$ *whose restriction on constant functions is not zero. Then, there is* $C_{p,Y} > 0$ *such that*

$$
(47)\qquad \|v\|_{W^{1,p}(Y)} \le C_{p,Y}\Big\{\|v\|_{L^p(Y)} + |f(v)|\Big\}, \quad \forall v \in W^{1,p}(Y).
$$

Define a space in such a way:

$$
\overline{W}^1_{per}(Q) = \{v \in H^1(Q) : \ v(\xi) \text{ is 1-periodic in } \xi \text{ and } \int_{Q\cap\omega} v(\xi)d\xi = 0\}
$$

which can be equipped with the norm

$$\|v\|_{\overline{W}^1_{per}(Q)} = \|\nabla v\|_{L^2(Q)},$$

because of the following lemma

Lemma 3.3. *There exists a constant C such that*

$$\|v\|_{H^1(Q)} \leq C\{\|\nabla v\|_{L^2(Q)} + |\int_{Q\cap\omega} v(\xi)d\xi|\} \tag{48}$$

Proof. Let $Q\cap\omega$ be a subset of Q of non-zero measure and set $f(v) = \frac{1}{|Q\cap\omega|}\int_{Q\cap\omega} v(\xi)d\xi$. It is clear that f is continuous, and if c is a constant function, $f(c)$ is zero if and only if c is zero. It follows from Lemma 3.2 (Also see, [7] p.115) that

$$\|v\|_{H^1(Q)} \leq C\{\|\nabla v\|_{L^2(Q)} + |\int_{Q\cap\omega} v(\xi)d\xi|\}.$$

Theorem 3.2. *Let $N_{m,\alpha_1}(\xi), N_{m,\alpha_1\alpha_2}$ be the weak solutions associated with problems (22) and (23), respectively, and let $\breve{N}^*_{m,\alpha_1}(\xi), \breve{N}^*_{m,\alpha_1\alpha_2}$ be respectively the weak solutions of problems (40) and (41). Under the assumptions of Theorem 3.1, $(B_1)-(B_3)$ and $(A_1)-(A_4)$, we obtain the following error estimates:*

$$\|N_{m,\alpha_1} - \breve{N}^*_{m,\alpha_1}\|_{H^1(Q\cap\omega)} \leq C\delta^{\frac{1}{4}}, \tag{49}$$

$$\|N_{m,\alpha_1\alpha_2} - \breve{N}^*_{m,\alpha_1\alpha_2}\|_{H^1(Q\cap\omega)} \leq C\delta^{\frac{1}{8}}. \tag{50}$$

where C is a constant independent of δ; $\delta > 0$ is a sufficiently small number, and the vector-valued functions

$$N_{m,\alpha_1}(\xi) = (N_{1m,\alpha_1}(\xi), \cdots, N_{nm,\alpha_1}(\xi))^T, N_{m,\alpha_1\alpha_2}(\xi) = (N_{1m,\alpha_1\alpha_2}(\xi), \cdots, N_{nm,\alpha_1\alpha_2}(\xi))^T;$$
$$\breve{N}^*_{m,\alpha_1}(\xi) = (\breve{N}^*_{1m,\alpha_1}(\xi), \cdots, \breve{N}^*_{nm,\alpha_1}(\xi))^T, \breve{N}^*_{m,\alpha_1\alpha_2}(\xi) = (\breve{N}^*_{1m,\alpha_1\alpha_2}(\xi), \cdots, \breve{N}^*_{nm,\alpha_1\alpha_2}(\xi))^T.$$

Proof. In Lemma 3.2, if we set $Y = Q\cap\omega$ and $f(v) = \int\limits_{Q\cap\omega} v(\xi)d\xi$, using (39), (22) and (40), then we deduce (49).

Due to Lemma 3.3 and (37), we have

$$\|\breve{N}^*_{m,\alpha_1} - \widetilde{N}_{m,\alpha_1}\|_{H^1(Q)} \leq C. \tag{51}$$

Let $\phi_i(\xi) = (\breve{N}^*_{im,\alpha_1\alpha_2} - \widetilde{N}_{im,\alpha_1\alpha_2})$. If we take $\phi_i(\xi)$ as the test functions, and then the variational forms of (41) and (23) are the following:

$$\begin{aligned}\int_Q a^*_{ijkh}\frac{\partial \breve{N}^*_{km,\alpha_1\alpha_2}}{\partial\xi_h}\frac{\partial\phi_i}{\partial\xi_j}d\xi &= -\int_Q a^*_{ijk\alpha_2}\breve{N}^*_{km,\alpha_1}\frac{\partial\phi_i}{\partial\xi_j}d\xi \\ &+ \int_Q a^*_{i\alpha_2kh}\frac{\partial\breve{N}^*_{km,\alpha_1}}{\partial\xi_h}\phi_i d\xi + \int_Q (a^*_{i\alpha_2m\alpha_1} - \frac{\eta(\xi)}{|Q\cap\omega|}\breve{\hat{a}}^*_{i\alpha_2m\alpha_1})\phi_i d\xi,\end{aligned} \tag{52}$$

$$\begin{aligned}\int_{Q\cap\omega} a_{ijkh}\frac{\partial N_{km,\alpha_1\alpha_2}}{\partial\xi_h}\frac{\partial\phi_i}{\partial\xi_j}d\xi &= -\int_{Q\cap\omega} a_{ijk\alpha_2}N_{km,\alpha_1}\frac{\partial\phi_i}{\partial\xi_j}d\xi \\ &+ \int_{Q\cap\omega} a_{i\alpha_2kh}\frac{\partial N_{km,\alpha_1}}{\partial\xi_h}\phi_i d\xi + \int_{Q\cap\omega}(a_{i\alpha_2m\alpha_1} - \widehat{a}_{i\alpha_2m\alpha_1})\phi_i d\xi.\end{aligned} \tag{53}$$

Subtracting (52) from (53), and using Lemma 3.3 and the Korn's inequality , we obtain

$$
\begin{aligned}
&C\|\breve{N}^*_{m,\alpha_1\alpha_2}-N_{m,\alpha_1\alpha_2}\|^2_{H^1(Q\cap\omega)}\le C\|\nabla(\breve{N}^*_{m,\alpha_1\alpha_2}-N_{m,\alpha_1\alpha_2})\|^2_{L^2(Q\cap\omega)}\\
&\le\int_{Q\cap\omega}a_{ijkh}\frac{\partial(\breve{N}^*_{km,\alpha_1\alpha_2}-N_{km,\alpha_1\alpha_2})}{\partial\xi_h}\frac{\partial(\breve{N}^*_{im,\alpha_1\alpha_2}-N_{im,\alpha_1\alpha_2})}{\partial\xi_j}d\xi\\
&=-\int_{Q\setminus\bar{\omega}}a^*_{ijkh}\frac{\partial(\breve{N}^*_{km,\alpha_1\alpha_2}-\widetilde{N}_{km,\alpha_1\alpha_2})}{\partial\xi_h}\frac{\partial(\breve{N}^*_{im,\alpha_1\alpha_2}-\widetilde{N}_{im,\alpha_1\alpha_2})}{\partial\xi_j}d\xi\\
&-\int_{Q\setminus\bar{\omega}}a^*_{ijkh}\frac{\partial\widetilde{N}_{km,\alpha_1\alpha_2}}{\partial\xi_h}\frac{\partial(\breve{N}^*_{im,\alpha_1\alpha_2}-\widetilde{N}_{im,\alpha_1\alpha_2})}{\partial\xi_j}d\xi\\
&-\int_{Q\cap\omega}a_{ijk\alpha_2}(\breve{N}^*_{km,\alpha_1}-N_{km,\alpha_1})\frac{\partial(\breve{N}^*_{im,\alpha_1\alpha_2}-N_{im,\alpha_1\alpha_2})}{\partial\xi_j}d\xi\\
&-\int_{Q\setminus\bar{\omega}}a^*_{ijk\alpha_2}\breve{N}^*_{km,\alpha_1}\frac{\partial(\breve{N}^*_{im,\alpha_1\alpha_2}-\widetilde{N}_{im,\alpha_1\alpha_2})}{\partial\xi_j}d\xi\\
&+\int_{Q\cap\omega}a_{i\alpha_2kh}\frac{\partial(\breve{N}^*_{km,\alpha_1}-N_{km,\alpha_1})}{\partial\xi_h}(\breve{N}^*_{im,\alpha_1\alpha_2}-N_{im,\alpha_1\alpha_2})d\xi\\
&+\int_{Q\setminus\bar{\omega}}a^*_{i\alpha_2kh}\frac{\partial\breve{N}^*_{km,\alpha_1}}{\partial\xi_h}(\breve{N}^*_{im,\alpha_1\alpha_2}-\widetilde{N}_{im,\alpha_1\alpha_2})d\xi\\
&+\int_{Q\setminus\bar{\omega}}a^*_{i\alpha_2m\alpha_1}(\breve{N}^*_{im,\alpha_1\alpha_2}-\widetilde{N}_{im,\alpha_1\alpha_2})d\xi\\
&+\int_{Q\cap\omega}(\frac{1}{|Q\cap\omega|}\widehat{a}^*_{i\alpha_2m\alpha_1}-\widehat{a}_{i\alpha_2m\alpha_1})(\breve{N}^*_{im,\alpha_1\alpha_2}-N_{im,\alpha_1\alpha_2})d\xi\\
&=I_0+I_1+I_2+I_3+I_4+I_5+I_6+I_7
\end{aligned}
\tag{54}
$$

where

$$
I_0=-\int_{Q\setminus\bar{\omega}}a^*_{ijkh}\frac{\partial(\breve{N}^*_{km,\alpha_1\alpha_2}-\widetilde{N}_{km,\alpha_1\alpha_2})}{\partial\xi_h}\frac{\partial(\breve{N}^*_{im,\alpha_1\alpha_2}-\widetilde{N}_{im,\alpha_1\alpha_2})}{\partial\xi_j}d\xi\le 0.
$$

Following the lines of the proof of Theorem 3.1, we can obtain

$$
\begin{aligned}
|I_1|&=|-\int_{Q\setminus\bar{\omega}}a^*_{ijkh}\frac{\partial\widetilde{N}_{km,\alpha_1\alpha_2}}{\partial\xi_h}\frac{\partial(\breve{N}^*_{im,\alpha_1\alpha_2}-\widetilde{N}_{im,\alpha_1\alpha_2})}{\partial\xi_j}d\xi|\\
&\le C\delta^{\frac{1}{2}}\|\nabla(\breve{N}^*_{m,\alpha_1\alpha_2}-\widetilde{N}_{m,\alpha_1\alpha_2})\|_{L^2(Q\setminus\bar{\omega})},
\end{aligned}
$$

$$
\begin{aligned}
|I_3|&=|-\int_{Q\setminus\bar{\omega}}a^*_{ijk\alpha_2}\breve{N}^*_{km,\alpha_1}\frac{\partial(\breve{N}^*_{im,\alpha_1\alpha_2}-\widetilde{N}_{im,\alpha_1\alpha_2})}{\partial\xi_j}d\xi|\\
&=|-\int_{Q\setminus\bar{\omega}}a^*_{ijk\alpha_2}(\breve{N}^*_{km,\alpha_1}-\widetilde{N}_{km,\alpha_1})\frac{\partial(\breve{N}^*_{im,\alpha_1\alpha_2}-\widetilde{N}_{im,\alpha_1\alpha_2})}{\partial\xi_j}d\xi\\
&\quad-\int_{Q\setminus\bar{\omega}}a^*_{ijk\alpha_2}\widetilde{N}_{km,\alpha_1}\frac{\partial(\breve{N}^*_{im,\alpha_1\alpha_2}-\widetilde{N}_{im,\alpha_1\alpha_2})}{\partial\xi_j}d\xi|\\
&\le C\delta^{\frac{1}{2}}\|\nabla(\breve{N}^*_{m,\alpha_1\alpha_2}-\widetilde{N}_{m,\alpha_1\alpha_2})\|_{L^2(Q\setminus\bar{\omega})},
\end{aligned}
$$

$$
\begin{aligned}
|I_5| &= |\int_{Q\setminus\bar{\omega}} a^*_{i\alpha_2 kh}\frac{\partial \breve{N}^*_{km,\alpha_1}}{\partial \xi_h}(\breve{N}^*_{im,\alpha_1\alpha_2}-\widetilde{N}_{im,\alpha_1\alpha_2})d\xi| \\
&= |\int_{Q\setminus\bar{\omega}} a^*_{i\alpha_2 kh}\frac{\partial (\breve{N}^*_{km,\alpha_1}-\widetilde{N}_{km,\alpha_1})}{\partial \xi_h}(\breve{N}^*_{im,\alpha_1\alpha_2}-\widetilde{N}_{im,\alpha_1\alpha_2})d\xi \\
&\quad + \int_{Q\setminus\bar{\omega}} a^*_{i\alpha_2 kh}\frac{\partial \widetilde{N}_{km,\alpha_1}}{\partial \xi_h}(\breve{N}^*_{im,\alpha_1\alpha_2}-\widetilde{N}_{im,\alpha_1\alpha_2})d\xi| \\
&\le C\delta^{\frac{1}{2}}\|\breve{N}^*_{m,\alpha_1\alpha_2}-\widetilde{N}_{m,\alpha_1\alpha_2}\|_{L^2(Q\setminus\bar{\omega})},
\end{aligned}
$$

$$
\begin{aligned}
|I_6| &= |\int_{Q\setminus\bar{\omega}} a^*_{i\alpha_2 m\alpha_1}(\breve{N}^*_{im,\alpha_1\alpha_2}-\widetilde{N}_{im,\alpha_1\alpha_2})d\xi| \\
&\le C\delta^{\frac{1}{2}}\|\breve{N}^*_{m,\alpha_1\alpha_2}-\widetilde{N}_{m,\alpha_1\alpha_2}\|_{L^2(Q\setminus\bar{\omega})},
\end{aligned}
$$

and

$$
\begin{aligned}
|I_2| &= |-\int_{Q\cap\omega} a_{ijk\alpha_2}(\breve{N}^*_{km,\alpha_1}-N_{km,\alpha_1})\frac{\partial(\breve{N}^*_{im,\alpha_1\alpha_2}-N_{im,\alpha_1\alpha_2})}{\partial \xi_j}d\xi| \\
&\le C\|\breve{N}^*_{m,\alpha_1}-N_{m,\alpha_1}\|_{L^2(Q\cap\omega)}\|\nabla(\breve{N}^*_{m,\alpha_1\alpha_2}-N_{m,\alpha_1\alpha_2})\|_{L^2(Q\cap\omega)} \\
&\le C\delta^{\frac{1}{4}}\|\nabla(\breve{N}^*_{m,\alpha_1\alpha_2}-N_{m,\alpha_1\alpha_2})\|_{L^2(Q\cap\omega)},
\end{aligned}
$$

$$
\begin{aligned}
|I_4| &= |\int_{Q\cap\omega} a_{i\alpha_2 kh}\frac{\partial(\breve{N}^*_{km,\alpha_1}-N_{km,\alpha_1})}{\partial \xi_h}(\breve{N}^*_{im,\alpha_1\alpha_2}-N_{im,\alpha_1\alpha_2})d\xi| \\
&\le C\|\nabla(\breve{N}^*_{m,\alpha_1}-N_{m,\alpha_1})\|_{L^2(Q\cap\omega)}\|\breve{N}^*_{m,\alpha_1\alpha_2}-N_{m,\alpha_1\alpha_2}\|_{L^2(Q\cap\omega)} \\
&\le C\delta^{\frac{1}{4}}\|\breve{N}^*_{m,\alpha_1\alpha_2}-N_{m,\alpha_1\alpha_2}\|_{L^2(Q\cap\omega)},
\end{aligned}
$$

$$
\begin{aligned}
|I_7| &= |\int_{Q\cap\omega}(\frac{1}{|Q\cap\omega|}\breve{\hat{a}}^*_{i\alpha_2 m\alpha_1}-\widehat{a}_{i\alpha_2 m\alpha_1})(\breve{N}^*_{im,\alpha_1\alpha_2}-\widetilde{N}_{im,\alpha_1\alpha_2})d\xi| \\
&\le C|\breve{\hat{a}}^*_{i\alpha_2 m\alpha_1}-|Q\cap\omega|\widehat{a}_{i\alpha_2 m\alpha_1}|\cdot\|\breve{N}^*_{m,\alpha_1\alpha_2}-\widetilde{N}_{m,\alpha_1\alpha_2}\|_{L^2(Q\cap\omega)} \\
&\le C\delta^{\frac{1}{4}}\|\breve{N}^*_{m,\alpha_1\alpha_2}-\widetilde{N}_{m,\alpha_1\alpha_2}\|_{L^2(Q\cap\omega)}.
\end{aligned}
$$

Using the above inequalities, (54) , Lemma 3.3 and the Korn's inequality, we have

$$
\begin{aligned}
&\delta^{\frac{1}{2}}\|\breve{N}^*_{m,\alpha_1\alpha_2}-\widetilde{N}_{m,\alpha_1\alpha_2}\|^2_{H^1(Q)} \le C\delta^{\frac{1}{2}}\|\nabla(\breve{N}^*_{m,\alpha_1\alpha_2}-\widetilde{N}_{m,\alpha_1\alpha_2})\|^2_{L^2(Q)} \\
&\le C\int_Q a^*_{ijkh}\frac{\partial(N^*_{km,\alpha_1\alpha_2}-\widetilde{N}_{km,\alpha_1\alpha_2})}{\partial \xi_h}\frac{\partial(N^*_{im,\alpha_1\alpha_2}-\widetilde{N}_{im,\alpha_1\alpha_2})}{\partial \xi_j}d\xi \\
&\le C\Big(|I_1|+|I_2|+|I_3|+|I_4|+|I_5|+|I_6|+|I_7|\Big) \\
&\le C\delta^{\frac{1}{4}}\|N^*_{m,\alpha_1\alpha_2}-\widetilde{N}_{m,\alpha_1\alpha_2}\|_{H^1(Q)}
\end{aligned}
$$

i.e.

$$
\|N^*_{m,\alpha_1\alpha_2}-\widetilde{N}_{m,\alpha_1\alpha_2}\|_{H^1(Q)} \le C\delta^{-\frac{1}{4}}.
$$

We thus obtain

$$
|I_1| \le C\delta^{\frac{1}{4}}, \quad |I_3| \le C\delta^{\frac{1}{4}}, \quad |I_5| \le C\delta^{\frac{1}{4}}, \quad |I_6| \le C\delta^{\frac{1}{4}}.
$$

On the other hand, using the inequality $ab \leq \gamma a^2 + \frac{1}{4\gamma}b^2$, $a, b, \gamma > 0$ and (43), we obtain

$$
\begin{aligned}
|I_2| &\leq C\|\breve{N}^*_{m,\alpha_1} - N_{m,\alpha_1}\|_{L^2(Q\cap\omega)}\|\nabla(\breve{N}^*_{m,\alpha_1\alpha_2} - N_{m,\alpha_1\alpha_2})\|_{L^2(Q\cap\omega)} \\
&\leq \frac{C}{4\gamma}\|\breve{N}^*_{m,\alpha_1} - N_{m,\alpha_1}\|^2_{L^2(Q\cap\omega)} + \gamma\|\nabla(\breve{N}^*_{m,\alpha_1\alpha_2} - N_{m,\alpha_1\alpha_2})\|^2_{L^2(Q\cap\omega)} \\
&\leq \frac{C}{4\gamma}\delta^{\frac{1}{2}} + \gamma\|\breve{N}^*_{m,\alpha_1\alpha_2} - N_{m,\alpha_1\alpha_2}\|^2_{H^1(Q\cap\omega)},
\end{aligned}
$$

$$
\begin{aligned}
|I_4| &\leq C\|\nabla(\breve{N}^*_{m,\alpha_1} - N_{m,\alpha_1})\|_{L^2(Q\cap\omega)}\|\breve{N}^*_{m,\alpha_1\alpha_2} - N_{m,\alpha_1\alpha_2}\|_{L^2(Q\cap\omega)} \\
&\leq \frac{C}{4\gamma}\|\nabla(\breve{N}^*_{m,\alpha_1} - N_{m,\alpha_1})\|^2_{L^2(Q\cap\omega)} + \gamma\|\breve{N}^*_{m,\alpha_1\alpha_2} - N_{m,\alpha_1\alpha_2}\|^2_{L^2(Q\cap\omega)} \\
&\leq \frac{C}{4\gamma}\delta^{\frac{1}{2}} + \gamma\|\breve{N}^*_{m,\alpha_1\alpha_2} - N_{m,\alpha_1\alpha_2}\|^2_{H^1(Q\cap\omega)},
\end{aligned}
$$

$$
\begin{aligned}
|I_7| &\leq C|\breve{\hat{a}}^*_{i\alpha_2 m\alpha_1} - |Q\cap\omega|\widehat{a}_{i\alpha_2 m\alpha_1}| \cdot \|\breve{N}^*_{m,\alpha_1\alpha_2} - N_{m,\alpha_1\alpha_2}\|_{L^2(Q\cap\omega)} \\
&\leq \frac{C}{4\gamma}|\breve{\hat{a}}^*_{i\alpha_2 m\alpha_1} - |Q\cap\omega|\widehat{a}_{i\alpha_2 m\alpha_1}|^2 + \gamma\|\breve{N}^*_{m,\alpha_1\alpha_2} - N_{m,\alpha_1\alpha_2}\|^2_{L^2(Q\cap\omega)} \\
&\leq \frac{C}{4\gamma}\delta^{\frac{1}{2}} + \gamma\|\breve{N}^*_{m,\alpha_1\alpha_2} - N_{m,\alpha_1\alpha_2}\|^2_{H^1(Q\cap\omega)}.
\end{aligned}
$$

From (54), choosing a sufficiently small number $\gamma > 0$ such that $(1-3\gamma) \geq \frac{1}{2}$, then we get

$$\|\breve{N}^*_{m,\alpha_1\alpha_2} - N_{m,\alpha_1\alpha_2}\|_{H^1(Q\cap\omega)} \leq C\delta^{\frac{1}{8}}.$$

Finally, we can obtain the following theorem:

Theorem 3.3. *Suppose that Ω^ε is a type-II perforated domain. Let $u^\varepsilon(x)$ be the unique weak solution of problem (1), and $\breve{U}^\varepsilon_1(x)$, $\breve{U}^\varepsilon_2(x)$ be as given in (46). Under the assumptions of Theorems 2.1, 3.1 and 3.2, $(B_1)-(B_3)$, $(A_1)-(A_4)$ and suppose u^0, $\breve{u}^{0,*} \in W^{1,\infty}(\Omega)\bigcap H^3(\Omega)$, we have*

$$\|u^\varepsilon - \breve{U}^\varepsilon_s\|_{H^1(\Omega^\varepsilon)} \leq C\Big\{\varepsilon^{\frac{1}{2}} + \delta^{\frac{1}{4}}\Big\}, \quad s = 1, 2 \tag{55}$$

where C is a constant independent of ε, δ.

Proof. Recalling Lemma 3.1, we have

$$\|u^\varepsilon - U^\varepsilon_s\|_{H^1(\Omega^\varepsilon)} \leq C\varepsilon^{\frac{1}{2}}, \quad s = 1, 2. \tag{56}$$

Using the expansions (20) and (46), by the triangle inequality, we have

$$
\begin{aligned}
&\|U^\varepsilon_1 - \breve{U}^\varepsilon_1\|_{H^1(\Omega^\varepsilon)} \leq \|u^0 - \breve{u}^{0,*}\|_{H^1(\Omega)} \\
&+\varepsilon\|N_{m,\alpha_1}\frac{\partial u^0_m}{\partial x_{\alpha_1}} - \breve{N}^*_{m,\alpha_1}\frac{\partial \breve{u}^{0,*}_m}{\partial x_{\alpha_1}}\|_{H^1(\Omega^\varepsilon)},
\end{aligned} \tag{57}
$$

$$
\begin{aligned}
\|U^\varepsilon_2 - \breve{U}^\varepsilon_2\|_{H^1(\Omega^\varepsilon)} &\leq \|U^\varepsilon_1 - \breve{U}^\varepsilon_1\|_{H^1(\Omega^\varepsilon)} \\
&+\varepsilon^2\|N_{m,\alpha_1\alpha_2}\frac{\partial^2 u^0_m}{\partial x_{\alpha_1}\partial x_{\alpha_2}} - \breve{N}^*_{m,\alpha_1\alpha_2}\frac{\partial^2 \breve{u}^{0,*}_m}{\partial x_{\alpha_1}\partial x_{\alpha_2}}\|_{H^1(\Omega^\varepsilon)}.
\end{aligned} \tag{58}
$$

We first give the error estimate of $\|u^0 - \breve{u}^{0,*}\|_{H^1(\Omega)}$. To do so, we subtract (45) from (25), and obtain

$$(59)\quad \begin{cases} -\frac{\partial}{\partial x_j}(\hat{a}_{ijkh}\frac{\partial(u_k^0 - \breve{u}_k^{0,*})}{\partial x_h}) = \frac{\partial}{\partial x_j}((\hat{a}_{ijkh} - \frac{|Q|}{|Q\cap\omega|}\breve{\hat{a}}^*_{ijkh})\frac{\partial \breve{u}_k^{0,*}}{\partial x_h}), \quad \text{in} \quad \Omega \\ u_k^0 - \breve{u}_k^{0,*} = 0, \quad \text{on} \quad \Gamma_1, \quad i = 1, \cdots, n \\ \hat{\sigma}_i(u^0 - \breve{u}^{0,*}) = \nu_j(\frac{|Q|}{|Q\cap\omega|}\breve{\hat{a}}^*_{ijkh} - \hat{a}_{ijkh})\frac{\partial \breve{u}_k^{0,*}}{\partial x_h}, \quad \text{on} \quad \Gamma_2. \end{cases}$$

The variational form is given by

$$(60)\int_\Omega \hat{a}_{ijkh}\frac{\partial(u_k^0 - \breve{u}_k^{0,*})}{\partial x_h}\frac{\partial \varphi_i}{\partial x_j}dx = -\int_\Omega (\hat{a}_{ijkh} - \frac{|Q|}{|Q\cap\omega|}\breve{\hat{a}}^*_{ijkh})\frac{\partial \breve{u}_k^{0,*}}{\partial x_h}\frac{\partial \varphi_i}{\partial x_j}dx,$$

where $\varphi_i \in H^1(\Omega, \Gamma_1)$.

If we take $\varphi_i = u_i^0 - \breve{u}_i^{0,*}$ as a test function and use the Korn's inequality, then we obtain

$$(61)\qquad \|u^0 - \breve{u}^{0,*}\|_{H^1(\Omega)} \le C\delta^{\frac{1}{4}}.$$

On the other hand, we know

$$(62)\qquad \begin{aligned} &\|N_{m,\alpha_1}\frac{\partial u_m^0}{\partial x_{\alpha_1}} - \breve{N}^*_{m,\alpha_1}\frac{\partial \breve{u}_m^{0,*}}{\partial x_{\alpha_1}}\|^2_{H^1(\Omega^\varepsilon)} \\ &\le \|N_{m,\alpha_1}(\xi)\frac{\partial u_m^0(x)}{\partial x_{\alpha_1}} - \breve{N}^*_{m,\alpha_1}(\xi)\frac{\partial \breve{u}_m^{0,*}(x)}{\partial x_{\alpha_1}}\|^2_{L^2(\Omega^\varepsilon)} \\ &+ \sum_{j=1}^n \|\frac{1}{\varepsilon}\frac{\partial N_{m,\alpha_1}(\xi)}{\partial \xi_j}\frac{\partial u_m^0(x)}{\partial x_{\alpha_1}} - \frac{1}{\varepsilon}\frac{\partial \breve{N}^*_{m,\alpha_1}(\xi)}{\partial \xi_j}\frac{\partial \breve{u}_m^{0,*}(x)}{\partial x_{\alpha_1}}\|^2_{L^2(\Omega^\varepsilon)} \\ &+ \sum_{j=1}^n \|N_{m,\alpha_1}(\xi)\frac{\partial^2 u_m^0(x)}{\partial x_{\alpha_1}\partial x_j} - \breve{N}^*_{m,\alpha_1}(\xi)\frac{\partial^2 \breve{u}_m^{0,*}(x)}{\partial x_{\alpha_1}\partial x_j}\|^2_{L^2(\Omega^\varepsilon)}. \end{aligned}$$

We observe that

$$(63)\qquad \begin{aligned} &\|N_{m,\alpha_1}(\xi)\frac{\partial u_m^0(x)}{\partial x_{\alpha_1}} - \breve{N}^*_{m,\alpha_1}(\xi)\frac{\partial \breve{u}_m^{0,*}(x)}{\partial x_{\alpha_1}}\|_{L^2(\Omega^\varepsilon)} \\ &\le \|N_{m,\alpha_1}\|_{L^\infty(Q\cap\omega)}\|u^0\|_{H^1(\Omega)} + \|\breve{N}^*_{m,\alpha_1}\|_{L^\infty(Q)}\|\breve{u}^{0,*}\|_{H^1(\Omega)}, \end{aligned}$$

$$(64)\qquad \begin{aligned} &\frac{1}{\varepsilon}\|\frac{\partial N_{m,\alpha_1}(\xi)}{\partial \xi_j}\frac{\partial u_m^0(x)}{\partial x_{\alpha_1}} - \frac{\partial \breve{N}^*_{m,\alpha_1}(\xi)}{\partial \xi_j}\frac{\partial \breve{u}_m^{0,*}(x)}{\partial x_{\alpha_1}}\|_{L^2(\Omega^\varepsilon)} \\ &\le \frac{1}{\varepsilon}\|(\frac{\partial N_{m,\alpha_1}(\xi)}{\partial \xi_j} - \frac{\partial \breve{N}^*_{m,\alpha_1}(\xi)}{\partial \xi_j})\frac{\partial u_m^0(x)}{\partial x_{\alpha_1}}\|_{L^2(\Omega^\varepsilon)} \\ &+ \frac{1}{\varepsilon}\|\frac{\partial \breve{N}^*_{m,\alpha_1}(\xi)}{\partial \xi_j}(\frac{\partial u_m^0(x)}{\partial x_{\alpha_1}} - \frac{\partial \breve{u}_m^{0,*}(x)}{\partial x_{\alpha_1}})\|_{L^2(\Omega^\varepsilon)} \\ &\le C\varepsilon^{-1}\delta^{\frac{1}{4}}\|u^0\|_{L^\infty(\Omega)} + C\varepsilon^{-1}\delta^{\frac{1}{4}}\|\frac{\partial \breve{N}^*_{m,\alpha_1}}{\partial \xi_j}\|_{L^\infty(Q)}, \end{aligned}$$

and

$$(65)\qquad \begin{aligned} &\|N_{m,\alpha_1}(\xi)\frac{\partial^2 u_m^0(x)}{\partial x_{\alpha_1}\partial x_j} - \breve{N}^*_{m,\alpha_1}(\xi)\frac{\partial^2 \breve{u}_m^{0,*}(x)}{\partial x_{\alpha_1}\partial x_j}\|_{L^2(\Omega^\varepsilon)} \\ &\le \|N_{m,\alpha_1}\|_{L^\infty(Q\cap\omega)}\|u^0\|_{H^2(\Omega)} + \|\breve{N}^*_{m,\alpha_1}\|_{L^\infty(Q)}\|\breve{u}^{0,*}\|_{H^2(\Omega)}, \end{aligned}$$

If we suppose that $N_{m,\alpha_1} \in W^{1,\infty}(Q\cap\omega)$, $\breve{N}^*_{m,\alpha_1} \in W^{1,\infty}(Q)$, and u^0, $\breve{u}^{0,*} \in W^{1,\infty}(\Omega)\cap H^2(\Omega)$, we then obtain

$$\varepsilon\|N_{m,\alpha_1}(\xi)\frac{\partial u^0_m(x)}{\partial x_{\alpha_1}} - \breve{N}^*_{m,\alpha_1}(\xi)\frac{\partial \breve{u}^{0,*}_m(x)}{\partial x_{\alpha_1}}\|_{H^1(\Omega^\varepsilon)} \le C\delta^{\frac{1}{4}}. \tag{66}$$

Similarly, we can get

$$\varepsilon^2\|N_{m,\alpha_1\alpha_2}\frac{\partial^2 u^0_m}{\partial x_{\alpha_1}\partial x_{\alpha_2}} - \breve{N}^*_{m,\alpha_1\alpha_2}\frac{\partial^2 \breve{u}^{0,*}_m}{\partial x_{\alpha_1}\partial x_{\alpha_2}}\|_{H^1(\Omega^\varepsilon)} \le C\varepsilon\delta^{\frac{1}{8}} \le C\varepsilon^{1/2}. \tag{67}$$

Combining (56)-(58), (61), (66) and (67) yields (55). Therefore we complete the proof of Theorem 3.3.

Remark 3.4. *Because of Theorem 3.3, contrary to the homogenization method (see, Remark 3.3), we use usually the multiscale asymptotic expansions (46) to compute the displacement and the stress fields instead of using the multiscale asymptotic expansions (27).*

4. Numerical Examples

In order to support the theoretical results presented in previous sections, we do some numerical experiments for problem (1) in a 3-D perforated domain Ω^ε as shown in Fig.3 (a). Let Ω^ε contain many small spheroid holes satisfying Conditions $(B_1)-(B_3)$, and we assume that the solid material is homogeneous and isotropic. In Section 2, we introduce the hole-filling method, see Definition 2.1, which its basic idea is to fill all holes with a very compliant material such that Ω^ε is changed into a nonperforated domain Ω, i.e. problem (3). We are interested in the asymptotic behaviour of the solution for problem (3) as the properties of one weak phase go to zero. In the previous sections, we have given the full mathematical justification of this limiting process.

In this section, we will give two numerical examples. In Example 4.1, we use the multiscale expansions (20) of the solution for problem (1) and the multiscale expansions (46) of the solution for the multiphase problem (3), respectively. We compare these numerical results, which support Theorem 3.3. In Example 4.2, we use the formulas (24) and (42) to calculate the corresponding homogenized coefficients , and observe the relationship between them, which validates Theorem 3.1 and (44).

Example 4.1. We consider the following relaxed problem associated with problem (1):

$$\begin{cases} -\dfrac{\partial}{\partial x_j}(a^*_{ijkh}(\frac{x}{\varepsilon})\dfrac{\partial u^{\varepsilon,*}_k(x)}{\partial x_h}) = \eta(\frac{x}{\varepsilon})f_i(x) & \text{in} \quad \Omega \\ u^{\varepsilon,*}(x) = g(x) \quad \text{on} \quad \partial\Omega, & i=1,2,\cdots,n \end{cases} \tag{68}$$

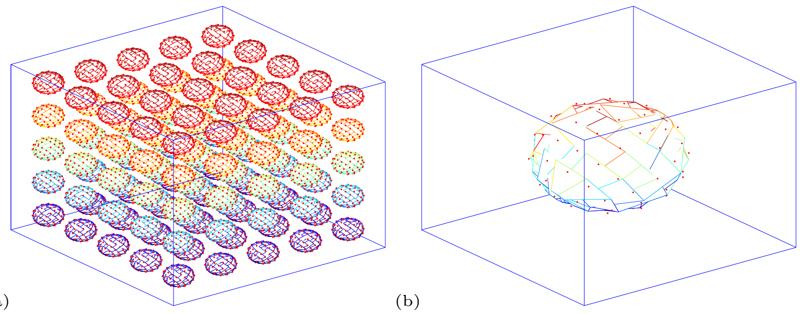
where a whole domain Ω and the unit cell Q are as shown in Fig. 2 (a)-(b), $f(x) = (0,0,-11050)$, $g(x) = (0,0,0)$, and let $\xi = \varepsilon^{-1}x$, $\varepsilon = \frac{1}{5}$,

$$a^*_{ijkh}(\xi) = \frac{E^*(\xi)}{2(1+\nu^*(\xi))}\delta_{ij}\delta_{kh} + \frac{\nu^*(\xi)E^*(\xi)}{(1+\nu^*(\xi))(1-2\nu^*(\xi))}(\delta_{ih}\delta_{jk}+\delta_{ik}\delta_{jh}), \tag{69}$$

$$E^*(\xi) = \begin{cases} 241000 & \text{in } Q\cap\omega \\ 1 & \text{in } Q\backslash\bar{\omega} \end{cases}, \ \nu^*(\xi) = \begin{cases} 0.25 & \text{in } Q\cap\omega \\ 0.30 & \text{in } Q\backslash\bar{\omega}, \end{cases}.$$

Some numerical results for Example 4.1 are shown in Fig.4, the comparison of numerical errors is as given in Tables 1 and 2.

For simplicity, without confusion let $u^0(x)$, $u^\varepsilon_1(x)$ and $u^\varepsilon_2(x)$ be respectively finite element solution of the homogenized equations, first-order multiscale finite



(a) (b)

FIGURE 3. In Example 4.1: (a) a whole domain Ω; (b) the unit cell Q

TABLE 1. Comparison of numerical errors

	$\frac{\|u^0-u^{0,*}\|_{0,\Omega}}{\|u^0\|_{0,\Omega}}$	$\frac{\|u_1^\varepsilon-u_1^{\varepsilon,*}\|_{0,\Omega^\varepsilon}}{\|u_1^\varepsilon\|_{0,\Omega^\varepsilon}}$	$\frac{\|u_2^\varepsilon-u_2^{\varepsilon,*}\|_{0,\Omega^\varepsilon}}{\|u_2^\varepsilon\|_{0,\Omega^\varepsilon}}$	$\frac{\|\sigma^\varepsilon-\sigma^{\varepsilon,*}\|_{0,\Omega^\varepsilon}}{\|\sigma^\varepsilon\|_{0,\Omega^\varepsilon}}$
	1.844824e-06	1.876889e-06	1.966307e-06	2.090513e-06

TABLE 2. Comparison of numerical errors

	$\frac{\|u^0-u^{0,*}\|_{1,\Omega}}{\|u^0\|_{1,\Omega}}$	$\frac{\|u_1^\varepsilon-u_1^{\varepsilon,*}\|_{1,\Omega^\varepsilon}}{\|u_1^\varepsilon\|_{1,\Omega^\varepsilon}}$	$\frac{\|u_2^\varepsilon-u_2^{\varepsilon,*}\|_{1,\Omega^\varepsilon}}{\|u_2^\varepsilon\|_{1,\Omega^\varepsilon}}$	$\frac{\|\sigma^\varepsilon-\sigma^{\varepsilon,*}\|_{1,\Omega^\varepsilon}}{\|\sigma^\varepsilon\|_{1,\Omega^\varepsilon}}$
	2.347352e-06	3.582470e-06	3.199034e-06	2.480323e-06

element solution , and second-order multiscale finite element solution for problem (1) in a perforated domain Ω^ε on the basis of (20) and (25). $u^{0,*}(x)$, $u_1^{\varepsilon,*}(x)$ and $u_2^{\varepsilon,*}(x)$ denote respectively finite element solution of the homogenized equations, first-order multiscale finite element solution , and second-order multiscale finite element solution for the hole-filling problem (3) in a domain Ω based on (45) and (46). $\sigma^\varepsilon(x)$, $\sigma^{\varepsilon,*}(x)$ denote respectively the stresses for problems (1) and (3) based on the second-order multiscale expansions. Set $\|v\|_{0,\Omega} = \|v\|_{L^2(\Omega)}$, $\|v\|_{0,\Omega^\varepsilon} = \|v\|_{L^2(\Omega^\varepsilon)}$, $\|v\|_{1,\Omega} = \|v\|_{H^1(\Omega)}$, $\|v\|_{1,\Omega^\varepsilon} = \|v\|_{H^1(\Omega^\varepsilon)}$.

Remark 4.1. *The numerical results presented in Tables 1 and 2(also see, Fig.4) clearly show that the hole-filling method is true.*

Example 4.2. In this example, we are going to compare the homogenized coefficients defined in (24) to the ones defined in (30). The periodic cell $Q\cap\omega$ is as shown in Fig.5 containing many small spheroid holes satisfying Conditions $(B_1)-(B_3)$.

We assume that the coefficients of multiphase problem (3) are given by:

$$a^*_{ijkh}(\xi) = \frac{E^*(\xi)}{2(1+\nu^*(\xi))}\delta_{ij}\delta_{kh} + \frac{\nu^*(\xi)E^*(\xi)}{(1+\nu^*(\xi))(1-2\nu^*(\xi))}(\delta_{ih}\delta_{jk}+\delta_{ik}\delta_{jh}),$$

where

$$E^*(\xi) = \begin{cases} 241000 & \text{in } Q\cap\omega \\ 1 & \text{in } Q\backslash\bar{\omega} \end{cases} \quad \nu^*(\xi) = \begin{cases} 0.25 & \text{in } Q\cap\omega \\ 0.30 & \text{in } Q\backslash\omega \end{cases}.$$

We use the formulas (24) and (30) to calculate the homogenized coefficients $\widehat{a}_{ijkh}$ and $\widehat{a}^*_{ijkh}$, respectively. The computational results for these homogenized

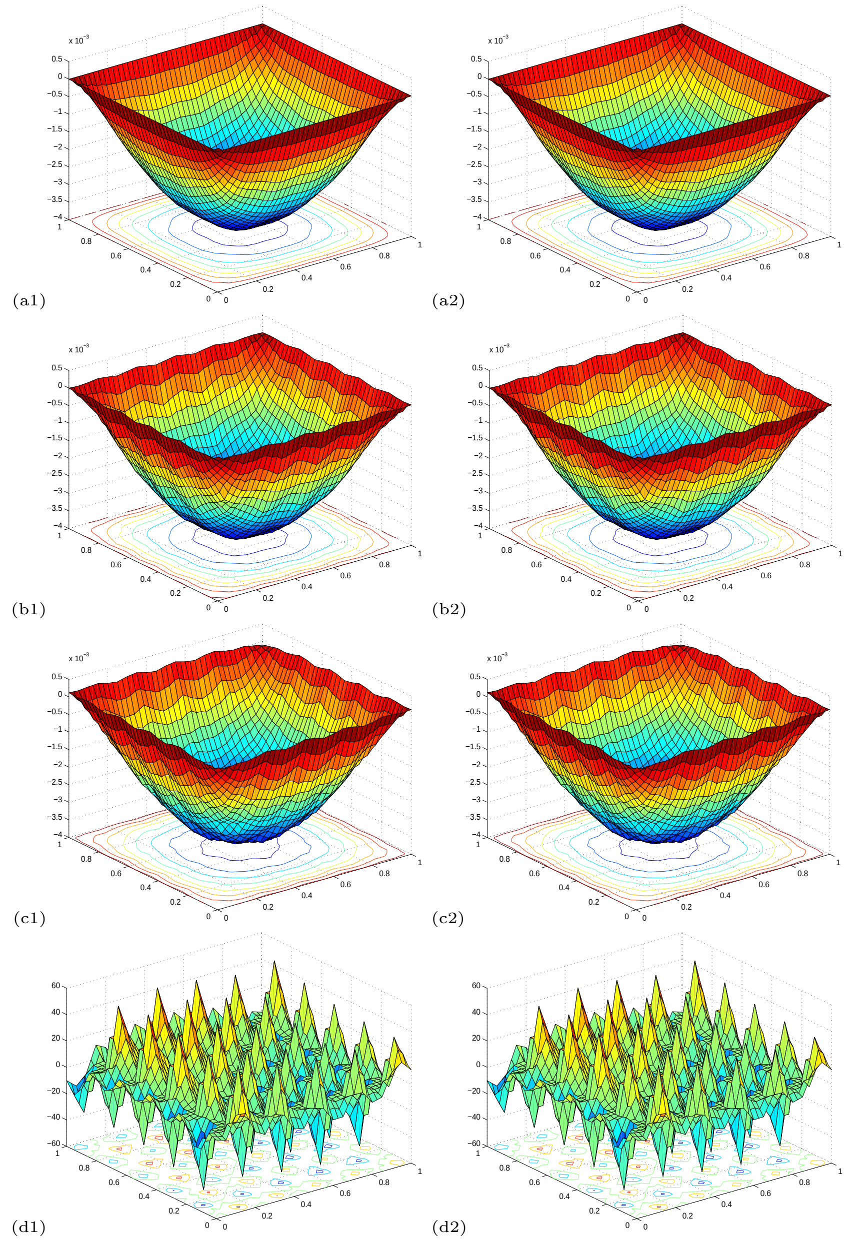


FIGURE 4. In Example 4.1: (a1), (a2) z components of the homogenized solutions; (b1), (b2) z components of first-order multiscale solutions; (c1), (c2) z components of second-order multiscale solutions; (d1), (d2) z components of the stress for the original problem (1) and the hole-filling problem (3), respectively.

coefficients are following

$$|Q \cap \omega|\{\widehat{a}_{ijkh}\} = \begin{bmatrix} 246229.77 & 78971.68 & 78227.98 & -497.45 & -375.59 & -438.31 \\ 78971.68 & 242934.80 & 78061.08 & -512.65 & -1079.87 & 15.09 \\ 78227.98 & 78061.08 & 243650.44 & -77.63 & -875.87 & -323.27 \\ -497.45 & -512.65 & -77.63 & 82701.43 & 3.39 & -415.68 \\ -375.59 & -1079.87 & -875.87 & 3.39 & 81628.89 & -109.78 \\ -438.31 & 15.09 & -323.27 & -415.68 & -109.78 & 81945.18 \end{bmatrix},$$

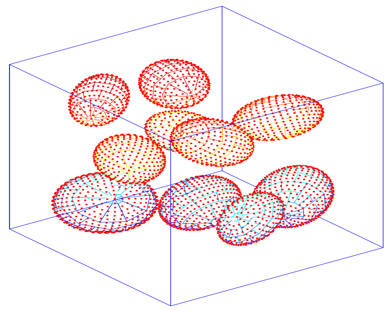


FIGURE 5. In Example 4.2: cell $Q \cap \omega$

$$\{\widehat{a}^*_{ijkh}\} = \begin{bmatrix} 246230.20 & 78971.92 & 78228.24 & -497.44 & -375.58 & -438.30 \\ 78971.92 & 242935.28 & 78061.34 & -512.64 & -1079.84 & 15.09 \\ 78228.24 & 78061.34 & 243650.92 & -77.62 & -875.85 & -323.27 \\ -497.44 & -512.64 & -77.62 & 82701.54 & 3.39 & -415.67 \\ -375.58 & -1079.84 & -875.85 & 3.39 & 81629.03 & -109.78 \\ -438.30 & 15.09 & -323.27 & -415.67 & -109.78 & 81945.30 \end{bmatrix}.$$

Remark 4.2. *Observing these numerical results, we can conclude that the homogenized coefficients for the hole-filling problem (3) with the coefficient a^*_{ijkh} defined in (2) provide a accurate approximation to the homogenized coefficient for the original perforated problem (1). The numerical results validate Theorem 3.1.*

5. Conclusions

This paper discusses the homogenization method and the multiscale analysis of linear elastic equations in a type-II perforated domain Ω^ε. We introduce the hole-filling method in a perforated domain, i.e. we replace all holes by a very compliant material, then we study the homogenization method and the multiscale analysis for the associated multiphase problem in a domain Ω without holes. We are interested in the asymptotic behavior of the solution for the multiphase problem as the material properties of one weak phase go to zero. The main contribution obtained in this paper is to give the full mathematical justification for this limiting process in general cases. Finally, some numerical results are reported, which support strongly the theoretical results of this paper.

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