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L^∞-ERROR ESTIMATES FOR GENERAL OPTIMAL CONTROL PROBLEM BY MIXED FINITE ELEMENT METHODS

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Abstract. In this paper, we investigate the L^∞-error estimates for the solutions of general optimal control problem by mixed finite element methods. The state and co-state are approximated by the lowest order Raviart-Thomas mixed finite element spaces and the control is approximated by piecewise constant functions. We derive L^∞-error estimates of optimal order both for the state variables and the control variable.

Key Words. L^∞-error estimates, mixed finite element, optimal control.

1. Introduction

Optimal control problems [17] have been extensively utilized in many aspects of the modern life such as social, economic, scientific and eigineering numerical simulation. Due to the wide applications of these problems, they must be solved successfully with efficient numerical methods. Among these numerical methods, finite element discretization of the state equation is widely applied though other methods are also used. There have been extensive studies in convergence of finite element approximation of optimal control problems, see, for example [1], [2], [7], [11], [14], [15], [18], [19] and [28]. A systematic introduction of finite element method for PDEs and optimal control can be found in, for example, [8], [13], [23] and [27].

Many contributions have been done to the L^∞ convergence theory, see [3], [9], [16], [20]. In [3], the authors studied L^∞-error estimates for a semilinear elliptic control problem with standard finite element methods. We also see the earlier work [16] in which L^∞ estimate was obtained for the solution of a semilinear second order elliptic problem by mixed methods. But it didn't focus on optimal control problem. More recently, C. Meyer and A. Rösch have studied the superconvergence property for linear-quadratic optimal control problem in [21], they also investigated the L^∞ estimates with standard finite element for this problem in [20]. Most recently, in [10], the authors studied L^∞-error estimates and superconvergence in maximum norm of mixed finite element methods for NonFickian flows in porous media. However, there doesn't seem to exist much work on theoretical analysis of mixed finite element approximation for optimal control problem in the literature.

In this paper, we will study the L^∞-error estimates for general convex optimal control problem with mixed methods. We have done some primary works on linear-quadratic optimal control problem in which L^∞ estimates for state variables and

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control variable were obtained with mixed methods. Here, in this paper, we will show that also for general convex optimal control problem the similar results can be obtained.

The problem that we will study is the following optimal control problem:

$$\min_{u\in K\subset L^\infty(\Omega)} \left\{ \int_\Omega (g_1(\boldsymbol{p}(x)) + g_2(y(x)) + h(u(x)))dx \right\} \tag{1}$$

subject to the state equation

$$-\mathrm{div}(A\mathbf{grad}y) = u, \qquad x \in \Omega, \tag{2}$$

$$y = 0, \qquad x \in \partial\Omega. \tag{3}$$

which can be written in the form of the first order system

$$\mathrm{div}\boldsymbol{p} = u, \qquad x \in \Omega \tag{4}$$

$$\boldsymbol{p} = -A\mathbf{grad}y, \qquad x \in \Omega \tag{5}$$

$$y = 0, \qquad x \in \partial\Omega \tag{6}$$

where $\Omega \subset R^2$ is a bounded domain with Lipschitz continuous boundary. Here, $g_1 = g_1(\cdot,\cdot)$, g_2 and h are strictly convex functionals which are continuously differentiable. In the rest of the paper, we shall simply write $g_1(\boldsymbol{p}(x)), g_2(y(x))$ and $h(u(x))$ as $g_1(\boldsymbol{p}), g_2(y)$ and $h(u)$. We further assume that $h(u) \to +\infty$ as $\| u \| \to \infty$. K denotes the admissible set of the control variable, defined by

$$K = \{u \in L^\infty(\Omega) : a \le u \le b \;\; a.e. \; in \; \Omega\}, \tag{7}$$

where a and b are real numbers.

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\| \cdot \|_{m,p}$ given by

$$\| \phi \|_{m,p}^p = \sum_{|\alpha|\le m} \| D^\alpha \phi \|_{L^p(\Omega)}^p,$$

a semi-norm $| \cdot |_{m,p}$ given by

$$| \phi |_{m,p}^p = \sum_{|\alpha| = m} \| D^\alpha \phi \|_{L^p(\Omega)}^p \; .$$

We set $W_0^{m,p}(\Omega) = \{\phi \in W^{m,p}(\Omega) : \phi\,|_{\partial\Omega} = 0\}$. For p=2, we denote

$$H^m(\Omega) = W^{m,2}(\Omega), H_0^m(\Omega) = W_0^{m,2}(\Omega),$$

and

$$\| \cdot \|_m = \| \cdot \|_{m,2}, \| \cdot \| = \| \cdot \|_{0,2} \; .$$

In addition we use $\| \cdot \|_{0,\,\infty}$ to denote the maximum norm in $L^2(\Omega)$.

2. Mixed finite element approximation of optimal control problems

In this section, we study the mixed finite element approximation of the problem (1) and (4)-(6). First, we assume that $A(x) = (a_{ij}(x))$ is a symmetric matrix with $a_{ij}(x) \in W^{1,\infty}(\Omega)$ and for any vector $\boldsymbol{X} \in R^2$, there is a constant $c > 0$, such that

$$\boldsymbol{X}^t A \boldsymbol{X} \ge c \, \| \boldsymbol{X} \|_{R^2}^2 \; .$$

Next, we introduce the co-state elliptic equation

$$-\mathrm{div}(A(x)(\mathbf{grad}z + g_1'(\boldsymbol{p}))) = g_2'(y), \qquad x \in \Omega, \tag{8}$$

with the boundary condition

$$z = 0, \qquad x \in \partial\Omega. \tag{9}$$

It is assumed that both the elliptic equations (2) and (8) have sufficient regularity. Let

$$\boldsymbol{V} = H(\mathrm{div};\Omega) = \{\boldsymbol{v} \in (L^2(\Omega))^2, \mathrm{div}\boldsymbol{v} \in L^2(\Omega)\}, \qquad W = L^2(\Omega).$$

In order to consider the finite element approximation of the above optimal control problem, we need a weak formulation of the problem (1)and (4)-(6): find $(\boldsymbol{p}, y, u) \in \boldsymbol{V} \times W \times K$ such that

$$\min_{u\in K}\Big\{\int_\Omega (g_1(\boldsymbol{p}(x)) + g_2(y(x)) + h(u(x)))dx\Big\} \tag{10}$$

$$(A^{-1}\boldsymbol{p}, \boldsymbol{v}) - (y, \mathrm{div}\boldsymbol{v}) = 0, \qquad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{11}$$

$$(\mathrm{div}\boldsymbol{p}, w) = (u, w), \qquad \forall w \in W, \tag{12}$$

where $(\ \cdot\ )$ denotes the inner product in $L^2(\Omega)$ or $(L^2(\Omega))^2$. Under the assumption on $g_1(\cdot,\cdot), g_2$ and h, it is well known that the convex control problem (10)-(12) has a unique solution $(\boldsymbol{p}, y, u)$, and that a triplet $(\boldsymbol{p}, y, u)$ is the solution of (10)-(12) if and only if there exists a co-state $(\boldsymbol{q}, z) \in \boldsymbol{V} \times W$ such that $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ satisfies the following optimality conditions:

$$(A^{-1}\boldsymbol{p}, \boldsymbol{v}) - (y, \mathrm{div}\boldsymbol{v}) = 0, \qquad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{13}$$

$$(\mathrm{div}\boldsymbol{p}, w) = (u, w), \qquad \forall w \in W, \tag{14}$$

$$(A^{-1}\boldsymbol{q}, \boldsymbol{v}) - (z, \mathrm{div}\boldsymbol{v}) = -(g_1'(\boldsymbol{p}), \boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{15}$$

$$(\mathrm{div}\boldsymbol{q}, w) = (g_2'(y), w), \qquad \forall w \in W, \tag{16}$$

$$(h'(u) + z, \tilde{u} - u) \geq 0, \qquad \forall \tilde{u} \in K, \tag{17}$$

where g_1' is the gradient of g_1, g_2', h' are the derivatives of g_2 and h, respectively. Moreover, we require:

- A1 $-g_1', g_2'$ and h' are locally Lipschitz continuous, that is

$$|h'(\tilde{u}(x_1)) - h'(\tilde{u}(x_2))| \leq C|x_1 - x_2|, \qquad \forall \tilde{u} \in K,\ x_1, x_2 \in \bar{\Omega};$$
$$|g_1'(\boldsymbol{q}_1) - g_1'(\boldsymbol{q}_2)| \leq C|\boldsymbol{q}_1 - \boldsymbol{q}_2|, \qquad \forall \boldsymbol{q}_1, \boldsymbol{q}_2 \in H(\mathrm{div};\Omega);$$
$$|g_2'(y_1) - g_2'(y_2)| \leq C|y_1 - y_2|, \qquad \forall y_1, y_2 \in L^2(\Omega),$$

  and we also assume that for all $x \in \bar{\Omega}$ and $\tilde{y} \in L^\infty(\Omega),\ \tilde{u} \in L^\infty(\Omega)$, the following estimates hold

$$|g_2'(\tilde{y})| \leq g(x), \quad |g_2''(\tilde{y})| \leq M, \quad h'(\tilde{u}) \leq M,$$

  where $g(x) \in L^\infty(\Omega),\ M > 0$ is a constant.

- A2 – There exists a positive constant m such that the following estimate holds

$$h''(\tilde{u}) \geq m, \qquad \forall \tilde{u} \in K.$$

- A3 – There exists a constant $c > 0$ such that

$$(h'(u_1) - h'(u_2), u_1 - u_2) \geq c \parallel u_1 - u_2 \parallel^2, \qquad \forall u_1, u_2 \in K.$$

Then, we recall some results from Bonnans and Casas [4].

Lemma 2.1. [4] *For every $\psi \in L^p(\Omega)$, the solution φ of*

$$-\mathit{div}(A(x)\mathbf{grad}\varphi) = \psi, \qquad x \in \Omega, \tag{18}$$

$$\varphi = 0, \qquad x \in \partial\Omega, \tag{19}$$

belongs to $H_0^1(\Omega) \cap W^{2,p}(\Omega)$ for every $p \geq 2$. Moreover, there exists a positive constant C, such that

$$\parallel \varphi \parallel_{2,p} \leq C \parallel \psi \parallel_{0,p}\ . \tag{20}$$

Due to Lemma 2.1, the state equation (2) and the co-state equation (8) admit unique solutions in $H_0^1(\Omega)\cap W^{2,p}(\Omega)$ if $g_2'(y)\in L^p(\Omega)$ for $p\geq 2$. This space is embedded in $C^{0,1}(\bar{\Omega})$.

By a detailed discussion of the variational inequality (17), we are able to derive a useful characterization of the optimal control. First, We introduce the projection [21]:

$$\Pi_{[a,b]}(f(x))=\max(a,\min(b,f(x))), \tag{21}$$

Similar as [3], we can prove the following Lemma.

Lemma 2.2. *Suppose that assumptions A1-A3 are satisfied. Then, for all $x\in\bar{\Omega}$,the equation*

$$h'(s(x))+z(x)=0, \tag{22}$$

has a unique solution $s(x)$. Moreover, $s(x)\in C^{0,1}(\bar{\Omega})$.

Proof. Existence and uniqueness of a solution $s(x)$ is obvious thanks to the assumption $h''(u)\geq m>0$. Let us prove that $s(x)\in C^{0,1}(\bar{\Omega})$. We observe that, due to the Lipschitz continuity of $z(x)$, by A2 and (22), we have

$$\begin{aligned}
&m|s(x)-s(x_0)|\\
&\leq\Big|\int_0^1 h''(\theta s(x)+(1-\theta)s(x_0))d\theta\cdot(s(x)-s(x_0))\Big|\\
&=|h'(s(x))-h'(s(x_0))|=|-z(x)+z(x_0)|\\
&\leq C|x-x_0|.
\end{aligned}$$

□

Now, we are able to formulate our characterization theorem, which is fundamental for our work. By means of Lemma 2.2, it can be proved in the same way as in [3].

Theorem 2.1. *Suppose that assumptions A1-A3 are satisfied. Let u be the optimal solution of (13)-(17), and let $s(x)$ be the associated solution of equation (22). Then*

$$u(x)=\Pi_{[a,b]}(s(x))=\max(a,\min(b,s(x))).$$

and u belongs to $C^{0,1}(\bar{\Omega})$.

The results of Theorem 2.1 can be easily extended to the case where a and b are functions of x. Then the Lipschitz continuity of the optimal control u is obtained under the assumption that a and b are Lipschitz continuous.

Now, we consider the finite element approximation of the control problem. Let $\mathcal{T}^h$ denotes a quasi-uniform (in the sense of [12])family of partition of Ω into triangles or rectangles, with boundary elements allowed to have one curved side. Here h is the maximum diameter of the element T in $\mathcal{T}^h$. Let $\boldsymbol{V}_h\times W_h\subset\boldsymbol{V}\times W$ denote the Raviart-Thomas space [25] of the lowest order associated with the triangulations or rectangulations $\mathcal{T}^h$ of Ω. P_k denotes polynomials of total degree at most k, $Q_{m,n}$ indicates the space of polynomials of degree no more than m and n in x_1 and x_2 variables respectively, where $x=(x_1,x_2)$. If $T\in\mathcal{T}^h$ is a triangle, Let

$$\boldsymbol{V}(T)=\boldsymbol{P}_0(T)\oplus\operatorname{span}(\boldsymbol{x}P_0(T)),\qquad W(T)=P_0(T).$$

Similarly, if $T\in\mathcal{T}^h$ is a rectangle, let

$$\boldsymbol{V}(T)=Q_{1,0}(T)\times Q_{0,1}(T),\qquad W(T)=P_0(T).$$

where $\boldsymbol{P}_0(T) = (P_0(T))^2$. Then we can define the finite dimensional spaces as follows

$$\mathbf{V}_h = \{\mathbf{v}_h \in \mathbf{V} : \mathbf{v}_h|_T \in \mathbf{V}(T),\ T \in \mathcal{T}^h\}, \tag{23}$$

$$W_h = \{w_h \in W : w_h|_T \in W(T),\ T \in \mathcal{T}^h\}, \tag{24}$$

$$K_h = \{\tilde{u}_h \in K : \tilde{u}_h|_T = \text{constant},\ T \in \mathcal{T}^h\}. \tag{25}$$

Then the finite element approximation of the problem (10)-(12) is to find $(\boldsymbol{p}_h, y_h, u_h) \in \mathbf{V}_h \times W_h \times K_h$ such that

$$\min_{u_h \in K_h} \Big\{ \int_\Omega (g_1(\boldsymbol{p}_h(x)) + g_2(y_h(x)) + h(u_h(x)))dx \Big\} \tag{26}$$

$$(A^{-1}\boldsymbol{p}_h, \boldsymbol{v}_h) - (y_h, \text{div}\boldsymbol{v}_h) = 0, \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{27}$$

$$(\text{div}\boldsymbol{p}_h, w_h) = (u_h, w_h), \qquad \forall w_h \in W_h. \tag{28}$$

The control problem (26)-(28) again has a unique solution $(\boldsymbol{p}_h, y_h, u_h)$ and that a triplet $(\boldsymbol{p}_h, y_h, u_h)$ is the solution of (26)-(28) if and only if there exists a co-state $(\boldsymbol{q}_h, z_h) \in \boldsymbol{V}_h \times W_h$ such that $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h)$ satisfies the following optimality conditions:

$$(A^{-1}\boldsymbol{p}_h, \boldsymbol{v}_h) - (y_h, \text{div}\boldsymbol{v}_h) = 0, \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{29}$$

$$(\text{div}\boldsymbol{p}_h, w_h) = (u_h, w_h), \qquad \forall w_h \in W_h, \tag{30}$$

$$(A^{-1}\boldsymbol{q}_h, \boldsymbol{v}_h) - (z_h, \text{div}\boldsymbol{v}_h) = -(g_1'(\boldsymbol{p}_h), \boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{31}$$

$$(\text{div}\boldsymbol{q}_h, w_h) = (g_2'(y_h), w_h), \qquad \forall w_h \in W_h, \tag{32}$$

$$(h'(u_h) + z_h, \tilde{u}_h - u_h)_U \geq 0, \qquad \forall \tilde{u}_h \in K_h. \tag{33}$$

Now, we can prove the following lemma which is the discrete counterpart of Lemma 2.1

Lemma 2.3. *Suppose that assumptions A1-A3 are satisfied. Let u_h be the optimal solution of (29)-(33), then there exists a unique function $s_h(x)$ such that $s_h(x) = s_T$ is a constant on each triangle $T \in \mathcal{T}^h$, and the equation*

$$\int_T (h'(s_T) + z_h(x))dx = 0 \tag{34}$$

is satisfied.

Then, for the approximate problem, Theorem 2.1 reads as follows:

Theorem 2.2. *Suppose that assumptions A1-A3 are satisfied. Let u_h is the optimal solution of (29)-(33), and let s_h be the solution of (34) corresponding to u_h. Then u_h is given by*

$$u_h(x) = \Pi_{[a,b]}(s_h(x)) = \max(a, \min(b, s_h(x))), \quad for \ \ a.e. \ \ x \in \Omega. \tag{35}$$

For the details, we can refer to [3].

In the rest of the paper, we shall use some intermediate variables. For any control function $\tilde{u} \in K$, we first define the state solution $(\boldsymbol{p}(\tilde{u}), y(\tilde{u}), \boldsymbol{q}(\tilde{u}), z(\tilde{u}))$ associated with $\tilde{u}$ that satisfies

$$(A^{-1}\boldsymbol{p}(\tilde{u}), \boldsymbol{v}) - (y(\tilde{u}), \text{div}\boldsymbol{v}) = 0, \qquad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{36}$$

$$(\text{div}\boldsymbol{p}(\tilde{u}), w) = (\tilde{u}, w), \qquad \forall w \in W, \tag{37}$$

$$(A^{-1}\boldsymbol{q}(\tilde{u}), \boldsymbol{v}) - (z(\tilde{u}), \text{div}\boldsymbol{v}) = -(g_1'(\boldsymbol{p}(\tilde{u})), \boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{38}$$

$$(\text{div}\boldsymbol{q}(\tilde{u}), w) = (g_2'(y(\tilde{u})), w), \qquad \forall w \in W. \tag{39}$$

Remark 2.1. *Obviously, if we chose $\tilde{u} = u_h$ in (36)-(39), then by the regularity of (18)-(19) and the uniqueness of the solution for (36)-(39), we can get that $y(u_h), z(u_h) \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$ satisfy the following relations:*

$$-div(A(x)\mathbf{grad}y(u_h)) = u_h, \tag{40}$$

$$-div(A(x)(\mathbf{grad}z(u_h) + g_1'(\boldsymbol{p}(u_h)))) = g_2'(y(u_h)), \tag{41}$$

and

$$\| y(u_h) \|_{2,p} \leq \| u_h \|_{0,p}, \tag{42}$$

$$\| z(u_h) \|_{2,p} \leq \| g_2'(y(u_h)) \|_{0,p} . \tag{43}$$

Then, we define the discrete state solution $(\boldsymbol{p}_h(\tilde{u}), y_h(\tilde{u}), \boldsymbol{q}_h(\tilde{u}), z_h(\tilde{u}))$ associated with $\tilde{u}$ that satisfies

$$(A^{-1}\boldsymbol{p}_h(\tilde{u}), \boldsymbol{v}_h) - (y_h(\tilde{u}), \mathrm{div}\boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{44}$$

$$(\mathrm{div}\boldsymbol{p}_h(\tilde{u}), w_h) = (\tilde{u}, w_h), \quad \forall w_h \in W_h, \tag{45}$$

$$(A^{-1}\boldsymbol{q}_h(\tilde{u}), \boldsymbol{v}_h) - (z_h(\tilde{u}), \mathrm{div}\boldsymbol{v}_h) = -(g_1'(\boldsymbol{p}_h(\tilde{u})), \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{46}$$

$$(\mathrm{div}\boldsymbol{q}_h(\tilde{u}), w_h) = (g_2'(y_h(\tilde{u})), w_h), \quad \forall w_h \in W_h. \tag{47}$$

Thus, as we defined, the exact solution and its approximation can be written in the following way:

$$\begin{aligned} (\boldsymbol{p}, y, \boldsymbol{q}, z) &= (\boldsymbol{p}(u), y(u), \boldsymbol{q}(u), z(u)), \\ (\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h) &= (\boldsymbol{p}_h(u_h), y_h(u_h), \boldsymbol{q}_h(u_h), z_h(u_h)). \end{aligned}$$

3. Error estimates for the intermediate error

In this section, we will give some error estimates for the intermediate error. First of all, we define the standard $L^2(\Omega)$-orthogonal projection $P_h : W \to W_h$ which satisfies: for any $w \in W$

$$(w - P_h w, w_h) = 0, \qquad \forall w_h \in W_h. \tag{48}$$

We also consider the Fortin projection ([5] and [9])$\Pi_h : \boldsymbol{V} \to \boldsymbol{V}_h$, which satisfies: for any $\boldsymbol{q} \in \boldsymbol{V}$,

$$(\mathrm{div}(\boldsymbol{q} - \Pi_h \boldsymbol{q}), w_h) = 0, \quad \forall w_h \in W_h. \tag{49}$$

For the projection defined above, we have the following relations(see [5], [9] and [16]):

(50) $\mathrm{div} \circ \Pi_h = P_h \circ \mathrm{div}$,

(51) $\| \boldsymbol{q} - \Pi_h \boldsymbol{q} \|_{0,r} \leq Ch|\boldsymbol{q}|_{1,r}$, for $\boldsymbol{q} \in (H^1(\Omega))^2,\ r > 1$,

(52) $\| \mathrm{div}(\boldsymbol{q} - \Pi_h \boldsymbol{q}) \|_{-s} \leq Ch^{1+s}|\mathrm{div}\boldsymbol{q}|_1$, $s = 0, 1$, for all $\mathrm{div}\boldsymbol{q} \in H^1(\Omega)$,

(53) $\| \phi - P_h \phi \|_{-s} \leq Ch^{1+s} \mid \phi \mid_1$, $s = 0, 1$, for $\phi \in H^1(\Omega)$.

Then, we recall the following existed result (see [6] and [9]) which is very useful for our work:

Lemma 3.1. *Suppose that assumptions A1-A3 are satisfied. Let $(\boldsymbol{p}, y, \boldsymbol{q}, z, u) \in (\boldsymbol{V} \times W)^2 \times K$ and $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h) \in (\boldsymbol{V}_h \times W_h)^2 \times K_h$ be the solution of (13)-(17) and (29)-(33) respectively. Then we have*

$$\| u - u_h \| \leq Ch. \tag{54}$$

Lemma 3.2. *Suppose that assumptions A1-A3 are satisfied. For any $\tilde u \in K$, let $(\boldsymbol{p}(\tilde u), y(\tilde u), \boldsymbol{q}(\tilde u), z(\tilde u))$ and $(\boldsymbol{p}_h(\tilde u), y_h(\tilde u), \boldsymbol{q}_h(\tilde u), z_h(\tilde u))$ be the solution of (36)-(39) and (44)-(47) respectively, then*

$$\| \boldsymbol{p}(\tilde u) - \boldsymbol{p}_h(\tilde u) \| + \| y(\tilde u) - y_h(\tilde u) \| \le Ch \| y(\tilde u) \|_2, \tag{55}$$

$$\| \boldsymbol{q}(\tilde u) - \boldsymbol{q}_h(\tilde u) \| + \| z(\tilde u) - z_h(\tilde u) \| \le Ch \| y(\tilde u) \|_2 . \tag{56}$$

Now, we will give the following lemma.

Lemma 3.3. *Assume that the regularity condition (42) and (43) hold , then for sufficiently small h,*

$$\| P_h y(u_h) - y_h \| \le Ch^2, \tag{57}$$

$$\| P_h z(u_h) - z_h \| \le Ch^2. \tag{58}$$

Proof. From equation (36)-(39) and (44)-(47) with the choice $\tilde u = u_h$ we can easily obtain the following error equations

$$(A^{-1}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), \boldsymbol{v}_h) - (y(u_h) - y_h, \mathrm{div}\boldsymbol{v}_h) = 0, \tag{59}$$

$$(\mathrm{div}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), w_h) = 0, \tag{60}$$

$$(A^{-1}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), \boldsymbol{v}_h) - (z(u_h) - z_h, \mathrm{div}\boldsymbol{v}_h) = -(g_1'(\boldsymbol{p}(u_h)) - g_1'(\boldsymbol{p}_h), \boldsymbol{v}_h), \tag{61}$$

$$(\mathrm{div}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), w_h) = (g_2'(y(u_h)) - g_2'(y_h), w_h), \tag{62}$$

for any $\boldsymbol{v}_h \in \boldsymbol{V}_h, w_h \in W_h$.

As a result of (48), we can rewrite (59)-(62) as

$$\begin{aligned} (A^{-1}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), \boldsymbol{v}_h) - (P_h y(u_h) - y_h, \mathrm{div}\boldsymbol{v}_h) &= 0, \\ (\mathrm{div}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), w_h) &= 0, \\ (A^{-1}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), \boldsymbol{v}_h) - (P_h z(u_h) - z_h, \mathrm{div}\boldsymbol{v}_h) &= -(g_1'(\boldsymbol{p}(u_h)) - g_1'(\boldsymbol{p}_h), \boldsymbol{v}_h), \\ (\mathrm{div}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), w_h) &= (g_2'(y(u_h)) - g_2'(y_h), w_h), \end{aligned}$$

for any $\boldsymbol{v}_h \in \boldsymbol{V}_h, w_h \in W_h$.

For sake of simplicity, we now denote

$$\tau = P_h y(u_h) - y_h, \qquad e = P_h z(u_h) - z_h. \tag{63}$$

Then, we estimate (57) and (58) in Part I and Part II respectively.

Part I. As we can see,

$$\| \tau \| = \sup_{\substack{\psi \in L^2(\Omega) \\ \psi \ne 0}} \frac{(\tau, \psi)}{\| \psi \|}, \tag{64}$$

we then need to bound (τ, ψ) for $\psi \in L^2(\Omega)$. Let $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the unique solution of (18)-(19). We can see from (49) and (63)

$$\begin{aligned} (\tau, \psi) &= (\tau, -\mathrm{div}(A\mathbf{grad}\varphi)) \\ &= -(\tau, \mathrm{div}(\Pi_h(A\mathbf{grad}\varphi))) \\ &= -(A^{-1}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), \Pi_h(A\mathbf{grad}\varphi)). \end{aligned} \tag{65}$$

Note that

$$(\mathrm{div}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), \varphi) + (A^{-1}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), A\mathbf{grad}\varphi) = 0, \tag{66}$$

so if we add the two equations (65)-(66) we can obtain

$$\begin{aligned} (\tau, \psi) = {} & (A^{-1}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), A\mathbf{grad}\varphi - \Pi_h(A\mathbf{grad}\varphi)) \\ & +(\mathrm{div}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), \varphi - P_h\varphi), \end{aligned} \tag{67}$$

where we have used (63).

We then estimate the two terms on the right side of (67). First, from Lemma 3.2, (51) and (42), it follows that

$$\begin{aligned}&(A^{-1}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h), A\mathbf{grad}\varphi-\Pi_h(A\mathbf{grad}\varphi))\\ &\quad\leq C\parallel \boldsymbol{p}(u_h)-\boldsymbol{p}_h\parallel\cdot\parallel A\mathbf{grad}\varphi-\Pi_h(A\mathbf{grad}\varphi)\parallel\\ &\quad\leq Ch^2\parallel y(u_h)\parallel_2\cdot\parallel\varphi\parallel_2\\ &\quad\leq Ch^2\parallel u_h\parallel\cdot\parallel\psi\parallel\\ &\quad\leq Ch^2\parallel\psi\parallel.\end{aligned}\tag{68}$$

Next, from equation (63) we can deduce that

$$P_h(\mathrm{div}\boldsymbol{p}(u_h))=\mathrm{div}\boldsymbol{p}_h,\tag{69}$$

then,

$$\mathrm{div}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h)=\mathrm{div}\boldsymbol{p}(u_h)-P_h(\mathrm{div}\boldsymbol{p}(u_h))=u_h-P_hu_h=0.\tag{70}$$

Here we have used the relation

$$\mathrm{div}\boldsymbol{p}(u_h)=u_h.$$

Now, it is easy to see that

$$(\mathrm{div}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h),\varphi-P_h\varphi)=0.\tag{71}$$

Inserting (68) and (71) into (67) and we can deduce that

$$\parallel\tau\parallel\leq Ch^2.\tag{72}$$

Part II. Since

$$\parallel e\parallel=\sup_{\substack{\psi\in L^2(\Omega)\\ \psi\neq 0}}\frac{(e,\psi)}{\parallel\psi\parallel},\tag{73}$$

we then need to bound (e,ψ) for $\psi\in L^2(\Omega)$. Let $\varphi\in H^2(\Omega)\cap H_0^1(\Omega)$ be the unique solution of (18)-(19). We can see from (49) and (63)

$$\begin{aligned}(e,\psi)&=(e,-\mathrm{div}(A\mathbf{grad}\varphi))\\ &=-(e,\mathrm{div}(\Pi_h(A\mathbf{grad}\varphi)))\\ &=-(A^{-1}(\boldsymbol{q}(u_h)-\boldsymbol{q}_h),\Pi_h(A\mathbf{grad}\varphi))\\ &\quad-(g_1'(\boldsymbol{p}(u_h))-g_1'(\boldsymbol{p}_h),\Pi_h(A\mathbf{grad}\varphi)).\end{aligned}\tag{74}$$

Note that

$$(\mathrm{div}(\boldsymbol{q}(u_h)-\boldsymbol{q}_h),\varphi)+(A^{-1}(\boldsymbol{q}(u_h)-\boldsymbol{q}_h),A\mathbf{grad}\varphi)=0,\tag{75}$$

so if we add the two equations (74) and (75) we can obtain

$$\begin{aligned}(e,\psi)&=(A^{-1}(\boldsymbol{q}(u_h)-\boldsymbol{q}_h),A\mathbf{grad}\varphi-\Pi_h(A\mathbf{grad}\varphi))\\ &\quad-(g_1'(\boldsymbol{p}(u_h))-g_1'(\boldsymbol{p}_h),\Pi_h(A\mathbf{grad}\varphi))\\ &\quad+(g_2'(y(u_h))-g_2'(y_h),P_h\varphi)+(\mathrm{div}(\boldsymbol{q}(u_h)-\boldsymbol{q}_h),\varphi-P_h\varphi),\end{aligned}\tag{76}$$

where we used (63).

Then, if we estimate each term on the right side of (76) in the same way as in Part I and apply the result in it, the following result can be easily obtained

$$\parallel e\parallel\leq Ch^2.$$

Thus, the Lemma has been proved. □

Lemma 3.4. *Assume that the regularity condition (42) and (43) hold , then for h sufficiently small,*

$$\| \, div(\boldsymbol{p}(u_h) - \boldsymbol{p}_h) \, \|_{0,\infty} = 0, \tag{77}$$

$$\| \, div(\boldsymbol{q}(u_h) - \boldsymbol{q}_h) \, \|_{0,\infty} \leq Ch. \tag{78}$$

Proof. First, from equation (70), (77) is obvious.

Then, we estimate (78). The error equation (62) can be written as

$$(\mathrm{div}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h), w_h) - (g_2'(y(u_h)) - g_2'(y_h), w_h) = 0, \qquad \forall w_h \in W_h.$$

It then follows that

$$P_h[\mathrm{div}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h) - (g_2'(y(u_h)) - g_2'(y_h))] = 0, \tag{79}$$

then, using (49), (50) and (79), we can see that

$$\begin{aligned} \mathrm{div}(\Pi_h \boldsymbol{q}(u_h) - \boldsymbol{q}_h) &= \mathrm{div} \circ \Pi_h(\boldsymbol{q}(u_h) - \boldsymbol{q}_h) \\ &= P_h \circ \mathrm{div}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h) \\ &= P_h(g_2'(y(u_h)) - g_2'(y_h)). \end{aligned} \tag{80}$$

Therefor, using (80), (52), Lemma 3.3 and (42)

$$\begin{aligned} &\| \, \mathrm{div}(\boldsymbol{q}(u_h) - \boldsymbol{q}_h) \, \|_{0,\infty} \\ &\quad \leq \| \, \mathrm{div}(\Pi_h \boldsymbol{q}(u_h) - \boldsymbol{q}_h) \, \|_{0,\infty} + \| \, \mathrm{div}(\Pi_h \boldsymbol{q}(u_h) - \boldsymbol{q}(u_h)) \, \|_{0,\infty} \\ &\quad \leq C\{\| \, y(u_h) - y_h \, \|_{0,\infty} + \| \, P_h \circ \mathrm{div}\boldsymbol{q}(u_h) - \mathrm{div}\boldsymbol{q}(u_h) \, \|_{0,\infty}\} \\ &\quad \leq C\{\| \, y(u_h) - P_h y(u_h) \, \|_{0,\infty} + \| \, P_h y(u_h) - y_h \, \|_{0,\infty} \\ &\qquad + \| \, P_h g_2'(y(u_h)) - g_2'(y(u_h)) \, \|_{0,\infty}\} \\ &\quad \leq Ch. \end{aligned} \tag{81}$$

□

Then, we introduce the weighted L^2-norms (see [22] and [24]) which will play an important role in our work to derive L^∞-error estimates. Let $x_0 \in \bar{\Omega}$ and $\rho > 0$. We define the weight function

$$\mu = |x - x_0|^2 + \rho^2, \quad x \in \bar{\Omega}. \tag{82}$$

And then for any $r \in R$ we define the r-weighted norm by

$$\| \, v \, \|_{r,\,\mu} = \| \, \mu^{-\frac{r}{2}} v \, \|, \qquad v \in L^2(\Omega) \ or \ (L^2(\Omega))^2. \tag{83}$$

In the following, we shall need some technical results which can be the special case of Lem 3.1 and Lem 3.2 in [16].

Lemma 3.5. *Let μ be given by (82), if $w \in (L^2(\Omega))^2$, then*

$$\| \, \mathbf{grad}\mu^{-1} \cdot w \, \| \leq C\rho^{-2} \, \| \, w \, \|_{1,\,\mu} \, . \tag{84}$$

Lemma 3.6. *If $w \in (L^\infty(\Omega))^2$, then*

$$\| \, w \, \| \leq C \, \| \, w \, \|_{1,\,\mu} \, . \tag{85}$$

We will also make use of the following relations between weighted L^2-norms and L^∞-norms [26]:

$$\| \, w \, \|_{1,\,\mu} \leq C|\ln h|^{\frac{1}{2}} \, \| \, w \, \|_{0,\,\infty}, \qquad w \in L^\infty(\Omega); \tag{86}$$

also, if $\omega \in W_h$ is a fixed element and $x_0 \in \bar{\Omega}$ is chosen so that $\| \, \omega \, \|_{0,\,\infty} = |\omega(x_0)|$, then

$$\| \, \omega \, \|_{0,\,\infty} \leq C_\kappa h^{-1}\rho \, \| \, \omega \, \|_{1,\,\mu}, \qquad \text{for } h^{-1}\rho \leq \kappa. \tag{87}$$

These results (86)-(87) can be extended to $\boldsymbol{v} \in (L^2(\Omega))^2$.

Furthermore, we need the following "super-approximability" result [26]: If η is an element of $\boldsymbol{V}_h$, then

$$\| \mu^{-1}\eta - \Pi_h(\mu^{-1}\eta) \|_{-1,\ \mu} \leq Ch\rho^{-1} \| \eta \|_{1,\ \mu}, \tag{88}$$

where C is independent of η and h.

Lemma 3.7. *Assume that the regularity condition (42) and (43) hold , then for h sufficiently small,*

$$\| \Pi_h \boldsymbol{p}(u_h) - \boldsymbol{p}_h \|_{0,\infty} \quad \leq \quad C|\ln h|^{\frac{1}{2}} h^{\frac{1}{2}}, \tag{89}$$

$$\| \Pi_h \boldsymbol{q}(u_h) - \boldsymbol{q}_h \|_{0,\infty} \quad \leq \quad C|\ln h|^{\frac{1}{2}} h^{\frac{1}{2}}. \tag{90}$$

Proof. Here, we only prove (89), (90) can be estimated in the same way.

For sake of simplicity, let us denote

$$\boldsymbol{\phi} = \Pi_h \boldsymbol{p}(u_h) - \boldsymbol{p}_h.$$

Note that

$$\begin{aligned} \| \boldsymbol{\phi} \|_{1,\mu}^2 &\leq C(A^{-1}\boldsymbol{\phi}, \mu^{-1}\boldsymbol{\phi}) \\ &\leq C\{(A^{-1}\boldsymbol{\phi}, \mu^{-1}\boldsymbol{\phi} - \Pi_h(\mu^{-1}\boldsymbol{\phi})) + (A^{-1}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), \Pi_h(\mu^{-1}\boldsymbol{\phi})) \\ &\quad +(A^{-1}(\Pi_h \boldsymbol{p}(u_h) - \boldsymbol{p}(u_h)), \Pi_h(\mu^{-1}\boldsymbol{\phi}))\}, \end{aligned} \tag{91}$$

for the first term of the right hand of (91), we can see that

$$\begin{aligned} (A^{-1}\boldsymbol{\phi}, \mu^{-1}\boldsymbol{\phi} - \Pi_h(\mu^{-1}\boldsymbol{\phi})) &= (A^{-1}\mu^{-\frac{1}{2}}\boldsymbol{\phi}, \mu^{\frac{1}{2}}(\mu^{-1}\boldsymbol{\phi} - \Pi_h(\mu^{-1}\boldsymbol{\phi}))) \\ &\leq C \| \mu^{-\frac{1}{2}}\boldsymbol{\phi} \| \cdot \| \mu^{\frac{1}{2}}(\mu^{-1}\boldsymbol{\phi} - \Pi_h(\mu^{-1}\boldsymbol{\phi})) \| \\ &= C \| \boldsymbol{\phi} \|_{1,\mu} \cdot \| \mu^{-1}\boldsymbol{\phi} - \Pi_h(\mu^{-1}\boldsymbol{\phi}) \|_{-1,\mu} \\ &\leq Ch\rho^{-1} \| \boldsymbol{\phi} \|_{1,\mu}^2, \end{aligned} \tag{92}$$

where it comes from (83) and (88). For the third term of the right side of (91), it is easy to obtain that

$$\begin{aligned} &(A^{-1}(\Pi_h \boldsymbol{p}(u_h) - \boldsymbol{p}(u_h)), \Pi_h(\mu^{-1}\boldsymbol{\phi})) \\ &\quad = (A^{-1}(\Pi_h \boldsymbol{p}(u_h) - \boldsymbol{p}(u_h)), \Pi_h(\mu^{-1}\boldsymbol{\phi}) - \mu^{-1}\boldsymbol{\phi}) \\ &\qquad +(A^{-1}(\Pi_h \boldsymbol{p}(u_h) - \boldsymbol{p}(u_h)), \mu^{-1}\boldsymbol{\phi}) \\ &\quad \leq C(\| \Pi_h \boldsymbol{p}(u_h) - \boldsymbol{p}(u_h) \|_{1,\mu} \cdot \| \Pi_h(\mu^{-1}\boldsymbol{\phi}) - \mu^{-1}\boldsymbol{\phi} \|_{-1,\mu} \\ &\qquad + \| \Pi_h \boldsymbol{p}(u_h) - \boldsymbol{p}(u_h) \|_{1,\mu} \cdot \| \boldsymbol{\phi} \|_{1,\mu}) \\ &\quad \leq C(1 + h\rho^{-1}) \| \Pi_h \boldsymbol{p}(u_h) - \boldsymbol{p}(u_h) \|_{1,\mu} \cdot \| \boldsymbol{\phi} \|_{1,\mu} . \end{aligned} \tag{93}$$

Combining (91)-(93) and then obtain

$$\begin{aligned} \| \boldsymbol{\phi} \|_{1,\mu}^2 &\leq C\{h\rho^{-1} \| \boldsymbol{\phi} \|_{1,\mu}^2 + (A^{-1}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), \Pi_h(\mu^{-1}\boldsymbol{\phi})) \\ &\quad +(1 + h\rho^{-1}) \| \Pi_h \boldsymbol{p}(u_h) - \boldsymbol{p}(u_h) \|_{1,\mu} \cdot \| \boldsymbol{\phi} \|_{1,\mu}\}, \end{aligned} \tag{94}$$

using ϵ-*Cauchy* inequality and for $h < \gamma\rho, \gamma$ sufficiently small, we then obtain

$$\| \boldsymbol{\phi} \|_{1,\mu}^2 \leq C\{\| \Pi_h \boldsymbol{p}(u_h) - \boldsymbol{p}(u_h) \|_{1,\mu}^2 + (A^{-1}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), \Pi_h(\mu^{-1}\boldsymbol{\phi}))\}. \tag{95}$$

Now, we estimate the two terms on the right side of (95) respectively. First, using (86) and (51), it follows that

$$\begin{aligned}\| \Pi_h \boldsymbol{p}(u_h) - \boldsymbol{p}(u_h) \|_{1,\mu}^2 &\le C|\ln h| \| \Pi_h \boldsymbol{p}(u_h) - \boldsymbol{p}(u_h) \|_{0,\infty}^2 \\ &\le C|\ln h| h^2 \| \boldsymbol{p}(u_h) \|_{1,\infty}^2 \\ &\le C|\ln h| h^2 \| y(u_h) \|_{2,\infty}^2 \\ &\le C|\ln h| h^2. \end{aligned} \tag{96}$$

For the second term, using equation (59), (50) and (48)

$$\begin{aligned}(A^{-1}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), \Pi_h(\mu^{-1}\boldsymbol{\phi})) &= (y(u_h) - y_h, \mathrm{div}(\Pi_h(\mu^{-1}\boldsymbol{\phi}))) \\ &= (y(u_h) - y_h, P_h \circ \mathrm{div}(\mu^{-1}\boldsymbol{\phi})) \\ &= (P_h y(u_h) - y_h, \mathrm{div}(\mu^{-1}\boldsymbol{\phi})) \\ &= (P_h y(u_h) - y_h, \mathbf{grad}\mu^{-1} \cdot \boldsymbol{\phi}) \\ &\quad +(P_h y(u_h) - y_h, \mu^{-1}\mathrm{div}\boldsymbol{\phi}). \end{aligned} \tag{97}$$

Note that

$$\mathrm{div}\boldsymbol{\phi} = \mathrm{div}(\Pi_h \boldsymbol{p}(u_h) - \boldsymbol{p}_h) = \mathrm{div} \circ \Pi_h(\boldsymbol{p}(u_h) - \boldsymbol{p}_h) = P_h \circ \mathrm{div}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h) = 0.$$

Thus, it comes from Lemma 3.5 and ϵ-*Cauchy* inequality that

$$\begin{aligned}(A^{-1}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h), \Pi_h(\mu^{-1}\boldsymbol{\phi})) &\le \| P_h y(u_h) - y_h \| \cdot \| \mathbf{grad}\mu^{-1} \cdot \boldsymbol{\phi} \| \\ &\le C(\rho^{-2} \| \boldsymbol{\phi} \|_{1,\mu} \cdot \| P_h y(u_h) - y_h \|) \\ &\le \epsilon \| \boldsymbol{\phi} \|_{1,\mu}^2 + C(\epsilon)(\rho^{-4} \| P_h y(u_h) - y_h \|^2). \end{aligned} \tag{98}$$

Combing (95), (96) and (98), we see that,

$$\| \boldsymbol{\phi} \|_{1,\mu} \le C\{|\ln h|^{\frac{1}{2}} h + \rho^{-2} \| P_h y(u_h) - y_h \|\}. \tag{99}$$

Note that Lemma 3.3 implies that

$$\begin{aligned}\| P_h y(u_h) - y_h \| &\le Ch \| \boldsymbol{p}(u_h) - \boldsymbol{p}_h \| \\ &\le Ch(\| \Pi_h \boldsymbol{p}(u_h) - \boldsymbol{p}_h \| + \| \boldsymbol{p}(u_h) - \Pi_h \boldsymbol{p}(u_h) \|) \\ &\le Ch(\| \boldsymbol{\phi} \| + h) \\ &\le Ch(\| \boldsymbol{\phi} \|_{1,\mu} + h). \end{aligned} \tag{100}$$

In the last step, we used Lemma 3.6. Then, inserting (100) into (99) yields the bound

$$\begin{aligned}\| \boldsymbol{\phi} \|_{1,\mu} &\le C\{|\ln h|^{\frac{1}{2}} h + h^2\rho^{-2} + h\rho^{-2} \| \boldsymbol{\phi} \|_{1,\mu}\} \\ &\le C\{(|\ln h|^{\frac{1}{2}} + h\rho^{-2})h + h\rho^{-2} \| \boldsymbol{\phi} \|_{1,\mu}\}, \end{aligned}$$

now, let

$$h\rho^{-2} = \frac{1}{2C},$$

that is to say

$$h = \frac{\rho^2}{2C}.$$

We then have

$$\| \boldsymbol{\phi} \|_{1,\mu} \le C|\ln h|^{\frac{1}{2}} h. \tag{101}$$

Using (87) and (101) we can obtain the bound

$$\| \boldsymbol{\phi} \|_{0,\infty} \le Ch^{-\frac{1}{2}} \| \boldsymbol{\phi} \|_{1,\mu} \le C|\ln h|^{\frac{1}{2}} h^{\frac{1}{2}}. \tag{102}$$

Thus, we completed the proof. □

4. L^∞-error estimates

In this section, we will give the L^∞-error estimates both for the control variable and the state, co-state variables.

First, we give the L^∞-error estimate for the scalar variables.

Theorem 4.1. *Assume that the regularity condition (42) and (43) hold , then for sufficiently small h,*

$$\| y - y_h \|_{0,\infty} \leq Ch, \tag{103}$$

$$\| z - z_h \|_{0,\infty} \leq Ch. \tag{104}$$

Proof. **Part I.** For (103), note that

$$\| y - y_h \|_{0,\infty} \leq \| y - y(u_h) \|_{0,\infty} + \| y(u_h) - y_h \|_{0,\infty} . \tag{105}$$

From equation (2) and (40), we have the following error equation

$$-\mathrm{div}(A\mathbf{grad}(y - y(u_h))) = u - u_h,$$

with the regularity result (20), Lemma 3.1 and the classical imbedding theorem, we can see that

$$\begin{aligned} \| y - y(u_h) \|_{0,\infty} &= \| y - y(u_h) \|_{C(\bar{\Omega})} \\ &\leq C \| y - y(u_h) \|_2 \\ &\leq C \| u - u_h \| \\ &\leq Ch. \end{aligned} \tag{106}$$

Then, for the second term of (105), using Lemma 3.3 and (53), it follows that

$$\begin{aligned} \| y(u_h) - y_h \|_{0,\infty} &\leq \| y(u_h) - P_h y(u_h) \|_{0,\infty} + \| P_h y(u_h) - y_h \|_{0,\infty} \\ &\leq C\{h \| y(u_h) \|_{1,\infty} + h^{-1} \| P_h y(u_h) - y_h \|\} \\ &\leq Ch, \end{aligned} \tag{107}$$

where we used the inverse inequality in finite dimensional space. Combing (106) and (107) we can obtain the (103).

Part II. For (104), also we have

$$\| z - z_h \|_{0,\infty} \leq \| z - z(u_h) \|_{0,\infty} + \| z(u_h) - z_h \|_{0,\infty} . \tag{108}$$

From equation (8) and (41), we can obtain that

$$-\mathrm{div}(A\mathbf{grad}(z - z(u_h)) + g_1'(\boldsymbol{p}) - g_1'(\boldsymbol{p}(u_h))) = g_2'(y) - g_2'(y(u_h)),$$

with the regularity result (20), Lemma 3.1 and the classical imbedding theorem, we can see that

$$\begin{aligned} \| z - z(u_h) \|_{0,\infty} &= \| z - z(u_h) \|_{C(\bar{\Omega})} \\ &\leq C \| z - z(u_h) \|_2 \\ &\leq C \| y - y(u_h) \| \\ &\leq C \| u - u_h \| \\ &\leq Ch. \end{aligned} \tag{109}$$

For the second term of (108), using Lemma 3.3 and (53), it follows that

$$\begin{aligned} \| z(u_h) - z_h \|_{0,\infty} &\leq \| z(u_h) - P_h z(u_h) \|_{0,\infty} + \| P_h z(u_h) - z_h \|_{0,\infty} \\ &\leq C\{h \| z(u_h) \|_{1,\infty} + h^{-1} \| P_h z(u_h) - z_h \|\} \\ &\leq Ch, \end{aligned} \tag{110}$$

where we used the inverse inequality. Combing (109) and (110) we can obtain the (104).

Thus, we have completed the proof. □

In the following, we will give the L^∞-error estimate for the control variable.

Theorem 4.2. *Let u and u_h be the optimal control of (13)-(17) and (29)-(33) respectively, then for h sufficiently small*

$$\| u - u_h \|_{0,\infty} \leq Ch. \tag{111}$$

Proof. Due to Lemma 2.2 and Lemma 2.3, there exist $s(x) \in C^{0,1}(\bar{\Omega})$ and $s_h(x) \in L^\infty(\Omega)$ such that for $\forall T \in \mathcal{T}^h$

$$h'(s(x)) + z(x) = 0, \qquad \forall x \in T, \tag{112}$$

$$s_h(x) = s_T, \quad \int_T (h'(s_T) + z_h(x))dx = 0, \qquad \forall x \in T. \tag{113}$$

From (113), we deduce that for every $T \in \mathcal{T}^h$, every $x \in T$,

$$h'(s_T) + z_h(x) = 0. \tag{114}$$

Suppose that $T \in \mathcal{T}^h$ is given fixed, and select an arbitrary $x \in T$. Note that the projection $\Pi_{[a,b]}$ defined in (21) is Lipschitz continuous, and due to the assumption on $h''(u)$, it then follows from (112) and (114) that

$$\begin{aligned} m|u(x) - u_h(x)| &= m|\Pi_{[a,b]}(s(x)) - \Pi_{[a,b]}(s_h(x))| \\ &\leq m|s(x) - s_h(x)| \\ &= m|s(x) - s_T| \\ &\leq |h'(s(x)) - h'(s_T)| \\ &= |z(x) - z_h(x)|. \end{aligned}$$

Hence

$$m|u(x) - u_h(x)| \leq \| z - z_h \|_{0,\infty,T},$$

along with the result of (104), we can easily deduce that

$$\begin{aligned} \| u - u_h \|_{0,\infty,\Omega} &= \sup_{T \in \mathcal{T}^h} \| u - u_h \|_{0,\infty,T} \\ &\leq C \| z - z_h \|_{0,\infty,\Omega} \\ &\leq Ch. \end{aligned}$$

□

Now, we give our third theorem.

Theorem 4.3. *Assume that the regularity condition (42) and (43) hold , then for h sufficiently small,*

$$\| \mathit{div}(\boldsymbol{p} - \boldsymbol{p}_h) \|_{0,\infty} \leq Ch, \tag{115}$$

$$\| \mathit{div}(\boldsymbol{q} - \boldsymbol{q}_h) \|_{0,\infty} \leq Ch. \tag{116}$$

Proof. **Part I.** For (115) , it is easy to see that

$$\| \mathrm{div}(\boldsymbol{p} - \boldsymbol{p}_h) \|_{0,\infty} \leq \| \mathrm{div}(\boldsymbol{p} - \boldsymbol{p}(u_h)) \|_{0,\infty} + \| \mathrm{div}(\boldsymbol{p}(u_h) - \boldsymbol{p}_h) \|_{0,\infty}, \tag{117}$$

so we only need to bound the first term. Note the following error equation which comes from (37) with $\tilde{u} = u$ and $\tilde{u} = u_h$ respectively

$$(\mathrm{div}\boldsymbol{p} - \boldsymbol{p}(u_h), w) = (u - u_h, w), \qquad \forall w \in L^2(\Omega), \tag{118}$$

then we have

$$\| \operatorname{div}(\boldsymbol{p}-\boldsymbol{p}(u_h)) \|_{0,\infty}=\| u-u_h \|_{0,\infty} . \tag{119}$$

Combining (119) and (77) we can get that

$$\| \operatorname{div}(\boldsymbol{p}-\boldsymbol{p}(u_h)) \|_{0,\infty} + \| \operatorname{div}(\boldsymbol{p}(u_h)-\boldsymbol{p}_h) \|_{0,\infty}\le Ch, \tag{120}$$

where we used Theorem 4.2.

Part II. We now estimate (116). First, note that

$$\| \operatorname{div}(\boldsymbol{q}-\boldsymbol{q}_h) \|_{0,\infty}\le\| \operatorname{div}(\boldsymbol{q}-\boldsymbol{q}(u_h)) \|_{0,\infty} + \| \operatorname{div}(\boldsymbol{q}(u_h)-\boldsymbol{q}_h) \|_{0,\infty} . \tag{121}$$

For the first term of the right hand of (121), note the following error equation which comes from (39) with $\tilde{u}=u$ and $\tilde{u}=u_h$ respectively

$$(\operatorname{div}\boldsymbol{q}-\boldsymbol{q}(u_h), w)=(g_2'(y)-g_2'(y(u_h)), w), \qquad \forall w\in L^2(\Omega), \tag{122}$$

then it is easy to know that

$$\| \operatorname{div}(\boldsymbol{q}-\boldsymbol{q}(u_h)) \|_{0,\infty}=\| g_2'(y)-g_2'(y(u_h)) \|_{0,\infty}\le Ch, \tag{123}$$

where we used (106). Then combining (123) and (78) it follows that

$$\| \operatorname{div}(\boldsymbol{q}-\boldsymbol{q}(u_h)) \|_{0,\infty} + \| \operatorname{div}(\boldsymbol{q}(u_h)-\boldsymbol{q}_h) \|_{0,\infty}\le Ch. \tag{124}$$

Thus, we completed the proof of the theorem. □

At last, we give the error estimate for the flux variables.

Theorem 4.4. *Assume that the regularity condition (42) and (43) hold , then for h sufficiently small,*

$$\| \boldsymbol{p}-\boldsymbol{p}_h \|_{0,\infty} \le Ch^{\frac{1}{2}}|\ln h|, \tag{125}$$

$$\| \boldsymbol{q}-\boldsymbol{q}_h \|_{0,\infty} \le Ch^{\frac{1}{2}}|\ln h|. \tag{126}$$

Proof. **Part I.** By triangular inequality

$$\| \boldsymbol{p}-\boldsymbol{p}_h \|_{0,\infty} \le \| \boldsymbol{p}-\boldsymbol{p}(u_h) \|_{0,\infty} + \| \boldsymbol{p}(u_h)-\boldsymbol{p}_h \|_{0,\infty} . \tag{127}$$

For the first term of the right side of (127), note that

$$\begin{aligned}
\| \boldsymbol{p}-\boldsymbol{p}(u_h) \|_{0,\infty} &= \| A(\mathbf{grad}y-\mathbf{grad}y(u_h)) \|_{0,\infty}\\
&\le C\| y-y(u_h) \|_{1,\infty}\\
&\le C\| u-u_h \|_{0,\infty}\\
&\le Ch, \qquad (128)
\end{aligned}$$

where we used (20) and Theorem 4.2.

For the second term, it is easy to see that

$$\begin{aligned}
\| \boldsymbol{p}(u_h)-\boldsymbol{p}_h \|_{0,\infty} &\le \| \boldsymbol{p}(u_h)-\Pi_h\boldsymbol{p}(u_h) \|_{0,\infty} + \| \Pi_h\boldsymbol{p}(u_h)-\boldsymbol{p}_h \|_{0,\infty}\\
&\le C\{h\| \boldsymbol{p}(u_h) \|_{1,\infty} +|\ln h|^{\frac{1}{2}}h^{\frac{1}{2}}\}\\
&\le Ch^{\frac{1}{2}}\{h^{\frac{1}{2}}\| y(u_h) \|_{2,\infty} +|\ln h|^{\frac{1}{2}}\}\\
&\le Ch^{\frac{1}{2}}|\ln h|, \qquad (129)
\end{aligned}$$

where we used (51), (89) and (20). Thus, combing (127), (128) and (129), we obtain that

$$\| \boldsymbol{p}-\boldsymbol{p}_h \|_{0,\infty}\le Ch^{\frac{1}{2}}|\ln h|. \tag{130}$$

Part II. Now, we estimate (126). Using (51), (128) , (90), (20) and Theorem 4.2, we can get that

$$
\begin{aligned}
\| \boldsymbol{q} - \boldsymbol{q}_h \|_{0,\infty} &\leq \| \boldsymbol{q} - \boldsymbol{q}(u_h) \|_{0,\infty} + \| \boldsymbol{q}(u_h) - \boldsymbol{q}_h \|_{0,\infty} \\
&\leq \| A(\mathbf{grad} z + g_1'(\boldsymbol{p})) - A(\mathbf{grad} z(u_h) + g_1'(\boldsymbol{p}(u_h)) \|_{0,\infty} \\
&\quad + \| \boldsymbol{q}(u_h) - \Pi_h \boldsymbol{q}(u_h) \|_{0,\infty} + \| \Pi_h \boldsymbol{q}(u_h) - \boldsymbol{q}_h \|_{0,\infty} \\
&\leq C\{\| z - z(u_h) \|_{1,\infty} + \| \boldsymbol{p} - \boldsymbol{p}(u_h) \|_{0,\infty} \\
&\quad + h \| \boldsymbol{q}(u_h) \|_{1,\infty} + |\ln h|^{\frac{1}{2}} h^{\frac{1}{2}}\} \\
&\leq C\{\| u - u_h \|_{0,\infty} + h + |\ln h|^{\frac{1}{2}} h^{\frac{1}{2}}\} \\
&\leq Ch^{\frac{1}{2}} |\ln h|. \qquad (131)
\end{aligned}
$$

Thus, we completed the proof. □

5. Conclusion and future works

In this paper, we give a complete estimate for control variable, state variables and co-state variables of optimal control problem (1.1) and (1.3)-(1.4) using mixed finite element methods. Our L^∞-error estimates for the co-state variables by mixed methods seem to be new. We have used piecewise constant functions to approximate the control variable. In our future work, we shall use the standard linear element space to approximate the control function. Furthermore, we shall consider the optimal boundary control problem.

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