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Partition of Unity for a Class of Nonlinear Parabolic Equation on Overlapping Non-Matching Grids†

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Abstract. A class of nonlinear parabolic equation on a polygonal domain $\Omega \subset \mathbb{R}^2$ is investigated in this paper. We introduce a finite element method on overlapping non-matching grids for the nonlinear parabolic equation based on the partition of unity method. We give the construction and convergence analysis for the semi-discrete and the fully discrete finite element methods. Moreover, we prove that the error of the discrete variational problem has good approximation properties. Our results are valid for any spatial dimensions. A numerical example to illustrate the theoretical results is also given.

Key words: Nonlinear parabolic equation; finite element method; overlapping non-matching grids; partition of unity.

AMS subject classifications: 65F10, 65N30, 65N15

1 Introduction

Since Huang and Xu [1] proposed a finite element method for overlapping non-matching grids based on partition of unity, the new finite element method has been attracting many authors' interest. Recently, there have been some studies of applying the finite element method to overlapping grids. These studies are within the framework of mortar finite elements or Lagrange multipliers [4-6]. The partition of unity method that has its roots in Babŭska and Melenk in [2,3], has been used for the numerical solutions of the parabolic problems [7-9]. Both linear elliptic and parabolic problems are studied [1,11]. However, the discrete case of the nonlinear parabolic problem has not been investigated when overlapping grids and non-matching grids are involved. In this paper, following the ideas of Huang and Xu, we propose a finite element method by introducing a conforming finite element space and by using an argument of the partition of unity type for a class of nonlinear parabolic problem.

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The rest of this paper is organized as follows. In Section 2, we give a brief description for the continuous nonlinear parabolic problem and the discretization of overlapping sub-domains. We also construct a globally conforming finite element space based on partition of unity. In Section 3, we give a few examples of the partition of unity function. We give the main results of the paper in Sections 4 and 5. They include the convergence analysis of the semi-discrete finite element solution based on partition of unity and the fully discrete finite element solution for the nonlinear parabolic problem. In Section 6, a numerical example is presented.

2 Construction of a global conforming subspace using the partition of unity

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with smooth boundary $\partial\Omega$, Γ be a closed subset of $\partial\Omega$. By $H_0^1(\Omega;\Gamma)$, we denote the closure in $H^1 - topology$ of $C^\infty(\overline{\Omega})$ functions that vanish in a neighborhood of Γ. Consider the following initial-boundary value problem for a class of nonlinear parabolic differential equation:

$$\begin{cases} \partial_t u - \nabla\cdot(a(u)\nabla u) = f(u), & \text{for } x\in\Omega, t\in(0,T],\\ u(x;t) = 0, & \text{for } x\in\partial\Omega, t\in(0,T],\\ u(x;0) = g(x), & \text{for } x\in\Omega, \end{cases} \tag{1}$$

where a and f are smooth functions defined on $\mathbb{R}$ such that

$$0 < \mu \le a(u) \le M, \quad |a'(u)| + |f'(u)| \le B, \quad \text{for } u \in \mathbb{R}. \tag{2}$$

Assume that the above problem admits a unique solution which is smooth enough for our purposes.

Now we begin our discussion of overlapping grids. We consider an overlapping domain decomposition of Ω, namely, we take $\Omega_1, \Omega_2, ..., \Omega_s$ to be overlapping sub-domains satisfying

$$\Omega = \bigcup_{i=1}^{s} \Omega_i.$$

We assume that each Ω_i is partitioned by a quasi-uniform finite element triangulation (or quadrilateral) J^{h_i} of maximal mesh size h_i, which are different from each other. Assume d_i is the minimal overlapping size of Ω_i with its neighboring sub-domains. Denote

$$J^h = \bigcup_{i=1}^{s} J^{h_i}, \quad h = \max_{1\le i\le s}\{h_i\}, \quad d = \min_{1\le i\le s}\{d_i\}.$$

We shall use the notation $\lesssim$ and $\gtrsim$, i.e., when we write $x_1 \lesssim y_1, x_2 \gtrsim y_2$, we mean that there exist constants c_1, c_2, such that

$$x_1 \le c_1 y_1, \quad x_2 \ge c_2 y_2,$$

where c_i $(i = 1, 2)$ are constants independent of mesh size h.

For every sub-domain Ω_i and partition $J^{h_i}(i = 1, 2, ..., s)$, we have the corresponding stationary finite element space:

$$V^{h_i}(\Omega_i) = \{v \in H_0^1(\Omega_i; \partial\Omega\cap\partial\Omega_i); v|_e \in P_{m_i+r-1}, e \in J^{h_i}, m_i \ge 1, r \ge 1\} \subset H^1(\Omega),$$

where P_{m_i+r-1} denotes the set of polynomials in two variables of degree at most m_i+r-1. The variational formulation of the problem (1) on Ω is: Find a $u(t) \in H_0^1(\Omega;\partial\Omega), t \in (0,T]$, such that

$$\begin{cases} (\partial_t u(t), v) + (a(u(t))\nabla u, \nabla v) = (f(u(t)), v), & \forall v \in H_0^1(\Omega;\partial\Omega), \\ u(x;0) = g(x), & \text{for } x \in \Omega, \end{cases} \tag{3}$$

where

$$(a(u)\nabla u, \nabla v) = \int_\Omega a(u)\nabla u \nabla v dx, \quad (f(u), v) = \int_\Omega f(u) v dx.$$

The semi-discrete approximate formulation of the problem (1) on Ω is: Find a $u_h(t) \in V^h(\Omega), t \in (0,T]$, such that

$$\begin{cases} (\partial_t u_h(t), v) + (a(u_h(t))\nabla u_h, \nabla v) = (f(u_h(t)), v), & \forall v \in V^h(\Omega), \\ u_h(x;0) = g_h(x) \in V^h(\Omega), & \text{for } x \in \Omega, \end{cases} \tag{4}$$

where g_h is the certain discrete approximation of g. Usually, by taking $g_h = I_h g$ (the interpolation function of g in $V^h(\Omega)$), we may assume

$$||g - g_h||_{l,\Omega} \le \sum_{i=1}^s ||g - g_h||_{l,\Omega_i} \lesssim \sum_{i=1}^s h_i^{m_i+r-l} ||g||_{m_i+r,\Omega_i}, \quad l = 0, 1. \tag{5}$$

Consider the discretization of time variable on $(0,T]$: $t_0 = 0 < t_1 < ... < t_N = T$. Define $I_j = (t_{j-1}, t_j), k_j = t_j - t_{j-1}, k = \max_{1\le j\le N}\{k_j\}$, and assume $U^j \approx u(t_j), U_h^j \approx u_h(t_j), \overline{\partial}_t U^j \approx \partial_t U^j, k_j \ge Ck$ (the constant C is independent of j and k). Then, the fully discrete finite element approximation of the problem (1) on Ω is: Find $U_h^j \in V^h(\Omega)$, such that

$$\begin{cases} (\overline{\partial}_t U_h^j, v) + (a(U_h^j)\nabla U_h^j, \nabla v) = (f(U_h^j), v), & \forall v \in V^h(\Omega), \\ U_h^0(x;0) = g_h(x) \in V^h(\Omega), j = 1, 2, ..., N, & \text{for } x \in \Omega. \end{cases} \tag{6}$$

The main question which attracts our interest is how to put these local finite element subspaces $V^{h_i}(\Omega_i)$ together to construct a global finite element subspaces of $H_0^1(\Omega)$. We would like to emphasize here that a new technique based on the partition of unity, unlike existing techniques such as Lagrange multiplier methods or mortar finite element methods, will be used to construct a globally conforming finite element space.

The main ingredient in our analysis and construction below is a partition of unity $\{\varphi_i\}_{i=1}^s$ associated with the overlapping sub-domains $\{\Omega_i\}_{i=1}^s$. It is easy to see that we can choose this partition of unity functions φ_i to satisfy the properties

$$\begin{cases} 0 \le \varphi_i(x) \le 1, \quad x \in \Omega, \\ \sum_{i=1}^s \varphi_i \equiv 1, \quad x \in \Omega, \\ supp(\varphi_i) \subset \overline{\Omega}_i, \quad \varphi_i \in W^{r,\infty}(\Omega), \\ |\nabla^k \varphi_i| \lesssim d_i^{-k}, \quad 1 \le k \le r, \end{cases} \tag{7}$$

where d_i is the minimal overlapping size of Ω_i with its neighboring subdomains.

Let $Q_i \subseteq H_0^1(\Omega_i; \partial\Omega \cap \partial\Omega_i)$ be given. Then the space

$$Q = \sum_{i=1}^s \varphi_i Q_i = \left\{ \sum_{i=1}^s \varphi_i v_i, v_i \in Q_i \right\} \tag{8}$$

is called the $PUFEM$ space (partition of unity finite element method).

By Theorem 2 of [1], and using the partition of unity described in (7), we can glue all the local subspaces $V^{h_i}(\Omega_i)$ together. Then the space

$$V^h(\Omega) = \sum_{i=1}^{s} \varphi_i V^{h_i}(\Omega_i) = \left\{ \sum_{i=1}^{s} \varphi_i v_i, v_i \in V^{h_i}(\Omega_i) \right\} \tag{9}$$

is called the $PUFEM$ space of the nonlinear parabolic problem (1). Therefore, the semi-discrete and fully discrete partition of unity finite element solution (or $PUFEM$ solution) of the problem (1) can be represented as follows:

$$u_h(t) = \sum_{i=1}^{s} \varphi_i u_h^i(t), \qquad u_h^i(t) \in V^{h_i}(\Omega_i), \tag{10}$$

and

$$U_h^j = \sum_{i=1}^{s} \varphi_i U_{h_i}^j, \qquad U_{h_i}^j \in V^{h_i}(\Omega_i). \tag{11}$$

3 Examples of the partition of unity functions

For simplicity and concreteness, we restrict our attention to the situation of two overlapping sub-domains with polygonal shapes. The analysis for many sub-domain cases is similar. Let Ω_1, Ω_2 be the overlapping sub-domains of Ω satisfying $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_0 = \Omega_1 \cap \Omega_2 \neq \emptyset$. Furthermore, we assume that Ω_1, Ω_2 are partitioned by quasi-uniform finite element triangulation (or quadrilateral) J^{h_1} and J^{h_2} of maximal mesh sizes h_1 and h_2 (may not match on Ω_0). Again, just for the sake of simplicity, we assume that Ω_0 is a stripe-type domain of width $d = \mathcal{O}(h_1)$ and $h_1 \geq h_2$.

Example 3.1. (The one dimensional case.) Let $\Omega = (0,1), \Omega_1 = (0, x_1), \Omega_2 = (x_2, 1)$, and $0 < x_2 < x_1 < 1, d = h = h_1 = x_1 - x_2$. Choose

$$\varphi_1^1(x) = \begin{cases} 1, & \text{for} \quad 0 < x \leq x_2, \\ \dfrac{x_1 - x}{x_1 - x_2}, & \text{for} \quad x_2 < x \leq x_1, \\ 0, & \text{for} \quad x_1 < x < 1, \end{cases} \tag{12a}$$

$$\varphi_2^1(x) = \begin{cases} 0, & \text{for} \quad 0 < x \leq x_2, \\ \dfrac{x - x_2}{x_1 - x_2}, & \text{for} \quad x_2 < x \leq x_1, \\ 1, & \text{for} \quad x_1 < x < 1. \end{cases} \tag{12b}$$

Then $\{\varphi_i^1(x)\}_{i=1}^2$ are a piecewise linear hat-functions which form a partition of unity.
Generally, let

$$\psi_1(x) = \begin{cases} 1, & \text{for} \quad 0 < x \leq x_2, \\ \alpha_1(x), & \text{for} \quad x_2 < x \leq x_1, \\ 0, & \text{for} \quad x_1 < x < 1, \end{cases} \quad \psi_2(x) = \begin{cases} 0, & \text{for} \quad 0 < x \leq x_2, \\ \alpha_2(x), & \text{for} \quad x_2 < x \leq x_1, \\ 1, & \text{for} \quad x_1 < x < 1, \end{cases}$$

where $\alpha_1(x), \alpha_2(x)$ are smooth functions satisfying

$$\begin{cases} \alpha_1(x_2) = 1, \ \alpha_1(x_1) = 0, \ \alpha_2(x_2) = 0, \ \alpha_2(x_1) = 1, \\ \alpha_1'(x) < 0, \ \alpha_2'(x) > 0, \quad x_2 < x < x_1. \end{cases}$$

Then the normalization

$$\varphi_i(x) = \frac{\psi_i(x)}{\sum_{j=1}^{2} \psi_j(x)}, \quad i = 1, 2, \tag{13}$$

yields a partition of unity subordinate to the cover $\{\Omega_i\}_{i=1}^2$. In particular, let

$$\alpha_1(x) = \frac{(x_1 - x)(x_1 + x - 2x_2)}{(x_1 - x_2)^2}, \quad \alpha_2(x) = \frac{(x - x_2)(2x_1 - x_2 - x)}{(x_1 - x_2)^2}.$$

Then the normalization functions

$$\varphi_1^2(x) = \begin{cases} 1, & \text{for} \quad 0 < x \le x_2, \\ \dfrac{(x_1 - x)(x_1 + x - 2x_2)}{(x_1 - x)(x_1 + x - 2x_2) + (x - x_2)(2x_1 - x_2 - x)}, & \text{for} \quad x_2 < x \le x_1, \\ 0, & \text{for} \quad x_1 < x < 1, \end{cases} \tag{14a}$$

and

$$\varphi_2^2(x) = \begin{cases} 0, & \text{for} \quad 0 < x \le x_2, \\ \dfrac{(x - x_2)(2x_1 - x_2 - x)}{(x_1 - x)(x_1 + x - 2x_2) + (x - x_2)(2x_1 - x_2 - x)}, & \text{for} \quad x_2 < x \le x_1, \\ 1, & \text{for} \quad x_1 < x < 1, \end{cases} \tag{14b}$$

are the partition of unity functions.

Example 3.2. (The two dimensional case of triangulation partition.) Let $\Omega = (0, 1)\times(0, 1), \Omega_1 = (0, x_1)\times(0, 1), \Omega_2 = (x_2, 1)\times(0, 1)$, and $0 < x_2 < x_1 < 1, d = x_1 - x_2, J^{h_1}$ be a member of a family of uniform triangulations of Ω_1 with $\max_{e\in J_{h_1}} diam\{e\} = h_1 = \sqrt{2}d$. Assume Ω_2 is partitioned by uniform triangulation (or quadrilateral) J^{h_2} of maximal mesh sizes h_2. Furthermore, we assume $h_1 \ge h_2$, and denote

$$\Gamma_1 : x = x_1 \ (0 \le y \le 1); \quad \Gamma_2 : x = x_2 \ (0 \le y \le 1),$$

$M_j = (x_1, y_j), N_j = (x_2, y_j), y_j = jd, j = 1, 2, ..., n, n = \frac{1}{d}$. Namely, $\Gamma_1 = \overline{M_0M_1...M_n}, \Gamma_2 = \overline{N_0N_1...N_n}$. On the element $e_{1j} = M_jN_jN_{j-1}$, let the functions $\alpha_{1j}, \alpha_{2j}, \alpha_{3j}$ $(j = 1, 2, ..., n)$ be the basic function (area coordinates) of nodes M_j, N_j, N_{j-1}, respectively. On the element $e_{2j} = N_{j-1}M_{j-1}M_j$, let the functions $\beta_{1j}, \beta_{2j}, \beta_{3j}$ $(j = 1, 2, ..., n)$ be the basic function (area coordinates) of nodes N_{j-1}, M_{j-1}, M_j, respectively. On the overlapping sub-domain Ω_0, we have

$$\sum_{j=1}^{n}\sum_{i=1}^{3}(\alpha_{ji} + \beta_{ji})(x, y) \equiv 1, \quad \text{for} \quad (x, y) \in \Omega_0. \tag{15}$$

Then the functions

$$\varphi_1(x, y) = \begin{cases} 1, & \text{for} \quad (x, y) \in \Omega \setminus \Omega_2, \\ \displaystyle\sum_{j=1}^{n}(\beta_{1j} + \alpha_{2j} + \alpha_{3j})(x, y), & \text{for} \quad (x, y) \in \Omega_1 \cap \Omega_2, \\ 0, & \text{for} \quad (x, y) \in \Omega \setminus \Omega_1, \end{cases} \tag{16a}$$

and

$$\varphi_2(x, y) = \begin{cases} 0, & \text{for} \quad (x, y) \in \Omega \setminus \Omega_2, \\ \displaystyle\sum_{j=1}^{n}(\alpha_{1j} + \beta_{2j} + \beta_{3j})(x, y), & \text{for} \quad (x, y) \in \Omega_1 \cap \Omega_2, \\ 1, & \text{for} \quad (x, y) \in \Omega \setminus \Omega_1, \end{cases} \tag{16b}$$

are the partition of unity functions.

Example 3.3. (The two dimensional case of quadrilateral partition.) Let $\Omega = (0,1)\times(0,1), \Omega_1 = (0,x_1)\times(0,1), \Omega_2 = (x_2,1)\times(0,1)$, and $0 < x_2 < x_1 < 1, d = h = h_1 = x_1 - x_2$, J^{h_1} be a member of a family of uniform quadrilateral of Ω_1 with $\max_{e\in J_{h_1}} diam\{e\} = h_1$. Assume Ω_2 is partitioned by uniform triangulation (or quadrilateral) J^{h_2} of maximal mesh sizes h_2. Furthermore, we assume $h_1 \geq h_2$. Similar to Example 3.2, $\Gamma_1 = \overline{M_0M_1...M_n}$ and $\Gamma_2 = \overline{N_0N_1...N_n}$. On the element $e_j = N_{j-1}N_jM_jM_{j-1}$, the basic functions of the four nodes $N_{j-1}, N_j, M_j, M_{j-1}$ can be expressed as

$$\begin{cases} \psi^1_{e_j}(x,y) = \dfrac{1}{h^2}(x_1 - x)(y_j - y), & \text{for } (x,y) \in e_j, \\ \psi^2_{e_j}(x,y) = \dfrac{1}{h^2}(x_1 - x)(y - y_{j-1}), & \text{for } (x,y) \in e_j, \\ \psi^3_{e_j}(x,y) = \dfrac{1}{h^2}(x - x_2)(y - y_{j-1}), & \text{for } (x,y) \in e_j, \\ \psi^4_{e_j}(x,y) = \dfrac{1}{h^2}(x - x_2)(y_j - y), & \text{for } (x,y) \in e_j, \end{cases} \tag{17}$$

respectively. On the overlapping sub-domain Ω_0, we have

$$\sum_{j=1}^{n}\sum_{i=1}^{4} \psi^i_{e_j}(x,y) \equiv 1, \quad \text{for } (x,y) \in \Omega_0. \tag{18}$$

Then the functions:

$$\varphi_1(x,y) = \begin{cases} 1, & \text{for } (x,y) \in \Omega \setminus \Omega_2, \\ \displaystyle\sum_{j=1}^{n}\sum_{i=1}^{2} \psi^i_{e_j}(x,y), & \text{for } (x,y) \in \Omega_1 \cap \Omega_2, \\ 0, & \text{for } (x,y) \in \Omega \setminus \Omega_1, \end{cases} \tag{19a}$$

and

$$\varphi_2(x,y) = \begin{cases} 0, & \text{for } (x,y) \in \Omega \setminus \Omega_2, \\ \displaystyle\sum_{j=1}^{n}\sum_{i=3}^{4} \psi^i_{e_j}(x,y), & \text{for } (x,y) \in \Omega_1 \cap \Omega_2, \\ 1, & \text{for } (x,y) \in \Omega \setminus \Omega_1, \end{cases} \tag{19b}$$

are the partition of unity functions.

4 Error estimate for the semi-discrete PUFEM solution

For every local sub-domain Ω_i, we may express the error as:

$$u^i_h(t) - u(t) = (u^i_h(t) - \widetilde{u}^i_h(t)) + (\widetilde{u}^i_h(t) - u(t)) = \theta_i(t) + \rho_i(t), \quad \forall t \in (0,T], \tag{20}$$

where $\widetilde{u}^i_h(t)$ is an elliptic projection of the exact solution $u(t)$ in $V^{h_i}(\Omega_i)$, defined by

$$a(u(t)) \cdot \nabla(\widetilde{u}^i_h(t) - u(t)), \nabla v_i) = 0, \quad \forall v_i \in V^{h_i}(\Omega_i). \tag{21}$$

Therefore,

$$\begin{aligned} u_h(t) - u(t) &= \sum_{i=1}^{s} \varphi_i(u^i_h(t) - u(t)) \\ &= \sum_{i=1}^{s} \varphi_i(u^i_h(t) - \widetilde{u}^i_h(t)) + \sum_{i=1}^{s} \varphi_i(\widetilde{u}^i_h(t) - u(t)) = \theta(t) + \rho(t), \end{aligned}$$

where

$$\theta(t) = \sum_{i=1}^{s} \varphi_i(u_h^i(t) - \widetilde{u}_h^i(t)), \quad \rho(t) = \sum_{i=1}^{s} \varphi_i(\widetilde{u}_h^i(t) - u(t)).$$

Now, we may show the following result for $\rho(t)$ and $\rho_t(t)$ under some appropriate regularity assumptions for u. In the rest of this paper, we will refrain the dependence of the constants in the error estimates of the regularity of the exact solution.

Lemma 4.1. *Assume $\rho(t) = \sum_{i=1}^{s} \varphi_i(\widetilde{u}_{h_i}(t) - u(t))$. Then under the appropriate regularity assumptions for u, we have*

$$||\rho(t)||_{0,\Omega} + h||\nabla\rho(t)||_{0,\Omega} \lesssim h^r \sum_{i=1}^{s} C_i(u) h_i^{m_i}, \quad \text{for} \quad t \in (0,T], \tag{22}$$

$$||\rho_t(t)||_{0,\Omega} + h||\nabla\rho_t(t)||_{0,\Omega} \lesssim h^r \sum_{i=1}^{s} C_i(u) h_i^{m_i}, \quad \text{for} \quad t \in (0,T], \tag{23}$$

where $\widetilde{u}_h^i(t)$ is defined by (21), $r \geq 1, m_i \geq 1\,(i = 1, 2, ..., s)$ are integers.

Proof According to Lemma 13.2 in [7], we have

$$||\rho_i(t)||_{0,\Omega_i} + h_i||\nabla\rho_i(t)||_{0,\Omega_i} \lesssim C_i(u) h_i^{m_i+r}, \quad \text{for} \quad t \in (0,T],$$

$$||\partial_t\rho_i(t)||_{0,\Omega_i} + h_i||\nabla\partial_t\rho_i(t)||_{0,\Omega_i} \lesssim C_i(u) h_i^{m_i+r}, \quad \text{for} \quad t \in (0,T].$$

Noting $|\varphi_i| \leq 1$, and $|\nabla\varphi_i| \lesssim d_i^{-1} \lesssim h_i^{-1}$, we obtain

$$\begin{aligned} ||\rho(t)||_{0,\Omega} &\leq \sum_{i=1}^{s} ||\varphi_i\rho_i(t)||_{0,\Omega_i} \leq \sum_{i=1}^{s} ||\rho_i(t)||_{0,\Omega_i} \\ &\lesssim \sum_{i=1}^{s} C_i(u) h_i^{m_i+r} \leq h^r \sum_{i=1}^{s} C_i(u) h_i^{m_i}, \end{aligned} \tag{24}$$

and

$$\begin{aligned} ||\nabla\rho(t)||_{0,\Omega} &\leq \sum_{i=1}^{s} ||\nabla\varphi_i \cdot \rho_i(t)||_{0,\Omega_i} + \sum_{i=1}^{s} ||\varphi_i \cdot \nabla\rho_i(t)||_{0,\Omega_i} \\ &\lesssim \sum_{i=1}^{s} h_i^{-1} ||\rho_i(t)||_{0,\Omega_i} + \sum_{i=1}^{s} ||\nabla\rho_i(t)||_{0,\Omega_i} \\ &\lesssim \sum_{i=1}^{s} C_i(u) h_i^{m_i+r-1} \leq h^{r-1} \sum_{i=1}^{s} C_i(u) h_i^{m_i}. \end{aligned} \tag{25}$$

By combining (24) and (25), it is easy to show that (22) hold.

Note that $\rho_t(t) = \sum_{i=1}^{s} \varphi_i \partial_t \rho_i(t)$ and $\nabla\rho_t(t) = \sum_{i=1}^{s} \varphi_i \cdot \partial_t \nabla\rho_i(t) + \sum_{i=1}^{s} \nabla\varphi_i \cdot \partial_t \rho_i(t)$. We can obtain two inequalities for ρ_t similar to (24) and (25), which can lead to (23). The proof is then complete. ■

Lemma 4.2. *Assume $\widetilde{u}_h^i(t)$ is defined by (21), and $\widetilde{u}_h(t) = \sum_{i=1}^{s} \varphi_i \widetilde{u}_h^i(t)$. Then*

$$||\nabla\widetilde{u}_h(t)||_{L_\infty,\Omega} \leq C(u), \quad \text{for } t \in (0,T]. \tag{26}$$

Proof By $\nabla \widetilde{u}_h(t) = \sum_{i=1}^{s} \nabla\varphi_i \cdot \widetilde{u}_{h_i}(t) + \sum_{i=1}^{s} \varphi_i \cdot \nabla\widetilde{u}_{h_i}(t)$, we obtain

$$\begin{aligned} ||\nabla(\widetilde{u}_h(t)||_{L_\infty,\Omega} &\leq \sum_{i=1}^{s} |\nabla\varphi_i| \cdot ||\widetilde{u}_{h_i}(t))||_{L_\infty,\Omega_i} + \sum_{i=1}^{s} |\varphi_i| \cdot ||\nabla\widetilde{u}_{h_i}(t)||_{L_\infty,\Omega_i} \\ &\lesssim \sum_{i=1}^{s} h_i^{-1}||\widetilde{u}_{h_i}(t))||_{L_\infty,\Omega_i} + \sum_{i=1}^{s} ||\nabla\widetilde{u}_{h_i}(t)||_{L_\infty,\Omega_i}. \end{aligned}$$

Using the inverse estimate, we have

$$\begin{aligned} h_i^{-1}||\widetilde{u}_{h_i}(t)||_{L_\infty,\Omega_i} &\leq h_i^{-2}||\widetilde{u}_{h_i}(t)||_{0,\Omega_i} \\ &\leq h_i^{-2}(||\widetilde{u}_{h_i}(t) - I_{h_i}u(t)||_{0,\Omega_i} + ||I_{h_i}u(t)||_{0,\Omega_i}) \lesssim C_i(u), \end{aligned}$$

and

$$\begin{aligned} ||\nabla\widetilde{u}_{h_i}(t)||_{L_\infty,\Omega_i} &\leq h_i^{-1}||\nabla\widetilde{u}_{h_i}(t)||_{0,\Omega_i} \\ &\leq h_i^{-1}(||\nabla(\widetilde{u}_{h_i}(t) - I_{h_i}u(t))||_{0,\Omega_i} + ||\nabla I_{h_i}u(t)||_{0,\Omega_i}) \lesssim C_i(u), \end{aligned}$$

where $C_i(u)$ is independent of h_i $(i = 1, 2, ..., s)$ and $t \in (0, T]$. Let $C(u) = \sum_{i=1}^{s} C_i(u)$. It is obvious that (26) hold. The proof is complete. ■

For the given initial function $g(x)$ on overlapping non-matching grids, the approximation of the partition of unity can be expressed as follows

$$g_h(x) = \sum_{i=1}^{s} \varphi_i g_{h_i}(x),$$

where $g_{h_i}(x)$ is an approximation of $g(x)$ in $V^{h_i}(\Omega_i)$. Similar to the proof of (24) and (25), we can obtain

$$\begin{aligned} ||g - g_h||_{l,\Omega} &\lesssim \sum_{i=1}^{s} h_i^{m_i+r-l}||g||_{m_i+r,\Omega_i} \\ &\lesssim h^{r-l}\sum_{i=1}^{s} h_i^{m_i}||g||_{m_i+r,\Omega_i}, \quad l = 0, 1. \end{aligned} \tag{27}$$

We are now ready to provide the error estimate for the semi-discrete PUFEM solution of (1).

Theorem 4.1. *Assume* $u_h(t) = \sum_{i=1}^{s} \varphi_i u_h^i(t)$ *is the semi-discrete PUFEM solution of* (1), *and* $u(t)$ *is an exact solution of* (1). *Then*

$$||u_h(t) - u(t)||_{0,\Omega} + h||\nabla(u_h(t) - u(t))||_{0,\Omega} \lesssim h^r \sum_{i=1}^{s} C_i(u)h_i^{m_i}, \quad \text{for} \quad t \in (0, T], \tag{28}$$

where $r \geq 1, m_i \geq 1$ $(i = 1, 2, ..., s)$ *are integers.*

Proof According to Lemma 4.1, we only need to prove

$$||\theta(t)||_{0,\Omega} + h||\nabla\theta(t)||_{0,\Omega} \lesssim h^r \sum_{i=1}^{s} C_i(u)h_i^{m_i}, \quad \text{for } t \in (0, T]. \tag{29}$$

On every sub-domain Ω_i, according to the definition of the elliptic projection in (21), for $\forall v_i \in V^{h_i}(\Omega_i)$, we have

$$\begin{aligned}
& (\partial_t\theta_i, v_i) + (a(u_{h_i})\nabla\theta_i, \nabla v_i) \\
= \ & (\partial_t u_{h_i}, v_i) + (a(u_{h_i})\nabla u_{h_i}, \nabla v_i) - (\partial_t \widetilde{u}_{h_i}, v_i) - (a(u_{h_i})\nabla\widetilde{u}_{h_i}, \nabla v_i) \\
= \ & (f(u_{h_i}), v_i) - (a(u)\nabla\widetilde{u}_{h_i}, \nabla v_i) + ((a(u) - g(u_{h_i}))\nabla\widetilde{u}_{h_i}, \nabla v_i) - (\partial_t\rho_i, v_i) - (\partial_t u, v_i) \\
= \ & ((f(u_{h_i}) - f(u)), v_i) + ((a(u) - a(u_{h_i}))\nabla\widetilde{u}_{h_i}, \nabla v_i) - (\partial_t\rho_i, v_i).
\end{aligned}$$

Choose $v_i = \theta_i$. It follows from Lemma 4.2 and Cauchy's inequality that

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}||\theta_i||^2_{0,\Omega_i} + \mu||\nabla\theta_i||^2_{0,\Omega_i} \\
\lesssim \ & ||u_{h_i} - u||_{0,\Omega_i}(||\theta_i||_{0,\Omega_i} + ||\nabla\theta_i||_{0,\Omega_i}) + ||\partial_t\rho_i||_{0,\Omega_i} \cdot ||\theta_i||_{0,\Omega_i} \\
\lesssim \ & \mu||\nabla\theta_i||^2_{0,\Omega_i} + ||\theta_i||^2_{0,\Omega_i} + ||\rho_i||^2_{0,\Omega_i} + ||\partial_t\rho_i||^2_{0,\Omega_i}.
\end{aligned}$$

Integrating both sides of the above gives

$$||\theta_i(t)||^2_{0,\Omega_i} \lesssim ||\theta_i(0)||^2_{0,\Omega_i} + \int_0^t (||\theta_i||^2_{0,\Omega_i} + ||\rho_i||^2_{0,\Omega_i} + ||\partial_t\rho_i||^2_{0,\Omega_i})d\tau.$$

It follows from Gronwall's lemma that

$$||\theta_i(t)||^2_{0,\Omega_i} \lesssim ||\theta_i(0)||^2_{0,\Omega_i} + \int_0^t (||\rho_i||^2_{0,\Omega_i} + ||\partial_t\rho_i||^2_{0,\Omega_i})d\tau.$$

Observe

$$\begin{aligned}
||\theta_i(0)||_{0,\Omega_i} & \leq ||g_{h_i} - g||_{0,\Omega_i} + ||\widetilde{u}_{h_i}(0) - g||_{0,\Omega_i} \\
& \lesssim h_i^{m_i+r}||g||_{m_i+r,\Omega_i}.
\end{aligned}$$

According to Lemma 4.1 and using the inverse estimate, we obtain

$$||\theta_i(t)||_{0,\Omega_i} + h_i||\nabla\theta_i(t)||_{0,\Omega_i} \lesssim h_i^{m_i+r}(||g||_{m_i+r,\Omega_i} + \widetilde{C_i}(u)) = C_i(u)h_i^{m_i+r}, \tag{30}$$

where $C_i(u) = ||g||_{m_i+r,\Omega_i} + \widetilde{C_i}(u)$. Note that $\theta = \sum_{i=1}^s \varphi_i\theta_i, \nabla\theta = \sum_{i=1}^s \nabla\varphi_i \cdot \theta_i + \sum_{i=1}^s \varphi_i \cdot \nabla\theta$. Therefore,

$$||\theta||_{0,\Omega} \leq \sum_{i=1}^s ||\varphi_i\theta_i||_{0,\Omega_i} \leq \sum_{i=1}^s ||\theta_i||_{0,\Omega_i} \lesssim \sum_{i=1}^s h_i^{m_i+r}C_i(u); \tag{31}$$

and

$$\begin{aligned}
||\nabla\theta||_{0,\Omega} & \leq \sum_{i=1}^s ||\nabla\varphi_i \cdot \theta_i||_{0,\Omega_i} + \sum_{i=1}^s ||\varphi_i \cdot \nabla\theta_i||_{0,\Omega_i} \\
& \lesssim \sum_{i=1}^s h_i^{-1}||\theta_i||_{0,\Omega_i} + \sum_{i=1}^s ||\nabla\theta_i||_{0,\Omega_i} \lesssim \sum_{i=1}^s h_i^{m_i+r-1}C_i(u).
\end{aligned} \tag{32}$$

The desired estimate (29) follows by combining (31) and (32). The proof is complete. ∎

5 Error estimate for the fully discrete PUFEM solution

We now consider the fully discrete schemes. We shall consider the backward Euler and the Crank-Nicolson Galerkin scheme. We first use the backward Euler Galerkin scheme:

$$\begin{cases} (\overline{\partial}_t U_h^j, v) + (a(U_h^j)\nabla U_h^j, \nabla v) = (f(U_h^j), v), & \forall v \in V^h(\Omega), \\ U_h^0 = g_h(x) \in V^h(\Omega), j = 1, 2, ..., N, & \text{for } x \in \Omega, \end{cases} \tag{33}$$

where U_h^j is the approximation of $u(t_j)$ in the subdomain Ω, and $\overline{\partial}_t U_h^j = (U_h^j - U_h^{j-1})/\tau$, $t_j = j\tau$, τ is the time step $(i = 1, 2, ..., s; j = 1, 2, ..., N)$.

The above method has the disadvantage that a nonlinear system of algebraic equations has to be solved at each time step. To avoid the presence of $a(U_h^j)$ and $f(U_h^j)$ in (33), we shall consider a linearized modification of the method by replacing U_h^j by U_h^{j-1} in these two places. This gives

$$\begin{cases} (\overline{\partial}_t U_h^j, v) + (a(U_h^{j-1})\nabla U_h^j, v) = (f(U_{h_i}^{j-1}), v_i), & \forall v_i \in V^{h_i}(\Omega_i), \\ U_h^0 = g_h(x) \in V^h(\Omega), j = 1, 2, ..., N, & \text{for } x \in \Omega. \end{cases} \tag{34}$$

The following theorem presents an error estimate for the linearized fully discrete PUFEM solution.

Theorem 5.1. *Assume $U_h^j = \sum_{i=1}^s \varphi_i U_h^j$ is a linearized fully discrete PUFEM solution of* (34) *at $t = t_j$. Let $u(t_j)$ be the solution of* (1) *at $t = t_j$. Then*

$$||U_h^j - u(t_j)||_{0,\Omega} + h||\nabla(U_h^j - u(t_j))||_{0,\Omega} \lesssim \sum_{i=1}^s C_i(u)(h_i^{m_i+r} + \tau), \tag{35}$$

where $r \geq 1, m_i \geq 1\,(i = 1, 2, ..., s; j = 1, 2, ..., N)$ are integers.

Proof Similar to (20), we may express the error as a sum of two terms:

$$U_h^j - u^j = (U_h^j - \widetilde{u}_h^j) + (\widetilde{u}_h^j - u^j) = \theta^j + \rho^j, \quad \forall t_j \in (0, T],$$

where $\widetilde{u}_h^j$ is an elliptic projection in $V^h(\Omega)$ of the exact solution $u(t_j)$ defined in (21). Set $\theta^j = \sum_{i=1}^s \varphi_i \theta^j$. Based on Lemma 4.1, we only need to prove

$$||\theta^j||_{0,\Omega} + h||\nabla\theta^j||_{0,\Omega} \lesssim \sum_{i=1}^s C_i(u)(h_i^{m_i+r} + \tau). \tag{36}$$

Observe that

$$\begin{aligned} & (\partial_t \theta_i^j, v) + (a(U_h^{j-1})\nabla\theta^j, \nabla v) \\ = \; & (f(U_h^{j-1}) - f(u(t_j)), v) - ((a(U^{j-1}) - a(u(t_j)))\nabla\widetilde{u}_h(t_j), \nabla v) \\ & -(\partial_t \rho^j, \nabla v) - (\overline{\partial_t} u(t_j) - \partial_t u(t_j), \nabla v), \end{aligned} \tag{37}$$

and

$$\begin{aligned} ||f(U_h^{j-1}) - f(u(t_j))||_{0,\Omega} & \lesssim ||U_h^{j-1} - u(t_j)||_{0,\Omega} \\ & \lesssim ||\theta^{j-1}||_{0,\Omega} + ||\rho^{j-1}||_{0,\Omega} + \tau||\widetilde{\partial}_t u(t_j)||_{0,\Omega}. \end{aligned}$$

Similarly, we can bound the term in $||a(U_h^{j-1}) - a(u(t_j))||_{0,\Omega}$. Choose $v = \theta^j$ in (37). Using Friedrich's inequality, we have

$$\begin{aligned} \frac{1}{2}\overline{\partial}_t||\theta^j||_{0,\Omega}^2 + \mu||\nabla\theta^j||_{0,\Omega}^2 \lesssim \; & (||\theta^{j-1}||_{0,\Omega} + ||\rho^{j-1}||_{0,\Omega} + \tau||\overline{\partial}_t u(t_j)||_{0,\Omega} \\ & + ||\overline{\partial}_t \rho^j||_{0,\Omega} + ||\overline{\partial}_t u(t_j) - \partial_t u(t_j)||_{0,\Omega}) \cdot ||\nabla\theta^j||_{0,\Omega}. \end{aligned}$$

Using Lemma 4.1 and Cauchy's inequality, we obtain

$$\overline{\partial}_t||\varphi_i\theta^j||_{0,\Omega_i}^2 \lesssim ||\varphi_i\theta^{j-1}||_{0,\Omega_i}^2 + C_i(u)(h_i^{m_i+r}+\tau)^2,$$

which leads to

$$||\varphi_i\theta^j||_{0,\Omega_i}^2 \lesssim (1+\tau)||\varphi_i\theta_i^{j-1}||_{0,\Omega_i}^2 + C_i(u)(h_i^{m_i+r}+\tau)^2.$$

By repeated application, it follows

$$||\varphi_i\theta^j||_{0,\Omega_i}^2 \lesssim ||\varphi_i\theta^0||_{0,\Omega_i}^2 + C_i(u)(h_i^{m_i+r}+\tau)^2.$$

Consequently,

$$||\varphi_i\theta^j||_{0,\Omega_i} \lesssim ||\varphi_i\theta^0||_{0,\Omega_i} + C_i(u)(h_i^{m_i+r}+\tau) \lesssim C_i(u)(h_i^{m_i+r}+\tau);$$
$$h_i||\nabla(\varphi_i\theta^j)||_{0,\Omega_i} \lesssim C_i(u)(h_i^{m_i+r}+\tau),$$

which yields

$$||\varphi_i\theta^j||_{0,\Omega_i} + h_i||\nabla(\varphi_i\theta^j)||_{0,\Omega_i} \lesssim C_i(u)(h_i^{m_i+r}+\tau).$$

Since $\theta^j = \sum_{i=1}^s \varphi_i\theta^j$, $\nabla\theta^j = \sum_{i=1}^s \nabla(\varphi_i\theta^j)$, we have

$$||\theta^j||_{0,\Omega} \le \sum_{i=1}^s ||\varphi_i\theta^j||_{0,\Omega_i} \lesssim \sum_{i=1}^s h_i^{m_i+r}(C_i(u)+\tau), \tag{38}$$

and

$$||\nabla\theta^j||_{0,\Omega} \le \sum_{i=1}^s ||\nabla(\varphi_i\theta^j)||_{0,\Omega_i} \lesssim \sum_{i=1}^s h_i^{-1}||\theta_i^j||_{0,\Omega_i} + \sum_{i=1}^s ||\nabla\theta_i^j||_{0,\Omega_i}. \tag{39}$$

Combining (38) and (39) gives (36). The proof is complete. ■

Now, we consider the Crank-Nicolson Galerkin scheme:

$$\begin{cases} (\overline{\partial}_t U_h^j, v) + (a(\widehat{U}_h^j)\nabla\widehat{U}_h^j, \nabla v) = (f(\widehat{U}_h^j), v), & \forall v \in V^h(\Omega), \\ U_h^0 = g_h(x) \in V^h(\Omega), j = 1, 2, ..., N, & \text{for } x \in \Omega, \end{cases} \tag{40}$$

where $\widehat{U}_h^j = (U_h^j + U_h^{j-1})/2$ is the approximation of $u(t_j)$, and $\overline{\partial}_t U_h^j = (U_h^j - U_h^{j-1})/\tau$. This equation is symmetric around the point $t = t_{j-\frac{1}{2}}$, however, according the first backward Euler method discussed above, the equation (40) is a nonlinear system. Thus, we shall consider it's modified linearized form:

$$\begin{cases} (\overline{\partial}_t U_h^j, v) + (a(\breve{U}_h^j)\nabla\widehat{U}_h^j, \nabla v) = (f(\breve{U}_h^j), v), & \forall v \in V^h(\Omega), \\ U_h^0 = g_h(x) \in V^h(\Omega), j = 1, 2, ..., N, & \text{for } x \in \Omega, \end{cases} \tag{41}$$

where $\breve{U}_h^j = \frac{3}{2}U_h^{j-1} - \frac{1}{2}U_h^{j-2}, j \ge 2, t_j = j\tau \in (0, T]$.

This method will require a separate prescription for calculating U_h^1. For the first approximate value $U_h^{1,0}$ determined by the case $j = 1$ of equation (41) with $\breve{U}_h^1$ replaced by U_h^0, we can obtain the final approximate result of the same equation by using $\breve{U}_h^1$ replaced by $(U_h^{1,0} + U_h^0)/2$. Thus, our starting procedure is to first let

$$U_h^0 = g_h, \tag{42}$$

and then

$$\left(\frac{U_h^{1,0} - U_h^0}{\tau}, v\right) + \left(a(U_h^0)\nabla\left(\frac{U_h^{1,0} + U_h^0}{2}\right), \nabla v\right) = (f(U_h^0), v). \tag{43}$$

Table 1: The error and relative error of PUEFM solution at $t = \pi$.

E, E_r	x_1	x_2	x_3	x_4
$\frac{\pi}{12}$	0.6972e-5	0.5055e-5	0.4355e-5	0.4863e-5
	0.3243e-3	0.2275e-3	0.1897e-3	0.2853e-3
$\frac{\pi}{14}$	0.6977e-5	0.5069e-5	0.4377e-5	0.4864e-5
	0.3249e-3	0.2282e-3	0.1907e-3	0.2054e-3
$\frac{\pi}{16}$	0.6979e-5	0.5075e-5	0.4383e-5	0.4868e-5
	0.3250e-3	0.2284e-3	0.1910e-3	0.2056e-3
$\frac{\pi}{18}$	0.6980e-5	0.5077e-5	0.4382e-5	0.4876e-5
	0.3252e-3	0.2285e-3	0.1909e-3	0.2059e-3

Finally,

$$(\overline{\partial}_t U_h^1, v) + \left(a\left(\frac{U_h^{1,0} + U_h^0}{2}\right) \nabla \breve{U}_h, \nabla v \right) = \left(f\left(\frac{U_h^{1,0} + U_h^0}{2}\right), v \right). \tag{44}$$

Theorem 5.2. *Assume U_h^j be a solution of* (41) *at $t = t_j$, U_h^0 and U_h^1 are defined by* (42)-(44). *Let $U^j = \sum_{i=1}^s \varphi_i U_h^j$ be a linearized fully discrete PUFEM solution, and $u(t_j)$ be the solution of* (1)*. Then*

$$||U_h^j - u(t_j)||_{0,\Omega} + h||\nabla(U_h^j - u(t_j))||_{0,\Omega} \lesssim \sum_{i=1}^s C_i(u)(h_i^{m_i+r} + \tau^2), \tag{45}$$

where $r \geq 1, m_i \geq 1\,(i = 1, \cdots, s; j = 1, \cdots, N)$ are integers.

Proof Our main observation is

$$\breve{U}_h^j = \frac{3}{2}U_h^{j-1} - \frac{1}{2}U_h^{j-2} = U_h^{j-\frac{1}{2}} + \mathcal{O}(\tau^2), \quad \text{as } \tau \to 0. \tag{46}$$

The rest of the proof is similar to that of Theorem 5.1, which will be omitted here. ■

6 Numerical example

Consider the following nonlinear parabolic initial-boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = -u^3 + e^{-3t}\sin^3 x, \quad 0 < x < \pi, 0 < t < T, \\ u(0,t) = 0, u(\pi,t) = 0, \quad 0 \leq t \leq T, \\ u(x,0) = \sin x, \quad 0 \leq x \leq \pi, \end{cases} \tag{47}$$

where $\Omega = [0,\pi], \Omega_1 = [0, \frac{3\pi}{5}], \Omega_2 = [\frac{\pi}{2}, \pi], \Omega_0 = \Omega_1 \cap \Omega_1 = [\frac{\pi}{2}, \frac{3\pi}{5}]$. Assume Ω_1 is partitioned by a uniform mesh of size $h_1 = \frac{\pi}{10}$, Ω_2 is partitioned by a uniform mesh of size $h_2 = \frac{\pi}{12}, \frac{\pi}{14}, \frac{\pi}{16}, \frac{\pi}{18}$ respectively. The error $E = |u - u_h|$ and the relative error $E_r = E/|u|$ are computed at the four sample points in $\Omega_0 : x_1 = 0.52\pi, x_2 = 0.54\pi, x_3 = 0.56\pi, x_4 = 0.58\pi$, and at $t = \pi, t = 5\pi$, respectively (see Tables 1 and 2). In Tables 1 and 2, $u = e^{-t}\sin x$ is the exact solution of (47), $u_h = \sum_{i=1}^2 \varphi_i u_h^i$ is the PUFEM solution of the semi-discrete problem, $u_h^i \in V^{h_i}(\Omega_i)$ is linear finite element solution of the semi-discrete problem. The partition of unity function is similar to the formulas (12a) and (12b) of Example 3.1.

It is observed in Tables 1 and 2 that the error function E of the PUFEM solution has good approximation properties, and the relative error function E_r of the PUFEM solution has good stability properties.

Table 2: The error and relative error of PUEFM solution at $t = 5\pi$.

E, E_r	x_1	x_2	x_3	x_4
$\frac{\pi}{12}$	0.2435e-10	0.1748e-10	0.1501e-10	0.1692e-10
	0.3252e-3	0.2256e-3	0.1875e-3	0.2048e-3
$\frac{\pi}{14}$	0.2438e-10	0.1756e-10	0.1515e-10	0.1696e-10
	0.3255e-3	0.2266e-3	0.1893e-3	0.2054e-3
$\frac{\pi}{16}$	0.2439e-10	0.1760e-10	0.1523e-10	0.1698e-10
	0.3257e-3	0.2272e-3	0.1902e-3	0.2056e-3
$\frac{\pi}{18}$	0.2440e-10	0.1763e-10	0.1524e-10	0.1701e-10
	0.3258e-3	0.2275e-3	0.1903e-3	0.2060e-3

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