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The new multi-order exact solutions of some nonlinear evolution equations

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Abstract. Based on the Lamé equation and Jacobi elliptic function, the perturbation method is applied to some nonlinear evolution equations. And there many multi-order solutions are derived to these nonlinear evolution equations. These multi-order solutions correspond to the different periodic solutions, which can degenerate to the different soliton solutions. The method can be also applied to many other nonlinear evolution equations.

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Key words: Lamé equation, Lamé function, multi-order exact solutions, Jacobi elliptic function, perturbation method, nonlinear evolution equations

1 Introduction

To find the exact solutions of the nonlinear evolution equations plays an important role in nonlinear studies. Applying some new methods, such as inverse scattering transformation [1], Bäcklund transformation [2], Darboux transformation [3], Hirota method [4], homogeneous balance method [5], Lie group method [6], sine-cosine method [7], homotopy perturbation method [8], variational method [9], tanh method [10], exp-method [11] and the JEFE method [12,13] and so on, many exact solutions are obtained, from which rich structures are shown to exist in different nonlinear wave equations. Furthermore, in order to discuss the stability of these solutions, one must superimpose a small disturbance on these solutions and analysis the evolution of the small disturbance. This is equivalent to the solutions of nonlinear evolution equations expanded as a power series in terms of a small parameter and derive multi-order exact solutions [14-17]. In the paper, using Jacobi elliptic function expansion method, the new multi-order periodic solutions of four nonlinear evolution equations are obtained by means of the Jacobi elliptic function and the new Lamé functions. They contain

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some previous exact periodic solutions. At the limit condition, the periodic solutions give corresponding solitary wave solutions.

This paper is organized as follows: In Section 2, we give the introduction of the Lamé function. In Section 3 and Section 4, we apply two kinds of Lamé functions $L_2(\xi)$ and $L_3(\xi)$ to solve nonlinear evolution equations and to derive their corresponding multi-order exact solutions, respectively. Conclusions will be presented in finally.

2 Lamé function

In general, the Lamé equation [18,19] in terms of $y(x)$ can be written as

$$\frac{d^2y(x)}{dx^2}+\left[\lambda-p(p+1)\mathrm{cs}^2(\xi)\right]y(x)=0, \tag{1}$$

where λ is an eigenvalue, p is a positive integer, $\mathrm{cs}(\xi)=\mathrm{cn}(\xi)/\mathrm{sn}(\xi)$ is a kind of Jacobi elliptic functions with its modulus m $(0<m<1)$.

Set

$$\eta=\mathrm{cs}^2(x). \tag{2}$$

Then the Lamé equation (1) becomes

$$\frac{d^2y}{d\eta^2}+\frac{1}{2}\left(\frac{1}{\eta}+\frac{1}{\eta+1}+\frac{h}{h\eta+h-1}\right)\frac{dy}{d\eta}-\frac{\mu+p(p+1)\eta h}{4\eta(\eta+1)(h\eta+h-1)}y=0, \tag{3}$$

where

$$h=m^{-2}>1, \qquad \mu=-h\lambda \tag{4}$$

Eq. (3) is a kind of the Fuchs-typed equations with four regular points $\eta=0$, -1, $h^{-1}-1$ and ∞, the solution of the Lamé equation (1) is known as Lamé function.

For example, when $p=2$, $\lambda=m^2-2$, i.e. $\mu=-h\lambda=-(1-2m^{-2})$, the Lamé function is

$$L_2(x)=(1-h^{-1}+\eta)^{\frac{1}{2}}(1+\eta)^{\frac{1}{2}}=\mathrm{ds}(x)\mathrm{ns}(x), \tag{5}$$

when $p=3$, $\lambda=4(m^2-2)$, i.e. $\mu=-h\lambda=-4(1-2m^{-2})$, the Lamé function is

$$L_3(x)=\eta^{\frac{1}{2}}(1-h^{-1}+\eta)^{\frac{1}{2}}(1+\eta)^{\frac{1}{2}}=\mathrm{cs}(x)\mathrm{ds}(x)\mathrm{ns}(x). \tag{6}$$

In (5) and (6), $\mathrm{ns}(x)=1/\mathrm{sn}(x)$ and $\mathrm{ds}(x)=\mathrm{dn}(x)/\mathrm{sn}(x)$ are two kinds of the Jacobi elliptic functions. In the next sections, we will apply these two kinds of Lamé functions $L_2(x)$ and $L_3(x)$ to solve nonlinear evolution equations and to derive their corresponding multi-order exact solutions.

3 Multi-order exact solutions with $L_2(x)$

In this case, the Lamé equation (1) reduces to

$$\frac{d^2y}{dx^2}+\Big[(m^2-2)-6\mathrm{cs}^2(x)\Big]y=0. \tag{7}$$

Here $p=2$ and $\lambda=m^2-2$ are chosen for (1) and the solution to (7) is (5). Next, we will illustrate the application of (7) to solve some nonlinear evolution equations.

3.1 The multi-order exact solutions of $MKdV$ equation

$MKdV$ equation reads

$$\frac{\partial u}{\partial t}+\alpha u^2\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}=0, \tag{8}$$

We seek its travelling wave solutions of the following form

$$u=u(\xi),\xi=k(x-ct). \tag{9}$$

where k and c are wave number and wave speed, respectively.

Substituting (9) into (8), we have

$$\beta k^2\frac{d^3u}{d\xi^3}+\alpha u^2\frac{du}{d\xi}-c\frac{du}{d\xi}=0. \tag{10}$$

Integrating (10) once with respect to ξ and taking the integration constants as zero, we get

$$\beta k^2\frac{d^2u}{d\xi^2}+\frac{\alpha}{3}u^2\frac{du}{d\xi}-cu=0. \tag{11}$$

Here we consider perturbation method and set

$$u=u_0+\epsilon u_1+\epsilon^2 u_2+\cdots, \tag{12}$$

where ϵ $(0<\epsilon\ll 1)$ is a small parameter, u_0, u_1 and u_2 represent the zeroth-order, first-order and second-order solutions, respectively.

Substituting (12) into (11), we derive the following systems of the zeroth-order, the first-order and the second-order equations

$$\epsilon^0:\beta k^2\frac{d^2u_0}{d\xi^2}+\frac{\alpha}{3}u_0^3-cu_0=0, \tag{13}$$

$$\epsilon^1:\beta k^2\frac{d^2u_1}{d\xi^2}+(\alpha u_0^2-c)u_1=0, \tag{14}$$

$$\epsilon^2:\beta k^2\frac{d^2u_2}{d\xi^2}+(\alpha u_0^2-c)u_2=-\alpha u_0u_1^2. \tag{15}$$

The zeroth-order equation (13) can be solved by the Jacobi elliptic function expansion method. The ansätz solution

$$u_0 = a_0 + a_1 \mathrm{cs}(\xi) \tag{16}$$

can be assumed.

Substituting (16) into (13), the expansion coefficients a_0 and a_1 can be easily determined as

$$a_0 = 0, a_1 = \pm\sqrt{-\frac{6\beta k^2}{\alpha}}, \qquad c = -(m^2-2)\beta k^2. \tag{17}$$

So the zeroth-order exact solution is

$$u_0 = \pm\sqrt{-\frac{6\beta k^2}{\alpha}}\mathrm{cs}(\xi). \tag{18}$$

Substituting the zeroth-order exact solution (18) into the first-order equation (14) yields

$$\frac{d^2 u_1}{d\xi^2} + \left[(m^2-2) - 6\mathrm{cs}^2(\xi)\right]u_1 = 0. \tag{19}$$

Obviously this is just the Lamé equation(7) with $p=2$ and $\lambda = m^2-2$, so its solution is

$$u_1 = AL_2(\xi) = A\mathrm{ds}(\xi)\mathrm{ns}(\xi), \tag{20}$$

where A is an arbitrary constant. (20) is the first-order exact solution of $MKdV$ equation (8).

In order to solve the second-order equation (15), the zeroth-order exact solution (18) and the first-order exact solution (20) have to be substituted into (15), thus the second-order equation (15) is rewritten as

$$\frac{d^2 u_1}{d\xi^2} + [(m^2-2) - 6\mathrm{cs}^2(\xi)]u_1 = \pm 3A^2\sqrt{\frac{-2\alpha}{3\beta k^2}}\mathrm{cs}(\xi)\mathrm{ds}^2(\xi)\mathrm{ns}^2(\xi) \tag{21}$$

Considering $\mathrm{ds}^2(\xi) = 1 - m^2 + \mathrm{cs}^2(\xi)$ and $\mathrm{ns}^2(\xi) = 1 + \mathrm{cs}^2(\xi)$, (21) can be written as

$$\begin{aligned} &\frac{d^2 u_1}{d\xi^2} + \left[(m^2-2) - 6\mathrm{cs}^2(\xi)\right]u_1 \\ &= \pm 3A^2\sqrt{\frac{-2\alpha}{3\beta k^2}}\left[(1-m^2)\mathrm{cs}(\xi) + (2-m^2)\mathrm{cs}^3(\xi) + \mathrm{cs}^5(\xi)\right]. \end{aligned} \tag{22}$$

It is obvious that (22) is an inhomogeneous Lamé equation with $p=2$ and $\lambda = m^2-2$. Its solution of homogeneous equation is just the same one as (20). And its special solution of inhomogeneous terms can be assumed to be

$$u_2 = b_1 \mathrm{cs}(\xi) + b_3 \mathrm{cs}^3(\xi). \tag{23}$$

Substituting (23) into (22), we can determine the expansion coefficients b_1 and b_3 as

$$b_1=\pm\frac{A^2(m^2-2)}{4}\sqrt{\frac{-2\alpha}{3\beta k^2}},b_3=\mp\frac{A^2}{2}\sqrt{\frac{-2\alpha}{3\beta k^2}}. \tag{24}$$

So the second-order exact solution of $MKdV$ equation (8) can be written as

$$u_2=\pm\frac{A^2(m^2-2)}{4}\sqrt{\frac{-2\alpha}{3\beta k^2}}\mathrm{cs}(\xi)\left[1-\frac{2}{m^2-2}\mathrm{cs}^2(\xi)\right]. \tag{25}$$

Finally, substituting (18), (20) and (25) into (12) and truncating the expansion, we can get a second-order asymptotic periodic solution of $MKdV$ equation (8) as follows

$$\tilde{u}=\pm\sqrt{-\frac{6\beta k^2}{\alpha}}\mathrm{cs}(\xi)+\epsilon A\mathrm{ds}(\xi)\mathrm{ns}(\xi)\pm\epsilon^2\frac{A^2(m^2-2)}{4}\sqrt{\frac{-2\alpha}{3\beta k^2}}\mathrm{cs}(\xi)\left[1-\frac{2}{m^2-2}\mathrm{cs}^2(\xi)\right], \tag{26}$$

where A is an arbitrary constant and ϵ a small parameter.

Remark 3.1. When $m\rightarrow 1$, the zero-order exact solution (18), first-order exact solution (20), second-order exact solution (25) and second-order asymptotic solution (26) of $MKdV$ equation (8) can respectively degenerate as follows

$$u_0=\pm\sqrt{-\frac{6\beta k^2}{\alpha}}\mathrm{csch}(\xi), \tag{27}$$

$$u_1=AL_2(\xi)=A\mathrm{csch}(\xi)\coth(\xi), \tag{28}$$

$$u_2=\mp\frac{A^2}{4}\sqrt{\frac{-2\alpha}{3\beta k^2}}\mathrm{csch}(\xi)\left[1+2\mathrm{csch}^2(\xi)\right], \tag{29}$$

$$\tilde{u}=\pm\sqrt{-\frac{6\beta k^2}{\alpha}}\mathrm{csch}(\xi)+\epsilon A\mathrm{csch}(\xi)\coth(\xi)\mp\epsilon^2\frac{A^2}{4}\sqrt{\frac{-2\alpha}{3\beta k^2}}\mathrm{csch}(\xi)\left[1+2\mathrm{csch}^2(\xi)\right]. \tag{30}$$

Remark 3.2. When $m\rightarrow 0$, the zero-order exact solution (18), first-order exact solution (20), second-order exact solution (25) and second-order asymptotic solution (26) of $MKdV$ equation (8) can respectively degenerate as follows

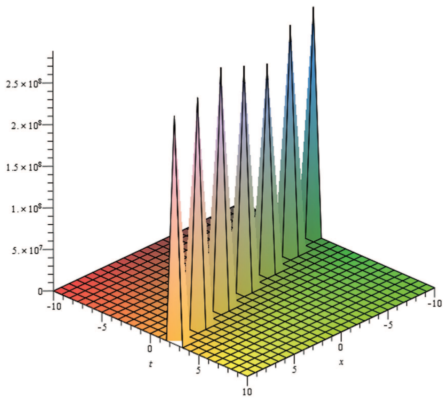
$$u_0=\pm\sqrt{-\frac{6\beta k^2}{\alpha}}\cot(\xi), \tag{31}$$

$$u_1=AL_2(\xi)=A\csc^2(\xi), \tag{32}$$

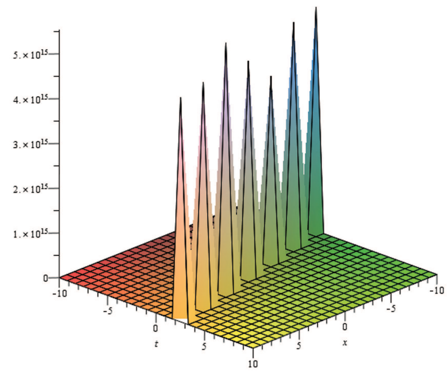
$$u_2=\mp\frac{A^2}{2}\sqrt{\frac{-2\alpha}{3\beta k^2}}\cot(\xi)\left[1+\cot^2(\xi)\right], \tag{33}$$

$$\tilde{u}=\pm\sqrt{-\frac{6\beta k^2}{\alpha}}\cot(\xi)+\epsilon A\csc^2(\xi)\mp\epsilon^2\frac{A^2}{2}\sqrt{\frac{-2\alpha}{3\beta k^2}}\cot(\xi)\left[1+\cot^2(\xi)\right]. \tag{34}$$

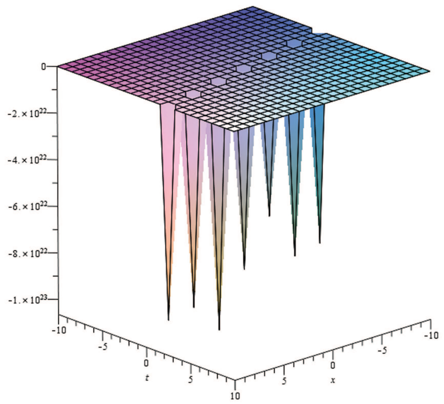
Remark 3.3. In order to have a better understanding of the properties of the solutions obtained above, four figures (Fig. 1) are plotted to illustrate the zero-order exact solution (18), first-order exact solution (20), second-order exact solution (25) and second-order asymptotic solution (26) of $MKdV$ equation (8) with $\alpha=4$, $\beta=-5$, $k=2$, $c=4$, $A=2$, $m=0.8$, $\epsilon=0.001$.



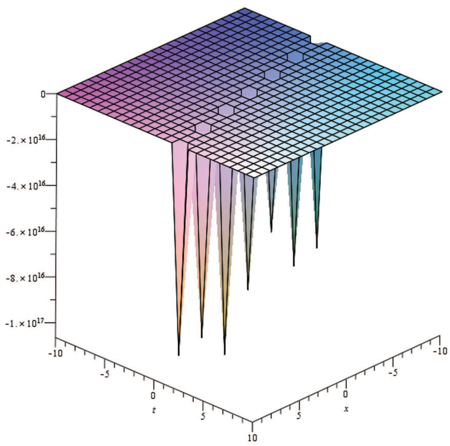
(a) The zero-order exact solution (18)



(b) The first-order exact solution (20)



(c) The second-order exact solution (25)



(d) The second-order asymptotic solution (26)

Figure 1: The multi-order exact solutions of $MKdV$ equation (8).

3.2 The multi-order exact solutions of mBBM equation

Modified Benjamin-Bona-Mahony (mBBM) equation reads

$$\frac{\partial u}{\partial t}+c_0\frac{\partial u}{\partial t}+\alpha u^2\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial^2 x\partial t}=0. \tag{35}$$

Substituting (9) into (35) yields

$$\beta k^2 c\frac{\partial^3 u}{\partial^3 \xi}-u^2\frac{\partial u}{\partial \xi}+(c-c_0)\frac{\partial u}{\partial \xi}=0. \tag{36}$$

Integrating (36) once with respect to ξ and taking integration constant as zero, we have

$$\beta k^2 c\frac{d^2 u}{d\xi^2}-\frac{1}{3}u^3+(c-c_0)u=0. \tag{37}$$

Substituting (12) into (37), we get the zeroth-order, the first-order and the second-order equations

$$\epsilon^0:\beta k^2 c\frac{d^2 u_0}{d\xi^2}+\frac{1}{3}u_0^3+(c-c_0)u_0=0, \tag{38}$$

$$\epsilon^1:\beta k^2 c\frac{d^2 u_1}{d\xi^2}+(-u_0^2+(c-c_0))u_1=0, \tag{39}$$

$$\epsilon^2:\beta k^2 c\frac{d^2 u_2}{d\xi^2}+(-u_0^2+(c-c_0))u_2=u_0u_1^2. \tag{40}$$

Applying (16) to (18), the zeroth-order exact solution can be easily obtained

$$u_0=\mp\sqrt{6\beta k^2 c}\,\mathrm{cs}(\xi), c-c_0=(m^2-2)\beta k^2 c. \tag{41}$$

Similarly, substituting (41) into the first-order equation (39) leads to

$$\frac{d^2 u_1}{d\xi^2}+\Big[(m^2-2)-6\mathrm{cs}^2(\xi)\Big]u_1=0. \tag{42}$$

Obviously this is the Lamé equation(7), its solution is

$$u_1=AL_2(\xi)=A\mathrm{ds}(\xi)\mathrm{ns}(\xi), \tag{43}$$

where A is an arbitrary constant.

Substituting the zeroth-order exact solution (41) and the first-order exact solution (43) into the second-order equation (40) results in

$$\frac{d^2 u_2}{d\xi^2}+\Big[(m^2-2)-6\mathrm{cs}^2(\xi)\Big]u_2=\pm A^2\sqrt{6}\beta k^2 c\,\mathrm{cs}(\xi)\mathrm{ds}^2(\xi)\mathrm{ns}^2(\xi). \tag{44}$$

Then combining (44) with (23) reaches the second-order exact solution of mBBM equation (35)

$$u_2=\pm\frac{A^2(m^2-2)}{4}\sqrt{\frac{2}{3\beta k^2c}}\mathrm{cs}(\xi)\left[1-\frac{2}{m^2-2}\mathrm{cs}^2(\xi)\right]. \tag{45}$$

Finally, substituting (41), (43) and (45) into (12) and truncating the expansion, we can get a second-order asymptotic periodic solution of mBBM equation (35) as follows

$$\tilde{u}=\pm\sqrt{6\beta k^2c}\,\mathrm{cs}(\xi)+\epsilon A\mathrm{ds}(\xi)\mathrm{ns}(\xi)\pm\epsilon^2\frac{A^2(m^2-2)}{4}\sqrt{\frac{2}{3\beta k^2c}}\mathrm{cs}(\xi)\left[1-\frac{2}{m^2-2}\mathrm{cs}^2(\xi)\right]. \tag{46}$$

where A is an arbitrary constant and ϵ a small parameter.

Remark 3.4. When $m\to 1$, the zero-order exact solution (41), first-order exact solution (43), second-order exact solution (45) and second-order asymptotic solution (46) of mBBM equation (35) can respectively degenerate as follows

$$u_0=\pm\sqrt{6\beta k^2c}\,\mathrm{csch}(\xi), \tag{47}$$

$$u_1=A\mathrm{csch}(\xi)\coth(\xi), \tag{48}$$

$$u_2=\mp\frac{A^2}{4}\sqrt{\frac{2}{3\beta k^2c}}\mathrm{csch}(\xi)\left[1+2\mathrm{csch}^2(\xi)\right], \tag{49}$$

$$\tilde{u}=\pm\sqrt{6\beta k^2c}\,\mathrm{csch}(\xi)+\epsilon A\mathrm{csch}(\xi)\coth(\xi)\mp\epsilon^2\frac{A^2}{4}\sqrt{\frac{2}{3\beta k^2c}}\mathrm{csch}(\xi)\left[1+2\mathrm{csch}^2(\xi)\right]. \tag{50}$$

Remark 3.5. When $m\to 0$, the zero-order exact solution (41), first-order exact solution (43), second-order exact solution (45) and second-order asymptotic solution (46) of mBBM equation (35) can respectively degenerate as follows

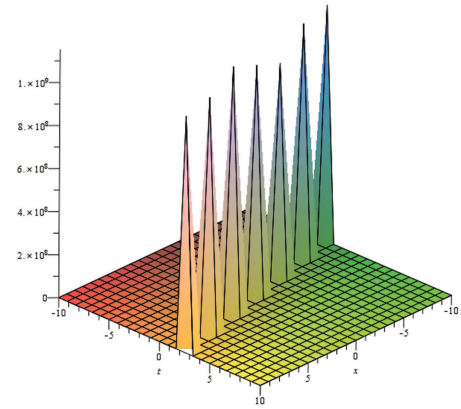
$$u_0=\pm\sqrt{6\beta k^2c}\cot(\xi), \tag{51}$$

$$u_1=A\csc^2(\xi), \tag{52}$$

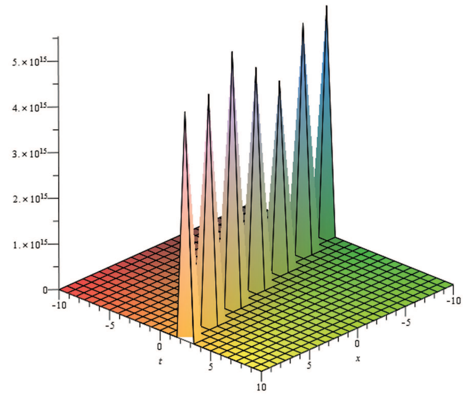
$$u_2=\mp\frac{A^2}{2}\sqrt{\frac{2}{3\beta k^2c}}\cot(\xi)\left[1+\cot^2(\xi)\right], \tag{53}$$

$$\tilde{u}=\pm\sqrt{6\beta k^2c}\cot(\xi)+\epsilon A\csc^2(\xi)\mp\epsilon^2\frac{A^2}{2}\sqrt{\frac{2}{3\beta k^2c}}\cot(\xi)\left[1+\cot^2(\xi)\right]. \tag{54}$$

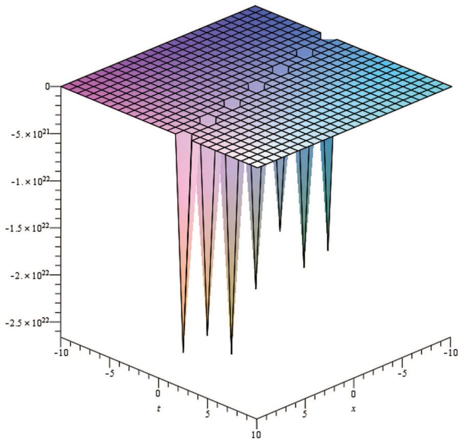
Remark 3.6. In order to have a better understanding of the properties of the solutions obtained above, four figures (Fig. 2) are plotted to illustrate the zero-order exact solution (41), first-order exact solution (43), second-order exact solution (45) and second-order asymptotic solution (46) of mBBM equation (35) with $\beta=5$, $k=2$, $c=4$, $A=2$, $m=0.8$, $\epsilon=0.001$.



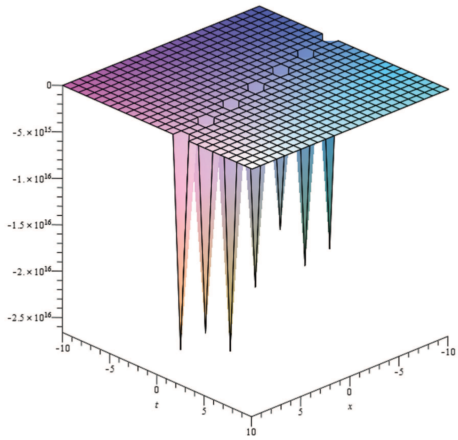
(a) The zero-order exact solution (41)



(b) The first-order exact solution (43)



(c) The second-order exact solution (45)



(d) The second-order asymptotic solution (46)

Figure 2: The multi-order exact solutions of mBBM equation (35).

4 Multi-order exact solutions with $L_3(x)$

In this case, the Lamé equation (1) reduces to

$$\frac{d^2y}{dx^2}+\left[4(m^2-2)-12\mathrm{cs}^2(x)\right]y=0. \tag{55}$$

Here $p=3$ and $\lambda=4(m^2-2)$ are chosen for (1) and the solution to (55) is (6). Next, we will illustrate the application of (55) to solve some nonlinear evolution equations.

4.1 The multi-order exact solutions of KdV equation

KdV equation reads

$$\frac{du}{dt}+\alpha u\frac{du}{dx}+\beta\frac{d^3u}{dx^3}=0. \tag{56}$$

Seeking its travelling wave solution in the frame of (9), so we have

$$\beta k^2\frac{d^3u}{d\xi^3}-c\alpha u\frac{du}{d\xi}+\alpha u\frac{du}{d\xi}=0. \tag{57}$$

which can be integrated once with respect to ξ and the integration constant is taken to be zero. So (57) can be rewritten

$$2\beta k^2\frac{d^2u}{d\xi^2}-2c\alpha u+\alpha u^2=0. \tag{58}$$

Considering the perturbation method and (12) and (58) can be expanded as multi-order equation and the first three order equation are

$$\epsilon^0: 2\beta k^2\frac{d^2u_0}{d\xi^2}+\alpha u_0^2-2cu_0=0, \tag{59}$$

$$\epsilon^1: 2\beta k^2\frac{d^2u_1}{d\xi^2}+(2\alpha u_0-2c)u_1=0, \tag{60}$$

$$\epsilon^2: 2\beta k^2\frac{d^2u_2}{d\xi^2}+(2\alpha u_0-2c)u_2=-\alpha u_1^2. \tag{61}$$

From the zeroth-order equation (59) and the ansätz solution

$$u_0=a_0+a_1\,\mathrm{cs}(\xi)+a_2\,\mathrm{cs}(\xi)^2. \tag{62}$$

So we can get the zeroth-order exact solution of KdV equation (56)

$$u_0=\frac{2\beta k^2(-4+2m^2\pm2\sqrt{1-m^2+m^4})}{\alpha}-\frac{12\beta k^2}{\alpha}\mathrm{cs}(\xi), \tag{63a}$$

$$c=-4\beta k^2(m^2-2)+2\beta k^2\left(-4+2m^2\pm2\sqrt{1-m^2+m^4}\right). \tag{63b}$$

Substituting the zeroth-order exact solution (63) into the first-order equation (60) leads to

$$\frac{d^2u_1}{d\xi^2}+\left[4(m^2-2)-12\mathrm{cs}^2(\xi)\right]u_1=0, \tag{64}$$

which takes the same form as the Lamé equation (55) with $p=3$ and $\lambda=4(m^2-2)$, so the first-order exact solution can be written as

$$u_1=AL_3(\xi)=A\mathrm{cs}(\xi)\mathrm{ns}(\xi)\mathrm{ds}(\xi), \tag{65}$$

which A is an arbitrary constant.

Substituting the zeroth-order exact solution (63) and the first-order exact solution (65) into the second-order equation (61) results in

$$\frac{d^2u_2}{d\xi^2}+\left[4(m^2-2)-12\mathrm{cs}(\xi)\right]u_2=-\frac{\alpha}{2\beta k^2}A^2\mathrm{cs}^2(\xi)\mathrm{ds}^2(\xi)\mathrm{ns}^2(\xi). \tag{66}$$

which is an inhomogeneous Lamé equation of the form (55), and it can be solved by introducing an ansätz solution.

$$u_2=b_0+b_2\mathrm{cs}^2(\xi)+b_4\mathrm{cs}^4(\xi). \tag{67}$$

Combining (66) with (67) reaches the second-order exact solution

$$u_2=\frac{A^2\alpha(m^2-1)}{48\beta k^2}\left[1+\frac{2(m^2-2)}{m^2-1}\mathrm{cs}^2(\xi)-\frac{3}{m^2-1}\mathrm{cs}^4(\xi)\right]. \tag{68}$$

Finally, substituting (63), (65) and (68) into (12) and truncating the expansion, we can get a second-order asymptotic periodic solution of KdV equation (56) as follows

$$\begin{aligned}\tilde{u}=&\frac{2\beta k^2(-4+2m^2\pm2\sqrt{1-m^2+m^4})}{\alpha}-\frac{12\beta k^2}{\alpha}\mathrm{cs}^2(\xi)\\&+\epsilon A\mathrm{cs}(\xi)\mathrm{ds}(\xi)+\epsilon^2\frac{A^2\alpha(m^2-1)}{48\beta k^2}\left[1+\frac{2(m^2-2)}{m^2-1}\mathrm{cs}^2(\xi)-\frac{3}{m^2-1}\mathrm{cs}^4(\xi)\right]\end{aligned} \tag{69}$$

where A is an arbitrary constant and ϵ a small parameter.

Remark 4.1. When $m\to1$, the zero-order exact solution (63), first-order exact solution (65), second-order exact solution (68) and second-order asymptotic solution (69) of KdV equation (56) can respectively degenerate as follows

$$u_0=-\frac{12\beta k^2}{\alpha}\mathrm{csch}^2(\xi)\quad\text{or}\quad u_0=-\frac{8\beta k^2}{\alpha}-\frac{12\beta k^2}{\alpha}\mathrm{csch}^2(\xi), \tag{70}$$

$$u_1=A\ \mathrm{csch}^2(\xi)\coth(\xi), \tag{71}$$

$$u_2=-\frac{A^2\alpha}{24\beta k^2}\mathrm{csch}^2(\xi)-\frac{A^2\alpha}{16\beta k^2}\mathrm{csch}^4(\xi), \tag{72}$$

$$\tilde{u}=-\frac{8\beta k^2}{\alpha}\left[1+\frac{3}{2}\mathrm{csch}^2(\xi)\right]+\epsilon A\ \mathrm{csch}^2(\xi)coth(\xi)-\epsilon^2\frac{A^2\alpha}{24\beta k^2}\mathrm{csch}^2(\xi)\left[1-\frac{3}{2}\mathrm{csch}^2(\xi)\right], \tag{73}$$

or

$$\tilde{u}=-\frac{12\beta k^2}{\alpha}\mathrm{csch}^2(\xi)+\epsilon A\ \mathrm{csch}^2(\xi)coth(\xi)-\epsilon^2\frac{A^2\alpha}{24\beta k^2}\mathrm{csch}^2(\xi)\left[1-\frac{3}{2}\mathrm{csch}^2(\xi)\right]. \tag{74}$$

Remark 4.2. When $m\to0$, the zero-order exact solution (63), first-order exact solution (65), second-order exact solution (68) and second-order asymptotic solution (69) of KdV equation

(56) can respectively degenerate as follows

$$u_0=-\frac{4\beta k^2}{\alpha}-\frac{12\beta k^2}{\alpha}\cot^2(\xi), \quad \text{or} \quad u_0=-\frac{12\beta k^2}{\alpha}\cot^2(\xi), \tag{75}$$

$$u_1=A\cot(\xi)\csc^2(\xi), \tag{76}$$

$$u_2=-\frac{a^2\alpha}{48\beta k^2}-\frac{2A^2\alpha}{24\beta k^2}\cot^2(\xi)-\frac{A^2\alpha}{16\beta k^2}\cot^4(\xi), \tag{77}$$

$$\tilde{u}=-\frac{4\beta k^2}{\alpha}-\frac{12\beta k^2}{\alpha}\cot^2(\xi)+\epsilon A\cot(\xi)\csc^2(\xi)-\epsilon^2\frac{A^2\alpha}{48\beta k^2}\left[1+4\cot^2(\xi)+3\cot^4(\xi)\right], \tag{78}$$

or

$$\tilde{u}=-\frac{12\beta k^2}{\alpha}-\frac{12\beta k^2}{\alpha}\cot^2(\xi)+\epsilon A\cot(\xi)\csc^2(\xi)-\epsilon^2\frac{A^2\alpha}{48\beta k^2}\left[1+4\cot^2(\xi)+3\cot^4(\xi)\right]. \tag{79}$$

Remark 4.3. In order to have a better understanding of the properties of the solutions obtained above, four figures (Fig. 3) are plotted to illustrate the zero-order exact solution (63), first-order exact solution (65), second-order exact solution (68) and second-order asymptotic solution (69) of KdV equation (56) with $\alpha=4$, $\beta=-5$, $k=2$, $c=4$, $A=2$, $m=0.8$, $\epsilon=0.001$.

4.2 The multi-order exact solutions of Boussinesq equation

Boussinesq equation reads

$$\frac{\partial^2 u}{\partial t^2}-c_0^2\frac{\partial^2 u}{\partial x^2}-\alpha\frac{\partial^4 u}{\partial x^4}-\beta\frac{\partial^2 u^2}{\partial x^2}=0. \tag{80}$$

In the frame of (9) and (80) can be written as

$$\alpha k^2\frac{d^2u}{d\xi^2}+\beta u^2+(c_0^2-c^2)u=0, \tag{81}$$

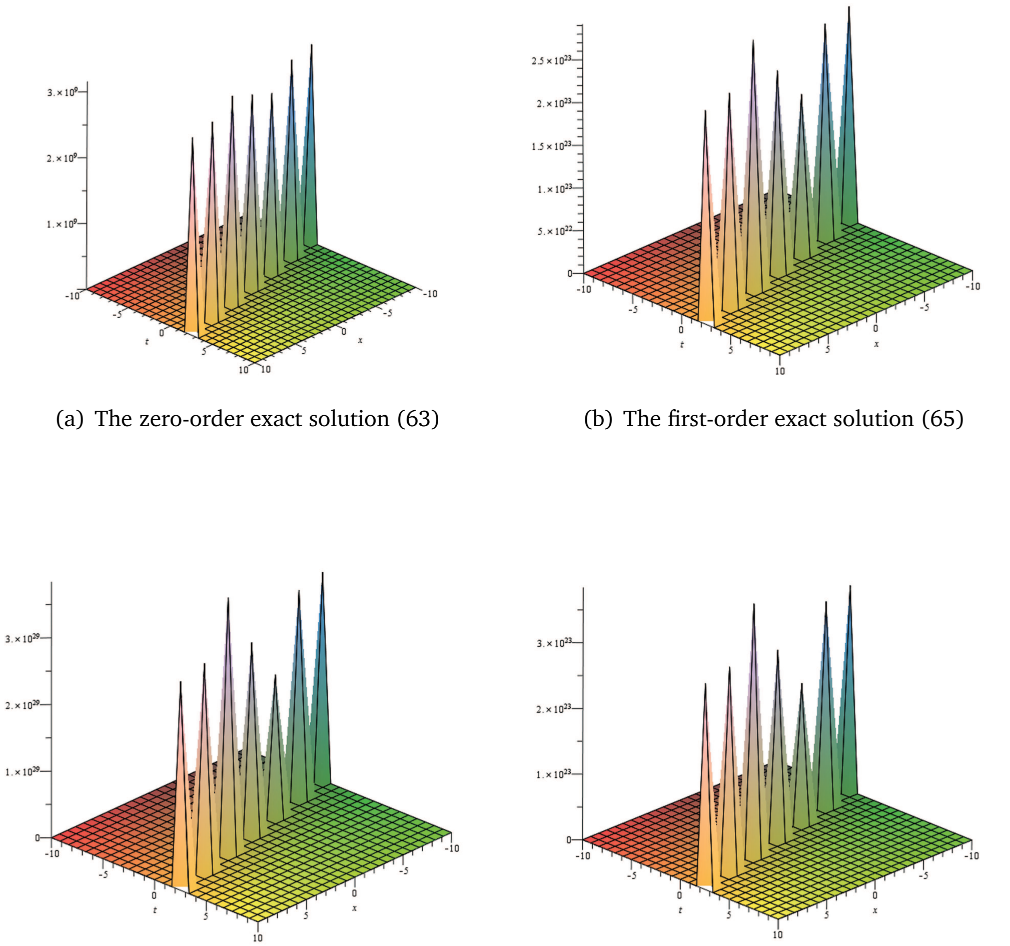
where integration with respect to ξ has been taken once and the integration constant is set as zero.

Applying the perturbation method to (81), we can derive the zeroth-order, the first-order and the second-order equations as

$$\epsilon^0:\alpha k^2\frac{d^2u_0}{d\xi^2}+\beta u_0^2+(c_0^2-c^2)u_0=0, \tag{82}$$

$$\epsilon^1:\alpha k^2\frac{d^2u_1}{d\xi^2}+(2\beta u_0+c_0^2-c^2)u_1=0, \tag{83}$$

$$\epsilon^2:\alpha k^2\frac{d^2u_2}{d\xi^2}+(2\beta u_0+c_0^2-c^2)u_2=-\beta u_1^2. \tag{84}$$



(a) The zero-order exact solution (63)

(b) The first-order exact solution (65)

(c) The second-order exact solution (68)

(d) The second-order asymptotic solution (69)

Figure 3: The multi-order solutions of KdV equation (56).

Similarly, from (62) and the zeroth-order equation (82), the zeroth-order exact solution is derived as

$$u_0=\frac{2\alpha k^2(-2+m^2\pm\sqrt{1-m^2+m^4})}{\beta}-\frac{6\alpha k^2}{\beta}\mathrm{cs}^2(\xi),\qquad c_0^2-c^2=\mp4\alpha k^2\sqrt{1-m^2+m^4}. \quad (85)$$

Substituting (85) into the first-order equation (83) leads to the first-order exact solution

$$u_1=A\mathrm{cs}(\xi)\mathrm{ns}(\xi)\mathrm{ds}(\xi), \quad (86)$$

where A is an arbitrary constant.

Combining (67), (85) and (86) with (84) gives the second-order exact solution of Boussinesq equation (80)

$$u_2 = \frac{A^2\beta(m^2-1)}{24\alpha k^2}\left[1+\frac{2(m^2-2)}{m^2-1}\mathrm{cs}^2(\xi)-\frac{3}{m^2-1}\mathrm{cs}^4(\xi)\right]. \tag{87}$$

Finally, substituting (85), (86) and (87) into (12) and truncating the expansion, we can get a second-order asymptotic periodic solution of Boussinesq equation (80) as follows

$$\begin{aligned}\tilde{u} = &\frac{2\alpha k^2(-2+m^2\pm\sqrt{1-m^2+m^4})}{\beta}-\frac{6\alpha k^2}{\beta}\mathrm{cs}^2(\xi)\\ &+\epsilon A\ \mathrm{cs}(\xi)\mathrm{ns}(\xi)\mathrm{ds}(\xi)+\epsilon^2\frac{A^2\beta(m^2-1)}{24\alpha k^2}\left[1+\frac{2(m^2-2)}{m^2-1}\mathrm{cs}^2(\xi)-\frac{3}{m^2-1}\mathrm{cs}^4(\xi)\right],\end{aligned} \tag{88}$$

where A is an arbitrary constant and ϵ a small parameter.

Remark 4.4. When $m\to 1$, the zero-order exact solution (85), the first-order exact solution (86), the second-order exact solution (87) and the second-order asymptotic solution (88) of Boussinesq equation (80) can respectively degenerate as follows

$$u_0 = -\frac{6\alpha k^2}{\beta}\mathrm{csch}^2(\xi), \quad \text{or} \quad u_0 = -\frac{4\alpha k^2}{\beta}-\frac{6\alpha k^2}{\beta}\mathrm{csch}^2(\xi), \tag{89}$$

$$u_1 = A\ \mathrm{csch}^2(\xi)\coth(\xi), \tag{90}$$

$$u_2 = -\frac{a^2\beta}{12\alpha k^2}\mathrm{csch}^2(\xi)-\frac{A^2\beta}{8\alpha k^2}\mathrm{csch}^4(\xi), \tag{91}$$

$$\begin{aligned}\tilde{u} = &\frac{2\alpha k^2(-2+m^2\pm\sqrt{1-m^2+m^4})}{\beta}-\frac{6\alpha k^2}{\beta}\mathrm{csch}^2(\xi)\\ &+\epsilon A\ \mathrm{csch}^2(\xi)\coth(\xi)-\frac{\epsilon^2 A^2\beta}{12\alpha k^2}\mathrm{csch}^2(\xi)\left[1+\frac{3}{2}\mathrm{csch}^2(\xi)\right].\end{aligned} \tag{92}$$

Remark 4.5. When $m\to 0$, the zero-order exact solution (85), the first-order exact solution (86), the second-order exact solution (87) and the second-order asymptotic solution (88) of Boussinesq equation (80) can respectively degenerate as follows

$$u_0 = -\frac{2\alpha k^2}{\beta}-\frac{6\alpha k^2}{\beta}\cot^2(\xi), \quad \text{or} \quad u_0 = -\frac{6\alpha k^2}{\beta}-\frac{6\alpha k^2}{\beta}\cot^2(\xi), \tag{93}$$

$$u_1 = A\ \cot(\xi)\csc^2(\xi), \tag{94}$$

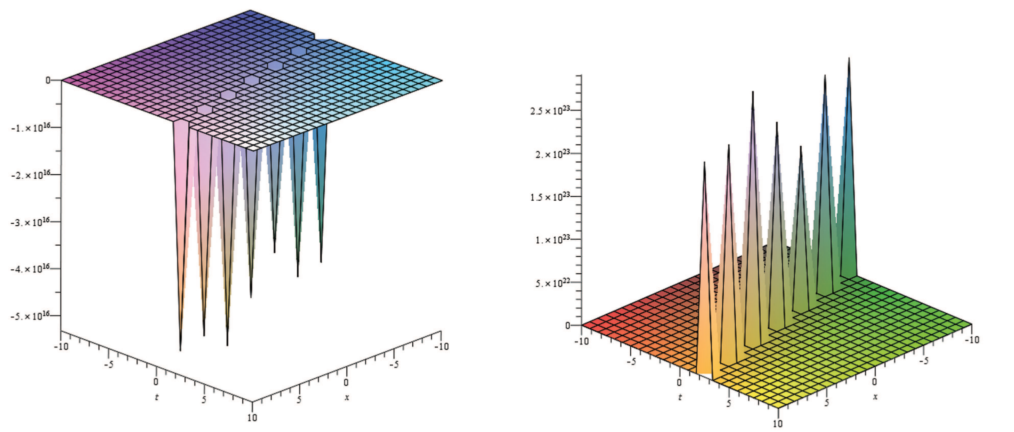
$$u_2 = -\frac{A^2\beta}{24\alpha k^2}\left[1+4\cot^2(\xi)+3\cot^4(\xi)\right], \tag{95}$$

$$\tilde{u} = -\frac{2\alpha k^2}{\beta}-\frac{6\alpha k^2}{\beta}\cot^2(\xi)+\epsilon A\ \cot(\xi)\csc^2(\xi)-\epsilon^2\frac{A^2\beta}{24\alpha k^2}\left[1+4\cot^2(\xi)+3\cot^4(\xi)\right], \tag{96}$$

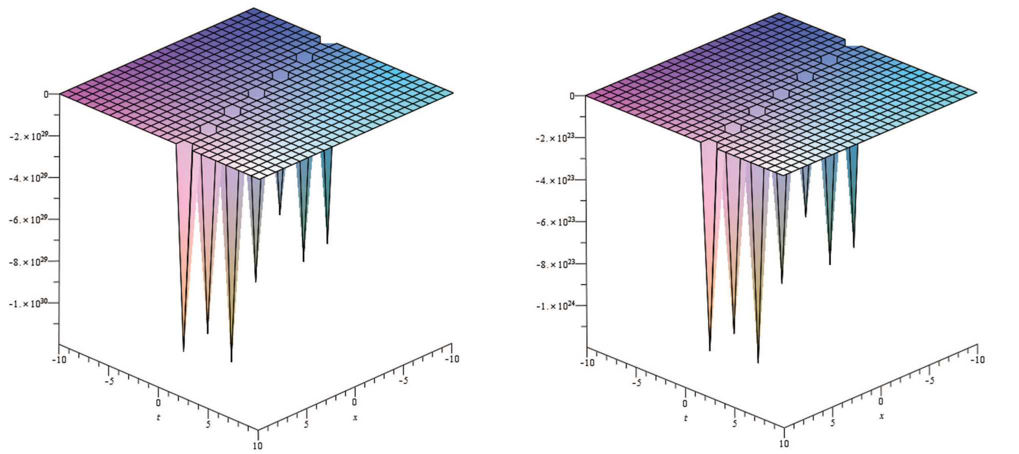
or

$$\tilde{u}=-\frac{6\alpha k^2}{\beta}-\frac{6\alpha k^2}{\beta}\cot^2(\xi)+\epsilon A\cot(\xi)\csc^2(\xi)-\epsilon^2\frac{A^2\beta}{24\alpha k^2}\Big[1+4\cot^2(\xi)+3\cot^4(\xi)\Big]. \quad (97)$$

Remark 4.6. In order to have a better understanding of the properties of the solutions obtained above, four figures (Fig. 4) are plotted to illustrate the zero-order exact solution (85), the first-order exact solution (86), the second-order exact solution (87) and the second-order asymptotic solution (88) of Boussinesq equation (80) with $\alpha=4$, $\beta=5$, $k=2$, $c=4$, $A=2$, $m=0.8$, $\epsilon=0.001$.



(a) The zero-order exact solution (83) (b) The first-order exact solution (84)



(c) The second-order exact solution (85) (d) The second-order asymptotic solution (86)

Figure 4: The multi-order solutions of Boussinesq equation (80).

5 Conclusion

In the paper, Jacobi elliptic function, the Lamé equation and Lamé functions are applied to solve nonlinear evolution equations. When perturbation method and two kinds of the Lamé functions $L_2(\xi)$ and $L_3(\xi)$ are considered, then the multi-order solutions are obtained for these nonlinear evolution equations. The results obtained in the paper is very important for nonlinear instability of nonlinear coherent structures of the nonlinear evolution equations. Additionally the method can be also applied to many other nonlinear evolution equations in mathematical physics.

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