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Approximation Solutions of Nonlinear Strongly Accretive Operator Equations by Ishikawa Iteration Procedure with Errors†

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Abstract. Let $1 < p \leq 2, E$ be a real p-uniformly smooth Banach space and $T : E \to E$ be a continuous and strongly accretive operator. The purpose of this paper is to investigate the problem of approximating solutions to the equation $Tx = f$ by the Ishikawa iteration procedure with errors

$$\begin{cases} x_{n+1} = a_n x_n + b_n(f - Ty_n + y_n) + c_n u_n, \\ y_n = a'_n x_n + b'_n(f - Tx_n + x_n) + c'_n v_n, \qquad n \geq 0 \end{cases}$$

where $x_0 \in E, \{u_n\}, \{v_n\}$ are bounded sequences in E and $\{a_n\}, \{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}, \{c'_n\}$ are real sequences in $[0, 1]$. Under the assumption of the condition $0 < \alpha \leq b_n + c_n, \forall n \geq 0$, it is shown that the iterative sequence $\{x_n\}$ converges strongly to the unique solution of the equation $Tx = f$. Furthermore, under no assumption of the condition $\lim_{n\to\infty}(b'_n + c'_n) = 0$, it is also shown that $\{x_n\}$ converges strongly to the unique solution of $Tx = f$.

Key words: Strongly accretive operator equation; Ishikawa iteration procedure with errors; solution; p-uniformly smooth Banach space.

AMS subject classifications: 47H05, 47H10, 47H17

1 Introduction and preliminaries

Let E be a real Banach space with norm $\|\cdot\|$, let E^* denote the dual space of E, and let $\langle\cdot,\cdot\rangle$ denote the generalized duality pairing between E and E^*. For $1 < p < \infty$, the mapping $J_p : E \to 2^{E^*}$ defined by

$$J_p(x) = \left\{u^* \in E^* : \langle x, u^*\rangle = \|x\|\|u^*\|, \|u^*\| = \|x\|^{p-1}\right\}, \qquad x \in E,$$

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is called the duality mapping with the gauge function $\phi(t) = t^{p-1}$. In particular, the duality mapping with the gauge function $\phi(t) = t$, denoted by J, is referred to be the normalized duality mapping. It is a well-known fact[17] that $J_p(x) = \|x\|^{p-2}J(x)$ for $x \in E \setminus \{0\}$ and $1 < p < \infty$. Equivalently, the duality mapping J_p can be defined as the subdifferential of the functional $\Psi(x) = p^{-1}\|x\|^p$, that is,

$$x^* \in J_p(x) \Leftrightarrow x^* \in \partial\Psi(x) = \left\{f \in E^* : p^{-1}\|y\|^p - p^{-1}\|x\|^p \geq \langle y - x, f\rangle, \forall y \in E\right\}. \tag{1}$$

In addition, it is also known that $J_p(\lambda x) = \lambda^{p-1}J_p(x), \forall\lambda \geq 0$.

An operator T with the domain $D(T)$ and range $R(T)$ in E is said to be strongly accretive if for $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that $\langle Tx - Ty, j(x-y)\rangle \geq k\|x-y\|^2$ for some constant $k > 0$; or equivalently, for $x, y \in D(T)$ there is $j_p(x-y) \in J_p(x-y)$ such that

$$\langle Tx - Ty, j_p(x-y)\rangle \geq k\|x-y\|^p \tag{2}$$

for some constant $k > 0$. In particular, T is said to be accretive if for $x, y \in D(T)$ there is $j(x-y) \in J(x-y)$ such that $\langle Tx - Ty, j(x-y)\rangle \geq 0$; or equivalently, for $x, y \in D(T)$ there exists $j_p(x-y) \in J_p(x-y)$ such that $\langle Tx - Ty, j_p(x-y)\rangle \geq 0$. Without loss of generality, we assume that $k \in (0,1)$. It is known that an operator T with the domain $D(T)$ and range $R(T)$ in E is accretive if and only if for all $x, y \in D(T)$ and $r > 0$ there holds the inequality

$$\|x - y\| \leq \|x - y + r(Tx - Ty)\|.$$

It is also known that T is strongly accretive if and only if there exists a positive number k such that $(T - kI)$ is accretive where I is the identity operator of $D(T)$. The accretive operators were introduced independently by Browder[1] and Kato[2] in 1967. An early fundamental result, due to Browder, in the theory of accretive operators states that the initial value problem $du/dt + Tu = 0, u(0) = u_0$ is solvable if T is a locally Lipschitzian and accretive operator on E. A strongly accretive operator is sometimes called the strictly accretive operator. These operators have been investigated previously by many authors; see [5-14, 18] for more details.

Now we remind the reader of the following fact: In most of the known results on the Ishikawa iteration procedure (with errors) for finding solutions to nonlinear equations $Tx = f$ of strongly accretive operators, generally, the Lipschitz continuity or uniform continuity is imposed on the strongly accretive operators T. Moreover, the sequences of the iteration parameters are assumed or possible to be convergent to zero. See, for example, [5-14, 18].

Now, let us recall the following iteration procedures due to Xu[5].

(I) The Ishikawa iteration procedure with errors is defined as follows: For a nonempty closed convex subset C of a Banach space E and an operator $T : C \subset E \to E$, the sequence $\{x_n\}$ in C is defined from an arbitrary $x_0 \in C$ by

$$\begin{cases} x_{n+1} = a_n x_n + b_n T y_n + c_n u_n, \\ y_n = a'_n x_n + b'_n T x_n + c'_n v_n, \quad n \geq 0, \end{cases}$$

where $\{u_n\}, \{v_n\}$ are two bounded sequences in C and $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ are real sequences in [0,1] satisfying certain restrictions.

(II) The Mann iteration procedure with errors is defined as follows: If $a'_n = 1, b'_n = c'_n = 0$ for all $n \geq 0$, then the above Ishikawa iteration procedure with errors is called the Mann iteration procedure with errors.

Let $1 < p \leq 2, E$ be a real p-uniformly smooth Banach space and $T : E \to E$ be a continuous and strongly accretive operator. In this paper, we investigate the problem of approximating solutions to the equation $Tx = f$ by the Ishikawa iteration procedure with errors

$$\begin{cases} x_{n+1} = a_n x_n + b_n(f - T y_n + y_n) + c_n u_n, \\ y_n = a'_n x_n + b'_n(f - T x_n + x_n) + c'_n v_n, \quad n \geq 0 \end{cases}$$

where $x_0 \in E, \{u_n\}, \{v_n\}$ are bounded sequences in E and $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ are real sequences in [0,1]. Under the assumption of the condition $0 < \alpha \leq b_n + c_n, \forall n \geq 0$, it is shown that the iterative sequence $\{x_n\}$ converges strongly to the unique solution of the equation $Tx = f$. Furthermore, under no assumption of the condition $b'_n + c'_n \to 0 (n \to \infty)$, it is also shown that $\{x_n\}$ converges strongly to the unique solution of $Tx = f$. The results presented in this paper improve and extend some earlier and recent results obtained previously by many authors, see, e.g., [5-14,18].

Next, we give some preliminaries. Let E be a real Banach space. Recall that the modulus $\rho_E(\cdot)$ of smoothness of E is defined by

$$\rho_E(\tau) = \sup \left\{ \left(\|x+y\| + \|x-y\| \right)/2 - 1 : x, y \in E, \|x\| = 1, \|y\| \leq \tau \right\}, \qquad \tau > 0,$$

and that E is said to be uniformly smooth if $\lim_{\tau \downarrow 0} \rho_E(\tau)/\tau = 0$. It is known (cf. [15]) that if E is uniformly smooth, then E is a smooth and reflexive Banach space, and J_p is single-valued, and uniformly continuous on any bounded subset of E. Recall that for a real number $1 < p \leq 2$, a Banach space E is said to be p-uniformly smooth if $\rho_E(\tau) \leq d\tau^p, \forall \tau > 0$, where $d > 0$ is a constant. It is known (cf. [16]) that for a real Hilbert space $H, \rho_H(\tau) = (1+\tau^2)^{1/2} - 1$ and hence H is 2-uniformly smooth. It is also known that if $1 < p < 2, L_p$ (or l_p) is p -uniformly smooth; while if $2 \leq p < \infty, L_p$ (or l_p) is 2-uniformly smooth. Xu[16] gave the following characterization for a real p-uniformly smooth Banach space: Let E be a real smooth Banach space and p be a fixed number in (1,2]. Then E is p-uniformly smooth if and only if there exists a constant $d_p > 0$such that

$$\|x+y\|^p \leq \|x\|^p + p\langle y, J_p(x)\rangle + d_p\|y\|^p, \qquad \forall x, y \in E.$$

Proposition 1.1 Let $1 < p \leq 2$ and E be a real p-uniformly smooth Banach space. Then

$$\|x+y\|^p \leq \|x\|^p + p\langle y, J_p(x+y)\rangle, \qquad \forall x, y \in X.$$

Proof. The conclusion follows from (1). ■

Proposition 1.2 Let $1 < p \leq 2$. Then
(i) $(a+b)^{p-1} \leq 2^{p-1}(a^{p-1} + b^{p-1}), \forall a, b, \in [0, \infty)$;
(ii) $(a+b+c)^{p-1} \leq 2^{2(p-1)}(a^{p-1} + b^{p-1} + c^{p-1}), \forall a, b, c, \in [0, \infty)$.

Proof. 1) If a, b are both zero, then the conclusion (i) is obviously true; if one is zero and the other is not zero, for example $a = 0, b \neq 0$, then the conclusion (i) is obviously true; if a, b are not zero, then

$$(a+b)^{p-1} = \frac{(a+b)^p}{a+b} \leq \frac{2^{p-1}(a^p + b^p)}{a+b} \leq 2^{p-1}(a^{p-1} + b^{p-1}).$$

This shows that the conclusion (i) is still true.

2) From the conclusion (i), it follows that

$$\begin{aligned} (a+b+c)^{p-1} &\leq 2^{p-1}\left(a+b)^{p-1} + c^{p-1}\right] \\ &\leq 2^{p-1}\left[2^{p-1}(a^{p-1} + b^{p-1}) + c^{p-1}\right] \\ &\leq 2^{2(p-1)}(a^{p-1} + b^{p-1} + c^{p-1}). \end{aligned}$$

This proves that the conclusion (ii) is true. ■

Lemma 1.1. *[15] Let $1 < p \leq 2$ and E be a real p-uniformly smooth Banach space. Then $J_p : E \to E^*$ is Holder continuous with power $(p-1)$, that is, there exists a constant $r > 0$ such that*

$$\|J_p(x) - J_p(y)\| \leq r\|x-y\|^{p-1}, \qquad \forall x, y \in E. \tag{3}$$

Lemma 1.2. *[14] Let $\{\sigma_n\}, \{\delta_n\}$ and $\{\gamma_n\}$ be three nonnegative real sequences satisfying*

$$\sigma_{n+1} \leq (1-\lambda_n)\sigma_n + \delta_n + \gamma_n$$

with $\{\lambda_n\} \subset [0,1], \sum_{n=0}^{\infty} \lambda_n = \infty, \delta_n = o(\lambda_n)$, and $\sum_{n=0}^{\infty} \gamma_n < \infty$. Then $\lim_{n\to\infty} \sigma_n = 0$.

Lemma 1.3. *[18] Suppose that $\{\mu_n\}$ and $\{\nu_n\}$ are two nonnegative real sequences satisfying the following inequality*

$$\mu_{n+1} \leq \gamma\mu_n + \nu_n, \qquad \forall n \geq 0,$$

where $\gamma \in [0,1)$ and $\lim_{n\to\infty} \nu_n = 0$. Then $\lim_{n\to\infty} \mu_n = 0$.

Browder[1] proved that if $T : E \to E$ is locally Lipschitzian and accretive then T is m-accretive; i.e., the operator $(I+T)$ where I denotes the identity operator of E is surjective. This result was subsequently generalized by Martin[3] to continuous accretive operators. It can be seen that the following lemma is an immediate consequence of Martin's result.

Lemma 1.4. *[4] If $T : E \to E$ is continuous and strongly accretive then T maps E onto E; that is, for each $f \in E$ the equation $Tx = f$ has a solution in E.*

2 Main results

Theorem 2.1. *Let $1 < p \leq 2, E$ be a real p-uniformly smooth Banach space and $T : E \to E$ be a continuous and strongly accretive operator. $S : E \to E$ is defined as $Sx = f - Tx + x$ for each $x \in E$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ be real sequences in (0,1) satisfying the following conditions*

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$;
(ii) $c_n \to 0 (n \to \infty), b'_n + c'_n \to 0 (n \to \infty)$;
(iii) $0 < \alpha \leq b_n + c_n \leq \min\left\{\frac{p\eta}{p-1}, \frac{1}{p(k-\eta)}, 1\right\}$ for some $\eta \in (0,k)$.

Let $\{x_n\}$ be the sequence in E generated from an arbitrary $x_0 \in E$ by the Ishikawa iteration procedure with errors:

$$\begin{cases} x_{n+1} = a_n x_n + b_n S y_n + c_n u_n, \\ y_n = a'_n x_n + b'_n S x_n + c'_n v_n, \qquad n \geq 0, \end{cases} \tag{ISE}$$

where $\{u_n\}, \{v_n\}$ are two bounded sequences in E. Assume that $\{Sx_n\}, \{Sy_n\}$ are both bounded. Then $\{x_n\}$ converges strongly to the unique solution x^* of the equation $Tx = f$ if and only if $\{Ty_n\}$ converges strongly to f.

Proof. At first, we observe that the equation $Tx = f$ has a unique solution which is denoted by x^*. Indeed, the existence follows from Lemma 1.4 and the uniqueness from the strong accretiveness of T. We also observe that for $x, y \in E$,

$$\begin{aligned} \langle Sx - Sy, J_p(x-y)\rangle &= -\langle Tx - Ty, J_p(x-y)\rangle + \|x-y\|^p \\ &\leq -k\|x-y\|^p + \|x-y\|^p = (1-k)\|x-y\|^p. \end{aligned}$$

Now, set $\alpha_n = b_n + c$ and $\beta_n = b'_n + c'_n$. Then (ISE) can be rewritten as

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n S y_n + c_n(u_n - S y_n), \\ y_n = (1-\beta_n)x_n + \beta_n S x_n + c'_n(v_n - S x_n), \qquad n \geq 0. \end{cases} \tag{ISE1}$$

Put $d = \|x_0 - x^*\| + \sup_{n\geq 0} \|Sy_n - x^*\| + \sup_{n\geq 0} \|u_n - x^*\|$. Then, by inductive reasoning, we get

$$\|x_n - x^*\| \leq d, \qquad \forall n \geq 0.$$

Since $\{x_n\}, \{Sx_n\}, \{v_n\}$ are bounded, it follows from (ISE1) that $\{y_n\}$ is bounded. Let

$$K = \{Sx_n\}_{n=0}^{\infty} \cup \{Sy_n\}_{n=0}^{\infty} \cup \{x_n\}_{n=0}^{\infty} \cup \{y_n\}_{n=0}^{\infty} \cup \{u_n\}_{n=0}^{\infty} \cup \{v_n\}_{n=0}^{\infty} \cup \{x^*\}.$$

Put $M = 1 + \sup_{x\in K} \|x\| + \mathrm{diam}K$, where $\mathrm{diam}K$ denotes the diameter of K.

Sufficiency. Suppose $\{Ty_n\}$ converges strongly to f. Then we assert that $\{x_n\}$ converges strongly to x^*. Indeed, it follows from Proposition 1.1 and the definition of M that

$$\begin{aligned} \|x_{n+1} - x^*\|^p &= \|(1-\alpha_n)x_n + \alpha_n S y_n - x^* + c_n(u_n - Sy_n)\|^p \\ &\leq \|(1-\alpha_n)x_n + \alpha_n S y_n - x^*\|^p + p\langle c_n(u_n - Sy_n), J_p(x_{n+1} - x^*)\rangle \\ &\leq \|(1-\alpha_n)x_n + \alpha_n S y_n - x^*\|^p + pM^p c_n. \end{aligned} \tag{4}$$

Now, set $w_n = (1-\alpha_n)x_n + \alpha_n S y_n$. Then it follows from (3) and Proposition 1.1 that

$$\begin{aligned} \|w_n - x^*\|^p &\leq (1-\alpha_n)^p \|x_n - x^*\|^p + p\alpha_n \langle Sy_n - x^*, J_p(w_n - x^*)\rangle \\ &= (1-\alpha_n)^p \|x_n - x^*\|^p + p\alpha_n \langle Sy_n - x^*, J_p(w_n - x^*) - J_p(y_n - x^*)\rangle \\ &\quad + p\alpha_n \langle Sy_n - x^*, J_p(y_n - x^*)\rangle \\ &\leq (1-\alpha_n)^p \|x_n - x^*\|^p + rp\alpha_n \|Sy_n - x^*\| \cdot \|w_n - y_n\|^{p-1} \\ &\quad + p(1-k)\alpha_n \|y_n - x^*\|^p \\ &\leq (1-\alpha_n)^p \|x_n - x^*\|^p + rp\alpha_n M \|w_n - y_n\|^{p-1} \\ &\quad + p(1-k)\alpha_n \|y_n - x^*\|^p. \end{aligned} \tag{5}$$

Utilizing Proposition 1.2, we have

$$\begin{aligned} \|w_n - y_n\|^{p-1} &= \|(1-\alpha_n)(x_n - y_n) + \alpha_n(Sy_n - y_n)\|^{p-1} \\ &\leq \{\|x_n - y_n\| + \|Sy_n - y_n\|\}^{p-1} \\ &= \{\|\beta_n(Sx_n - x_n) + c'_n(v_n - Sx_n)\| + \|Sy_n - y_n\|\}^{p-1} \\ &\leq 2^{2(p-1)}\{\beta_n^{p-1}\|Sx_n - x_n\|^{p-1} + c_n'^{p-1}\|v_n - Sx_n\|^{p-1} + \|Sy_n - y_n\|^{p-1}\} \\ &\leq (4M)^{p-1}(\beta_n^{p-1} + c_n'^{p-1} + \|Ty_n - f\|^{p-1}). \end{aligned} \tag{6}$$

As in the proof of (4), we obtain

$$\begin{aligned} \|y_n - x^*\|^p &= \|(1-\beta_n)x_n + \beta_n S x_n + c'_n(v_n - Sx_n) - x^*\|^p \\ &\leq \|(1-\beta_n)(x_n - x^*) + \beta_n(Sx_n - x^*)\|^p + pc'_n\langle v_n - Sx_n, J_p(y_n - x^*)\rangle \\ &\leq (1-\beta_n)^p\|x_n - x^*\|^p + p\beta_n\langle Sx_n - x^*, J_p((1-\beta)(x_n - x^*) + \beta_n(Sx_n - x^*))\rangle \\ &\quad - p\beta_n\langle Sx_n - x^*, J_p((1-\beta_n)(x_n - x^*))\rangle \\ &\quad + p\beta_n\langle Sx_n - x^*, J_p((1-\beta_n)(x_n - x^*))\rangle + pM^p c'_n \\ &\leq (1-\beta_n)^p\|x_n - x^*\|^p + rp\|\beta_n(Sx_n - x^*)\|^p + p(1-k)\beta_n\|x_n - x^*\|^p + pM^p c'_n \\ &\leq [(1-\beta_n)^p + p(1-k)\beta_n]\,\|x_n - x^*\|^p + (r\beta_n^p + c'_n)pM^p. \end{aligned} \tag{7}$$

Substituting (6) and (7) into (5), we derive

$$\begin{aligned} \|w_n - x^*\|^p &\le (1-\alpha_n)^p\|x_n - x^*\|^p \\ &\quad + rp\alpha_n M\cdot(4M)^{p-1}(\beta_n^{p-1} + c_n'^{p-1} + \|Ty_n - f\|^{p-1}) + p(1-k)\alpha_n \\ &\quad \cdot\{[(1-\beta_n)^p + p(1-k)\beta_n]\|x_n - x^*\|^p + (r\beta_n^p + c_n')pM^p\} \\ &\le [(1-\alpha_n)^p + p(1-k)\alpha_n]\|x_n - x^*\|^p + 4^{p-1}rpM^p \\ &\quad \cdot(\beta_n^{p-1} + c_n'^{p-1} + \|Ty_n - f\|^{p-1}) + p^2M^p(\beta_n + r\beta_n^p + c_n'). \end{aligned} \tag{8}$$

Substituting (8) into (4), we deduce

$$\begin{aligned} \|x_{n+1} - x^*\|^p &\le [(1-\alpha_n)^p + p(1-k)\alpha_n]\,\|x_n - x^*\|^p \\ &\quad + D\left\{\beta_n^{p-1} + \beta_n + \beta_n^p + c_n'^{p-1} + c_n' + c_n\right\} + D\|Ty_n - f\|^{p-1}, \end{aligned} \tag{9}$$

where $D = \max\{4^{p-1}rpM^p, rp^2M^p, p^2M^p\}$. Note that for $1 < p \le 2$ we have

$$(1-t)^{p-1} \le 1 - (p-1)t, \forall t \in [0,1].$$

Since $0 < \alpha \le \alpha_n \le \min\{p\eta/p-1, 1/(p(k-\eta)), 1\}$ for some $\eta \in (0,k)$, we deduce

$$\begin{aligned} (1-\alpha_n)^p + p(1-k)\alpha_n &\le (1-(p-1)\alpha_n)(1-\alpha_n) + p(1-k)\alpha_n \\ &= 1 - pk\alpha_n + (p-1)\alpha_n^2 \\ &\le 1 - pk\alpha_n + (p-1)\cdot\frac{p\eta}{p-1}\alpha_n \\ &= 1 - p(k-\eta)\alpha_n, \end{aligned} \tag{10}$$

and $0 \le 1 - p(k-\eta)\alpha_n \le 1 - p(k-\eta)\alpha < 1$. Substituting (10) into (9), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\|^p &\le (1-p(k-\eta)\alpha_n)\|x_n - x^*\|^p \\ &\quad + D\{\beta_n^{p-1} + \beta_n + \beta_n^p + c_n'^{p-1} + c_n' + c_n\} + D\|Ty_n - f\|^{p-1} \\ &\le (1-p(k-\eta)\alpha)\|x_n - x^*\|^p \\ &\quad + D\{\beta_n^{p-1} + \beta_n + \beta_n^p + c_n'^{p-1} + c_n' + c_n\} + D\|Ty_n - f\|^{p-1} \\ &= (1-p(k-\eta)\alpha)\|x_n - x^*\|^p + \theta_n, \end{aligned} \tag{11}$$

where

$$\theta_n = D\{\beta_n^{p-1} + \beta_n + \beta_n^p + c_n'^{p-1} + c_n' + c_n\} + D\|Ty_n - f\|^{p-1}.$$

Since $\|Ty_n - f\| \to (n \to \infty)$, by virtue of the condition (ii) and by using Lemma 1.3 for (11), we infer that $\|x_n - x^*\| \to 0$ as $n \to \infty$.

Necessity. Suppose that $\{x_n\}$ converges strongly to the unique solution x^* of the equation $Tx = f$. Then, it follows from the continuity of S that $\{Sx_n\}$ converges strongly to $Sx^* = x^*$. Since

$$\|y_n - x_n\| = \|\beta_n(Sx_n - x_n) + c_n'(v_n - Sx_n)\| \le 2M\beta_n \to 0, \quad n \to \infty.$$

So, we have $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} x_n = x^*$. Hence, this implies that $\lim\limits_{n\to\infty} Sy_n = Sx^* = x^*$. Thus, we have

$$\|Ty_n - f\| = \|Sy_n - y_n\| \le \|Sy_n - x^*\| + \|y_n - x^*\| \to 0(n \to \infty).$$

The proof is thus complete. ■

Remark 2.1. By the careful analysis of the proof of Theorem 2.1, we readily see that if the condition (iii) in Theorem 2.1 is replaced by the following condition: $\liminf_{n\to\infty}(b_n + c_n) > \alpha > 0$, and

$$\limsup_{n\to\infty}(b_n + c_n) < \min\left\{\frac{p\eta}{p-1}, \frac{1}{p(k-\eta)}, \right\} \quad \text{for some} \quad \eta \in (0,k),$$

then Theorem 2.1 is still valid.

Theorem 2.2. *Let E, T, S be as in Theorem 2.1. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ be real sequences in (0,1) satisfying the conditions:*

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$;

(ii) $\sum_{n=0}^{\infty}(b_n + c_n) = \infty, c_n = o(b_n)$;

(iii) $\sum_{n=0}^{\infty}(b_n + c_n)c_n'^{p-1} < \infty$.

Let $\{x_n\}$ be the sequence in E generated from an arbitrary $x_0 \in E$ by the Ishikawa iteration procedure (ISE) with errors, where $\{u_n\}, \{v_n\}$ are two bounded sequences in E. Assume that $\{Sx_n\}, \{Sy_n\}$ are both bounded. Then $\{x_n\}$ converges strongly to the unique solution of the equation $Tx = f$ if and only if $\{Tx_n\}$ converges strongly to f.

Proof. Following the idea of the proof in Theorem 2.1, we know that the equation $Tx = f$ has a unique solution which is denoted by x^* and that $\{x_n\}, \{y_n\}$ are both bounded. Let $K, M, \{\alpha_n\}, \{\beta_n\}$ be as in the proof of Theorem 2.1. Next, we still need to use the rewritten version of (ISE).

Sufficiency. Suppose that $\{Tx_n\}$ converges strongly to f. Set $w_n = (1-\alpha_n)x_n + \alpha_n Sy_n$. Then, observe that

$$\begin{aligned}
& \|x_{n+1} - x^*\|^p = \|(1-\alpha_n)x_n + \alpha_n Sy_n + c_n(u_n - Sy_n) - x^*\|^p \\
\leq\ & \|(1-\alpha_n)(x_n - x^*) + \alpha_n(Sy_n - x^*)\|^p + pc_n M^p \\
\leq\ & (1-\alpha_n)^p\|x_n - x^*\|^p + p\alpha_n\langle Sy_n - x^*, J_p(w_n - x^*)\rangle + pc_n M^p \\
& -p\alpha_n\langle Sy_n - x^*, J_p(y_n - x^*)\rangle + p\alpha_n\langle Sy_n - x^*, J_p(y_n - x^*)\rangle \\
\leq\ & (1-\alpha)^p\|x_n - x^*\|^p + rp\alpha_n\|Sy_n - x^*\|\|w_n - y_n\|^{p-1} \\
& +p(1-k)\alpha_n\|y_n - x^*\|^p + pM^p c_n \\
\leq\ & (1-\alpha_n)^p\|x_n - x^*\|^p + rp\alpha_n M\|w_n - y_n\|^{p-1} \\
& +p(1-k)\alpha_n\|y_n - x^*\|^p + pM^p c_n.
\end{aligned} \tag{14}$$

Utilizing the estimates (6) and (7), we have

$$\begin{aligned}
& \|w_n - y_n\|^{p-1} = \|\alpha_n(Sy_n - x_n) - \beta_n(Sx_n - x_n) - c'_n(v_n - Sx_n)\|^{p-1} \\
\leq\ & 2^{2(p-1)}\left\{\alpha_n^{p-1}\|Sy_n - x_n\|^{p-1} + \beta_n^{p-1}\|Sx_n - x_n\|^{p-1} + c_n'^{p-1}\|v_n - Sx_n\|^{p-1}\right\} \\
\leq\ & 2^{2(p-1)}\left\{\alpha_n^{p-1}M^{p-1} + \|Sx_n - x_n\|^{p-1} + c_n'^{p-1}M^{p-1}\right\} \\
\leq\ & (4M)^{p-1}\left\{\alpha_n^{p-1} + \|Tx_n - f\|^{p-1} + c_n'^{p-1}\right\},
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
& \|y_n - x^*\|^p = \|x_n - x^* + \beta_n(Sx_n - x_n) + c'_n(v_n - Sx_n)\|^p \\
\leq\ & \|x_n - x^* + \beta_n(Sx_n - x_n)\|^p + pM^p c'_n \\
\leq\ & \|x_n - x^*\|^p + p\beta_n M^{p-1}\|Sx_n - x_n\| + pM^p c'_n \\
\leq\ & \|x_n - x^*\|^p + pM^p\|Tx_n - f\| + pM^p c'_n.
\end{aligned} \tag{16}$$

Now, substituting (15) and (16) into (14), we have

$$\begin{aligned}
& \|x_{n+1} - x^*\|^p \\
\leq\ & (1-\alpha_n)^p\|x_n - x^*\|^p + rp\alpha_n M\cdot(4M)^{p-1}\left\{\alpha_n^{p-1} + \|Tx_n - f\|^{p-1} + c_n'^{p-1}\right\} \\
& +p(1-k)\alpha_n\cdot\{\|x_n - x^*\|^p + pM^p\|Tx_n - f\| + pM^p c'_n\} + pM^p c_n \\
\leq\ & [(1-\alpha_n)^p + p(1-k)\alpha_n]\|x_n - x^*\|^p + rp\alpha_n M \\
& \cdot(4M)^{p-1}\left\{\alpha_n^{p-1} + \|Tx_n - f\|^{p-1} + c_n'^{p-1}\right\} \\
& +p^2M^p\alpha_n\{\|Tx_n - f\| + c'_n\} + pM^p c_n \\
\leq\ & [(1-\alpha_n)^p + p(1-k)\alpha_n]\|x_n - x^*\|^p \\
& +D_0\alpha_n\left\{\alpha_n^{p-1} + \|Tx_n - f\|^{p-1} + \|Tx_n - f\| + c_n'^{p-1} + c'_n\right\} + D_0 c_n,
\end{aligned} \tag{17}$$

where $D_0 = \max\{rpM(4M)^{p-1}, p^2M^p\}$.

Since for $1 < p \leq 2$ we have $(1-t)^{p-1} \leq 1-(p-1)t, \forall t \in [0,1]$, so, it is easy to see that

$$\begin{aligned} & (1-\alpha_n)^p + p(1-k)\alpha_n \\ \leq \; & (1-(p-1)\alpha_n)(1-\alpha_n) + p(1-k)\alpha_n = 1 - pk\alpha_n + (p-1)\alpha^2. \end{aligned} \tag{18}$$

Substituting (18) into (17), we obtain

$$\begin{aligned} & \|x_{n+1} - x^*\|^p \\ \leq \; & [1-pk\alpha_n]\|x_n - x^*\|^p + (p-1)\alpha_n^2\|x_n - x^*\|^p \\ & + D_0\alpha_n \left\{\alpha_n^{p-1} + \|Tx_n - f\|^{p-1} + \|Tx_n - f\| + c_n'^{p-1} + c_n'\right\} + D_0 c_n, \\ \leq \; & [1-pk\alpha_n]\|x_n - x^*\|^p + D_0\alpha_n \left\{\alpha_n^{p-1} + \alpha_n + \|Tx_n - f\|^{p-1} + \|Tx_n - f\|\right\} \\ & + D_0 c_n + D_0(\alpha_n c_n'^{p-1} + \alpha_n c_n'). \end{aligned} \tag{19}$$

Now, set $\sigma_n = \|x_n - x^*\|^p, \delta_n = D_0\alpha_n\left\{\alpha_n^{p-1} + \alpha_n + \|Tx_n - f\|^{p-1} + \|Tx_n - f\|\right\} + D_0 c_n$, $\gamma_n = D_0(\alpha_n c_n'^{p-1} + \alpha_n c_n')$ and $\lambda_n = pk\alpha_n$. Then (19) reduces to

$$\sigma_{n+1} \leq (1-\lambda_n)\sigma_n + \delta_n + \gamma_n.$$

Since $c_n = o(b_n)$ and $\sum_{n=0}^{\infty} \alpha_n c_n'^{p-1} < \infty$, we conclude that $c_n = o(\alpha_n)$ and $\sum_{n=0}^{\infty} \alpha_n c_n' < \infty$. Therefore, according to the conditions (ii), (iii), we can see that

$$\sum_{n=0}^{\infty} \lambda_n = \infty, \quad \delta_n = o(\lambda_n) \quad \text{and} \quad \sum_{n=0}^{\infty} \gamma_n < \infty.$$

Hence, by using Lemma 1.2, we know that $\sigma_n \to 0 (n \to \infty)$ i.e., $x_n \to x^* (n \to \infty)$.

Necessity. Suppose that $\{x_n\}$ converges strongly to the unique solution x^* of the equation $Tx = f$. Then it follows from the continuity of T that $\{Sx_n\}$ converges strongly to $Sx^* = x^*$. Thus, it is readily seen that

$$\|Tx_n - f\| = \|Sx_n - x_n\| \leq \|Sx_n - x^*\| + \|x_n - x^*\| \to 0 (n \to \infty).$$

The proof is for this theorem complete. ■

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