J. At. Mol. Sci.
doi: 10.4208/jams.101410.110810a

Vol. **2**, No. 3, pp. 255-261
August 2011

Solution to the master equation for nondegenerate parametric amplification with thermal reservoir

Zhong-Jie Wang* and Xiao-Dong Zhang

Department of Physics, Anhui Normal University, Wuhu 241000, China

Received 14 October 2010; Accepted (in revised version) 8 November 2010
Published Online 28 March 2011

Abstract. By using Lie dynamical algebra representation theory, we have solved the master equation for the nondegenerate parametric amplifier model in a thermal reservoir. Applying a series of transformations, we show that the master equation has multiple commuted $SU(1,1)$ Lie algebra structures. The explicit solution to the master equation has been obtained.

PACS: 03.65.Fd, 42.65.Lm

Key words: nondegenarate parametric amplifier, Lie algebra, master equation

1 Introduction

The quantum decoherence theory is a very important aspect in quantum optics and quantum engineers such as quantum computation and quantum communication [1,2]. The origin of decoherence is that all quantum systems cannot be completely isolated from their environment, their damping processes will result in loss of coherence. The decohernce of a quantum system can be modeled by a large number of harmonic oscillators. With application of the Markovian approximation, the evolution of the reduced density operator of the system can be governed by the master equation. Therefore, it is important to solve the master equation. In general, One uses the Glauber-Sudarshan representation or coherence states representation to solve the master equation [3,4].

Rather recently, in order to solve the master equation, the dynamical Lie algebra method is proposed. This method is firstly applied to solve Lovellian-Bolch equation [5]. Hou *et al.* [6] and Wang *et al.* [7] have introduced the left and right algebras based on the location of the destruction and creation operators, which is used to construct the $SU(1,1)$ algebra. Yang *et al.* [8] have proposed an elegant solution to the master equation based on Lie algebra representation theory for super-operators. However these methods are only used to solve

*Corresponding author. *Email address:* `wuliwzj@mail.ahnu.edu.cn` (Z. J. Wang)

the master equation for single mode damped harmonic oscillator with linear driving. Rather recently, Xu *et al.* [9] analyzed the master equation on phase diffusion model by using the entangled state representation method.

The nondegenerate parametric amplifier (NPA) has attracted much attention because of its important role in quantum optics. BjÖrk *et al.* [10] proposed a method for producing light field Fock state based on NPA model. Roid *et al.* [11] obtained EPR entanglement state of light field using NPA technique. Zhang *et al.* [12] studied entanglement swapping applying NPA model. In this paper, we study the master equation for NPA model in a thermal reservoir. With application of a series of transformations, the master equation is shown to have multiple commuted $SU(1,1)$ Lie algebra structures, and the solution to it is obtained.

2 The master equation and its solution

We consider a nondegenerate parametric amplifier. In the interaction picture, the master equation is given by [13]

$$\frac{d\rho}{dt}=-i[H_d,\rho]+\sum_{i=1}^{2}\Gamma_i\left(2a_i\rho a_i^+-a_i^+a_i\rho-\rho a_i^+a_i\right)+\sum_{i=1}^{2}\Gamma_i\left(2a_i^+\rho a_i-a_ia_i^+\rho-\rho a_ia_i^+\right), \quad (1)$$

where $H_d=i\epsilon(t)[a_1a_2-a_1^+a_2^+]$, $a_1(a_1^+)$ and $a_2(a_2^+)$ is the annihilation (creation)operator of the signal and idler modes, respectively, Γ_1,Γ_2 are the decay rate of two modes, respectively. In what follows we assume that two modes have the same the decay rate, namely, $\Gamma_1=\Gamma_2=\Gamma$. It should be pointed out that this master equation can also be used to describe dynamics of two-mode damped harmonic oscillator with nonlinear driving field.

Alebachew [13] considered the effect of the squeezed reservoir and studied steady solution of Eq. (1).Here we neglect the effect of the squeezed reservoir. The method we will give in this paper can be used to solve the master equation with the squeezed reservoir, too. In order to solve exactly the master Eq. (1), we use the Lie algebra representation theory for superoperators. For this goal, we, first of all, make a quantum rotation transformation

$$\rho^{(1)}=e^{\Gamma t(a_1^+a_1+a_2^+a_2)}\rho e^{\Gamma t(a_1^+a_1+a_2^+a_2)}, \quad (2)$$

Eq. (1) is transformed into

$$\begin{aligned}\frac{d\rho^{(1)}}{dt}=&\epsilon_1\left(a_1^+a_2^+\rho^{(1)}+\rho^{(1)}a_1a_2\right)-\epsilon_2\left(a_1a_2\rho^{(1)}+\rho^{(1)}a_1^+a_2^+\right)\\&+\Gamma_1\left(a_1\rho^{(1)}a_1^++a_2\rho^{(1)}a_2^+\right)+\Gamma_2\left(a_1^+\rho^{(1)}a_1+a_2^+\rho^{(1)}a_2\right)-2\Gamma\rho^{(1)}, \quad (3a)\end{aligned}$$

where

$$\epsilon_1=\epsilon(t)e^{2\Gamma t}, \qquad \epsilon_2=\epsilon(t)e^{-2\Gamma t}, \quad (3b)$$

$$\Gamma_1=\Gamma e^{-2\Gamma t}, \qquad \Gamma_2=\Gamma e^{-2\Gamma t}. \quad (3c)$$

Next we make the transformation that is represented as

$$\rho^{(2)}=e^{\alpha a_1^+ a_2^+}\rho^{(1)}e^{\alpha a_1 a_2}, \tag{4a}$$

and considering that

$$e^{\alpha a_1^+ a_2^+}a_1 e^{-\alpha a_1 a_2}=a_1-\alpha a_2^+, \tag{4b}$$

$$e^{\alpha a_1^+ a_2^+}a_2 e^{-\alpha a_1 a_2}=a_2-\alpha a_1^+, \tag{4c}$$

$$e^{-\alpha a_1 a_2}a_1^+ e^{\alpha a_1^+ a_2^+}=a_1^+-\alpha a_2, \tag{4d}$$

$$e^{-\alpha a_1 a_2}a_2^+ e^{\alpha a_1^+ a_2^+}=a_2^+-\alpha a_1, \tag{4e}$$

Eq. (3a) is then transformed into

$$\begin{aligned}\frac{d\rho^{(2)}}{dt}=&\Big(\dot{\alpha}+\epsilon_1-\alpha^2\epsilon_2-\epsilon_1\Big)\Big(a_1^+a_2^+\rho^{(2)}+\rho^{(2)}a_1a_2\Big)-\epsilon_2\Big(a_1a_2\rho^{(2)}+\rho^{(2)}a_1^+a_2^+\Big)\\&+\Big(\Gamma_2+\alpha^2\Gamma_1\Big)\Big(a_1^+\rho^{(3)}a_1+a_2^+\rho^{(3)}a_2\Big)+\Big(\Gamma_2+\alpha^2\Gamma_1\Big)\Big(a_1^+\rho^{(2)}a_1+a_2^+\rho^{(2)}a_2\Big)\\&-\Gamma_1\alpha\Big(a_1\rho^{(2)}a_2+a_2\rho^{(2)}a_1+a_1^+\rho^{(2)}a_2^++a_2^+\rho^{(2)}a_1^+\Big)+\Gamma_1\Big(a_1\rho^{(2)}a_1^++a_2\rho^{(2)}a_2^+\Big)\\&+\epsilon_2\alpha\Big(a_1a_1^+\rho^{(2)}+a_2^+a_2\rho^{(2)}+\rho^{(2)}a_1^+a_1+\rho^{(2)}a_2^+a_2\Big)-2\Gamma\rho^{(2)},\end{aligned} \tag{5}$$

where $\dot{\alpha}$ denotes the differential with time. We proceed to make the transformation that is represented as

$$\rho^{(3)}=e^{\beta a_1 a_2}\rho^{(2)}e^{\beta a_1^+ a_2^+}. \tag{6}$$

Eq. (5) is then transformed into

$$\begin{aligned}\frac{d\rho^{(3)}}{dt}=&\Big(\dot{\beta}+\beta^2(\dot{\alpha}+\epsilon_1-\alpha^2\epsilon_2)+2\alpha\beta\epsilon_2\Big)(a_1a_2\rho^{(3)}+\rho^{(3)}a_1^+a_2^+)\\&+(\dot{\alpha}+\epsilon_1-\alpha^2\epsilon_2)(a_1^+a_2^+\rho^{(3)}+\rho^{(3)}a_1a_2)-2\Big(\Gamma-\beta(\dot{\alpha}+\epsilon_1-\alpha^2\epsilon_2\Big)\rho^{(3)}\\&+\Big(\Gamma_1-2\Gamma_2\alpha\beta+\beta^2(\Gamma_2+\alpha^2\Gamma_1)\Big)(a_1\rho^{(3)}a_1^++a_2\rho^{(3)}a_2^+)\\&+\Big(\beta(\Gamma_2+\alpha^2\Gamma_1)-\Gamma_1\alpha\Big)(a_1\rho^{(3)}a_2+a_2\rho^{(3)}a_1+a_1^+\rho^{(3)}a_2^++a_2^+\rho^{(3)}a_1^+)\\&+\Big(\beta(\dot{\alpha}+\epsilon_1-\alpha^2\epsilon_2)+\epsilon_2\alpha\Big)(a_1^+a_1\rho^{(3)}+a_2^+a_2\rho^{(3)}+\rho^{(3)}a_1^+a_1+\rho^{(3)}a_2^+a_2).\end{aligned} \tag{7}$$

Finally we make the transformation

$$\rho^{(4)}=e^{\lambda(a_1^+a_1+a_2^+a_2)}\rho^{(3)}e^{\lambda(a_1^+a_1+a_2^+a_2)}, \tag{8a}$$

where the parameter $\lambda(t)$ satisfies the following differential equation

$$-\dot{\lambda}=\beta(\dot{\alpha}+\epsilon_1-\alpha^2\epsilon_2)+\epsilon_2\alpha, \tag{8b}$$

we then obtain

$$\begin{aligned}\frac{d\rho^{(4)}}{dt}=&f_1(t)\Big(a_1a_2\rho^{(4)}+\rho^{(4)}a_1^+a_2^+\Big)+f_2(t)\Big(a_1^+a_2^+\rho^{(4)}+\rho^{(3)}a_1a_2\Big)\\&+g_1(t)\Big(a_1\rho^{(3)}a_1^++a_2\rho^{(3)}a_2^+\Big)+g_2(t)\Big(a_1^+\rho^{(4)}a_1+a_2^+\rho^{(4)}a_2\Big)\\&+h(t)\Big(a_1\rho^{(4)}a_2+a_1^+\rho^{(4)}a_2^++a_2\rho^{(4)}a_1+a_2^+\rho^{(4)}a_1^+\Big)-\Gamma'\rho^{(4)},\end{aligned}\tag{9a}$$

where

$$f_1(t)=\Big(\dot\beta+\beta^2(\dot\alpha+\epsilon_1-\alpha^2\epsilon_2)+2\alpha\beta\epsilon_2-\epsilon_1\Big)e^{-2\lambda},\tag{9b}$$

$$f_2(t)=(\dot\alpha+\epsilon_1-\alpha^2\epsilon_2)e^{2\lambda},\tag{9c}$$

$$g_1(t)=\Big(\Gamma_1-2\Gamma_2\alpha\beta+\beta^2(\Gamma_2+\alpha^2\Gamma_1)\Big)e^{-2\lambda},\tag{9d}$$

$$g_2(t)=\Big(\Gamma_2+\alpha^2\Gamma_1\Big)e^{2\lambda},\tag{9e}$$

$$h(t)=\beta\Big(\Gamma_2+\alpha^2\Gamma_1\Big)-\Gamma_1\alpha,\tag{9f}$$

$$\Gamma'=2\Big(\Gamma-\beta(\dot\alpha+\epsilon_1-\alpha^2\epsilon_2)-\alpha\epsilon_2\Big).\tag{9g}$$

We can select the parameters α,β satisfy the following equations,

$$f_2(t)=-f_1(t)\equiv f(t),\qquad g_2(t)=g_1(t)\equiv g(t).\tag{10}$$

Eq. (9a) is changed into the following form

$$\begin{aligned}\frac{d\rho^{(4)}}{dt}=&f(t)\Big(a_1^+a_2^+\rho^{(4)}+\rho^{(3)}a_1a_2-a_1a_2\rho^{(4)}-\rho^{(4)}a_1^+a_2^+\Big)-\Gamma'\rho^{(4)}\\&+g(t)\Big(a_1\rho^{(3)}a_1^++a_2\rho^{(3)}a_2^++a_1^+\rho^{(4)}a_1+a_2^+\rho^{(4)}a_2\Big)\\&\times h(t)\Big(a_1\rho^{(4)}a_2+a_1^+\rho^{(4)}a_2^++a_2\rho^{(4)}a_1+a_2^+\rho^{(4)}a_1^+\Big).\end{aligned}\tag{11}$$

In order to write the master Eq. (11) in the form of the Schrodinger-like, we introduce the super-operators:

$$K_i^+\rho=a_i\rho a_i^+,\qquad K_i^-\rho=a_i^+\rho a_i,\tag{12a}$$

$$K_i^0\rho=-\frac{1}{2}(a_i^+a_i\rho+\rho a_i^+a_i+\rho),\tag{12b}$$

$$R_1^+\rho=a_1\rho a_2,\qquad R^-\rho=a_1^+\rho a_2^+,\tag{12c}$$

$$R_2^+\rho=a_2\rho a_1,\qquad R_2^-\tilde\rho=a_2^+\rho a_1^+,\tag{12d}$$

$$R_1^0\tilde\rho=\frac{1}{2}(\rho a_1^+a_1-a_1^+a_1\rho),\qquad R_2^0\tilde\rho=\frac{1}{2}(\rho a_2^+a_2-a_2^+a_2\rho),\tag{12e}$$

$$Q_1^+\rho=\rho a_1a_2,\qquad Q_1^-\rho=\rho a_1^+a_2^+,\tag{12f}$$

$$Q_2^+\rho=a_1^+a_2^+\rho,\qquad Q_2^-\rho=a_1a_2\rho,\tag{12g}$$

$$Q_1^0\rho=\frac{1}{2}\rho(a_2^+a_2+a_1^+a_1+1),\qquad Q_2^0\rho=\frac{1}{2}(a_2^+a_2+a_1^+a_1+1)\rho.\tag{12h}$$

It is easy to see that K_i^+, K_i^-, K_i^0, R_i^+, R_i^-, R_i^0 and Q_i^+, Q_i^-, Q_i^0 $(i=1,2)$ obey the commutation relation of $SU(1,1)$, respectively

$$[X_i^+, X_i^-]\rho = -2X_i^0\rho, \tag{13a}$$
$$[X_i^0, X_i^+]\rho = X_i^+\rho, \tag{13b}$$
$$[X_i^0, X_i^-]\rho = -X_i^-\rho, \tag{13c}$$

where $X = K, R, Q$.

By means of above $SU(1,1)$ superoperators, the master Eq. (11) can be written as

$$\frac{d\rho^{(4)}}{dt} = f(t)Q\rho^{(4)} + g(t)K\rho^{(4)} + h(t)R\rho^{(4)} - \Gamma'\rho^{(4)}, \tag{14a}$$

where

$$Q\rho^{(4)} = (Q_1^+ + Q_2^+ - Q_1^- - Q_2^-)\rho^{(4)}, \tag{14b}$$
$$K\rho^{(4)} = (K_1^+ + K_2^+ + K_1^- + K_2^-)\rho^{(4)}, \tag{14c}$$
$$R\rho^{(4)} = (R_1^+ + R_2^+ + R_1^- + R_2^-)\rho^{(4)}, \tag{14d}$$

It is easy to see that the superoperators K, R and Q also obey the commutation relation of $SU(1,1)$.

$$[R, K]\rho = 2Q\rho, \tag{15a}$$
$$[R, Q]\rho = 2K\rho, \tag{15b}$$
$$[Q, K]\rho = -2R\rho. \tag{15c}$$

With application of $SU(1,1)$ algebra property of K, R and Q, the solution to the master Eq. (12a) can be obtained as [5]:

$$\begin{aligned}\rho^{(4)}(t) &= e^{-\int \Gamma' dt} e^{-\mu_1(t)R} e^{-\mu_2(t)K} e^{-\mu_3(t)Q} \rho^{(4)}(0) \\ &= e^{-\int \Gamma' dt} e^{-\mu_1(t)(R_1^+ + R_2^+ + R_1^- + R_2^-)} \\ &\qquad \times e^{-\mu_2(t)(K_1^+ + K_2^+ + K_1^- + K_2^-)} e^{-\mu_3(t)(Q_1^+ + Q_2^+ - Q_1^- - Q_2^-)} \rho^{(4)}(0),\end{aligned} \tag{16a}$$

where

$$2\dot{\mu}_1 - 2\dot{\mu}_3\sin(2\mu_2) = -2h(t), \tag{16b}$$
$$2\dot{\mu}_2\cos(2\mu_1) + 2\dot{\mu}_3\sin(2\mu_1)\cos(2\mu_2) = -2g(t), \tag{16c}$$
$$2\dot{\mu}_3\cos(2\mu_1)\cos(2\mu_2) - 2\dot{\mu}_2\sin(2\mu_1) = -2f(t), \tag{16d}$$
$$\mu_1(0) = \mu_2(0) = \mu_3(0) = 0. \tag{16e}$$

Noticing that the superoperators $X_1^+ \pm X_1^-$ and $X_2^+ \pm X_2^-$ $(X=K,R,Q)$ commute each other, we can obtain

$$\rho^{(4)}(t)=e^{-\int \Gamma' dt}e^{-\mu_1(t)(R_1^+ + R_1^-)}e^{-\mu_1(t)(R_2^+ + R_2^-)}e^{-\mu_2(t)(K_1^+ + K_1^-)} \times e^{-\mu_2(t)(K_2^+ + K_2^-)}e^{-\mu_3(t)(Q_1^+ - Q_1^-)}e^{-\mu_3(t)(Q_2^+ - Q_2^-)}\rho^{(4)}(0). \tag{17}$$

Applying $SU(1,1)$ algebra property of $K_i^\pm$, $R_i^\pm$ and $Q_i^\pm$, we can obtain an explicit form of Eq. (14a):

$$\begin{aligned}\rho^{(4)}(t)=e^{-\int \Gamma' dt}&e^{x_0(t)R_1^0}e^{x_1(t)R_1^+}e^{x_2(t)R_1^-}e^{x_0(t)R_2^0}e^{x_1(t)R_2^+}e^{x_2(t)R_2^-}\\ &\times e^{y_0(t)K_1^0}e^{y_1(t)K_1^+}e^{y_2(t)K_1^-}e^{y_0(t)K_2^0}e^{y_1(t)K_2^+}e^{y_2(t)K_2^-}\\ &\times e^{z_0(t)Q_1^0}e^{z_1(t)Q_1^+}e^{z_2(t)Q_1^-}e^{z_0(t)Q_2^0}e^{z_1(t)Q_2^+}e^{z_2(t)Q_2^-},\end{aligned} \tag{18a}$$

where

$$\dot{x}_0 - 2x_0\dot{x}_2 = 0, \tag{18b}$$
$$\dot{x}_1 e^{-x_0} - \dot{x}_2 x_1^2 e^{-x_0} = -\mu_1, \tag{18c}$$
$$\dot{x}_2 e^{x_0} = -\mu_1, \tag{18d}$$
$$\dot{y}_0 - 2y_0\dot{y}_2 = 0, \tag{18e}$$
$$\dot{y}_1 e^{-y_0} - \dot{y}_2 y_1^2 e^{-y_0} = -\mu_2, \tag{18f}$$
$$\dot{y}_2 e^{y_0} = -\mu_2, \tag{18g}$$
$$\dot{z}_0 - 2z_0\dot{z}_2 = 0, \tag{18h}$$
$$\dot{z}_1 e^{-z_0} - \dot{z}_2 z_1^2 e^{-z_0} = -\mu_3, \tag{18i}$$
$$\dot{z}_2 e^{z_0} = \mu_3. \tag{18j}$$

In the end, the solution to the master Eq. (1) can be represented as

$$\begin{aligned}\rho(t)=&e^{-\Gamma t(a_1^+a_1+a_2^+a_2)}e^{-\alpha a_1^+a_2^+}e^{-\beta a_1a_2}e^{-\lambda(a_1^+a_1+a_2^+a_2)}\rho^{(4)}(t)\\ &\times e^{-\lambda(a_1^+a_1+a_2^+a_2)}e^{-\beta a_1^+a_2^+}e^{-\alpha a_1a_2}e^{-\Gamma t(a_1^+a_1+a_2^+a_2)}.\end{aligned} \tag{19}$$

3 Conclusion

In conclusion, we have studied the master equation for the nondegenerate parametric amplifier with the thermal reservoir. By applying a series of transformations and the superoperator theory, we have obtained an exact solution to the master equation.

Acknowledgments. This work was supported by the Natural Science Foundation of Anhui Province of China under Grant No. 090412060 and the Natural Science Foundation of the Education Committee of Anhui Province of China under Grant No. KJ2008A029.

References

[1] A. Garg, Phys. Rev. Lett. 77 (1996) 964.
[2] S. J. Van Enk and O. Hirda, Phys. Rev. A 64 (2001) 022310.
[3] D. F. Walls, G. J. Milburn, Quantum Optics (Springer, Berlin,1994).
[4] R. Feynman and A. Hibbs, Quantum Mechanics and Path Integrals (MoGraw-Hill, New York, 1965).
[5] A. R. P. Rau, Phys. Rev. Lett. 81 (1998) 4785.
[6] B. P. Hou, S. J. Wang, and W. L. Yu, Chinese Phys. Lett. 20 (2003) 979.
[7] S. J. Wang, D. Zhao, H. G. Luo, *et al.*, Phys. Rev. A 64 (2001) 052102.
[8] J. Yang, H. X. Lu, B. Zhao, *et al.*, Chinese Phys. Lett. 20 (2003) 796.
[9] X. L. Xu, H. Q. Li, and H. Y. Fan, Chinese Phys. B 18 (2009) 918.
[10] G. BjÖrk and Y. Yamamoto, Phys. Rev. A 37 (1988) 125.
[11] M. D. Roid, Phys. Rev. A 40 (1989) 913.
[12] J. Zhang, C. D Xie, and K. C. Peng, Phys. Lett. A 299 (2002) 427.
[13] E. Alebachew, Phys, Rev. A 76 (2007) 023808.