J. At. Mol. Sci.
doi: 10.4208/jams.031309.042309a

Vol. **1**, No. 1, pp. 54-61
February 2010

Entropy squeezing of a three-level atom interacting with a cavity field

M. M. Nassar[a], H. Eleuch[b], M. Abdel-Aty[c,d,*] and A.-S. F. Obada[e]

[a] *Mathematics Department, College of Science and Arts at Shagra, King Saud University, Saudi Arabia*
[b] *Institute for Quantum Studies and Department of Physics, Texas A&M University, Texas, USA*
[c] *Mathematics Department, College of Science, Bahrain University, Bahrain*
[d] *Mathematics Department, College of Science, Sohag University, Egypt*
[e] *Mathematics Department, Faculty of Science, Al-Azher University, Cairo, Egypt*

Received 13 March 2009; Accepted (in revised version) 23 April 2009;
Available online 15 February 2010

Abstract. We study the field entropy squeezing as a measure of the entanglement in a three-level system interacting with a cavity field. Numerical calculations under current experimental conditions are performed and it is found that the initial state setting and atom-field coupling present changes of the general features of the field entropy squeezing dramatically.

PACS: 32.80.-t, 42.50.Ct

Key words: entropy squeezing, three-level atom, entanglement

---

1 Introduction

Great progress has recently been made in quantum information theory [1]. Also, entropy becomes a fundamental quantity to describe not only uncertainty or chaos of a system but also information carried by the system [2]. Compared to the long history of the theoretical understanding of entropy and entanglement of atom-field systems extending over many decades [3], intensive experimental investigations started only recently involving different systems [4].

To identify the fundamentally inequivalent ways quantum systems can be entangled is a major goal of quantum information theory [4]. It might be thought that there is nothing new to be said about bipartite entanglement if the shared state is pure, but in a

---

*Corresponding author. *Email address:* `mabdelaty@sci.uob.bh` (M. Abdel-Aty)

recent paper [5] it has been shown that exact coherence of the atom is in general never regained for a two-level model with a general initial pure quantum state of the radiation field. Also, it has been shown that the purification of the atomic state is actually independent of the nature of the initial pure state of the radiation field. An analysis of both analytical and numerical investigations of the process of atomic information entropy in the three-level systems has been presented [6].

Our motivation is to discuss the entropy squeezing from another point of view by considering the entropy squeezing for the field instead of the atom. Using an appropriate representation, an explicit expression for entropy squeezing when the system starts from a mixed state is derived. The physical situation which we shall refer to, belongs to the experimental domains of cavity quantum electrodynamics.

2 The model and its solution

We start by devoting this section to a brief discussion on the 3-level atom [7,8] being it the model describing the interaction between a single multi-level atom and a quantized cavity field. To set the stage, we first begin by describing the multilevel-atom model. Therefore, the physical system on which we focus is an 3-level. The atom interacts with a high Q-cavity which sustain a single cavity field with frequencies Ω. We denote by $\hat{a}$ and $\hat{a}^{\dagger}$ the annihilation and creation operators for the field mode, and ω_j is the frequency associated with the level of the atom. Therefore in the rotating wave approximation we can cast the Hamiltonian of the system in the form [8] ($\hbar=1$)

$$\hat{H}=\hat{H}_0+\hat{H}_1, \tag{1}$$

where the Hamiltonian for the interacting system $\hat{H}_0$ is given by

$$\hat{H}_0=\Omega\hat{a}^{\dagger}\hat{a}+\sum_{i=1,2,3}\omega_i|i\rangle\langle i|.$$

The interaction Hamiltonian between the atomic system and the cavity field is given by

$$\hat{H}_1=\sum_{j=1}^{2}\lambda_j(\hat{S}_{1,j+1}\hat{a}+h.c.).$$

The transition in the 3-level atom is characterized by the coupling λ_j. The operator $\widehat{S}_{ii}$ describes the atomic population of level $|i\rangle_A$ with energy $\omega_i,(i=1,2,3)$ and the operator $\widehat{S}_{ij}=|i\rangle\langle j|,(i\neq j)$ describes the transition from level $|i\rangle_A$ to level $|j\rangle_A$.

We have applied the rotating wave approximation discarding the rapidly oscillating terms and selecting the terms that oscillate with minimum frequency [9]. The resulting effective Hamiltonian may be written as

$$\hat{H}_0 = (\omega_1 - \Delta)\, I + \sum_{j=1}^{2} \Omega_j \left(\hat{a}^\dagger \hat{a} - \hat{S}_{j+1,j+1} \right),$$
$$\hat{H}_1 = \Delta \hat{S}_{11} + \sum_{j=1}^{2} \lambda_j \left(\hat{S}_{1,j+1} \hat{a} + \hat{S}_{j+1,1} \hat{a}^\dagger \right). \tag{2}$$

Here we assume that the detuning parameter Δ is given by $\Delta = \omega_1 - \omega_{j+1} - \Omega$, $j = 1,2$. It can be shown that $\hat{H}_0$ and $\hat{H}_1$ are constants of motion,

$$[\hat{H}_0, \hat{H}_1] = [\hat{H}, \hat{H}_0] = 0.$$

We assume that, before entering the cavity, the atom is prepared in a mixed state. Mixed states arise when there is some ignorance with respect to the system, so that consideration has to be given to the possibility that the system is in any one of several possible states, S_{ii}, each with some probability, γ_i, of being realized. To this end, the initial state of the atom can be written in the following form

$$\rho = \left(\gamma_1 \widehat{S}_{11} + \gamma_2 \widehat{S}_{22} + \gamma_3 \widehat{S}_{33} \right) \in S_A, \tag{3}$$

where $\gamma_i \geq 0$, and $\sum_{i=1}^{3} \gamma_i = 1$. In terms of quantum information processes, an understanding of mixed states is essential, as it is almost inevitable that the ideal pure states will interact with the environment at some stage.

Also we suppose that the initial state of the field is given by

$$|\varpi_1\rangle = \sum_{n=0}^{\infty} b_n |n\rangle \in S_F, \tag{4}$$

where $b_n = \langle \varpi_1 | n \rangle$, b_n^2 being the probability distribution of photon number for the initial state. The continuous map $\mathcal{E}_t^*$ describing the time evolution between the atom and the field is defined by the unitary operator generated by $\hat{H}$ such that

$$\begin{aligned} &\mathcal{E}_t^* : S_A \longrightarrow S_A \otimes S_F, \\ &\mathcal{E}_t^* \rho = \hat{U}_t (\rho \otimes \varpi) \hat{U}_t^*, \\ &\hat{U}_t \equiv \exp\left(-\frac{i}{\hbar} \int \hat{H}(t) dt \right). \end{aligned} \tag{5}$$

where $\varpi = |\varpi_1\rangle\langle\varpi_1|$. Bearing these facts in mind we find that the evolution operator $\hat{U}_t$ takes the next from

$$\hat{U}_t \equiv \exp(-(\omega_1 - \Delta)\, t) \left[\prod_{j=1}^{2} \exp\left(-i \Omega \widehat{N}_j t \right) \right] \exp\left(-i \int_0^t \hat{H}_1 dt \right), \tag{6}$$

where $\widehat{N}_j = \hat{a}^{\dagger}\hat{a} - S_{j+1,j+1}$. The first two factors in equation (6) produce phases that will not affect the results that follow, while calculations of the third factor show that it takes the following compact matrix form

$$\exp\left(-i\hat{H}_1 t\right) = \exp\left(-\frac{i}{2}\Delta t\right)\begin{bmatrix} \widehat{U}_0 & \widehat{U}_1 \\ -\widehat{U}_1^* & \widehat{U}_2 \end{bmatrix}, \tag{7}$$

where $\widehat{U}_0$ is the single element matrix $\{\widehat{U}_{11}\}$ which takes the following form

$$\widehat{U}_{11} = \cos\widehat{\mu}_n t - \frac{i\Delta}{2}\frac{\sin\widehat{\mu}_n t}{\widehat{\mu}_n}. \tag{8}$$

The matrix $\widehat{U}_1$ is the 1×2 row matrix $\{\widehat{U}_{1k}\}$, where

$$\widehat{U}_{1k} = -i\frac{\sin\widehat{\mu}_n t}{\widehat{\mu}_n}\lambda_k \widehat{a}_k, \qquad k = 1,2 \tag{9}$$

and $\widehat{U}_1^*$ its Hermitian conjugate. Finally the matrix $\widehat{U}_2$ is the $(2)\times(2)$ square matrix $\{\widehat{U}_{ij}\}$ of which the elements can be written as

$$\widehat{U}_{ij} = \delta_{ij}\exp\left(-\frac{i}{2}\Delta t\right) - \lambda_i \hat{a}^{\dagger} v^{-1}\left(\cos\widehat{\mu}_n t - \exp\left(-\frac{i}{2}\Delta t\right) + \frac{i\Delta}{2}\frac{\sin\widehat{\mu}_n t}{\widehat{\mu}_n}\right)\lambda_j \hat{a}, \tag{10}$$

with $i,j = 1,2$ and

$$\widehat{\mu}_n = \left(\frac{\Delta^2}{4} + \sum_{i=1}^{2}\lambda_i^2 \hat{a}\hat{a}^{\dagger}\right)^{\frac{1}{2}}, \qquad v^{-1} = \sum_{i=1}^{2}\lambda_i^2 \hat{a}\hat{a}^{\dagger}. \tag{11}$$

Having obtained the explicit form of the unitary operator U_t, we are therefore able to discuss the total correlations of the system.

3 Derivation of the field entropy squeezing

There has recently been interesting developments of the information entropies [11] and particularly for the position and momentum for the single-slit and double-slit diffraction experiments [12, 13]. It is well known that [14, 15] position and momentum components along one direction are a pair of complementary observables, satisfying the commutation rule $[x,p] = i\hbar$. Considering various experiments to determine the position and momentum of an electron, Heisenberg found that $[x_1,p_1] = \hbar$, where x_1 and p_1 are the "imprecisions" with which the values of x and p are determined. Heisenberg estimated x_1 and p_1 by some plausible measure in each case separately, but did not gave an exact definition for the "imprecisions" x_1 and p_1. The inequality $\Delta X\Delta Y \geqslant \frac{\hbar}{2}$ has been presented by [16], where ΔA denotes the standard deviation of observable A.This equation was adopted by Heisenberg as the true mathematical expression of the uncertainty principle for the position-momentum pair. An alternative mathematical formulation of the uncertainty principle is provided by the inequality [17,18],

$$\delta X \delta Y \geqslant \pi e \hbar,$$

where δA is defined as the exponential of the differential entropy corresponding to the observable A.

Here, we apply the results obtained previously to derive the general form of the entropy squeezing for a single 3-level atom interacting with a cavity field. With a certain unitary operator, the final state after the interaction between the atom and the field is given by

$$\mathcal{E}_t^* \rho = U_t (\rho \otimes \varpi) U_t^*. \tag{12}$$

Taking the partial trace over the atomic system, we obtain

$$\rho_t^F = \mathrm{tr}_A \mathcal{E}_t^* \rho. \tag{13}$$

We define the position entropy and momentum entropy of the field as

$$S_x(t) = -\int \langle x|\rho_t^F|x\rangle \ln\langle x|\rho_t^F|x\rangle dx,$$
$$S_p(t) = -\int \langle p|\rho_t^F|p\rangle \ln\langle p|\rho_t^F|p\rangle dp.$$

The density matrix elements can be obtained using equations (13). The entropy uncertainty relation of position and momentum is given by

$$\exp[S_x(t)] \exp\left[S_p(t)\right] \geqq \pi e, \tag{14}$$

where $\delta_x = \exp[S_x(t)] - \sqrt{\pi e}$, and $\delta_p = \exp\left[S_p(t)\right] - \sqrt{\pi e}$. When $\delta_{x,p} < 0$, the position (momentum) of the field is squeezed.

Figures 1-3 depict the behavior of the entropy squeezing of the the position and momentum, δ_x and δ_p, as a function of the scaled time λt as it evolves under different coupling parameters. It can be clearly seen that the comparison between δ_x and δ_p, as the λt increases, the oscillation amplitude broadens thereby indicating increasing squeezing with time development. The entropy squeezing of the the position increases further than the entropy squeezing of momentum with increasing the scaled time (see Fig. 1), as is evident from a comparison of the solid and the dotted curves. The entropy squeezing factor δ_x exhibits larger squeezing properties than entropy squeezing factor δ_p.

Also by comparing Fig. 1 with Fig. 2 indicating that, in a strong coupling regime the squeezing doesn't occur for the momentum while small squeezing is observed for the position component, it can be seen that an increase in the interaction time causes a corresponding decrease in entropy squeezing of the position in this case (see Fig. 2). The

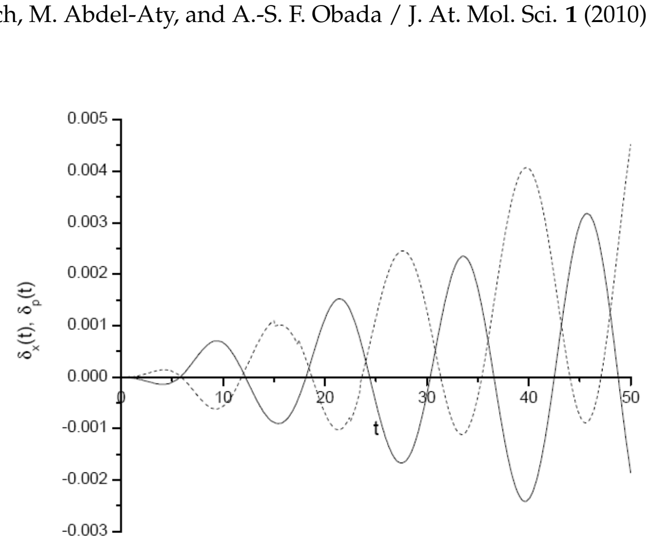

Figure 1: The time evolution of the entropy squeezing factors for position $\delta_x(t)$, and momentum $\delta_p(t)$ (solid and dashed curves), respectively. The parameters are $\gamma_1 = 0.9$, $\gamma_2 = 0$, $\lambda_i = 0.01\lambda$. The detuning parameter is a $\Delta/\lambda = 0.5$ and the initial mean photon number of the coherent state is $\bar{n} = 20$.

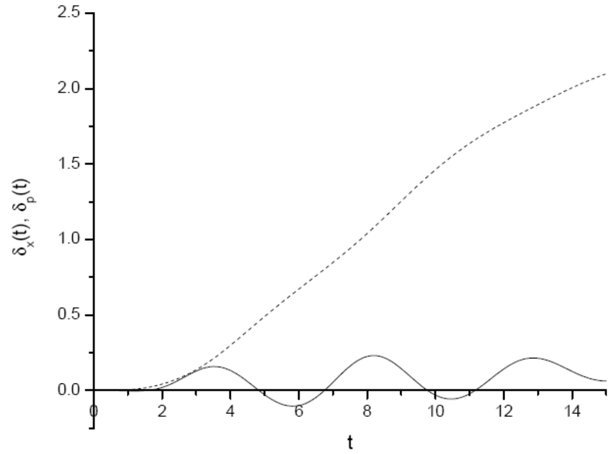

Figure 2: The same as Fig. 1, but $\lambda_i = 0.1\lambda$.

broadening of the curves in all cases takes place in such a fashion that the exchange of the squeezing between position and momentum is preserved. One can see clearly from Fig. 2 that there exists a situation at the first period of time where entropy squeezing of the momentum occurs but very small and then the effect disappear. Furthermore, when we consider the long-time evolution behaviors of the squeezing factor δ_x shown in Fig. 2, one can see that the squeezing getting smaller and can not eventually approach a stable value.

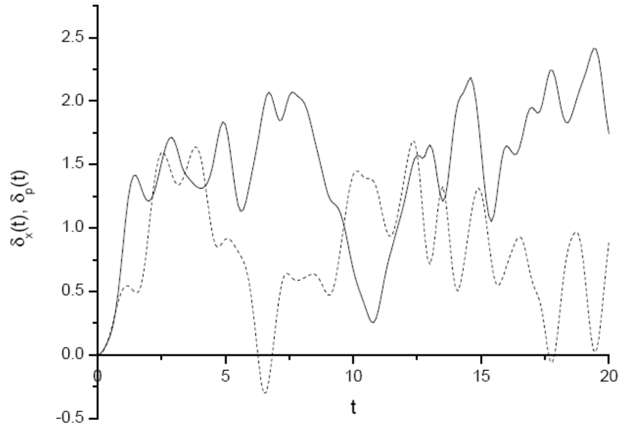

Figure 3: The same as figure 1, but $\lambda_i = \lambda$ and $\bar{n} = 1$.

As a comparison, in Fig. 3, the time evolution of the entropy squeezing factors are plotted for the cases of strong coupling regime $\lambda_2/\lambda_1 = 1$ and for small value of the mean photon number $\overline{n} = 1$. One can see that the entropy squeezing of the position is oscillating with time and has positive values only. However the entropy squeezing for the momentum shows squeezing for a short period of time. This squeezing reappears again but its amplitude is smaller than the first one and the squeezing disappears completely once the time goes on further. What means that, in the weak coupling regime, both entropy squeezing of the position and momentum occur. However, when the strong regime is considered, we find that the momentum shows squeezing only and the entropy squeezing has an apparently decay in the short time range and only occurs at the first period of time.

4 Conclusion

In this paper we have analyzed the field entropy squeezing of physically interesting systems interacting with the cavity field. We have explicitly evaluated the entropy squeezing factors and worked out the effects of different values of the coupling parameter on the dynamics of the system starting with atomic mixed state and coherent state of the field. It is shown that both position and momentum show entropy squeezing in the weak coupling regime. However, in a strong coupling regime, we have considered two different values of initial mean photon number of the the initial state of the system. The entropy squeezing corresponding to the dynamics evolution in the second case is more tilted than that in the first case, which is a signature of the squeezing inherent in the momentum. For small values of the initial mean photon number the entropy squeezing occurs only for the position component. In the strong coupling regimes a broadening of the entropy squeezing curve, indicating a decrease in squeezing, results with an increase in the interaction time.

Acknowledgments. One of us (HE) would like to thank Prof. Howard Carmichael at the Auckland University in New Zealand, Prof. M. Suhail Zubairy and Prof. M. Scully for stimulating discussions and hospitality. MA gratefully acknowledges the financial support from Paul-Sabatier University.

References

[1] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, 2000).
[2] M. Ohya and D. Petz, Quantum Entropy and Its Use (Springer-Verlag, New York, 1993).
[3] K.-H. Fichtner and M. Ohya, Commun. Math. Phys. 222 (2001) 229; M. Horodecki, P. Horodecki and R. Horodecki, Phys. Rev. Lett. 78 (1997) 574.
[4] J. Eisert and M. B. Plenio, J. Mod. Opt. 46 (1999) 145; Q. C. Zhou and S. N. Zhu, Chinese Phys. 14 (2005) 336; F. N. M. Al-Showaikh, Appl. Math. Inf. Sci. 2 (2008) 21; L. M. Arévalo-Aguilar, R. Juárez-Amaro, J. M. Vargas-Martínez, O. Aquilar-Loreto, and H. Moya-Cessa, Appl. Math. Inf. Sci. 2 (2008) 43.
[5] P. K. Rekdal, B. K. Skagerstam, and P. L. Knight, J. Mod. Opt. 51 (2004) 75.
[6] M. Abdel-Aty, J. Phys. A: Math. Gen. 38 (2005) 8589.
[7] A. M. Abdel-Hafez, A.-S. F. Obada, and M. M. A. Ahmed, Physica A 144 (1987) 530.
[8] A. M. Abdel-Hafez, A. M. M. Abu-Sitta, and A.-S. F. Obada, Physica A 156 (1989) 689.
[9] W. Vogel and R. L. de Matos Filho, Phys. Rev. A 52 (1995) 4214 .
[10] I. M. Gelfand and A. M. Yaglom, Amer. Math. Soc. Transl. 12 (1959) 199.
[11] R. M. Gray, Entropy and Information Theory (Springer-Verlag, New York, 2008).
[12] A. Orlowski, Phys. Rev. A 56 (1997) 2545.
[13] S. R. Jorge, Phys. Rev. A 57 (1998) 1519.
[14] W. Heisenberg, Z. Phys. 43 (1927) 172.
[15] W. Heisenberg, The Physical Principles of the Quantum Theory (University of Chicago Press, Chicago, 1930).
[16] E. H. Kennard, Z. Phys. 44 (1927) 326.
[17] W. Beckner, Ann. Math. 102 (1975) 159.
[18] I. Bialynicki-Birula and J. Mycielski, Commun. Math. Phys. 44 (1975) 129.