INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 5, Number 3, Pages 386–406

ON THE CONVERGENCE OF DIFFERENCE SCHEMES FOR PARABOLIC PROBLEMS WITH CONCENTRATED DATA

BOŠKO S. JOVANOVIĆ AND LUBIN G. VULKOV

Abstract. Parabolic equations with unbounded coefficients and even generalized functions (in particular Dirac–delta functions) model large–scale of problems in the heat–mass transfer. This paper provides estimates for the convergence rate of difference scheme in discrete Sobolev like norms, compatible with the smoothness of the differential problems solutions, i.e with the smoothness of the input data.

Key Words. concentrated capacity, Sobolev spaces, generalized solution, difference scheme, rate of convergence.

1. Introduction

The present paper continues the study for convergence of finite difference schemes of the model heat equation with concentrated capacity in [11], [12]. In the heat capacity coefficient the Dirac–delta distribution is involved and as a result, the jump of the heat flow at the interface point is proportional to the time derivative of the temperature. Dynamical boundary conditions correspond to concentrated capacity on the boundary [5], [7], [19]. These problems are nonstandard and the classical analysis is difficult to be applied for error estimates and convergence proof. The finite difference method for parabolic problems with discontinuous data (coefficients, initial and boundary conditions) is based on associated weak solutions [17], [20], [30]. For these problems the most used tool for studying convergence of the difference solutions is the Bramble–Hilbert lemma and its generalizations [4], [6]. The theory of difference scheme convergence rate estimates **compatible** with the smoothness of the differential problem solutions was developed first for elliptic problems in papers of Samarskii, Lazarov and Makarov, cf. the monograph [23]. Further development of this theory is presented in [8], and especially, results for parabolic problems. The basic physical model corresponding to the parabolic problems considered in the present paper is that of heat–transfer, where the process take places in two adjoining bodies at different scale in each body. The diffusion through thin layers, divided the bodies has high specific heat. We consider the limiting case, when the thickness of the layers goes to zero and where the specific heat goes to infinity. The simplest mathematical model of this phenomena is derived in [26] and its further development in [5], [19]. Our aim is to treat these problems as a first order abstract–evolution equation (1), with selfadjoint positive linear operators A, B, defined in Hilbert space H and then to use energy methods

---

Received by the editors January 1, 2007 and, in revised form, March 22, 2007.
2000 *Mathematics Subject Classification.* 65M15.
The research of the first author was supported by Ministry of Science of Republic of Serbia under project 144005A and Bulgarian National Fund of Sciences under project HS-MI-106/2005, while the research of the second author was supported by Bulgarian National Fund of Sciences under project HS-MI-106/2005.

from the theory of Hilbert spaces. Discrete analysis of appropriate subspaces of the Sobolev spaces are used and yet that allow the discrete operators to be selfadjoint of the space involved. In this first stage we obtain a priori estimates for the discrete solutions. The second important idea of the method consists in constructing the special integral representations of the error of the difference schemes. This allows us by applying imbedding Sobolev's theorems to obtain more accurate estimates. We do not use the Bramble–Hilbert lemma. The remainder of this paper is organized as follows: energy estimates for the solutions of an abstract Cauchy problem for a first order evolution equation and for an operator–difference scheme can be found in the next section. These auxiliary results are used in the next sections for obtaining a priori estimates to the derivation of convergence rate estimates in special discrete Sobolev norms of difference schemes approximations to heat equation with discontinuous coefficients and dynamical conditions of conjugation, i.e in which the time derivative of the solution is involved. We also treat parabolic equations with dynamical boundary condition and elliptic equation with dynamical condition of conjugation. The method proposed here is applied to analogous hyperbolic problem in [13], see also [25]. Energy stability for a class of two-dimensional interface parabolic problems is investigated in [15], while the stability of difference schemes for parabolic equations with dynamical boundary conditions and conditions on conjugation is analyzed in [16]. Two–dimensional elliptic problems in which the Dirac–delta function appears in the lowest coefficients are treated in [9] and [14], while finite-difference approximation for Poisson's equation with a dynamic boundary condition is given in [29]. Convergence of difference schemes on classical solutions for parabolic and hyperbolic equations with dynamical boundary conditions or dynamical conditions of conjugation are studied in [1], [2], [3], [28].

2. Preliminary Results

Let H be a real separable Hilbert space endowed with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$ and S – unbounded selfadjoint positive definite linear operator, with domain $D(S)$ dense in H. The product $(u,\, v)_S = (Su,\, v)$ $(u,\ v \in D(S))$ satisfies the inner product axioms. Reinforcing $D(S)$ in the norm $\|u\|_S = (u,\, u)_S^{1/2}$ we obtain a Hilbert space $H_S \subset H$. The inner product $(u,\, v)$ continuously extends to $H_S^* \times H_S$, where H_S^* is the adjoint space for H_S. Operator S extends to mapping $S : H_S \to H_S^*$. There exists unbounded selfadjoint positive definite linear operator $S^{1/2}$, such that $D(S^{1/2}) = H_S$ and $(u,\, v)_S = (Su,\, v) = (S^{1/2}u,\, S^{1/2}v)$ (see [17], [21]). We also define the Sobolev spaces $W_2^s(a,b;\, H)$, $W_2^0(a,b;\, H) = L_2(a,b;\, H)$, of the functions $u = u(t)$ mapping interval $(a,b) \subset R$ into H [17]. Let A and B be unbounded selfadjoint positive definite linear operators, not depending on t, in Hilbert space H, with $D(A)$ – dense in H_B. In general, A and B are noncommutative. We consider an abstract Cauchy problem [20], [30].

$$B\frac{du}{dt} + Au = f(t), \quad 0 < t < T; \qquad u(0) = u_0, \tag{2.1}$$

where u_0 is a given element in H_B, $f(t) \in L_2(0,T;\, H_{A^{-1}})$ – given function and $u(t)$ – unknown function from (0,T) into H_A. Setting in (1) $f(t) = dg(t)/dt$ we get the Cauchy problem

$$B\frac{du}{dt} + Au = \frac{dg}{dt}, \quad 0 < t < T; \qquad u(0) = u_0. \tag{2.2}$$

The following proposition holds.

Lemma 1. *The solution u of the problem (2.1) satisfies a priori estimates:*

$$\int_0^T \left(\|Au(t)\|^2_{B^{-1}} + \left\|\frac{du(t)}{dt}\right\|^2_B\right) dt \le C\left(\|u_0\|^2_A + \int_0^T \|f(t)\|^2_{B^{-1}}\, dt\right),$$

if $u_0 \in H_A$ and $f \in L_2(0,T;\, H_{B^{-1}})$;

$$\int_0^T \|u(t)\|^2_A\, dt + \int_0^T \int_0^T \frac{\|u(t) - u(t')\|^2_B}{|t-t'|^2}\, dt\, dt' \le C\left(\|u_0\|^2_B + \int_0^T \|f(t)\|^2_{A^{-1}}\, dt\right),$$

if $u_0 \in H_B$ and $f \in L_2(0,T;\, H_{A^{-1}})$; and

$$\int_0^T \|u(t)\|^2_B\, dt \le C\left(\|Bu_0\|^2_{A^{-1}} + \int_0^T \|A^{-1} f(t)\|^2_B\, dt\right),$$

if $u_0 \in H_{BA^{-1}B}$ and $f \in L_2(0,T;\, H_{A^{-1}BA^{-1}})$. The solution u of the problem (2.2) satisfies a priori estimates:

$$\int_0^T \|u(t)\|^2_A\, dt + \int_0^T \int_0^T \frac{\|u(t) - u(t')\|^2_B}{|t-t'|^2}\, dt\, dt' \le C\Bigg[\|u_0\|^2_B +$$
$$+ \int_0^T \int_0^T \frac{\|g(t) - g(t')\|^2_{B^{-1}}}{|t-t'|^2}\, dt\, dt' + \int_0^T \left(\frac{1}{t} + \frac{1}{T-t}\right)\|g(t)\|^2_{B^{-1}}\, dt\Bigg],$$

if $u_0 \in H_B$ and $g \in W_2^{1/2}(0,T;\, H_{B^{-1}})$; and

$$\int_0^T \|u(t)\|^2_B\, dt \le C\left(\|Bu_0 - g(0)\|^2_{A^{-1}} + \int_0^T \|g(t)\|^2_{B^{-1}}\, dt\right),$$

if $Bu_0 - g(0) \in H_{A^{-1}}$ and $g \in L_2(0,T;\, H_{B^{-1}})$. *Proof:* Using energy method and Fourier expansion. ☐ Analogous results hold for operator–difference schemes. Let H_h be finite dimensional real Hilbert space with inner product $(\cdot,\cdot)_h$ and norm $\|\cdot\|_h$. Let A_h and B_h be constant selfadjoint positive linear operators in H_h, in general case noncommutative. By H_{S_h}, where $S_h = S_h^* > 0$, we denote the space $H_{S_h} = H_h$ with inner product $(v,\, w)_{S_h} = (S_h v,\, w)_h$ and norm $\|v\|_{S_h} = (S_h v,\, v)_h^{1/2}$. Let ω_τ be an uniform mesh on $(0,T)$ with the step size $\tau = T/m$, $\omega_\tau^- = \omega_\tau \cup \{0\}$, $\omega_\tau^+ = \omega_\tau \cup \{T\}$ and $\bar\omega_\tau = \omega_\tau \cup \{0,T\}$. Further we shall use standard denotation of the theory of difference schemes [22]. We consider the simplest two–level operator–difference scheme

$$B_h v_{\bar t} + A_h v = \varphi(t), \quad t \in \omega_\tau^+; \qquad v(0) = v_0, \tag{2.3}$$

where v_0 is a given element in H_h, $\varphi(t)$ is also given and $v(t)$ – unknown function with values in H_h. Let us also consider the scheme

$$B_h v_{\bar t} + A_h v = \psi_{\bar t}, \quad t \in \omega_\tau^+; \qquad v(0) = v_0, \tag{2.4}$$

where $\psi(t)$ is a given function with values in H_h. The following analogue of Lemma 1 holds true.

Lemma 2. *The solution v of the problem (2.3) satisfies a priori estimates:*

$$\tau \sum_{t\in\omega_\tau^+} \|A_h v(t)\|^2_{B_h^{-1}} + \tau \sum_{t\in\omega_\tau^+} \|v_{\bar t}(t)\|^2_{B_h} \le C\left(\|v_0\|^2_{A_h} + \tau \sum_{t\in\omega_\tau^+} \|\varphi(t)\|^2_{B_h^{-1}}\right),$$

$$\tau \sum_{t\in\bar\omega_\tau} \|v(t)\|^2_{A_h} + \tau^2 \sum_{t\in\bar\omega_\tau} \sum_{t'\in\bar\omega_\tau,\ t'\ne t} \frac{\|v(t) - v(t')\|^2_{B_h}}{|t-t'|^2} \le$$
$$\le C\left(\|v_0\|^2_{B_h} + \tau\, \|v_0\|^2_{A_h} + \tau \sum_{t\in\omega_\tau^+} \|\varphi(t)\|^2_{A_h^{-1}}\right),$$

$$\tau \sum_{t\in\omega_\tau^+} \|v(t)\|^2_{B_h} \le C\Big(\|B_h v_0\|^2_{A_h^{-1}} + \tau \sum_{t\in\omega_\tau^+} \|A_h^{-1}\varphi(t)\|^2_{B_h}\Big).$$

The solution v of the problem (2.4) satisfies a priori estimates:

$$\tau \sum_{t\in\bar\omega_\tau} \|v(t)\|^2_{A_h} + \tau^2 \sum_{t\in\bar\omega_\tau} \sum_{t'\in\bar\omega_\tau,\ t'\ne t} \frac{\|v(t)-v(t')\|^2_{B_h}}{|t-t'|^2} \le C\Big[\|v_0\|^2_{B_h} + \tau\,\|v_0\|^2_{A_h} +$$

$$+\tau^2 \sum_{t\in\bar\omega_\tau} \sum_{t'\in\bar\omega_\tau,\ t'\ne t} \frac{\|\psi(t)-\psi(t')\|^2_{B_h^{-1}}}{|t-t'|^2} + \tau \sum_{t\in\omega_\tau} \Big(\frac{1}{t} + \frac{1}{T-t}\Big) \|\psi(t)\|^2_{B_h^{-1}}\Big],$$

$$\tau \sum_{t\in\omega_\tau^+} \|v(t)\|^2_{B_h} \le C\Big(\|B_h v_0 - \psi(0)\|^2_{A_h^{-1}} + \tau \sum_{t\in\omega_\tau^+} \|\psi(t)\|^2_{B_h^{-1}}\Big).$$

3. Heat equation with concentrated capacity

Let us consider the initial boundary value problem for the heat equation with concentrated capacity at the interior point $x = \xi$ [5], [18], [19]:

$$\big[c(x) + K\,\delta(x-\xi)\big]\,\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\Big(a(x)\,\frac{\partial u}{\partial x}\Big) = f(x,t), \quad (x,t)\in Q, \tag{3.1}$$

$$u(0,\,t) = 0, \quad u(1,\,t) = 0, \quad 0 < t < T \tag{3.2}$$

$$u(x,\,0) = u_0(x), \quad x \in (0,\,1), \tag{3.3}$$

where $Q = (0,\,1)\times(0,\,T)$, $K > 0$, $0 < c_1 \le a(x) \le c_2$, $0 < c_3 \le c(x) \le c_4$ and $\delta(x)$ is the Dirac distribution [27]. It follows from (3.1) that the solution of this problem satisfies at $(x,t)\in Q_1 = (0,\,\xi)\times(0,\,T)$ and $(x,t)\in Q_2 = (\xi,\,1)\times(0,\,T)$ the equation

$$c(x)\,\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\Big(a(x)\,\frac{\partial u}{\partial x}\Big) = f(x,t)\,,$$

and at $x = \xi$ – the conditions

$$[u]_{x=\xi} \equiv u(\xi+0,t) - u(\xi-0,t) = 0\,, \qquad \Big[a\,\frac{\partial u}{\partial x}\Big]_{x=\xi} = K\,\frac{\partial u(\xi,t)}{\partial t}\,.$$

It is easy to see that the initial boundary value problem (3.1)-(3.3) can be reduced in the form (2.1) letting $H = L_2(0,1)$,

$$Au = -\frac{\partial}{\partial x}\Big(a(x)\,\frac{\partial u}{\partial x}\Big) \qquad \text{and} \qquad Bu = \big[c(x) + K\,\delta(x-\xi)\big]\,u(x,\,t)\,.$$

Then $H_A = \overset{\circ}{W}{}^1_2(0,1)$,

$$\|w\|^2_A = \int_0^1 a(x)\,[w'(x)]^2\,dx\,, \tag{3.4}$$

$$\|w\|^2_B = \int_0^1 c(x)\,w^2(x)\,dx + K\,w^2(\xi)\,. \tag{3.5}$$

Further we assume that the function $c(x)$ is continuous on $[0,1]$ and $a(x)$ has finite jump in the point $x = \xi$.

3.1. The functional spaces $\widetilde{W}_2^k(0,1)$ and $\widetilde{W}_2^{k,\,k/2}(Q)$. By $\widetilde{L}_2(0,1)=\widetilde{W}_2^0(0,1)$ we denote the closure of $C[0,1]$ in the norm $\|w\|_{\widetilde{L}_2(0,1)}$ induced by inner product

$$(u,w)_{\widetilde{L}_2(0,1)}=\int_0^1 u(x)\,w(x)\,dx+u(\xi)\,w(\xi)\,.$$

Further we let $\widetilde{W}_2^1(0,1)=\overset{\circ}{W}{}_2^1(0,1)$ and $\widetilde{W}_2^k(0,1)=\overset{\circ}{W}{}_2^1(0,1)\cap W_2^k(0,\xi)\cap W_2^k(\xi,1)$, $k=2,\,3,\,\dots$. The following assertion holds true.

Lemma 3. *Let $Aw=-(a(x)\,w'(x))'$ and $Bw=(c(x)+K\,\delta(x-\xi))\,w(x)$, where $a,\,c\in L_\infty(0,1)$. Then the norm $\|w\|_B$ is equivalent to the norm $\|w\|_{\widetilde{L}_2(0,1)}$. The norm $\|w\|_A$ is equivalent to the norm $\|w\|_{\widetilde{W}_2^1(0,1)}$, at $w\in\widetilde{W}_2^1(0,1)$. If in addition $a'\in L_2(0,\xi)\cap L_2(\xi,1)$ then the norm $\|Aw\|_{B^{-1}}$ is equivalent to the norm $\|w\|_{\widetilde{W}_2^2(0,1)}$, at $w\in\widetilde{W}_2^2(0,1)$.* *Proof:* The first assertion immediately follows from (3.5) and the boundness of $c(x)$. The second one follows from (3.4), the boundness of $a(x)$ and the Friedrich's inequality

$$\|w\|_{L_2(0,1)}\le 0.5\,\|w'\|_{L_2(0,1)},\qquad w\in\overset{\circ}{W}{}_2^1(0,1).$$

We prove the third assertion. Using the equality

$$\|Aw\|_{B^{-1}}=\sup_{v\neq 0}\frac{|(Aw,\,v)|}{\|v\|_B}$$

we obtain

$$\int_0^\xi\frac{[Aw(x)]^2}{c(x)}\,dx+\int_\xi^1\frac{[Aw(x)]^2}{c(x)}\,dx\le\|Aw\|_{B^{-1}}^2\le$$
$$\le\int_0^\xi\frac{[Aw(x)]^2}{c(x)}\,dx+\int_\xi^1\frac{[Aw(x)]^2}{c(x)}\,dx+\frac{[aw']_{x=\xi}^2}{K}\,.$$

Hence, using the equality $Aw=-a\,w''-a'\,w'$, the boundness of a and c, and the imbedding $W_2^1\subset C$ we get

$$\|Aw\|_{B^{-1}}\le C_1\,\|w\|_{\widetilde{W}_2^2(0,1)}\,. \tag{3.6}$$

Further

$$\|w\|_{\widetilde{W}_2^2(0,1)}^2=\|w\|_{L_2(0,1)}^2+\|w'\|_{L_2(0,1)}^2+\|w''\|_{L_2(0,\xi)}^2+\|w''\|_{L_2(\xi,1)}^2\,.$$

We already have proved

$$\|w\|_{L_2(0,1)}^2+\|w'\|_{L_2(0,1)}^2=\|w\|_{\widetilde{W}_2^1(0,1)}^2\le C_2\,\|w\|_A^2\,. \tag{3.7}$$

Let us estimate $\|w''\|_{L_2(0,\xi)}$:

$$\|w''\|_{L_2(0,\xi)}^2=\int_0^\xi[w''(x)]^2\,dx=\int_0^\xi\left[\frac{1}{a(x)}\Big(Aw(x)+a'(x)\,w'(x)\Big)\right]^2dx\le$$

$$\le C\bigg\{\int_0^\xi\frac{[Aw(x)]^2}{c(x)}\,dx+\max_{x\in[0,\xi]}[w'(x)]^2\bigg\}.$$

Further, using imbedding $W_2^1 \subset C$ we get

$$\max_{x\in[0,\xi]} [w'(x)]^2 \le \frac{1}{c_1^2} \max_{x\in[0,\xi]} [a(x)\, w'(x)]^2 \le$$
$$\le C\bigg\{ \int_0^\xi \Big[\big(a(x)\, w'(x)\big)'\Big]^2 dx + \int_0^\xi \big[a(x)\, w'(x)\big]^2 dx \bigg\} \le$$
$$\le C_3\bigg\{ \int_0^\xi \frac{\big[\big(a(x)\, w'(x)\big)'\big]^2}{c(x)}\, dx + \int_0^\xi a(x)\, \big[w'(x)\big]^2 dx \bigg\}.$$

¿From here it follows

$$\|w''\|^2_{L_2(0,\xi)} \le C_4\Big(\|Aw\|^2_{B^{-1}} + \|w\|^2_A\Big)\,. \tag{3.8}$$

In an analogous way one can estimate $\|w''\|_{L_2(\xi,1)}$. Let consider the eigenvalue problem

$$Aw = \lambda\, Bw\,, \quad w \in \overset{\circ}{W}{}_2^1\,(0,1)\,.$$

As it is known its spectrum is discrete, all eigenvalues λ_i are positive, and the eigenfunctions w_i are orthogonal in H_B [26]. The inequality holds

$$\|Aw\|_{B^{-1}} \ge \sqrt{\lambda_{min}}\, \|w\|_A\,, \tag{3.9}$$

where

$$\lambda_{min} = \inf_{v\ne 0} \frac{(Av,\, v)}{(Bv,\, v)} = \inf_{v\ne 0} \frac{\int_0^1 a(x)\, [v'(x)]^2\, dx}{\int_0^1 c(x)\, v^2(x)\, dx + K\, v^2(\xi)} \ge \frac{4\, c_1}{c_4 + K} > 0\,.$$

Now, the third assertion follows from (3.6)–(3.9). □ We also define the spaces $\widetilde{W}_2^{k,\,k/2}(Q) = L_2(0,T;\, \widetilde{W}_2^k(0,1)) \cap W_2^{k/2}(0,T;\, \widetilde{L}_2(0,1))$, $k = 0,\, 1,\, 2,\, \ldots$. Differentiating equation (3.1) on x and t and applying Lemmas 1 and 3 we obtain the following assertion.

Lemma 4. *(i) If* $a \in W_2^1(0,\xi) \cap W_2^1(\xi,1)$, $c \in W_2^1(0,1)$, $u_0 \in \widetilde{W}_2^1(0,1)$ *and* $f \in L_2(Q_1) \cap L_2(Q_2)$, *then the problem (3.1)–(3.3) has unique solution* $u \in \widetilde{W}_2^{2,1}(Q)$. *(ii) If* $a \in W_2^2(0,\xi) \cap W_2^2(\xi,1)$, $c \in W_2^2(0,1)$, $u_0 \in \widetilde{W}_2^2(0,1)$, $f \in W_2^{1,1/2}(Q_1) \cap W_2^{1,1/2}(Q_2)$ *and the compatibility conditions hold*

$$U_t\,(0) = U_t\,(1) = 0, \quad [U_t]_{x=\xi} = 0, \quad [au_0']_{x=\xi} = K \lim_{x\to\xi} U_t(x),$$

where

$$U_t(x) = \frac{a(x)u_0''(x) + a'(x)u_0'(x) + f(x,0)}{c(x)},$$

then the problem (3.1)–(3.3) has unique solution $u \in \widetilde{W}_2^{3,3/2}(Q)$. *(iii) If* $a \in W_2^3(0,\xi) \cap W_2^3(\xi,1)$, $c \in W_2^3(0,1)$, $u_0 \in \widetilde{W}_2^3(0,1)$, $f \in W_2^{2,1}(Q_1) \cap W_2^{2,1}(Q_2)$ *and the last compatibility conditions hold, then the problem (3.1)–(3.3) has unique solution* $u \in \widetilde{W}_2^{4,2}(Q)$.

3.2. The difference scheme. Let $\omega_h = \{x_1, x_2, \ldots, x_{n-1}\}$ be an nonuniform mesh in (0, 1), chosen, so that ξ is a node. We let $\omega_h^- = \omega_h \cup \{x_0\}$, $\omega_h^+ = \omega_h \cup \{x_n\}$, $\bar\omega_h = \omega_h \cup \{x_0,\, x_n\}$, $x_0 = 0$, $x_n = 1$ and $h_i = x_i - x_{i-1}$. Also we let

$$v_x = (v_+ - v)/h_+\,,\quad v_{\bar x} = (v - v_-)/h\,,\quad v_{\hat x} = (v_+ - v)/\hbar\,,$$

$$v = v(x)\,,\quad v_\pm = v(x_\pm)\,,\quad x = x_i\,,\quad x_\pm = x_{i\pm 1}\,,\quad \hbar = (h + h_+)/2\,.$$

We assume that the condition holds

$$1/c_0 \le h_+/h \le c_0\,,\quad c_0 = \mathrm{const} \ge 1\,.$$

We approximate the problem (3.1)–(3.3) on the mesh $\bar\omega_h \times \bar\omega_\tau$ by the implicit difference scheme with averaged right hand side

$$(c + K\,\delta_h)\, v_{\bar t} - (\tilde a\, v_{\bar x})_{\hat x} = T_x^2 T_t^- f,\quad (x,t) \in \omega_h \times \omega_\tau^+, \tag{3.10}$$

$$v(0,t) = 0,\quad v(1,t) = 0,\quad t \in \omega_\tau^+, \tag{3.11}$$

$$v(x,0) = u_0(x),\quad x \in \bar\omega_h, \tag{3.12}$$

where $\tilde a(x) = \big[a(x) + a(x-h)\big]/2$, for $x \ne \xi,\ \xi_+$, $\tilde a(\xi) = \big[a(\xi-0) + a(\xi_-)\big]/2$, $\tilde a(\xi_+) = \big[a(\xi_+) + a(\xi+0)\big]/2$,

$$\delta_h = \delta_h(x-\xi) = \begin{cases} 0, & x \in \omega_h \setminus \{\xi\} \\ 1/\hbar, & x = \xi \end{cases}$$

is the mesh Dirac function and T_x^2, T_t^- are the Steklov averaging operators [8], [23]:

$$T_t^- f(x,t) = T_t^+ f(x, t-\tau) = \frac{1}{\tau}\int_{t-\tau}^{t} f(x,t')\,dt',$$

$$T_x^- f(x,t) = \frac{1}{h}\int_{x_-}^{x} f(x',t)\,dx',\quad T_x^+ f(x,t) = \frac{1}{h_+}\int_{x}^{x_+} f(x',t)\,dx',$$

$$T_x^2 f(x,t) = \frac{1}{\hbar}\int_{x_-}^{x_+} \kappa(x,x')\, f(x',t)\,dx',\quad \kappa(x,x') = \begin{cases} 1 + (x'-x)/h, & x_- < x' < x, \\ 1 - (x'-x)/h_+, & x < x' < x_+. \end{cases}$$

We note that these operators are commutative and map derivatives into difference relations, for example,

$$T_x^2 \frac{\partial^2 u}{\partial x^2} = u_{\bar x \hat x}\,,\qquad T_t^- \frac{\partial u}{\partial t} = u_{\bar t}\,.$$

Let H_h be the set of all mesh functions on the mesh $\bar\omega_h$ equal to zero at $x = 0$ and $x = 1$. We define the inner products

$$(v,\, w)_h = \sum_{x \in \omega_h} v(x)\, w(x)\, \hbar,\qquad (v,\, w]_{h\star} = \sum_{x \in \omega_h^+} v(x)\, w(x)\, h,$$

with corresponding norms

$$\|w\|_h = \|w\|_{L_{2,h}} = (w,\, w)_h^{1/2}\,,\qquad \|w]|_{h\star} = (w,\, w]_{h\star}^{1/2}\,.$$

The difference scheme (3.10)–(3.12) can be written in the form (2.3) letting $A_h v = -(\tilde a\, v_{\bar x})_{\hat x}$ and $B_h v = (c + K\,\delta_h)\, v$. For $w \in H_h$ we have

$$\|w\|_{A_h}^2 = (A_h w,\, w)_h = \sum_{x \in \omega_h^+} \tilde a(x)\, w_{\bar x}^2(x)\, h \asymp \|w_{\bar x}]|_{h\star}^2,$$

$$\|w\|_{B_h}^2 = (B_h w,\, w)_h = \sum_{x \in \omega_h} c(x)\, w^2(x)\, \hbar + K\, w^2(\xi) \asymp \|w\|_{B_{0h}}^2$$

and

$$\|w\|^2_{B_h^{-1}} = (B_h^{-1}w,\, w)_h = \sum_{x\in\omega_h\setminus\{\xi\}} \frac{w^2(x)}{c(x)}\,\hbar + \frac{\hbar^2(\xi)}{K+\hbar\, c(\xi)}\, w^2(\xi) \asymp \|w\|^2_{B_{0h}^{-1}},$$

where $B_{0h}\, w = (1+\delta_h)\, w$. We define the mesh spaces $\widetilde{W}^k_{2,h}$ and $\widetilde{W}^{k,k/2}_{2,h\tau}$ $(k=0,\,1,\,2)$, with norms:

$$\|w\|^2_{\widetilde{L}_{2,h}} = \|w\|^2_{B_{0h}} = \|w\|^2_{L_{2,h}} + w^2(\xi)\,, \qquad \|w\|^2_{\widetilde{W}^1_{2,h}} = \|w_{\bar{x}}]|^2_{h\star} + \|w\|^2_h\,,$$

$$\|w\|^2_{\widetilde{W}^2_{2,h}} = \sum_{x\in\omega_h\setminus\{\xi\}} w^2_{\bar{x}\hat{x}}(x)\,\hbar + \|w_{\bar{x}}]|^2_{h\star} + \|w\|^2_h\,; \qquad \|v\|^2_{\widetilde{L}_{2,h\tau}} = \tau \sum_{t\in\bar{\omega}_\tau} \|v(\,\cdot\,,t)\|^2_{\widetilde{L}_{2,h}}\,,$$

$$\|v\|^2_{\widetilde{W}^{1,\,1/2}_{2,h\tau}} = \tau \sum_{t\in\bar{\omega}_\tau} \|v(\,\cdot\,,t)\|^2_{\widetilde{W}^1_{2,h}} + \tau^2 \sum_{t\in\bar{\omega}_\tau}\ \sum_{t'\in\bar{\omega}_\tau,\ t'\neq t} \frac{\|v(\,\cdot\,,t) - v(\,\cdot\,,t')\|^2_{\widetilde{L}_{2,h}}}{|t-t'|^2}\,,$$

$$\|v\|^2_{\widetilde{W}^{2,1}_{2,h\tau}} = \tau \sum_{t\in\bar{\omega}_\tau} \|v(\,\cdot\,,t)\|^2_{\widetilde{W}^2_{2,h}} + \tau \sum_{t\in\omega_\tau^+} \|v_{\bar{t}}(\,\cdot\,,t)\|^2_{\widetilde{L}_{2,h}}\,.$$

Lemma 5. *Let $A_h w = -(\tilde{a}\, w_{\bar{x}})_{\hat{x}}$ and $B_h w = (c + K\,\delta_h)\, w$, where $\tilde{a}$ and c are bounded mesh functions and $w \in H_h$. Then the norm $\|w\|_{B_h}$ is equivalent to the mesh norm $\widetilde{L}_{2,h}$ and the norm $\|w\|_{A_h}$ to the mesh norm $\widetilde{W}^1_{2,h}$. If in addition, $\tilde{a}_x$ is a bounded mesh function for $x \neq \xi$, then the norm $\|A_h w\|_{B_h^{-1}}$ is equivalent to the norm $\widetilde{W}^2_{2,h}$.* *The proof* is analogous to the proof of Lemma 3. □

3.3. Convergence in the norm $\widetilde{W}^{1,\,1/2}_{2,h\tau}$. Let u be the solution of the problem (3.1)–(3.3) and v the solution of (3.10)–(3.12). The error $z = u - v$ satisfies difference scheme

(3.13) $$(c + K\,\delta_h)\, z_{\bar{t}} - (\tilde{a}\, z_{\bar{x}})_{\hat{x}} = \psi_{\bar{t}} - \chi_{\hat{x}}, \quad (x,t) \in \omega_h \times \omega_\tau^+,$$

(3.14) $$z(0,t) = 0, \quad z(1,t) = 0, \quad t \in \omega_\tau^+,$$

(3.15) $$z(x,0) = 0, \quad x \in \bar{\omega}_h,$$

where

$$\psi = c\,u - T_x^2(c\,u) + \Big(\frac{h^2}{6}\,(c\,u)_{\bar{x}}\Big)_{\hat{x}} \quad \text{and} \quad \chi = \tilde{a}\, u_{\bar{x}} - T_x^- T_t^- \Big(a\,\frac{\partial u}{\partial x}\Big) + \frac{h^2}{6}\,(c\,u)_{\bar{x}\bar{t}}.$$

From Lemmas 2 and 5, and the inequality

(3.16) $$\|\chi_{\hat{x}}\|_{A_h^{-1}} = \max_{w\in H_h} \frac{|(\chi_{\hat{x}},\, w)_h|}{\|w\|_{A_h}} = \max_{w\in H_h} \frac{|-(\chi,\, w_{\bar{x}}]_{h\star}|}{\|w\|_{A_h}} \le \le \max_{w\in H_h} \frac{\|\chi]|_{h\star}\, \|w_{\bar{x}}]|_{h\star}}{\|w\|_{A_h}} \le \frac{1}{c_1}\, \|\chi]|_{h\star}\,,$$

the a priori estimate for the problem (3.13)–(3.15) follows

(3.17)
$$\|z\|_{\widetilde{W}^{1,\,1/2}_{2,h\tau}} \le C\Bigg\{\tau \sum_{t\in\omega_\tau^+} \|\chi(\,\cdot\,,\,t)]|^2_{h\star} + \tau^2 \sum_{t\in\bar{\omega}_\tau}\ \sum_{t'\in\bar{\omega}_\tau,\ t'\neq t} \frac{\|\psi(\,\cdot\,,\,t) - \psi(\,\cdot\,,\,t')\|^2_{B_{0h}^{-1}}}{|t-t'|^2} + +\tau \sum_{t\in\omega_\tau} \Big(\frac{1}{t} + \frac{1}{T-t}\Big)\, \|\psi(\,\cdot\,,\,t)\|^2_{B_{0h}^{-1}}\Bigg\}^{1/2}.$$

Therefore, in order to estimate the rate of convergence of the difference scheme (3.10)–(3.12) in the norm $\widetilde{W}_{2,h\tau}^{1,\,1/2}$ it is sufficient to be estimated the right hand side of (3.17). We let

$$\chi = \chi_1 + \chi_2\,, \quad \chi_1 = \tilde{a}\, u_{\bar{x}} - T_x^- T_t^- \Big(a\,\frac{\partial u}{\partial x}\Big)\,, \quad \chi_2 = \frac{h^2}{6}\,(c\,u)_{\bar{x}\bar{t}}\,.$$

Using the integral representation

$$\chi_2(x,t) = \frac{h^2}{6\,h\tau}\int_{x_-}^{x}\int_{t-\tau}^{t}\frac{\partial^2 (c\,u)(x',\,t')}{\partial x\partial t}\,dt'\,dx', \tag{3.18}$$

and summing on the mesh nodes, we immediately get

$$\begin{aligned}\Big\{\tau\sum_{t\in\omega_\tau^+}\|\chi_2(\,\cdot\,,\,t)]|_{h\star}^2\Big\}^{1/2} &\le C\,h_{max}^2\,\Big\|\frac{\partial^2 (c\,u)}{\partial x\partial t}\Big\|_{L_2(Q)} \le \\ &\le C\,h_{max}^2\,\|c\|_{W_2^1(0,1)}\,\Big\|\frac{\partial^2 u}{\partial x\partial t}\Big\|_{L_2(Q)}\,.\end{aligned} \tag{3.19}$$

Next, we let

$$\chi_1 = \chi_{11} + \chi_{12} + \chi_{13}\,, \qquad \chi_{11} = \tilde{a}\,\Big(T_x^-\frac{\partial u}{\partial x} - T_x^- T_t^-\frac{\partial u}{\partial x}\Big)\,,$$

$$\chi_{12} = (\tilde{a} - T_x^- a)\,\Big(T_x^- T_t^-\frac{\partial u}{\partial x}\Big)\,, \qquad \chi_{13} = (T_x^- a)\,\Big(T_x^- T_t^-\frac{\partial u}{\partial x}\Big) - T_x^- T_t^-\Big(a\,\frac{\partial u}{\partial x}\Big)\,.$$

The integral formulas

$$\chi_{11}(x,t) = \frac{a(x_-) + a(x)}{2\,h\tau}\int_{x_-}^{x}\int_{t-\tau}^{t}\int_{t'}^{t}\frac{\partial^2 u(x',\,t'')}{\partial x\partial t}\,dt''\,dt'\,dx'\,,$$

$$\chi_{12}(x,t) = \Big(\frac{1}{2h}\int_{x_-}^{x}\int_{x'}^{x}\int_{x'}^{x''} a''(x''')\,dx'''\,dx''\,dx'\Big)\Big(\frac{1}{h\tau}\int_{x_-}^{x}\int_{t-\tau}^{t}\frac{\partial u(x',t')}{\partial x}\,dt'\,dx'\Big)\,,$$

and

$$\chi_{13}(x,t) = \frac{1}{2\,h^2\tau}\int\limits_{t-\tau}^{t}\int\limits_{x_-}^{x}\int\limits_{x_-}^{x}\Big(\int\limits_{x''}^{x'} a'(x''')\,dx'''\Big)\Big(\int\limits_{x'}^{x''}\frac{\partial^2 u(x''',t')}{\partial x^2}\,dx'''\Big)\,dx''\,dx'\,dt'$$

implies

$$\begin{aligned}\Big\{\tau\sum_{t\in\omega_\tau^+}\|\chi_1(\,\cdot\,,\,t)]|_{h\star}^2\Big\}^{1/2} &\le C\,\tau\,\Big(\|a\|_{W_2^1(0,\xi)} + \|a\|_{W_2^1(\xi,1)}\Big)\,\Big\|\frac{\partial^2 u}{\partial x\partial t}\Big\|_{L_2(Q)} \\ &+C\,h_{max}^2\,\Big(\|a\|_{W_2^2(0,\xi)}\,\|u\|_{W_2^{2,0}(Q_1)} + \|a\|_{W_2^2(\xi,1)}\,\|u\|_{W_2^{2,0}(Q_2)}\Big)\,.\end{aligned} \tag{3.20}$$

The addendum ψ can be presented in two forms:

$$\begin{aligned}\psi(x,t) = \frac{1}{\hbar}\int_{x_-}^{x_+}\int_{x'}^{x}\Big(1 - \frac{|x'-x|}{h}\Big)\,\frac{\partial (c\,u)(x'',\,t)}{\partial x}\,dx''\,dx' - \\ -\frac{h}{6\,\hbar}\int_{x_-}^{x}\frac{\partial (c\,u)(x',\,t)}{\partial x}\,dx' + \frac{h_+}{6\,\hbar}\int_{x}^{x_+}\frac{\partial (c\,u)(x',\,t)}{\partial x}\,dx'\end{aligned} \tag{3.21}$$

and also

$$
\begin{aligned}
\psi(x,t) = \frac{1}{\hbar}\int_{x_-}^{x}\int_{x'}^{x}\int_{x}^{x''}\Big(1+\frac{x'-x}{h}\Big)\frac{\partial^2(cu)(x''',t)}{\partial x^2}\,dx'''\,dx''\,dx' + \\
+\frac{1}{\hbar}\int_{x}^{x_+}\int_{x'}^{x}\int_{x}^{x''}\Big(1-\frac{x'-x}{h_+}\Big)\frac{\partial^2(cu)(x''',t)}{\partial x^2}\,dx'''\,dx''\,dx' + \\
+\frac{h}{6\,\hbar}\int_{x_-}^{x}\int_{x'}^{x}\frac{\partial^2(cu)(x'',t)}{\partial x^2}\,dx''dx' + \frac{h_+}{6\,\hbar}\int_{x}^{x_+}\int_{x}^{x'}\frac{\partial^2(cu)(x'',t)}{\partial x^2}\,dx''dx'.
\end{aligned}
\tag{3.22}
$$

Representation (3.22) implies the estimate

$$
\begin{aligned}
\Big\{\tau\sum_{t\in\omega_\tau}\Big(\frac{1}{t}+\frac{1}{T-t}\Big)\|\psi(t\,\cdot\,,\,)\|^2_{B_{0h}^{-1}}\Big\}^{1/2} \le C\,h_{max}^2\sqrt{\log 1/\tau}\,\|c\|_{W_2^2(0,1)}\times \\
\times\Big(\max_{t\in[0,T]}\|u(\,\cdot\,,t)\|_{W_2^2(0,\xi)} + \max_{t\in[0,T]}\|u(\,\cdot\,,t)\|_{W_2^2(\xi,1)}\Big).
\end{aligned}
\tag{3.23}
$$

Next

$$
\begin{aligned}
\tau^2\sum_{t\in\bar\omega_\tau}\sum_{t'\in\bar\omega_\tau,\,t'\ne t}\frac{\|\psi(\cdot,t)-\psi(\cdot,t')\|^2_{B_{0h}^{-1}}}{|t-t'|^2} \le 4\tau^2\sum_{t=\tau}^{T}\sum_{t'=0}^{t-\tau}\frac{\|T_t^-\psi(\cdot,t)-T_t^+\psi(\cdot,t')\|^2_{B_{0h}^{-1}}}{|t-t'|^2} + \\
+8\tau^2\sum_{t=\tau}^{T}\sum_{t'=0}^{t-\tau}\frac{\|\psi(\cdot,t)-T_t^-\psi(\cdot,t)\|^2_{B_{0h}^{-1}}}{|t-t'|^2} + 8\tau^2\sum_{t=\tau}^{T}\sum_{t'=0}^{t-\tau}\frac{\|\psi(\cdot,t')-T_t^+\psi(\cdot,t')\|^2_{B_{0h}^{-1}}}{|t-t'|^2}.
\end{aligned}
$$

Using again (3.22), we get

$$
\begin{aligned}
\Big\{\tau^2\sum_{t=\tau}^{T}\sum_{t'=0}^{t-\tau}\frac{\|T_t^-\psi(\,\cdot\,,t)-T_t^+\psi(\,\cdot\,,t')\|^2_{B_{0h}^{-1}}}{|t-t'|^2}\Big\}^{1/2} \le \\
\le C\,h_{max}^2\Big(\Big\|\frac{\partial^2(cu)}{\partial x^2}\Big\|_{W_2^{1/2}(0,T;\,L_2(0,\xi))} + \Big\|\frac{\partial^2(cu)}{\partial x^2}\Big\|_{W_2^{1/2}(0,T;\,L_2(\xi,1))}\Big) \le \\
\le C\,h_{max}^2\Big(\|u\|_{W_2^{3,\,3/2}(Q_1)} + \|u\|_{W_2^{3,\,3/2}(Q_2)}\Big).
\end{aligned}
\tag{3.24}
$$

Now (3.21) gives

$$
\begin{aligned}
\Big\{\tau^2\sum_{t=\tau}^{T}\sum_{t'=0}^{t-\tau}\frac{\|\psi(\cdot,t)-T_t^-\psi(\cdot,t)\|^2_{B_{0h}^{-1}}}{|t-t'|^2}\Big\}^{1/2} \le C\,h_{max}\sqrt{\tau}\,\Big\|\frac{\partial^2(cu)}{\partial x\partial t}\Big\|_{L_2(Q)} \\
\le C\,(h_{max}^2+\tau)\,\|c\|_{W_2^1}(0,1)\,\Big\|\frac{\partial^2 u}{\partial x\partial t}\Big\|_{L_2(Q)}.
\end{aligned}
\tag{3.25}
$$

In an analogous way can be estimated the expression

$$
\tau^2\sum_{t=\tau}^{T}\sum_{t'=0}^{t-\tau}\|\psi(\,\cdot\,,t')-T_t^+\psi(\,\cdot\,,t')\|^2_{B_{0h}^{-1}}/|t-t'|^2.
$$

Finally, from (3.17), (3.19), (3.20), (3.23), (3.24), (3.25) and the imbedding theorem we get the desired convergence rate estimate of the difference scheme (3.10)–(3.12).

Theorem 1. *Let the assumptions of the second part (ii) of Lemma 4 hold. Then*

$$
\|z\|_{\widetilde W_{2,h\tau}^{1,1/2}} \le C\Big(\tau+h_{max}^2\sqrt{\log\tfrac{1}{\tau}}\Big)\Big(\|a\|_{W_2^2(0,\xi)}+\|a\|_{W_2^2(\xi,1)}+\|c\|_{W_2^2(0,1)}\Big)\|u\|_{\widetilde W_2^{3,3/2}(Q)}.
\tag{3.26}
$$

3.4. Convergence of the difference scheme in $\widetilde{L}_{2,h\tau}$. We approximate the initial condition (3.3) as follows

$$v(x,0)=\begin{cases} \frac{T_x^2(c\,u_0)(x)}{c(x)}\,, & x\in\omega_h\setminus\{\xi\}\\ \frac{K\,u_0(\xi)+\hbar\,T_x^2(c\,u_0)(\xi)}{K+\hbar\,c(\xi)}\,, & x=\xi\,. \end{cases} \tag{3.27}$$

Let u be the solution of the initial boundary value problem (3.1)–(3.3) and v – the solution of the difference problem (3.10), (3.11), (3.27). The error $z=u-v$ satisfies the conditions

$$(c+K\,\delta_h)\,z_{\bar t}-(\tilde a\,z_{\bar x})_{\hat x}=\psi_{\bar t}-(\tilde a\,\mu_{\bar x})_{\hat x}-\alpha_{\hat x}-\beta_{\hat x\bar t}\,,\quad (x,t)\in\omega_h\times\omega_\tau^+, \tag{3.28}$$

$$z(0,t)=0,\quad z(1,t)=0,\quad t\in\bar\omega_\tau, \tag{3.29}$$

$$(c(x)+K\,\delta_h(x-\xi))\,z(x,0)=\psi(x,0)-\beta_{\hat x}(x,0)\,,\quad x\in\omega_h, \tag{3.30}$$

where ψ is the same in the Section 3.3,

$$\mu=u-T_t^-u,\quad \alpha=\chi_{12}+\chi_{13}=\tilde a\,T_x^-T_t^-\Big(\frac{\partial u}{\partial x}\Big)-T_x^-T_t^-\Big(a\,\frac{\partial u}{\partial x}\Big),\quad \beta=\frac{h^2}{6}\,(c\,u)_{\bar x}\,.$$

Lemma 6. *The solution of the difference scheme*

$$(c+K\,\delta_h)\,z_{\bar t}-(\tilde a\,z_{\bar x})_{\hat x}=-\beta_{\hat x\bar t}\,,\quad (x,t)\in\omega_h\times\omega_\tau^+,$$

with initial and boundary conditions (3.14), (3.15), satisfies the a priori estimate:

$$\|z\|'_{\widetilde{L}_{2,h\tau}}=\Big\{\tau\sum_{t\in\omega_\tau^+}\Big\|\frac{z(\,\cdot\,,t)+z(\,\cdot\,,t-\tau)}{2}\Big\|^2_{\widetilde{L}_{2,h}}\Big\}^{1/2}\le$$

$$\le C\Big\{\tau^2\sum_{t\in\bar\omega_\tau}\sum_{t'\in\bar\omega_\tau,\,t'\neq t}\frac{\|\beta(\,\cdot\,,t)-\beta(\,\cdot\,,t')]|^2_{h\star}}{|t-t'|^2}+\tau\sum_{t\in\omega_\tau}\Big(\frac{1}{t}+\frac{1}{T-t}\Big)\,\|\beta(\,\cdot\,,t)]|^2_{h\star}\Big\}^{1/2}.$$

The proof is analogous to the proof of Lemmas 1 and 2. ☐ Using lemmas 2,5 and 6 for the difference scheme (3.28)–(3.30) we get the a priori estimate

$$\begin{aligned}\|z\|'_{\widetilde{L}_{2,h\tau}}\le C\Big\{&\tau\sum_{t\in\omega_\tau^+}\|\psi(\cdot,t)\|^2_{B_{0h}^{-1}}+\tau\sum_{t\in\omega_\tau^+}\|\mu(\cdot,t)\|^2_{B_{0h}}+\tau\sum_{t\in\omega_\tau^+}\|\alpha(\cdot,t)]|^2_{h\star}\\ &+\tau^2\sum_{t\in\bar\omega_\tau}\sum_{t'\in\bar\omega_\tau,\,t'\neq t}\frac{\|\beta(\cdot,t)-\beta(\cdot,t')]|^2_{h\star}}{|t-t'|^2}+\tau\sum_{t\in\omega_\tau}\Big(\frac{1}{t}+\frac{1}{T-t}\Big)\|\beta(\cdot,t)]|^2_{h\star}\Big\}^{1/2}.\end{aligned} \tag{3.31}$$

Using the integral representations (3.21) and (3.22), the decomposition $\psi=T_t^-\psi+(\psi-T_t^-\psi)$ and the technique described in [10], we obtain the estimate

$$\begin{aligned}\Big\{\tau\sum_{t\in\omega_\tau^+}\|\psi(\cdot,t)\|^2_{B_{0h}^{-1}}\Big\}^{1/2}&\le C(h_{max}^2+\tau)\Big(\|c\,u\|_{W_2^{2,1}(Q_1)}+\|c\,u\|_{W_2^{2,1}(Q_2)}\Big)\\ &\le C\,(h_{max}^2+\tau)\,\|c\|_{W_2^2(0,1)}\,\Big(\|u\|_{W_2^{2,1}(Q_1)}+\|u\|_{W_2^{2,1}(Q_2)}\Big)\,.\end{aligned} \tag{3.32}$$

The integral formula

$$\mu(x,t)=\frac{1}{\tau}\int_{t-\tau}^{t}\int_{t'}^{t}\frac{\partial u(x,t'')}{\partial t}\,dt''\,dt'\,,$$

the decomposition $\mu = T_x^- \mu + (\mu - T_t^- \mu)$, at $x < \xi$, respectively $\mu = T_x^+ \mu + (\mu - T_t^+ \mu)$, at $x > \xi$, and again the technique in [10], leads to the estimate

$$\begin{aligned}\Big\{\tau \sum_{t\in\omega_\tau^+} \|\mu(\,\cdot\,,\,t)\|_{B_{0h}}^2\Big\}^{1/2} \le \\ \le C\,(h_{max}^2+\tau)\left(\|u\|_{W_2^{2,1}(Q_1)}+\|u\|_{W_2^{2,1}(Q_2)}\right)+C\,\tau\left\|\frac{\partial u(\xi,\,\cdot)}{\partial t}\right\|_{L_2(0,T)}.\end{aligned} \tag{3.33}$$

From the estimates χ_{12} and χ_{13} (see (3.20)) we immediately find

$$\begin{aligned}\Big\{\tau \sum_{t\in\omega_\tau^+} \|\alpha(\,\cdot\,,\,t)]|_{h\star}^2\Big\}^{1/2} \le \\ \le C\,h_{max}^2\left(\|a\|_{W_2^2(0,\xi)}\,\|u\|_{W_2^{2,0}(Q_1)}+\|a\|_{W_2^2(\xi,1)}\,\|u\|_{W_2^{2,0}(Q_2)}\right).\end{aligned} \tag{3.34}$$

Next, the formula $\beta = \frac{h^2}{6}\,T_x^-\left(\frac{\partial(c\,u)}{\partial c}\right)$ implies the estimate

$$\begin{aligned}\Big\{\tau \sum_{t\in\omega_\tau}\Big(\frac{1}{t}+\frac{1}{T-t}\Big)\,\|\beta(\,\cdot\,,t)]|_{h\star}^2\Big\}^{1/2} \le \\ \le C\,h_{max}^2\,\sqrt{\log 1/\tau}\,\max_{t\in[0,T]}\left\|\frac{\partial(c\,u)(\,\cdot\,,\,t)}{\partial x}\right\|_{L_2(0,1)} \le \\ \le C\,h_{max}^2\,\sqrt{\log 1/\tau}\,\|c\|_{W_2^1(0,1)}\left(\|u\|_{W_2^{2,1}(Q_1)}+\|u\|_{W_2^{2,1}(Q_2)}\right).\end{aligned} \tag{3.35}$$

Finally, by an analogous way as at the estimation of ψ, we get

$$\begin{aligned}\Big\{\tau^2 \sum_{t\in\bar\omega_\tau}\sum_{t'\in\bar\omega_\tau,\ t'\ne t}\frac{\|\beta(\,\cdot\,,t)-\beta(\,\cdot\,,t')]|_{h\star}^2}{|t-t'|^2}\Big\}^{1/2} \le \\ \le C\,(h_{max}^2+\tau)\,\|c\|_{W_2^2(0,1)}\Big(\|u\|_{W_2^{2,1}(Q_1)}+\|u\|_{W_2^{2,1}(Q_2)}\Big).\end{aligned} \tag{3.36}$$

Now, from (3.31)–(3.36) we obtain the desired convergence rate estimate of the difference scheme (3.10), (3.11), (3.27):

Theorem 2. *Let the following assumptions hold:* $a \in W_2^2(0,\xi)\cap W_2^2(\xi,1)$, $c \in W_2^2(0,1)$, $u_0 \in \widetilde{W}_2^1(0,1)$, $f \in L_2(Q_1)\cap L_2(Q_2)$. *Then*

$$\|z\|_{\widetilde{L}_{2,h\tau}}' \le C\Big(h_{max}^2\,\sqrt{\log\tfrac{1}{\tau}}+\tau\Big)\Big(\|c\|_{W_2^2(0,1)}+\|a\|_{W_2^2(0,\xi)}+\|a\|_{W_2^2(\xi,1)}+1\Big)\|u\|_{\widetilde{W}_2^{2,1}(Q)}. \tag{3.37}$$

Remark. In the estimate (3.37) the requirements for the smoothness of a and c in the differential equation (3.1) can be relaxed. An analogous estimate in the case $a \in W_2^1(0,\xi)\cap W_2^1(\xi,1)$ can be obtained using the so called "exact" difference scheme [22] for the approximation of $\frac{\partial}{\partial x}\Big(a(x)\,\frac{\partial u}{\partial x}\Big)$.

3.5. Approximation and convergence in $\widetilde{W}_{2,h\tau}^{2,1}$. Following [24] we approximate the equation (3.1) as follows

$$(c+K\,\delta_h)\,v_{\bar t}+\frac{h_+-h}{3}\,(c\,v)_{x\bar t}-(\tilde a\,v_{\bar x})_{\hat x}-\frac{h_+-h}{6}\,\big(a_x\,v_{\bar x\hat x}-a_{\bar x\hat x}\,v_{\bar x}\big)=T_x^2T_t^-f, \tag{3.38}$$

for $(x,t)\in\omega_h\times\omega_\tau^+$. In the expressions $a_x(\xi_-)$ and $a_{\bar x\hat x}(\xi_-)$ the value $a(\xi)$ must be replaced with $a(\xi-0)$ and in the expressions $a_x(\xi)$ and $a_{\bar x\hat x}(\xi_+)$ the value $a(\xi)$ must be replaced by $a(\xi+0)$. We approximate the boundary and the initial conditions as above with (3.11) and (3.12). With respect to the mesh $\bar\omega_h$ we suppose that $c_0 \le 2$ and $h_+ = h$ at $x = \xi$. The error $z = u - v$, where u is the solution of the problem

(3.1)–(3.3) and v – the solution of difference problem (3.38), (3.11), (3.12), satisfies the difference scheme

$$(c+K\,\delta_h)\,z_{\bar t}+\frac{h_+-h}{3}\,(c\,z)_{x\bar t}-(\tilde a\,z_{\bar x})_{\hat x}-\frac{h_+-h}{6}\big(a_x\,z_{\bar x\hat x}-a_{\bar x\hat x}\,z_{\bar x}\big)=\varphi, \tag{3.39}$$

$(x,t)\in\omega_h\times\omega_\tau^+$, with homogeneous boundary and initial conditions (3.14) and (3.15). Here

$$\varphi=\varphi_1+\varphi_2=T_t^-\bigg[c\,\frac{\partial u}{\partial t}-T_x^2\Big(c\,\frac{\partial u}{\partial t}\Big)+\frac{h_+-h}{3}\,\Big(c\,\frac{\partial u}{\partial t}\Big)_x\bigg]-$$
$$-\bigg[(\tilde a\,u_{\bar x})_{\hat x}+\frac{h_+-h}{6}\,\big(a_x\,u_{\bar x\hat x}-a_{\bar x\hat x}\,u_{\bar x}\big)-T_x^2T_t^-\,\frac{\partial}{\partial x}\Big(a\,\frac{\partial u}{\partial x}\Big)\bigg].$$

We also denote

$$A_{1h}z=-\frac{h_+-h}{6}\,\big(a_x\,z_{\bar x\hat x}-a_{\bar x\hat x}\,z_{\bar x}\big)\qquad\text{and}\qquad B_{1h}z=\frac{h_+-h}{3}\,(c\,z)_x\,.$$

It is easy to be verified the assertions.

Lemma 7. *If $c\in C^1[0,1]$ and the maximal step size of $\bar\omega_h$ is sufficiently small ($h_{max}\le(1/6-\varepsilon)/\|c\|_{C^1[0,1]}$, $0<\varepsilon<1/6$) then the inequality holds*

$$|(B_{1h}z,\,z)_h|\le(1-\varepsilon)\,\|z\|^2_{B_h}\,,\quad z\in H_h\,.$$

Lemma 8. *If $a\in C^2[0,\xi]\cap C^2[\xi,1]$ then the inequality holds*

$$\|A_{1h}z\|_{B_{0h}^{-1}}\le C\,h_{max}\Big(\|a\|_{C^2[0,\xi]}+\|a\|_{C^2[\xi,1]}\Big)\,\|z\|_{\widetilde W^2_{2,h}}\,,\quad z\in H_h\,,$$

where C is constant depending on ξ. It follows from lemmas 2, 7 and 8 that for sufficiently small h_{max} the a priori estimate is valid

$$\|z\|_{\widetilde W^{2,1}_{2,h\tau}}\le C\Big\{\tau\sum_{t\in\omega_\tau^+}\|\varphi(\,\cdot\,,t)\|^2_{B_{0h}^{-1}}\Big\}^{1/2}. \tag{3.40}$$

Summing the integral representations

$$\varphi_1(x,t)=\frac{1}{\hbar\tau}\int_{t-\tau}^{t}\int_{x_-}^{x_+}\int_{x'}^{x}\int_{x}^{x''}\kappa(x,x')\,\frac{\partial^3(c\,u)(x''',\,t')}{\partial x^2\partial t}\,dx'''\,dx''\,dx'\,dt'+$$
$$+\frac{h_+-h}{3\,\hbar\tau}\int_{t-\tau}^{t}\int_{x}^{x_+}\int_{x}^{x'}\frac{\partial^3(c\,u)(x'',\,t')}{\partial x^2\partial t}\,dx''\,dx'\,dt'\,,\quad x\ne\xi$$

and

$$\varphi_1(\xi,t)=\frac{1}{\hbar\tau}\int_{t-\tau}^{t}\int_{\xi_-}^{\xi}\int_{x'}^{\xi}\int_{\xi}^{x''}\kappa(\xi,x')\,\frac{\partial^3(c\,u)(x''',\,t')}{\partial x^2\partial t}\,dx'''\,dx''\,dx'\,dt'+$$
$$+\frac{1}{\hbar\tau}\int\limits_{t-\tau}^{t}\int_{\xi}^{\xi_+}\int\limits_{x'}^{\xi}\int\limits_{\xi}^{x''}\kappa(\xi,x')\,\frac{\partial^3(c\,u)(x''',t')}{\partial x^2\partial t}\,dx'''dx''dx'dt'-\frac{h}{6\tau}\int\limits_{t-\tau}^{t}\bigg[\frac{\partial^2(c\,u)}{\partial x\partial t}\bigg]_{(\xi,t')}dt'$$

we find the estimate

$$\Big\{\tau\sum_{t\in\omega_\tau^+}\|\varphi_1(\,\cdot\,,t)\|^2_{B_{0h}^{-1}}\Big\}^{1/2}\le C\,h^2_{max}\,\|c\|_{W_2^2(0,1)}\,\bigg\{\Big\|\frac{\partial^2u}{\partial x\partial t}\Big\|_{L_2(Q)}+$$
$$+\Big\|\frac{\partial^3u}{\partial x^2\partial t}\Big\|_{L_2(Q_1)}+\Big\|\frac{\partial^3u}{\partial x^2\partial t}\Big\|_{L_2(Q_2)}+\Big\|\,\Big[\frac{\partial^2u}{\partial x\partial t}\Big]_{(\xi,\,\cdot\,)}\Big\|_{L_2(0,T)}\bigg\}. \tag{3.41}$$

For $x \neq \xi$ we decompose the addendum φ_2 as follows:

$$\varphi_2 = \varphi_{21} + \varphi_{22} + \varphi_{23} + \varphi_{24} + \varphi_{25} + \varphi_{26} + \varphi_{27} =$$

$$= -\frac{1}{2}\left(T_x^2\,\frac{\partial^2(a\,u)}{\partial x^2} - T_x^2 T_t^-\,\frac{\partial^2(a\,u)}{\partial x^2}\right) - \frac{1}{2}\left(a + \frac{h_+ - h}{3}\,a_x\right)\left(T_x^2\,\frac{\partial^2 u}{\partial x^2} - T_x^2 T_t^-\,\frac{\partial^2 u}{\partial x^2}\right)$$

$$-\frac{1}{2}\Big(a - T_x^2 a + \frac{h_+ - h}{3}\,a_x\Big)T_x^2 T_t^-\,\frac{\partial^2 u}{\partial x^2} - \frac{1}{2}\left[\left(T_x^2 a\right)\left(T_x^2 T_t^-\,\frac{\partial^2 u}{\partial x^2}\right) - T_x^2 T_t^-\left(a\,\frac{\partial^2 u}{\partial x^2}\right)\right]$$

$$+\frac{1}{2}\left(T_x^2 a''\right)\left(u + \frac{h_+ - h}{3}\,u_{\bar x} - T_t^- u - \frac{h_+ - h}{3}\,T_t^- u_{\bar x}\right) +$$

$$+\frac{1}{2}\left(T_x^2 a''\right)T_t^-\Big(u - T_x^2 u + \frac{h_+ - h}{3}\,u_{\bar x}\Big) + \frac{1}{2}\Big[\left(T_x^2 a''\right)\left(T_x^2 T_t^- u\right) - T_x^2 T_t^-\left(a''\,u\right)\Big].$$

The integral representation

$$\varphi_{21}(x,t) = -\frac{1}{2\,\hbar\tau}\int_{x_-}^{x_+}\int_{t-\tau}^{t}\int_{t'}^{t}\kappa(x,x')\,\frac{\partial^3(a\,u)(x',\,t'')}{\partial x^2\partial t}\,dt''\,dt'\,dx', \quad x \neq \xi$$

implies

$$\Big\{\tau\sum_{t\in\omega_\tau^+}\;\sum_{x\in\omega_h,\;x<\xi}\varphi_{21}^2(x,t)\,\hbar\Big\}^{1/2} \leq \tag{3.42}$$
$$\leq C\,\tau\,\|a\|_{W_2^2(0,\xi)}\left(\Big\|\frac{\partial^3 u}{\partial x^2\partial t}\Big\|_{L_2(Q_1)} + \Big\|\frac{\partial^2 u}{\partial x\partial t}\Big\|_{L_2(Q_1)}\right).$$

In an analogous way, we obtain

$$\Big\{\tau\sum_{t\in\omega_\tau^+}\;\sum_{x\in\omega_h,\;x<\xi}\varphi_{22}^2(x,t)\,\hbar\Big\}^{1/2} \leq C\,\tau\,\|a\|_{W_2^1(0,\xi)}\,\Big\|\frac{\partial^3 u}{\partial x^2\partial t}\Big\|_{L_2(Q_1)}. \tag{3.43}$$

Using a known estimate for an expression of the form $a - T_x^2 a + \frac{h_+ - h}{3}\,a_x$ (cf. φ_1) we find

$$\Big\{\tau\sum_{t\in\omega_\tau^+}\;\sum_{x\in\omega_h,\;x<\xi}\varphi_{23}^2(x,t)\,\hbar\Big\}^{1/2} \leq C\,h_{max}^2\,\|a\|_{W_2^3(0,\xi)}\,\Big\|\frac{\partial^2 u}{\partial x^2}\Big\|_{L_2(Q_1)}. \tag{3.44}$$

The formula

$$\varphi_{24}(x,t) = -\frac{1}{4\,\hbar^2\tau}\int_{x_-}^{x_+}\int_{x_-}^{x_+}\int_{t-\tau}^{t}\kappa(x,x')\,\kappa(x,x'')\bigg(\int_{x''}^{x'} a'(x''')\,dx'''\bigg)\times$$
$$\times\bigg(\int_{x'}^{x''}\frac{\partial^3 u(x''',\,t')}{\partial x^3}\,dx'''\bigg)dt'\,dx''\,dx', \quad x \neq \xi$$

and the imbedding theorem give

$$\Big\{\tau\sum_{t\in\omega_\tau^+}\;\sum_{x\in\omega_h,\;x<\xi}\varphi_{24}^2(x,t)\,\hbar\Big\}^{1/2} \leq C\,h_{max}^2\,\|a\|_{W_2^2(0,\xi)}\,\Big\|\frac{\partial^3 u}{\partial x^3}\Big\|_{L_2(Q_1)}. \tag{3.45}$$

¿From the obvious inequality

$$|\varphi_{25}| \leq C\,\|a''\|_{C[0,\xi]}\,\max_{x\in[0,\xi]}|u - T_t^- u|,$$

by applying imbedding theorem, we find

$$\Big\{\tau\sum_{t\in\omega_\tau^+}\;\sum_{x\in\omega_h,\;x<\xi}\varphi_{25}^2(x,t)\,\hbar\Big\}^{1/2} \leq C\,\tau\,\|a\|_{W_2^3(0,\xi)}\,\Big\|\frac{\partial^2 u}{\partial x\partial t}\Big\|_{L_2(Q_1)}. \tag{3.46}$$

The addendum φ_{26} can be estimated in a similar way as φ_1

$$\Big\{\tau \sum_{t\in\omega_\tau^+}\ \sum_{x\in\omega_h,\ x<\xi} \varphi_{26}^2(x,t)\,\hbar\Big\}^{1/2} \le C\,h_{max}^2\,\|a\|_{W_2^3(0,\xi)}\,\Big\|\frac{\partial^2 u}{\partial x^2}\Big\|_{L_2(Q_1)}\,. \tag{3.47}$$

The integral formula

$$\varphi_{27}(x,t)=\frac{1}{4\,\hbar^2\tau}\int_{x_-}^{x_+}\int_{x_-}^{x_+}\int_{t-\tau}^{t}\kappa(x,x')\,\kappa(x,x'')\Big(\int_{x''}^{x'} a'''(x''')\,dx'''\Big)\times$$
$$\times\Big(\int_{x'}^{x''}\frac{\partial u(x''',t')}{\partial x}\,dx'''\Big)\,dt'\,dx''\,dx'\,,\quad x\ne\xi$$

implies

$$\Big\{\tau \sum_{t\in\omega_\tau^+}\ \sum_{x\in\omega_h,\ x<\xi} \varphi_{27}^2(x,t)\,\hbar\Big\}^{1/2} \le C\,h_{max}^2\,\|a\|_{W_2^3(0,\xi)}\,\|u\|_{W_2^{2,0}(Q_1)}\,. \tag{3.48}$$

For $x=\xi$ we set

$$\varphi_2=\varphi_{28}+\varphi_{29}=-\Big[(\tilde a u_{\bar x})_{\hat x}-T_t^-(\tilde a u_{\bar x})_{\hat x}\Big]-\Big[T_t^-(\tilde a u_{\bar x})_{\hat x}-T_x^2T_t^-\frac{\partial}{\partial x}\Big(a\,\frac{\partial u}{\partial x}\Big)\Big]\,.$$

¿From the integral representation

$$\hbar\,\varphi_{28}(\xi,t)=\frac{a(\xi-0)+a(\xi-h)}{2h\tau}\int_{\xi_-}^{\xi}\int_{t-\tau}^{t}\int_{t'}^{t}\frac{\partial^2 u(x',t'')}{\partial x\,\partial t}\,dt''\,dt'\,dx'-$$
$$-\frac{a(\xi+0)+a(\xi+h)}{2h\tau}\int_{\xi}^{\xi_+}\int_{t-\tau}^{t}\int_{t'}^{t}\frac{\partial^2 u(x',t'')}{\partial x\,\partial t}\,dt''\,dt'\,dx'\,,$$

by applying of imbedding theorem we obtain

$$\Big\{\tau \sum_{t\in\omega_\tau^+}\varphi_{28}^2(\xi,t)\,\hbar^2\Big\}^{1/2} \le C\,\tau\Big[\|a\|_{W_2^1(0,\xi)}\Big(\Big\|\frac{\partial^3 u}{\partial^2 x\,\partial t}\Big\|_{L_2(Q_1)}+\Big\|\frac{\partial^2 u}{\partial x\,\partial t}\Big\|_{L_2(Q_1)}\Big)+$$
$$+\|a\|_{W_2^1(\xi,1)}\Big(\Big\|\frac{\partial^3 u}{\partial^2 x\,\partial t}\Big\|_{L_2(Q_2)}+\Big\|\frac{\partial^2 u}{\partial x\,\partial t}\Big\|_{L_2(Q_2)}\Big)\Big]\,. \tag{3.49}$$

In a similar way, from the formula

$$\hbar\,\varphi_{29}(\xi,t)=\frac{1}{2h\tau}\int_{\xi_-}^{\xi}\int_{x'}^{\xi}\int_{x'}^{x''}\int_{t-\tau}^{t}\Big(a''(x''')\frac{\partial u(x'',t')}{\partial x}-a'(x'')\frac{\partial^2 u(x''',t')}{\partial x^2}\Big)dx'''dx''dx'dt'$$
$$-\frac{1}{2h\tau}\int_{\xi}^{\xi_+}\int_{x'}^{\xi_+}\int_{x'}^{x''}\int_{t-\tau}^{t}\Big(a''(x''')\,\frac{\partial u(x'',t')}{\partial x}-a'(x'')\,\frac{\partial^2 u(x''',t')}{\partial x^2}\Big)dx'''dx''dx'dt'$$

we find

$$\Big\{\tau \sum_{t\in\omega_\tau^+}\varphi_{29}^2(\xi,t)\,\hbar^2\Big\}^{1/2} \le C h_{max}^2\Big(\|a\|_{W_2^3(0,\xi)}\,\|u\|_{W_2^{3,0}(Q_1)}+\|a\|_{W_2^3(\xi,1)}\,\|u\|_{W_2^{3,0}(Q_2)}\Big). \tag{3.50}$$

From (3.42)–(3.48), analogous estimates for $x > \xi$, (3.49) and (3.50), we obtain the estimate for φ_2 :

(3.51)

$$\Big\{\tau \sum_{t\in\omega_\tau^+} \|\varphi_2(\,\cdot\,,t)\|^2_{B_{0h}^{-1}}\Big\}^{1/2} \le C\,(h^2_{max}+\tau)\,\Big(\|a\|_{W_2^3(0,\xi)} + \|a\|_{W_2^3(\xi,1)}\Big)\,\|u\|_{\widetilde{W}_2^{4,2}(Q)}.$$

Finally, from (3.40), (3.41) and (3.51) we get the required estimate for the rate of convergence of the difference scheme (3.37), (3.11), (3.12):

Theorem 3. *Let the assumptions of the third part (iii) of Lemma 4 hold. Then*

$$(3.52)\quad \|z\|_{\widetilde{W}_{2,h\tau}^{2,1}} \le C\,(h^2_{max}+\tau)\,\Big(\|a\|_{W_2^3(0,\xi)}+\|a\|_{W_2^3(\xi,1)}+\|c\|_{W_2^2(0,1)}\Big)\,\|u\|_{\widetilde{W}_2^{4,2}(Q)}.$$

4. Problem with dynamical boundary condition

Let us consider the initial-boundary value problem for the heat equation with dynamical boundary condition at $x = 0$ (cf. [7], [28]):

$$(4.1)\qquad c(x)\,\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\Big(a(x)\,\frac{\partial u}{\partial x}\Big) = f(x,t), \quad x \in (0,\,1), \quad 0 < t < T,$$

$$(4.2)\qquad K\,\frac{\partial u(0,\,t)}{\partial t} = a(0)\,\frac{\partial u(0,\,t)}{\partial x}\,, \quad u(1,\,t) = 0, \quad 0 < t < T$$

$$(4.3)\qquad u(x,\,0) = u_0(x), \quad x \in (0,\,1),$$

where, as in Section 3, $K > 0$, $0 < c_1 \le a(x) \le c_2$ and $0 < c_3 \le c(x) \le c_4$. The problem (4.1)–(4.3) can be reduced to a problem of the form (3.1)–(3.3) using even extension of the input data: $c(x) = c(-x)$, $a(x) = a(-x)$, $u_0(x) = u_0(-x)$, $f(x,t) = f(-x,t)$, for $x \in (-1,0)$. It easily follows that the solution $u(x,t)$ also can be extended by even fashion on $(-1,0)\times(0,T)$ and it satisfies the conditions

$$(4.4)\quad \big[c(x) + 2K\,\delta(x)\big]\,\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\Big(a(x)\,\frac{\partial u}{\partial x}\Big) = f(x,t), \quad x \in (-1,1), \quad 0 < t < T,$$

$$(4.5)\qquad u(-1,t) = 0, \quad u(1,t) = 0, \quad 0 < t < T,$$

$$(4.6)\qquad u(x,\,0) = u_0(x), \quad x \in (-1,1).$$

The problem (4.4)–(4.6) can be written in the form (2.1) if one lets $H = L_2(-1,1)$,

$$Au = -\frac{\partial}{\partial x}\Big(a(x)\,\frac{\partial u}{\partial x}\Big) \qquad \text{and} \qquad Bu = \big[c(x) + 2K\,\delta(x)\big]\,u(x,\,t)\,.$$

If $w(x)$ is an even function on the segment $(-1,1)$, then

$$\|w\|_A^2 = \int_{-1}^1 a(x)\,[w'(x)]^2\,dx = 2\int_0^1 a(x)\,[w'(x)]^2\,dx\,,$$

$$\|w\|_B^2 = \int_{-1}^1 c(x)\,w^2(x)\,dx + 2K\,w^2(0) = 2\int_0^1 c(x)\,w^2(x)\,dx + 2K\,w^2(0)\,.$$

Further, we assume that the functions $c(x)$ and $a(x)$ are continuous on $[0,1]$. By $\widehat{L}_2(0,1) = \widehat{W}_2^0(0,1)$ we denote the closure of $C[0,1]$ in the norm $\|w\|_{\widehat{L}_2(0,1)}$ induced by inner product

$$(u,w)_{\widehat{L}_2(0,1)} = \int_0^1 u(x)\,w(x)\,dx + u(0)\,w(0)\,.$$

We let $\widehat{W}_2^1(0,1) = \big\{w \in W_2^1(0,1)\,:\,w(1) = 0\big\}$ and $\widehat{W}_2^k(0,1) = \widehat{W}_2^1(0,1)\cap W_2^k(0,1)$, $k = 2,\,3,\,\ldots$. The following analog of Lemma 3 holds.

Lemma 9. *If $Aw = -(a(x)\,w'(x))'$ and $Bw = (c(x) + K\,\delta(x))\,w(x)$, where $a,\, c \in L_\infty(0,1)$. Then the norm $\|w\|_B$ is equivalent to the norm $\|w\|_{\widehat{L}_2(0,1)}$. The norm $\|w\|_A$ is equivalent to the norm $\|w\|_{\widehat{W}_2^1(0,1)}$, for $w \in \widehat{W}_2^1(0,1)$. If in addition $a' \in L_2(0,1)$ then the norm $\|Aw\|_{B^{-1}}$ is equivalent to the norm $\|w\|_{\widehat{W}_2^2(0,1)}$, for $w \in \widehat{W}_2^2(0,1)$.* We define also the spaces $\widehat{W}_2^{k,\,k/2}(Q) = L_2(0,T;\, \widehat{W}_2^k(0,1)) \cap W_2^k(0,T;\, \widehat{L}_2(0,1))$, $k = 0,\, 1,\, 2,\, \dots$.

Lemma 10. *(i) If $a \in W_2^1(0,1)$, $c \in W_2^1(0,1)$, $u_0 \in \widehat{W}_2^1(0,1)$ and $f \in L_2(Q)$, then the problem (4.1)–(4.3) has unique solution $u \in \widehat{W}_2^{2,1}(Q)$. (ii) If $a \in W_2^2(0,1)$, $c \in W_2^2(0,1)$, $u_0 \in \widehat{W}_2^2(0,1)$, $f \in W_2^{1,1/2}(Q)$ and the compatibility conditions hold*

$$U_t(1) = 0, \quad a(0)u_0'(0) = K\,U_t(0),$$

where $U_t(x)$ is defined as in Lemma 4, then the problem (4.1)–(4.3) has unique solution $u \in \widehat{W}_2^{3,3/2}(Q)$. (iii) If $a \in W_2^3(0,1)$, $c \in W_2^3(0,1)$, $u_0 \in \widehat{W}_2^3(0,1)$, $f \in W_2^{2,1}(Q)$ and the last compatibility conditions hold, then the problem (4.1)–(4.3) has unique solution $u \in \widehat{W}_2^{4,2}(Q)$. On the segment $[0,1]$ we introduce nonuniform mesh $\bar\omega_h$. Let $\widehat{H}_h$ be the space of the mesh functions, equal to zero at $x = 1$. We define the following inner product

$$[v,\, w)_h = \frac{h_1}{2}\, v(0)\, w(0) + \sum_{x\in\omega_h} v(x)\, w(x)\, \hbar,$$

and the corresponding norm $|[w\|_h = |[w\|_{L_{2,h}} = [w,\, w)_h^{1/2}$. We also define the mesh norms

$$|[w\|^2_{\widehat{L}_{2,h}} = |[w\|^2_{L_{2,h}} + w^2(0)\,, \quad |[w\|^2_{\widehat{W}^1_{2,h}} = \|w_{\bar x}]|^2_{h\star} + |[w\|^2_h\,,$$

$$|[w\|^2_{\widehat{W}^2_{2,h}} = \|w_{\bar x\hat x}\|^2_h + \|w_{\bar x}]|^2_{h\star} + |[w\|^2_h\,;$$

$$|[v\|^2_{\widehat{L}_{2,h\tau}} = \tau \sum_{t\in\bar\omega_\tau} |[v(\,\cdot\,,t)\|^2_{\widehat{L}_{2,h}}, \quad |[v\|'_{\widehat{L}_{2,h\tau}} = \left\{\tau \sum_{t\in\omega_\tau^+} \left\|\left[\frac{v(\,\cdot\,,t) + v(\,\cdot\,,t-\tau)}{2}\right\|^2_{\widehat{L}_{2,h}}\right\}^{1/2},$$

$$|[v\|^2_{\widehat{W}^{1,\,1/2}_{2,h\tau}} = \tau \sum_{t\in\bar\omega_\tau} |[v(\,\cdot\,,t)\|^2_{\widehat{W}^1_{2,h}} + \tau^2 \sum_{t\in\bar\omega_\tau}\ \sum_{t'\in\bar\omega_\tau,\ t'\neq t} \frac{|[v(\,\cdot\,,t) - v(\,\cdot\,,t')\|^2_{\widehat{L}_{2,h}}}{|t-t'|^2}\,,$$

$$|[v\|^2_{\widehat{W}^{2,1}_{2,h\tau}} = \tau \sum_{t\in\bar\omega_\tau} |[v(\,\cdot\,,t)\|^2_{\widehat{W}^2_{2,h}} + \tau \sum_{t\in\omega_\tau^+} |[v_{\bar t}(\,\cdot\,,t)\|^2_{\widehat{L}_{2,h}}\,.$$

We approximate the problem (4.1)–(4.3) by the difference scheme

$$(4.7) \qquad (c + 2K\,\delta_h)\, v_{\bar t} - (\tilde a\, v_{\bar x})_{\hat x} = T_x^2 T_t^- f, \quad (x,t) \in \omega_h^- \times \omega_\tau^+\,,$$

$$(4.8) \qquad v(1,t) = 0, \quad t \in \bar\omega_\tau,$$

$$(4.9) \qquad v(x,0) = u_0(x), \quad x \in \omega_h^-\,,$$

where

$$(\tilde a\, v_{\bar x})_{\hat x}\Big|_{x=0} = \frac{2}{h_1}\,(\tilde a\, v_{\bar x})\Big|_{x=x_1}\,, \qquad \delta_h = \delta_h(x) = \begin{cases} 0, & x \in \omega_h^+ \\ 1/h_1, & x = 0\,, \end{cases}$$

$$T_x^2 f(0,t) = \frac{2}{h_1}\int_0^{x_1} \left(1 - \frac{x'}{h_1}\right) f(x',t)\, dx'\,.$$

We also consider higher order approximation of (4.1)

$$
\begin{aligned}
&(c+2K\,\delta_h)\,v_{\bar t}+\theta\,\frac{h_+-h}{3}\,(c\,v)_{x\bar t}-(\tilde a\,v_{\bar x})_{\hat x}- \\
&-\theta\,\frac{h_+-h}{6}\,\big(a_x\,v_{\bar x\hat x}-a_{\bar x\hat x}\,v_{\bar x}\big)=T_x^2T_t^-\,f,\qquad (x,t)\in\omega_h^-\times\omega_\tau^+,
\end{aligned}
\tag{4.10}
$$

where $\theta(x) = 1$ for $x \in \omega_h$ and $\theta(0) = 0$, and the following approximation of initial condition (4.3)

$$
v(x,0)=\begin{cases}\frac{T_x^2(c\,u_0)(x)}{c(x)}\,, & x\in\omega_h\\ \frac{2\,K\,u_0(0)+h_1\,T_x^2(c\,u_0)(0)}{2\,K+h_1\,c(0)}\,, & x=0\,.\end{cases}
\tag{4.11}
$$

Using results obtained in Section 3, we immediately obtain the following result.

Theorem 4. *Let the assumptions of the first part (i) of Lemma 10 hold and* $a \in W_2^2(0,1)$, $c \in W_2^2(0,1)$. *Then difference scheme (4.7), (4.8), (4.11) converges and the following error bound holds*

$$
|[u-v\|'_{\widehat L_{2,h\tau}}\le C\left(h_{max}^2\sqrt{\log 1/\tau}+\tau\right)\left(\|c\|_{W_2^2(0,1)}+\|a\|_{W_2^2(0,1)}+1\right)\|u\|_{\widehat W_2^{2,1}(Q)}\,.
$$

If the assumptions of the second part (ii) of Lemma 10 hold then difference scheme (4.7)–(4.9) converges and the following error bound holds

$$
|[u-v\|_{\widehat W_{2,h\tau}^{1,\,1/2}}\le C\left(h_{max}^2\sqrt{\log 1/\tau}+\tau\right)\left(\|a\|_{W_2^2(0,1)}+\|c\|_{W_2^2(0,1)}\right)\|u\|_{\widehat W_2^{3,\,3/2}(Q)}\,.
$$

If the assumptions of the third part (iii) of Lemma 10 hold then difference scheme (4.10), (4.8), (4.9) converges and the following error bound holds

$$
|[u-v\|_{\widehat W_{2,h\tau}^{2,1}}\le C\,(h_{max}^2+\tau)\left(\|a\|_{W_2^3(0,1)}+\|c\|_{W_2^2(0,1)}\right)\|u\|_{\widehat W_2^{4,2}(Q)}\,.
$$

5. Weakly-parabolic equation

Let us consider the initial–boundary value problem:

$$
K\,\delta(x-\xi)\,\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\Big(a(x)\,\frac{\partial u}{\partial x}\Big)=f(x,t),\quad x\in(0,\,1),\quad 0<t<T,
\tag{5.1}
$$

$$
u(0,\,t)=0,\quad u(1,\,t)=0,\quad 0<t<T
\tag{5.2}
$$

$$
u(\xi,\,0)=u_0=\text{const},
\tag{5.3}
$$

where $K > 0$, $0 < c_1 \le a(x) \le c_2$ and $\delta(x)$ is the Dirac–delta function. From (5.1) follows, that the solution for $(x,t) \in Q_1$ and $(x,t) \in Q_2$ satisfies the equation

$$
-\frac{\partial}{\partial x}\Big(a(x)\,\frac{\partial u}{\partial x}\Big)=f(x,t)\,,
$$

and for $x = \xi$ – the conjugation conditions

$$
[u]_{x=\xi}\equiv u(\xi+0,t)-u(\xi-0,t)=0\,,\qquad \Big[a\,\frac{\partial u}{\partial x}\Big]_{x=\xi}=K\,\frac{\partial u(\xi,t)}{\partial t}\,.
$$

Therefore, at fixed t, the equation is elliptic on $(0,\xi)$ and $(\xi,1)$, and its parabolic character exhibits only in the point $x = \xi$. The problem (5.1)–(5.3) also has the form (2.1), where $Au = -\frac{\partial}{\partial x}\Big(a(x)\,\frac{\partial u}{\partial x}\Big)$ and $Bu = K\,\delta(x-\xi)\,u(x,t)$. The operator A is positively definite in the space $H_A = \overset{\circ}{W}{}_2^1(0,1)$. The operator B is nonnegative in H_A and

$$
\|w\|_B=\sqrt{K}\,|w(\xi)|\,.
$$

It is easy to see, that in this case the estimates of Lemma 1 in which doesn't participate B^{-1} are valid. From Lemma 1 and well known results for elliptic equations it follows the following assertion.

Lemma 11. *(i) If $a \in L_\infty(0,1)$, $f = f_0 + \frac{\partial f_1}{\partial x}$ and $f_0, f_1 \in L_2(Q)$, then the problem (5.1)–(5.3) has unique solution $u \in W_2^{1,0}(Q)$ and $u(\xi,\,\cdot\,) \in W_2^{1/2}(0,T)$. (ii) If $a \in W_2^1(0,\xi) \cap W_2^1(\xi,1)$ and $f \in L_2(Q)$ then the problem (5.1)–(5.3) has unique solution $u \in W_2^{1,0}(Q) \cap W_2^{2,0}(Q_1) \cap W_2^{2,0}(Q_2)$ and $u(\xi,\,\cdot\,) \in W_2^1(0,T)$. (iii) If $a \in W_2^2(0,\xi) \cap W_2^2(\xi,1)$ and $f \in W_2^1(Q)$ then the problem (5.1)–(5.3) has unique solution $u \in W_2^{1,0}(Q) \cap W_2^{3,0}(Q_1) \cap W_2^{3,0}(Q_2)$, $\frac{\partial^2 u}{\partial t \partial x} \in L_2(Q)$ and $u(\xi,\,\cdot\,) \in W_2^{3/2}(0,T)$.* Holding back the notations from Section 3, we approximate the problem (5.1)–(5.3) by the implicit difference scheme with averaged right–hand side

$$K\,\delta_h\,v_{\bar t} - (\tilde a\,v_{\bar x})_{\hat x} = T_x^2 T_t^- f, \quad (x,t) \in \omega_h \times \omega_\tau^+, \tag{5.4}$$

$$v(0,t) = 0, \quad v(1,t) = 0, \quad t \in \bar\omega_\tau, \tag{5.5}$$

$$v(\xi,0) = u_0. \tag{5.6}$$

The error $z = u - v$ satisfies the discrete problem

$$K\,\delta_h\,z_{\bar t} - (\tilde a\,z_{\bar x})_{\hat x} = -\chi_{1,\,\hat x}, \quad (x,t) \in \omega_h \times \omega_\tau^+, \tag{5.7}$$

$$z(0,t) = 0, \quad z(1,t) = 0, \quad t \in \bar\omega_\tau, \tag{5.8}$$

$$z(\xi,0) = 0, \tag{5.9}$$

where, as in Section 3.3

$$\chi_1 = \tilde a\,u_{\bar x} - T_x^- T_t^- \Big(a\,\frac{\partial u}{\partial x}\Big)\,.$$

The difference scheme (5.7)–(5.9) can be written in the form (2.3) where $A_h v = -(\tilde a\,v_{\bar x})_{\hat x}$ is positive linear operator in H_h and $B_h v = K\,\delta_h v$ – nonnegative linear operator in H_h. Also,

$$\|w\|_{A_h} = \Big\{\sum_{x\in\omega_h^+} \tilde a\,w_{\bar x}^2\,\hbar\Big\}^{1/2} = \|w_{\bar x}]|_{h\star}\,, \qquad \|w\|_{B_h} = \sqrt{K}\,|w(\xi)|\,.$$

We need the norm

$$\|v\|^2_{\overline{W}_{2,h\tau}^{1,\,1/2}} = \tau \sum_{t\in\omega_\tau^+}\Big(\|v(\,\cdot\,,t)\|_h^2 + \|v_{\bar x}(\,\cdot\,,t)]|^2_{h\star}\Big) + \tau^2 \sum_{t\in\bar\omega_\tau}\ \sum_{t'\in\bar\omega_\tau,\ t'\neq t} \frac{|v(\xi,t)-v(\xi,t')|^2}{|t-t'|^2}\,.$$

From Lemma 2, using the discrete Friedrichs inequality, we get the a priori estimate

$$\|z\|_{\overline{W}_{2,h\tau}^{1,\,1/2}} \le C\Big\{\tau \sum_{t\in\omega_\tau^+} \|\chi_1(\,\cdot\,,t)]|^2_{h\star}\Big\}^{1/2}. \tag{5.10}$$

Using the estimate (3.20) for χ_1 we immediately obtain the following estimate for the rate of convergence of the difference scheme (5.4)–(5.6):

$$\begin{aligned} \|z\|_{\overline{W}_{2,h\tau}^{1,\,1/2}} &\le C\,(h_{max}^2 + \tau)\Big(\|a\|_{W_2^2(0,\xi)} + \|a\|_{W_2^2(\xi,1)}\Big)\times \\ &\times\Big(\Big\|\frac{\partial^2 u}{\partial x \partial t}\Big\|_{L_2(Q)} + \|u\|_{W_2^{2,0}(Q_1)} + \|u\|_{W_2^{2,0}(Q_2)}\Big)\,. \end{aligned} \tag{5.11}$$

From Lemma 2 it follows also the estimate in the "weak" (semi)norm

$$|z|^2_{\overline{L}_{2,h\tau}} \equiv \tau \sum_{t\in\omega_\tau^+} |z(\xi,\,t)|^2 = \frac{\tau}{K} \sum_{t\in\omega_\tau^+} \|z(\,\cdot\,,\,t)\|^2_{B_h} \le C\,\tau \sum_{t\in\omega_\tau^+} \|A_h^{-1}\chi_{1,\,\hat{x}}(\,\cdot\,,\,t)\|^2_{B_h}\,. \tag{5.12}$$

Letting

$$-\chi_{1,\,\hat{x}} = A_h\mu - \alpha_{\hat{x}}\,,$$

where as in Section 3.4

$$\mu = u - T_t^- u \qquad \text{and} \qquad \alpha = \tilde{a}\,T_x^- T_t^-\Big(\frac{\partial u}{\partial x}\Big) - T_x^- T_t^-\Big(a\,\frac{\partial u}{\partial x}\Big)\,.$$

We find

$$\|A_h^{-1}\chi_{1,\,\hat{x}}\|^2_{B_h} \le \|\mu\|^2_{B_h} + \|A_h^{-1}(\alpha_{\hat{x}})\|^2_{B_h}\,. \tag{5.13}$$

We set

$$A_h^{-1}(\alpha_{\hat{x}}) = \nu\,.$$

Then, using the imbedding, [22]:

$$\|\nu\|_{C(\bar{\omega}_h)} = \max_{x\in\bar{\omega}_h} |\nu(x)| \le 0.5\,\|\nu_{\bar{x}}]|_{h\star}\,,$$

we get

$$\begin{aligned} &\|A_h^{-1}(\alpha_{\hat{x}})\|^2_{B_h} = \|\nu\|^2_{B_h} = K\,\nu^2(\xi) \le K\,\|\nu\|^2_{C(\bar{\omega}_h)} \le \\ &\le \frac{K}{4}\,\|\nu_{\bar{x}}]|^2_{h\star} \le C\,\|\nu\|^2_{A_h} = C\,\|\alpha_{\hat{x}}\|^2_{A_h^{-1}} \le C\,\|\alpha]|^2_{h\star}\,. \end{aligned} \tag{5.14}$$

Therefore, from (5.12)–(5.14) it follows

$$\tau \sum_{t\in\omega_\tau^+} |z(\xi,\,t)|^2 \le C\,\tau \sum_{t\in\omega_\tau^+} \Big(|\mu(\xi,t)|^2 + \|\alpha(\,\cdot\,,\,t)]|^2_{h\star}\Big)\,. \tag{5.15}$$

From (5.15) and the estimates for μ and α (3.33) and (3.34) we get the following rate of convergence estimate.

$$\begin{aligned} |z|_{\overline{L}_{2,h\tau}} \le C\,\tau\,\Big\|\frac{\partial u(\xi,\,\cdot\,)}{\partial t}\Big\|_{L_2(0,T)} + \\ +C\,h^2_{max}\Big(\|a\|_{W_2^2(0,\xi)} + \|a\|_{W_2^2(\xi,1)}\Big)\Big(\|u\|_{W_2^{2,0}(Q_1)} + \|u\|_{W_2^{2,0}(Q_2)}\Big)\,. \end{aligned} \tag{5.16}$$

In such a way, the following assertion is valid.

Theorem 5. *Let $a \in W_2^2(0,\xi) \cap W_2^2(\xi,1)$. If $f \in W_2^1(Q)$ then the difference scheme (5.4)–(5.6) converges in discrete $\overline{W}^{1,1/2}_{2,h\tau}$ norm and the convergence rate estimate (5.11) is satisfied. If $f \in L_2(Q)$ then the convergence rate estimate (5.16) holds.*

References

[1] I. Braianov, Convergence of a Crank-Nicolson difference scheme for heat equation with interface in the heat flow and concentrated heat capacity. Lect. Notes Comput. Sci. 1196 (1997), 58-65.

[2] I. Braianov, L. Vulkov, Finite Difference Schemes with Variable Weights for Parabolic Equations with Concentrated Capacity. Notes on Numerical Fluid Dynamics, Vieweg, 62 (1998), 208-216.

[3] I. Braianov, L. Vulkov, Homogeneous difference schemes for the heat equation with concentrated capacity. Zh. vychisl. mat. mat. fiz. **39** (1999), 254-261 (Russian).

[4] H. Bramble, R. Hilbert, Bounds for a class of linear functionals with application to Hermite interpolation. Numer. Math. **16** (1971), 362–369.

[5] P. Colli, J. Rodrigues, Diffusion through thin layers with high specific heat. Asymptotic Analysis 3 (1990), 249-263.
[6] T. Dupont, R. Scott, Polynomial approximation of functions in Sobolev spaces. Math. Comput. 34 (1980), 441–463.
[7] J. Escher, Quasilinear parabolic systems with dynamical boundary conditions. Communs. Partial Differential Equations 19 (1993), 1309-1364.
[8] B. Jovanović, Finite difference method for boundary value problems with weak solutions. Posebna izdanja Mat. Instituta 16, Mat. Institut, Belgrade 1993.
[9] B. Jovanović, J. Kandilarov and L. Vulkov, Construction and convergence of difference scheme for a model elliptic equation with Dirac–delta function coefficient. Lect. Notes Comput. Sci. 1988 (2001), 431–438.
[10] B. Jovanović, P. Matus and V. Shcheglik, Difference schemes on nonuniform meshes for the heat equation with variable coefficients and generalized solutions. Doklady NAN Belarusi 42, No 6 (1998), 38-44 (Russian).
[11] B. Jovanović, L. Vulkov, On the convergence of finite difference schemes for the heat equation with concentrated capacity. Numer. Math. 89, No 4 (2001), 715–734.
[12] B. Jovanović, L. Vulkov, Operator's approach to the problems with concentrated factors. Lect. Notes Comput. Sci. 1988 (2001), 439–450.
[13] B. Jovanović, L. Vulkov, On the convergence of difference schemes for hyperbolic problems with concentrated data. SIAM J. Numer. Anal. 41, No 2 (2003), 516–538.
[14] B. Jovanović, L. Vulkov, Finite difference approximation of an elliptic interface problem with variable coefficients. Lect. Notes Comput. Sci. 3401 (2005), 46–55.
[15] B. Jovanović, L. Vulkov, Energy stability for a class of two-dimensional interface linear parabolic problems. J. Math. Anal. Appl. 311 (2005), 120–138.
[16] B. Jovanović, L. Vulkov, Stability of difference schemes for parabolic equations with dynamical boundary conditions and conditions on conjugation. Appl. Math. Comput. 163 (2005), 849–868.
[17] J. Lions, E. Magenes, Non homogeneous boundary value problems and applications. Springer–Verlag, Berlin and New York, 1972.
[18] A. Lykov, Heat-mass transfer. Energiya, Moscow 1978 (Russian).
[19] E. Magenes, Some new results on a Stefan problem in a concentrated capacity. Rend. Mat. Acc. Lincei. **9** (1992), 323–334.
[20] M. Renardy, R. Rogers, An introduction to partial differential equations. Springer–Verlag, Berlin and New York, 1993.
[21] F. Riesz, B. Sz.–Nagy, Leçons d'analyse fonctionelle. Akadémiai Kiadó, Budapest 1972.
[22] A. Samarskii, Theory of difference schemes. Nauka, Moscow 1989 (Russian).
[23] A. Samarskii, R. Lazarov and V. Makarov, Difference schemes for differential equations with generalized solutions. Vysshaya shkola, Moscow 1987 (Russian).
[24] A. Samarskii, V. Mazhukin, D. Malafei and P. Matus, Difference schemes of high order of approximation on nonuniform in space meshes. Doklady RAN **367**, No 3 (1999), 1–4 (Russian).
[25] Z. Sun, Y. Zhu, A second order accurate difference scheme for the heat equation with concentrated capacity. Numer. Math. 97, No 2 (2004), 379–395.
[26] A. Tikhonov, A. Samarskiĭ, Equations of mathematical physics. Nauka, Moscow 1966 (Russian).
[27] V. Vladimirov, Equations of mathematical physics. Nauka, Moscow 1988 (Russian).
[28] L. Vulkov, Application of Steklov–type eigenvalues problems to convergence of difference schemes for parabolic and hyperbolic equation with dynamical boundary conditions. Lect. Notes Comput. Sci. **1196** (1997), 557–564.
[29] L. Vulkov, B. Jovanović, Convergence of finite-difference schemes for Poisson's equation with a dynamic boundary condition. Comput. Math. Math. Phys. 45, No 2 (2005), 275–284.
[30] J. Wloka, Partial differential equations. Cambridge Univ. Press, Cambridge 1987.

University of Belgrade, Faculty of Mathematics, 11000 Belgrade, Serbia
E-mail: `bosko@matf.bg.ac.yu`

University of Rousse, Department of Mathematics, 7017 Rousse, Bulgaria
E-mail: `vulkov@ami.ru.acad.bg`