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# Superconvergence of Continuous Finite Elements with Interpolated Coefficients for Initial Value Problems of Nonlinear Ordinary Differential Equation†

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**Abstract.** In this paper, $n$-degree continuous finite element method with interpolated coefficients for nonlinear initial value problem of ordinary differential equation is introduced and analyzed. An optimal superconvergence $u - u_h = \mathcal{O}(h^{n+2})$, $n \geq 2$, at $(n+1)$-order Lobatto points in each element respectively is proved. Finally the theoretical results are tested by a numerical example.

**Key words**: Nonlinear ordinary differential equation; continuous finite element with interpolated coefficients; Lobatto points; superconvergence.

**AMS subject classifications:** 65M60

## 1 Introduction

Consider the initial value problem of nonlinear ordinary differential equation

$$u' = f(t,u), \quad t \in I = [0,T],\ u(0) = u_0, \tag{1}$$

where $f(t,u)$ is a sufficiently smooth function.

Let $J_h$ be a partition of $I$ such that $J_h : 0 = t_0 < t_1 < \cdots < t_N = T$. Set element $I_j = [t_{j-1}, t_j]$, midpoint $\bar{t}_j = (t_j + t_{j-1})/2$ and half-step $h_j = (t_j - t_{j-1})/2, h = \max(h_j), j = 1, \cdots, N$. Assume that $J_h$ is quasi-uniform, i.e., there is a $C > 0$ such that $h \leq Ch_j$. Define for the partition $J_h$ the finite element space

$$S^h = \{u \in C(I) : u|_{I_j} \in \mathbf{P}_n(I_j),\ j = 1, \cdots, N\}$$

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where $\mathbf{P}_n(I_j)$ denotes the space of all univariable polynomials of degree $\leq n$ in $I_j$.

On the element $I_j$, an $n$-degree polynomial has $n+1$ parameters. The value of the left endpoint is known on the element $I_j$ for initial value problems, so the finite element on this element has $n$ degrees of freedom. Classical continuous finite element solution $\bar{u}_h$ of (1) can be expressed as $\bar{u}_h = \sum \varphi_\alpha(t)\bar{u}_h(t_\alpha) \in S^h$ satisfying

$$\int_{I_j} (\bar{u}_h' - f(t, \bar{u}_h))v dt = 0, \quad v \in \mathbf{P}_{n-1},\ \bar{u}_h(0) = u_0. \tag{2}$$

For the sake of simplicity, we now define $n$-degree continuous finite element with interpolated coefficients, $u_h \in S^h$, by

$$\int_{I_j} (u_h' - I_h f(t, u_h))v dt = 0, \quad v \in \mathbf{P}_{n-1},\ u_h(0) = u_0, \tag{3}$$

where $I_h$ denotes the Lagrangian interpolating operator on $(n+1)$-order Lobatto points and $u_h$ and $I_h f(t, u_h)$ satisfy

$$u_h = \sum \varphi_\alpha(t) u_h(t_\alpha) \in S^h,$$
$$I_h f(t, u_h) = \sum \varphi_\alpha(t) f(t_\alpha, u_h(t_\alpha)) \in S^h,$$

where $\varphi_\alpha(t)$ are basis functions in element $I_j$. Note that the exact solution of (1) satisfies, for smooth function $v$,

$$\int_{I_j} (u' - f(t, u))v dt = 0, \tag{4}$$

and hence, subtracting (4) from (3) gives

$$\int_{I_j} (e' - f(t, u) + I_h f(t, u_h))v dt = 0, \quad v \in \mathbf{P}_{n-1},\ e(0) = 0 \tag{5}$$

where $e = u - u_h$.

For continuous finite elements, Chen [1] and Pan et al. [2] proved superconvergence for linear case $f(t, u) = au + b$ by a new element orthogonality analysis. In virtue of a simple argument Yang et al. [3] obtained superconvergence of classical finite element for nonlinear problems. The finite element method with interpolated coefficients is an economic and graceful method. This method was introduced and analyzed for semilinear parabolic problems in Zlamal et al. [4]. Later Larsson et al. [5] studied the semidiscrete linear triangular finite element $u_h$ and obtained the following error estimate

$$\|(u_h - u)(t)\|_{L^2(\Omega)} = \mathcal{O}(h), \qquad \text{for } 0 \leq t \leq T.$$

Chen et al. [6] derived almost optimal order of convergence

$$\|(u_h - u)(t)\|_{L^2(\Omega)} = \mathcal{O}(h^2 \ln h), \qquad \text{for } 0 \leq t \leq T,$$

on piecewise uniform triangular meshes by using superconvergence techniques. Recently, Xiong et al. [7] studied superconvergence of triangular quadratic finite elements for semilinear elliptic problems. By the compendious argument we shall study superconvergence of continuous finite element with interpolated coefficients for the initial value problems of nonlinear ordinary differential equation (1). Finally, the theoretical results are tested by a numerical example.

For our analysis, we introduce [1] Legendre's polynomials in the interval $E = [-1, 1]$

$$l_0 = 1, l_1 = s, l_2 = \frac{1}{2}(3s^2 - 1), l_3 = \frac{1}{2}(5s^3 - 3s), \cdots, l_n = \frac{1}{2^n n!}\partial_s^n (s^2 - 1)^n, \cdots, \qquad (6)$$

where the inner product $(l_i, l_j) = 0$ if $i \neq j$, otherwise $(l_i, l_j) = \frac{2}{2j+1}, l(\pm 1) = (\pm 1)^j$. The polynomial $l_n(s)$ has $n$ distinct roots ($n$ order Gauss points) in $(-1, 1)$. Integrating $l_n$ gives another family of polynomials

$$M_0 = 1, M_1 = s, M_2 = \frac{1}{2}(s^2 - 1), M_3 = \frac{1}{2}(s^3 - s), \cdots, M_{n+1} = \frac{1}{2^n n!}\partial_s^{n-1}(s^2 - 1)^n \cdots, \qquad (7)$$

which has the quasiorthogonal property: $(M_i, M_j) \neq 0$ if $i - j = 0$ or $\pm 2$, otherwise $(M_i, M_j) = 0$. Obviously $M_j(\pm 1) = 0$ for $j \geq 2$. $M_{n+1}(s)$ has $n+1$ distinct roots ($(n+1)$ order Lobatto points): $-1 = z_1 < z_2 < \cdots < z_{n+1} = 1$ in $E$. Denotes set of $(n+1)$-order Lobatto points in all elements in Partition $J^h$ by

$$Z_0 = \{t_{ji} = \bar{t}_j + h_j z_i, j = 1, 2, \cdots, N, i = 1, 2, \cdots, n+1\}.$$

Here and below, denote Sobolev space and its norm by $W^{k,p}(I)$ and $\|u\|_{k,p,I}$, respectively. If $p = 2$, simply use $H^k(I)$ and $\|u\|_{k,I}$.

Our main result about continuous finite element with interpolated coefficients for nonlinear initial value problem of ordinary differential equation is the following.

**Theorem 1.1.** *Assume that the partition of the interval $I = [0, T]$ is quasiuniform and let $u_h \in S^h$ be $n$-degree continuous finite element solution with interpolated coefficients for Eq.* (1)*. Then, at $z \in Z_0$, there is superconvergence estimate*

$$(u - u_h)(z) = \mathcal{O}(h^{n+2}), \qquad n \geq 2. \qquad (8)$$

**Remark 1.1.** By using the classical finite element methods (CFEM), one can solve nonlinear problems absolutely. To solve discrete system with Newton method, one needs to compute its tangent matrix. However, the tangent matrix dependents on every iterative value. As a result, one has to calculate this tangent matrix many times, which is very time consuming. The finite element methods with interpolated coefficients (ICFEM) is to directly substitute the interpolation $I_h f(u_h)$ for $f(u_h)$. Therefore, its tangent matrix can simply be calculated. The computational cost of ICFEM is greatly decreased and thus it is a high effective algorithm. Theorem 1.1 indicates that ICFEM for nonlinear ordinary differential equations has same superconvergence as that of CFEM, which is still valid for the case that the nonlinear term contains variable $t$.

## 2 Proof of Theorem 1.1

In order to show superconvergence of finite element with interpolated coefficients for nonlinear problems, we construct an auxiliary linear projection, $\tilde{u}_h \in S^h$ of $u$, such that

$$\begin{cases} \int_{I_j} ((u' - \tilde{u}_h') - f_u(t, u)(u - \tilde{u}_h))\, v dt = 0, & v \in \mathbf{P}_{n-1}, \\ \tilde{u}_h(0) = u_0. \end{cases} \qquad (9)$$

Recalling the result of continuous finite element for linear problems of ordinary differential equations [1], we obtain the following superconvergence estimate.

**Lemma 2.1.** *Let* $u_I$ *and* $\tilde{u}_h$ *be the interpolation of exact solution* $u$ *of* (1) *and auxiliary linear projection defined by* (9)*, respectively. Then at* $z \in Z_0$

$$(u - \tilde{u}_h)(z) = \mathcal{O}(h^{n+2}), \quad n \geq 2. \tag{10}$$

*Moreover, the following uniform estimate holds*

$$\|u_I - \tilde{u}_h\| = \mathcal{O}(h^{n+1}), \quad n \geq 2. \tag{11}$$

**Proof of Theorem 1.1** Subtracting (9) from (5), we get

$$\int_{I_j} [(\tilde{u}_h' - u_h') - (f(t,u) - I_h f(t,u_h)) - f_u(t,u)(\tilde{u}_h - u)]vdt = 0. \tag{12}$$

Denote by $w_I$ and $u_I$ the Lagrangian interpolation of $w$ and $u$ with respect to the variable $t$, respectively, and let $\theta = \tilde{u}_h - u_h, \rho = u_I - u_h, w(t) = f(t, u(t))$. Rewrite (12) as following

$$\begin{aligned} & \int_{I_j} (\theta' - f_u(t,u)\theta)vdt \\ = & \int_{I_j} (w - w_I + I_h(f(t,u) - f(t,u_h)) - f_u(t,u)(u - u_h))vdt. \end{aligned} \tag{13}$$

By Taylor's expansion on the element $I_j$,

$$\begin{aligned} & I_h(f(t,u) - f(t,u_h)) = \sum (f(t_k,u_k) - f(t_k,u_{hk}))\varphi_k \\ = & \sum [f_u(t_k,u_k)(u_k - u_{hk}) + 0.5 f_u(t_k,\xi)(u_k - u_{hk})^2]\varphi_k \\ = & f_u(t,u)(u_I - u_h) + \mathcal{O}(h)\max|\rho| + \mathcal{O}(1)\max|\rho|^2, \end{aligned}$$

where $u_k = u(t_k), u_{hk} = u_h(t_k)$. Hence, this yields the important formula

$$\begin{aligned} & \int_{I_j} (\theta' - f_u(t,u)\theta)vdt \\ = & \int_{I_j} (w - w_I - f_u(t,u)(u - u_I))vdt + [\mathcal{O}(h)\max|\rho| + \mathcal{O}(1)\max|\rho|^2]h^{1/2}\|v\|_{I_j}. \end{aligned} \tag{14}$$

In the following arguments we shall use the inverse inequality $\max|\rho| \leq Ch^{-1/2}\|\rho\|_{I_j}$.

By summation for $j$ for (14), and noticing that $R = u - u_I = \mathcal{O}(h^{n+1})$ where $u_I$ is the Lagrangian interpolation associated with the $(n+1)$-order Lobatto points, we have

$$\begin{aligned} & \int_0^{t_j} (\theta' - f_u(t,u)\theta)vdt \\ \leq & \int_0^{t_j} (w - w_I - f_u(t,u)(u - u_I))vdt + C(h\|\rho\| + h^{-1/2}\|\rho\|^2)\|v\|, \end{aligned} \tag{15}$$

or

$$\int_0^{t_j} (\theta' - f_u(t,u)\theta)v \leq C(h^{n+1} + h\|\rho\| + h^{-1/2}\|\rho\|^2)\|v\|.$$

In particular, above formula is valid for $n = N, t_N = T$. Choosing $v = \theta'$, we have

$$\begin{aligned}\|\theta'\|^2 &= \int_0^T |\theta'|^2 dt \\ &\leq \int_0^T f_u(t,u)\theta\theta' dt + C(h^{n+1} + h\|\rho\| + h^{-1/2}\|\rho\|^2)\|\theta'\| \\ &\leq C\|\theta\|\|\theta'\| + C(h^{n+1} + h\|\rho\| + h^{-1/2}\|\rho\|^2)\|\theta'\|.\end{aligned}$$

Consequently, the superconvergence estimate of derivative follows:

$$\|\theta'\| \leq C\|\theta\| + Ch^{n+1} + Ch\|\rho\| + Ch^{-1/2}\|\rho\|^2. \tag{16}$$

It is known, see Chen [1], that for the interpolation $u_I$ at $(n+1)$-order points the remainder $R = u - u_I$ has good approximate orthogonal property

$$\int_{I_k} Rvdt = \mathcal{O}(h^{n+2})\|u\|_{n+2,p,I_k}\|v\|_{1,p',I_k}, \qquad \forall v \in H^1.$$

Letting $p = p' = 2$ and summing from 1 to $j$, we have

$$\int_0^{t_j} Rvdt = \mathcal{O}(h^{n+2})\|u\|_{n+2}\|v\|_1. \tag{17}$$

In order to bound $\|\theta\|$, we shall use dual argument. To begin with, construct inverse initial value problem

$$-\psi_t - f_u(t,u)\psi = g,\ t \leq T = t_N, \quad \psi(T) = 0. \tag{18}$$

For solutions of the first order ordinary differential equation, the following regular estimate holds:

$$\|\psi\|_1 \leq C\|g\|.$$

By integration by parts, noting $\theta(0) = \psi(T) = 0$, we have

$$\begin{aligned}J &= \int_0^T (\theta' - f_u(t,u)\theta)\psi dt \\ &= \theta\psi|_0^T - \int_0^T \theta(\psi' + f_u(t,u)\psi)dt = (\theta, g).\end{aligned}$$

On the other hand, let $\psi_I$ be the $(n-1)$-degree piecewise polynomial approximation such that $R = \psi - \psi_I = \mathcal{O}(h^n), R(t_j) = 0$. Hence, using (15) and (17) gives

$$\begin{aligned}J = &\int_0^T (\theta' - f_u(t,u)\theta)(\psi - \psi_I)dt + \int_0^T (\theta' - f_u(t,u))\theta)\psi_I dt \\ &\leq C\|\theta\|_1 h\|\psi\|_1 + Ch^{n+2}\|\psi_I\|_1 + (Ch\|\rho\| + Ch^{-1/2}\|\rho\|^2)\|\psi_I\|.\end{aligned}$$

Choose $g = \theta$ and use the regularity estimates $\|\psi\|_1 \leq C\|\theta\|$ and $\|\psi_I\|_1 \leq C\|\psi\|_1$. Then using the above two formulas of $J$ we have

$$\|\theta\|^2 \leq (Ch\|\theta\|_1 + Ch^{n+2} + Ch\|\rho\| + Ch^{-1/2}\|\rho\|^2)\|\theta\|,$$

which yields

$$\|\theta\| \leq Ch\|\theta\|_1 + Ch^{n+2} + Ch\|\rho\| + Ch^{-1/2}\|\rho\|^2.$$

Substituting the superconvergence estimate (16) for derivatives into above formula gives

$$\|\theta\| \le Ch\|\theta\| + Ch^{n+2} + Ch\|\rho\| + Ch^{-1/2}\|\rho\|^2.$$

Consequently, for sufficiently small $h$, canceling the first term of right hand side, we obtain

$$\|\theta\| \le Ch^{n+2} + Ch\|\rho\| + Ch^{-1/2}\|\rho\|^2. \tag{19}$$

Recalling Lemma 2.1 gives

$$\|\rho\| = \|u_I - u_h\| \le \|u_I - \tilde{u}_h\| + \|\theta\| \le Ch^{n+1} + Ch\|\rho\| + Ch^{-1/2}\|\rho\|^2,$$

and for sufficiently small $h$, we have

$$\|\rho\| \le C_1 h^{n+1} + C_2 h^{-1/2}\|\rho\|^2. \tag{20}$$

Now adopting a simplified continuity argument, temporarily assume that there exits $h_1 > 0$ such that

$$\|\rho\| \le 2C_1 h^{n+1} \tag{21}$$

holds for any $h < h_1$. Substituting it for $\rho$ on the right hand side in (20), we have

$$\|\rho\| \le C_1 h^{n+1} + 4C_2C_1^2 h^{2n+3/2} \le C_1(1 + 4C_1C_2h^{n+1/2})h^{n+1}.$$

If taking $h < h_2$ such that $4C_1C_2h^{n+1/2} < 1$, then $\|\rho\| \le 2C_1h^{n+1}$ still holds for all $h < \min(h_1, h_2)$. This shows that the assumption (21) is right.

Substituting (21) into (19), we obtain

$$\|\theta\| \le Ch^{n+2}.$$

Substituting the above estimate and (21) into (16), we obtain

$$\|\theta'\| \le Ch^{n+1}.$$

In order to obtain pointwise estimate, we again use the dual argument. We construct following inverse initial value problem

$$\psi' + f_u(t,u)\psi = 0, \qquad \psi(t_j) = \theta(t_j), \tag{22}$$

which has better regularity estimate

$$\|\psi\|_l \le C|\theta(t_j)|, \quad l \ge 0.$$

By integration by part we have

$$\int_0^{t_j} (\theta' - f_u(t,u)\theta)\psi dt = \theta\psi\Big|_0^{t_j} - \int_0^{t_j} \theta(\psi' + f_u(t,u)\psi)dt = \theta^2(t_j).$$

On the other hand, let again $\psi_I$ be the $(n-1)$-degree piecewise polynomial approximation such that $R = \psi - \psi_I = \mathcal{O}(h^n), R(t_j) = 0$. This, together with (15),(17) and (21), implies

$$\begin{aligned}\int_0^{t_j} (\theta' - f_u(t,u)\theta)\psi dt &= \int_0^{t_j} (\theta' - f_u(t,u)\theta)(\psi - \psi_I)dt + \int_0^{t_j} (\theta' - f_u(t,u)\theta)\psi_I dt \\ &\le C\|\theta\|_1 h\|\psi\|_1 + Ch^{n+2}\|\psi_I\|_1 \le (Ch\|\theta\|_1 + Ch^{n+2})|\theta(t_j)|.\end{aligned}$$

From the above two formulas, we can obtain the following superconvergence estimate

$$|\theta(t_j)| \le Ch\|\theta\|_1 + Ch^{n+2} \le Ch^{n+2}.$$

It remains to prove uniform estimate $|\theta(t)| \le Ch^{n+2}$. Choosing $v = \theta'$ in (14) yields

$$\int_{I_j} \theta'^2 dt \quad \le \quad C\int_{I_j} \theta^2 dt + C\int_{I_j} (w - w_I)^2 dt + C\int_{I_j} (u - u_I)^2 dt + Ch^{2n+4}.$$

Using the approximation properties $\int_{I_j}(w - w_I)^2 dt = \mathcal{O}(h^{2n+3})$ and $\int_{I_j}(u - u_I)^2 dt = \mathcal{O}(h^{2n+3})$, the above formula becomes

$$\int_{I_j} \theta'^2 dt \le C\int_{I_j} \theta^2 dt + Ch^{2n+3}. \tag{23}$$

Noting $\theta(0) = 0$, hence, $\theta(t) = \int_0^t \theta'(t)dt$. An application of Schwarz inequality yields

$$\int_{I_j} \theta^2 dt \le Th_j \int_0^{t_j} \theta'^2 dt \le Ch \int_0^{t_j} \theta^2 dt + Ch^{2n+3}.$$

Canceling the first term of right hand side by the discrete Gronwall inequality, we obtain

$$\int_{I_j} \theta^2 dt \le Ch^{2n+3}.$$

Recalling (23), we obtain

$$\int_{I_j} \theta'^2 dt \le Ch^{2n+3}.$$

In the element $I_j$, we have

$$\begin{aligned} |\theta(t)| \quad &\le \quad \left|\theta(t_{j-1}) + \int_{t_{j-1}}^{t} \theta' dt\right| \le |\theta(t_{j-1})| + \int_{I_j} |\theta'| dt \\ &\le \quad Ch^{n+2} + Ch^{1/2}\left(\int_{I_j} \theta'^2 dt\right)^{1/2} \le Ch^{n+2}. \end{aligned} \tag{24}$$

Finally we decompose the error as

$$u - u_h = u - \tilde{u}_h + \tilde{u}_h - u_h = u - \tilde{u}_h + \mathcal{O}(h^{n+2}),$$

and an application of Lemma 2.1 completes the proof of this theorem. ■

# 3 Numerical example

Consider the initial value problem of the nonlinear ordinary differential equation

$$u'(t) = e^{t-u}, \qquad 0 < t < T = 1, \quad u(0) = 1.$$

Its exact solution is $u = \ln(e^t + e - 1)$. We compute the approximate solution by the quadratic continuous finite element method with interpolated coefficients. Divide uniformly interval $I$ into $N$ elements. The errors $e_N(t_j) = u(t_j) - u_h(t_j)$ are listed in Table 1 where the right two columns are their ratios.

From Table 1, we see that the quadratic continuous finite element with interpolated coefficients has high accuracy of $\mathcal{O}(h^4)$ and good stability which conforms our theoretical analysis.

Table 1: The errors and ratios of the quadratic interpolating coefficient continuous finite element.

| | $e_{10}$ | $e_{20}$ | $e_{40}$ | $e_{10}/e_{20}$ | $e_{20}/e_{40}$ |
|---|---|---|---|---|---|
| $t=1$ | $9.8129E-10$ | $6.1329E-11$ | $3.8338E-12$ | 16.001 | 15.997 |
| $t=2$ | $1.9742E-9$ | $1.2338E-10$ | $7.7121E-12$ | 16.002 | 15.998 |
| $t=3$ | $2.9578E-9$ | $1.8484E-10$ | $1.1554E-11$ | 16.003 | 15.997 |
| $t=4$ | $3.9096E-9$ | $2.4431E-10$ | $1.5272E-11$ | 16.003 | 15.997 |
| $t=5$ | $4.8067E-9$ | $3.0035E-10$ | $1.8775E-11$ | 16.004 | 15.997 |
| $t=6$ | $5.6261E-9$ | $3.5154E-10$ | $2.1976E-11$ | 16.005 | 15.997 |
| $t=7$ | $6.3468E-9$ | $3.9655E-10$ | $2.4789E-11$ | 16.006 | 15.997 |
| $t=8$ | $6.9501E-9$ | $4.3422E-10$ | $2.7144E-11$ | 16.006 | 15.997 |
| $t=9$ | $7.4207E-9$ | $4.6361E-10$ | $2.8981E-11$ | 16.007 | 15.997 |
| $t=10$ | $7.7478E-9$ | $4.8402E-10$ | $3.0256E-11$ | 16.007 | 15.997 |

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