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Construction of Real Band Anti-Symmetric Matrices from Spectral Data

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Abstract. In this paper, we describe how to construct a real anti-symmetric $(2p-1)$-band matrix with prescribed eigenvalues in its p leading principal submatrices. This is done in two steps. First, an anti-symmetric matrix B is constructed with the specified spectral data but not necessary a band matrix. Then B is transformed by Householder transformations to a $(2p-1)$-band matrix with the prescribed eigenvalues. An algorithm is presented. Numerical results are presented to demonstrate that the proposed method is effective.

Key words: anti-symmetric; eigenvalues; inverse problem.

AMS subject classifications: 65F10, 15A09

1 Introduction

This work deals with inverse eigenvalue problems for real banded anti-symmetric matrices. The solution of inverse eigenvalue problems is currently attracting a great interest due to their importance in many applications. In particular, real banded matrices play an important role in areas as applied mechanics [1,2], structure design [3], circuit theory and inverse Sturm-Liouville problem [4].

Let $p, n \in N, 0 < p \leq n$ and $\{\lambda_j^{(k)}\}_{j=1}^{k} (k = n-p+1, \cdots, n)$ be a set of real numbers with

$$\lambda_j^{(k)} = -\lambda_{k-j+1}^{(k)}, \ j = 1, \cdots, k; k = n-p+1, \cdots, n. \tag{1}$$

$$\lambda_j^{(k)} \leq \lambda_j^{(k-1)} \leq \lambda_{j+1}^{(k)}, \ j = 1, \cdots, k-1; k = n-p+2, \cdots, n. \tag{2}$$

The problem is to determine a real anti-symmetric $n \times n$ matrix A with eigenvalues $\{\lambda_j^{(k)} i\}_{j=1}^{k}$ $(i^2 = -1)$ in the leading $k \times k$ principal submatrix of $A(k = n-p+1, \cdots, n)$ and $a_{st} = 0$ for $|s-t| \geq p$. In this paper a matrix A is called real anti-symmetric if $A \in R^{n\times n}, A^T = -A$. A similar problem with symmetric matrices has been studied in many papers, (see [5–10]). For anti-symmetric matrices, the case $p = 2$ has been studied by He Chengcai [11], but the complex numbers were used there, so that the computation is rather complicated .

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In Section 2 the eigen-properties of real anti-symmetric matrices were studied. In Section 3 an anti-symmetric matrix B is constructed where B has the specified spectral data, but it is not necessary a banded matrix. In Section 4 B is transformed to a $(2p-1)$-band matrix with the prescribed spectra. In Section 5, an algorithm is presented with numerical examples which show that the method is effective.

2 Some properties of real anti-symmetric matrices

In order to prove our main results, let us first investigate the eigen- properties of real anti-symmetric matrices. Some of them are well known so the proof is omitted.

Let A be a real anti-symmetric $n \times n$ matrix, i.e. $A \in R^{n\times n}, A^T = -A$. Then $-iA \in C^{n\times n}, (-iA)^H = iA^T = -iA$, hence $-iA$ is Hermitian and its eigenvalues are real. Let $\{\lambda_j^{(k)}\}_{j=1}^k$ be the eigenvalues of the $k \times k$ leading principal submatrix of $-iA (k = 1, \cdots, n)$ satisfying

$$\lambda_1^{(k)} \le \lambda_2^{(k)} \le \cdots \le \lambda_k^{(k)}. \tag{3}$$

According to Cauchy interlacing theorem, we have

$$\lambda_j^{(k)} \le \lambda_j^{(k-1)} \le \lambda_{j+1}^{(k)},\ j = 1, \cdots, k-1; k = 2, \cdots, n. \tag{4}$$

Noting that $A = i(-iA)$, we assert that $\{\lambda_j^{(k)} i\}_{j=1}^k$ be the eigenvalues of $k\times k$ leading submatrix of A and (4) hold. Furthermore, because $\{\lambda_j^{(k)} i\}_{j=1}^k$ are roots of a polynomial with real coefficients, so

$$\lambda_j^{(k)} = -\lambda_{k-j+1}^{(k)},\ j = 1, \cdots, k; k = 1, \cdots, n. \tag{5}$$

Lemma 2.1. *The eigenvalues of real anti-symmetric $n\times n$ matrix A are either zeroes or conjugate imaginaries. Let $\{\lambda_j^{(k)} i\}_{j=1}^k$ be the eigenvalues of the $k \times k$ leading submatrices of A satisfying* (3)*, then* (4) *and* (5) *hold.*

Lemma 2.2. *[12, 2.5.14] $A \in R^{n\times n}$ is anti-symmetric if and only if there exist an orthogonal matrix $U \in R^{n\times n}$ such that*

$$T = U^T A U = \begin{bmatrix} 0 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & 0 & & & & & & & \\ & & & 0 & \beta_1 & & & & & \\ & & & -\beta_1 & 0 & & & & & \\ & & & & & 0 & \beta_2 & & & \\ & & & & & -\beta_2 & 0 & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & 0 & \beta_r \\ & & & & & & & & -\beta_r & 0 \end{bmatrix} \tag{6}$$

and $\pm\beta_1 i, \cdots, \pm\beta_r i$ are all non-real eigenvalues of A. In this paper, T is referred to as the normal canonical form of A if $\beta_1 \le \beta_2 \le \cdots \le \beta_r$ in (6).

Remark 2.1. The orthogonal matrix U can be chosen as follow: the first $n-2r$ columns are the orthonormal eigenvectors corresponding to zero eigenvalues of A, the remaining columns are the orthonormal imagine part and real part of the eigenvectors corresponding to eigenvalues $\beta_1 i, \cdots, \beta_r i$ respectively. The orthonormalization is needed when there are multiple eigenvalues.

3 Construction of real anti-symmetric matrices from spectral data

One of the main results, Theorem 3.1, will be given in this section, whose proof depends on several lemmas.

Theorem 3.1. *Let $\{\lambda_j^{(k)}\}_{j=1}^{k}(k = n-p+1, \cdots, n, 0 < p \le n)$ be a set of real numbers satisfying (1), (2), then there exists a real anti-symmetric $n \times n$ matrix B with eigenvalues $\{\lambda_j^{(k)} i\}_{j=1}^{k}$ in the $k \times k$ leading principal submatrix of $B(k = n-p+1, \cdots, n)$.*

3.1 Some lemmas

To prove theorem 3.1, we need following three lemmas.

Lemma 3.1. *Let*

$$\mu_1 < a_1 < \mu_2 < \cdots \mu_k < a_k < 0, \tag{7}$$

then there exist $b_1, \cdots, b_k \in R$ such that

$$T_{2k+1} = \begin{bmatrix} 0 & a_1 & & & & & & b_1 \\ -a_1 & 0 & & & & & & 0 \\ & & 0 & a_2 & & & & b_2 \\ & & -a_2 & 0 & & & & 0 \\ & & & & \ddots & & & \vdots \\ & & & & & 0 & a_k & b_k \\ & & & & & -a_k & 0 & 0 \\ -b_1 & 0 & -b_2 & 0 & \cdots & -b_k & 0 & 0 \end{bmatrix} \tag{8}$$

has eigenvalues $\pm\mu_1 i, \cdots, \pm\mu_k i, 0$.

Proof Let

$$b_l = \left(\frac{\prod_{j=1}^{k}(\mu_j^2 - a_l^2)}{\prod_{t \ne l}(a_t^2 - a_l^2)} \right)^{\frac{1}{2}}, \quad l = 1, \cdots, k. \tag{9}$$

It follows from (7) that $b_l \in R$. Direct calculation gives

$$p(\lambda) = \det(\lambda I - T_{2k+1}) = \lambda \left[\prod_{j=1}^{k}(\lambda^2 + a_j^2) + \sum_{j=1}^{k} b_j^2 \prod_{t \ne j}(\lambda^2 + a_t^2) \right] = \lambda q(\lambda),$$

where

$$q(\lambda) = \prod_{j=1}^{k}(\lambda^2 + a_j^2) + \sum_{j=1}^{k} b_j^2 \prod_{t \ne j}(\lambda^2 + a_t^2).$$

Let $g(\lambda) = \prod_{j=1}^{k}(\lambda^2 + \mu_j^2)$, then both $q(\lambda)$ and $g(\lambda)$ are monic polynomials of degree $2k$, while by (9),

$$q(\pm a_l i) = 0 + b_l^2 \prod_{t \ne l}(a_t^2 - a_l^2) = \prod_{j=1}^{k}(\mu_j^2 - a_l^2) = g(\pm a_l i), \qquad l = 1, \cdots, k.$$

So $q(\lambda) \equiv g(\lambda)$ and therefore $p(\lambda) = \lambda \prod_{j=1}^{k}(\lambda^2 + \mu_j^2)$ which means that T_{2k+1} has eigenvalues $\pm\mu_1 i, \cdots, \pm\mu_k i, 0$. ■

Remark 3.1. For the purpose of the later use in constructing of a real anti-symmetric banded matrix numerically, we need to find an orthogonal matrix U so that $U^T T_{2k+1} U$ is the normal canonical form of T_{2k+1}. Note that the eigenvector corresponding to eigenvalue $\mu_j i$ can be taken as

$$\xi_j = \left(\frac{-b_1\mu_j i}{\mu_j^2 - a_1^2}, \frac{a_1 b_1}{\mu_j^2 - a_1^2}, \cdots, \frac{-b_k\mu_j i}{\mu_j^2 - a_k^2}, \frac{a_k b_k}{\mu_j^2 - a_k^2}, 1 \right)^T = v_j + w_j i,$$

where $v_j, w_j \in R^{2k+1}$ are defined by

$$v_j = \left(0, \frac{a_1 b_1}{\mu_j^2 - a_1^2}, \cdots, 0, \frac{a_k b_k}{\mu_j^2 - a_k^2}, 1\right)^T, \quad w_j = \left(\frac{-b_1\mu_j}{\mu_j^2 - a_1^2}, 0, \cdots, 0, \frac{-b_k\mu_j}{\mu_j^2 - a_k^2}, 0, 0\right)^T. \tag{10}$$

The eigenvector corresponding to the zero eigenvalue can be taken as

$$\xi_0 = \left(0, -\frac{b_1}{a_1}, \cdots, 0, -\frac{b_k}{a_k}, 1\right)^T. \tag{11}$$

By (9), it can be verified that $\xi_0, w_1, v_1, \cdots, w_k, v_k$ are orthogonal vectors. They can be taken as the columns of the matrix U after being normalized.

Lemma 3.2. *Let*

$$\mu_1 < a_1 < \mu_2 < \cdots < a_{k-1} < \mu_k < 0. \tag{12}$$

Then there exist $b_0, b_1, \cdots, b_k \in R$ *such that*

$$T_{2k} = \begin{bmatrix} 0 & & & & & & & & b_0 \\ & 0 & a_1 & & & & & & b_1 \\ & -a_1 & 0 & & & & & & 0 \\ & & & 0 & a_2 & & & & b_2 \\ & & & -a_2 & 0 & & & & 0 \\ & & & & & \ddots & & & \vdots \\ & & & & & & 0 & a_{k-1} & b_{k-1} \\ & & & & & & -a_{k-1} & 0 & 0 \\ -b_0 & -b_1 & 0 & -b_2 & 0 & \cdots & -b_{k-1} & 0 & 0 \end{bmatrix} \tag{13}$$

has eigenvalues $\pm\mu_1 i, \cdots, \pm\mu_k i$.

Proof Let

$$b_0 = \prod_{j=1}^{k} \mu_j \Bigg/ \prod_{j=1}^{k-1} a_j , \tag{14}$$

$$b_l = \left(-\prod_{j=1}^{k}(\mu_j^2 - a_l^2) \Bigg/ a_l^2 \prod_{t\neq l}(a_t^2 - a_l^2) \right)^{\frac{1}{2}}, \qquad l = 1, \cdots, k-1. \tag{15}$$

Then $b_0, b_1, \cdots, b_k \in R$ because of (12). It is easy to verified that

$$p(\lambda) = det(\lambda I - T_{2k}) = (\lambda^2 + b_0^2)\prod_{j=1}^{k-1}(\lambda^2 + a_j^2) + \lambda^2 \sum_{j=1}^{k-1} b_j^2 \prod_{t\neq j}(\lambda^2 + a_t^2).$$

Noticing that $g(\lambda) \equiv p(\lambda) - \prod_{j=1}^{k}(\lambda^2 + \mu_j^2)$ is a polynomial of degree not greater than $2k-2$, while by (14) and (15),

$$g(0) = p(0) - \prod_{j=1}^{k} \mu_j^2 = b_0^2 \prod_{j=1}^{k-1} a_j^2 - \prod_{j=1}^{k} \mu_j^2 = 0,$$
$$g(\pm a_l i) = 0 - a_l^2 b_l^2 \prod_{t \neq l}(a_t^2 - a_l^2) - \prod_{j=1}^{k}(\mu_j^2 - a_l^2) = 0. \quad l = 1, \cdots, k-1.$$

So $g(\lambda) = 0$ and therefore $p(\lambda) = \prod_{j=1}^{k}(\lambda^2 + \mu_j^2)$ which implies that T_{2k} has eigenvalues $\pm\mu_1 i, \cdots,$ $\pm\mu_k i$. ■

Lemma 3.3. *Let $\{a_j\}_{j=1}^{n-1}$, $\{\mu_j\}_{j=1}^{n}$ be two sets of real numbers with*

$$a_j = -a_{n-j}, \; j = 1, \cdots, n-1. \tag{16}$$
$$\mu_j = -\mu_{n-j+1}, \; j = 1, \cdots, n. \tag{17}$$
$$\mu_1 \le a_1 \le \mu_2 \le \cdots \le a_{n-1} \le \mu_n. \tag{18}$$

Let

$$T_{n-1} = \begin{bmatrix} 0 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & 0 & & & & & & & \\ & & & 0 & a_1 & & & & & \\ & & & -a_1 & 0 & & & & & \\ & & & & & 0 & a_2 & & & \\ & & & & & -a_2 & 0 & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & 0 & a_r \\ & & & & & & & & -a_r & 0 \end{bmatrix} \tag{19}$$

be the normal canonical form of a real anti-symmetric $(n-1) \times (n-1)$ matrix with eigenvalues $\{a_j i\}_{j=1}^{n-1}$. Then there exists $c \in R^{n-1}$ such that matrix

$$T_n = \begin{bmatrix} T_{n-1} & c \\ -c^T & 0 \end{bmatrix} \tag{20}$$

has eigenvalues $\{\mu_j i\}_{j=1}^{n}$.

Proof Lemma 3.1 and Lemma 3.2 guarantee the existence of $c \in R^{n-1}$ when the strict inequalities hold in (18) and n is odd or even respectively. If there are some equalities in (18), we may take some a_is and μ_is out so that the remainder of (18) satisfies strict inequalities and (16)-(17) still hold. For example, if

$$\mu_1 < a_1 = \mu_2 = \cdots = \mu_s < a_s \le \cdots \le \mu_n,$$

then because of (17), (18), we have

$$\mu_1 < a_1 = \mu_2 = \cdots = \mu_s < a_s \le \cdots \le a_{n-s} < \mu_{n-s+1} = \cdots = \mu_{n-1} = a_{n-1} < \mu_n.$$

In this case, we may take $a_1 = \mu_2 = \cdots = \mu_s$ and $\mu_{n-s+1} = \cdots = \mu_{n-1} = a_{n-1}$ out, remaining

$$\mu_1 < a_s \le \cdots \le a_{n-s} < \mu_n.$$

For another example, if

$$\mu_1 < a_1 = \mu_2 = \cdots = a_s < \mu_{s+1} \le \cdots \le \mu_n,$$

then (18) must be

$$\mu_1 < a_1 = \mu_2 = \cdots = a_s < \mu_{s+1} \le \cdots \le \mu_{n-s} < a_{n-s} = \cdots = \mu_{n-1} = a_{n-1} < \mu_n.$$

In this case, we may take $a_1 = \mu_2 = \cdots = \mu_s$ and $\mu_{n-s+1} = \cdots = \mu_{n-1} = a_{n-1}$ out, remaining

$$\mu_1 < a_s < \mu_{s+1} \le \cdots \le \mu_{n-s} < a_{n-s} < \mu_n.$$

Repeat the above process till the remainder satisfies strict inequalities, denote by

$$\hat{\mu}_1 < \hat{a}_1 < \hat{\mu}_2 < \cdots < \hat{a}_r < 0 < -\hat{a}_r < \cdots < -\hat{\mu}_2 < -\hat{a}_1 < -\hat{\mu}_1, \tag{21}$$

where, without loss of generality, suppose n is odd. According to Lemma 3.1, there exist $\hat{b}_1, \cdots, \hat{b}_r \in R$ such that

$$T_{2r+1} = \begin{bmatrix} 0 & \hat{a}_1 & & & & & & \hat{b}_1 \\ -\hat{a}_1 & 0 & & & & & & 0 \\ & & 0 & \hat{a}_2 & & & & \hat{b}_2 \\ & & -\hat{a}_2 & 0 & & & & 0 \\ & & & & \ddots & & & \vdots \\ & & & & & 0 & \hat{a}_r & \hat{b}_r \\ & & & & & -\hat{a}_r & 0 & 0 \\ -\hat{b}_1 & 0 & -\hat{b}_2 & 0 & \cdots & -\hat{b}_r & 0 & 0 \end{bmatrix} \tag{22}$$

has eigenvalues $\pm\hat{\mu}_1 i, \cdots, \pm\hat{\mu}_r i, 0$.

Now, denote the numbers that have been taken out from (18) by $\pm\overline{a}_1,\cdots, \pm\overline{a}_s,0,\cdots,0$. It is obvious that there exists a permutation matrix P such that

$$P^T T_{n-1} P = \begin{bmatrix} 0 & & & & & & & & & & & & \\ & \ddots & & & & & & & & & & & \\ & & 0 & & & & & & & & & & \\ & & & 0 & \overline{a}_1 & & & & & & & & \\ & & & -\overline{a}_1 & 0 & & & & & & & & \\ & & & & & \ddots & & & & & & & \\ & & & & & & 0 & \overline{a}_s & & & & & \\ & & & & & & -\overline{a}_s & 0 & & & & & \\ & & & & & & & & 0 & \hat{a}_1 & & & \\ & & & & & & & & -\hat{a}_1 & 0 & & & \\ & & & & & & & & & & \ddots & & \\ & & & & & & & & & & & 0 & \hat{a}_r \\ & & & & & & & & & & & -\hat{a}_r & 0 \end{bmatrix}. \tag{23}$$

On the other hand, from (22) we see that

$$\hat{T}_n = \begin{bmatrix} 0 & & & & & & & & & & & & & 0 \\ & \ddots & & & & & & & & & & & & \vdots \\ & & 0 & & & & & & & & & & & \vdots \\ & & & 0 & \overline{a}_1 & & & & & & & & & \vdots \\ & & & -\overline{a}_1 & 0 & & & & & & & & & \vdots \\ & & & & & \ddots & & & & & & & & \vdots \\ & & & & & & 0 & \overline{a}_s & & & & & & \vdots \\ & & & & & & -\overline{a}_s & 0 & & & & & & 0 \\ & & & & & & & & 0 & \hat{a}_1 & & & & \hat{b}_1 \\ & & & & & & & & -\hat{a}_1 & 0 & & & & 0 \\ & & & & & & & & & & \ddots & & & \vdots \\ & & & & & & & & & & & 0 & \hat{a}_r & \hat{b}_r \\ & & & & & & & & & & & -\hat{a}_r & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -\hat{b}_1 & 0 & \cdots & -\hat{b}_r & 0 & 0 \end{bmatrix} \tag{24}$$

has eigenvalues $0,\cdots,0$, $\pm\overline{a}_1 i,\cdots$, $\pm\overline{a}_s i,\pm\hat{\mu}_1 i,\cdots,\pm\hat{\mu}_r i,0$, which are actually $\{\mu_j i\}_{j=1}^n$ by the definition of $\overline{a}_1,\cdots,\overline{a}_s,\hat{\mu}_1,\cdots,\hat{\mu}_r$.

Now let

$$c = P\left[0,\cdots,0,\hat{b}_1,0,\cdots,\hat{b}_r,0\right]^T, \tag{25}$$

then $c \in R^{n-1}$ and

$$T_n = \begin{bmatrix} T_{n-1} & c \\ -c^T & 0 \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} P^T T_{n-1} P & P^T c \\ -c^T P & 0 \end{bmatrix}\begin{bmatrix} P^T & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix}\hat{T}_n\begin{bmatrix} P^T & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, T_n has eigenvalues $\{\mu_j i\}_{j=1}^n$ as $\hat{T}_n$. ■

3.2 The proof of Theorem 3.1

Let

$$B^{(n-p+1)} = \begin{bmatrix} 0 & & & & & & & \\ & \ddots & & & & & & \\ & & 0 & & & & & \\ & & & 0 & \lambda_1^{(n-p+1)} & & & \\ & & & -\lambda_1^{(n-p+1)} & 0 & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & \lambda_{r_{n-p+1}}^{(n-p+1)} \\ & & & & & & -\lambda_{r_{n-p+1}}^{(n-p+1)} & 0 \end{bmatrix} \tag{26}$$

be the normal canonical form of a real anti-symmetric $(n-p+1)\times(n-p+1)$ matrix having eigenvalues $\{\lambda_j^{(n-p+1)} i\}_{j=1}^{n-p+1}$ with $\lambda_1^{(n-p+1)} \le \lambda_2^{(n-p+1)} \le \cdots \le \lambda_{n-p+1}^{(n-p+1)}$. We shall construct a sequence of matrices $B^{(n-p+1)},\cdots,B^{(n)} = B$ by embedding a last row and column to preceding matrix, step by step. We now describe how to construct $B^{(m+1)}$ from $B^{(m)}$. Suppose that $B^{(m)}$ be real anti-symmetric matrix with its leading $k\times k$ principal submatrix having eigenvalues

$\{\lambda_j^{(k)}\}_{j=1}^k$ where $\lambda_1^{(k)} \leq \lambda_2^{(k)} \leq \cdots \leq \lambda_k^{(k)}$,$(k = n-p+1, \cdots, m)$. By Lemma 2.2, there exists an unitary matrix U_m such that

$$U_m B^{(m)} U_m^T = T^{(m)} = \begin{bmatrix} 0 & & & & & & & \\ & \ddots & & & & & & \\ & & 0 & & & & & \\ & & & 0 & \lambda_1^{(m)} & & & \\ & & & -\lambda_1^{(m)} & 0 & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & \lambda_{r_m}^{(m)} \\ & & & & & & -\lambda_{r_m}^{(m)} & 0 \end{bmatrix}, \tag{27}$$

where $T^{(m)}$ is normal canonical form of $B^{(m)}$. By Lemma 3.3, there exists $c^{(m)} \in R^m$ so that

$$\overline{B}^{(m+1)} = \begin{bmatrix} T^{(m)} & c^{(m)} \\ -c^{(m)^T} & 0 \end{bmatrix} \tag{28}$$

with eigenvalues $\{\lambda_j^{(m+1)} i\}_{j=1}^{m+1}$. Let

$$\overline{U}_{m+1} = \begin{bmatrix} U_m & 0 \\ 0 & 1 \end{bmatrix}. \tag{29}$$

Then

$$\begin{aligned} B^{(m+1)} &= \overline{U}_{m+1}^T \overline{B}^{(m+1)} \overline{U}_{m+1} = \begin{bmatrix} U_m^T T^{(m)} U_m & U_m^T c^{(m)} \\ -c^{(m)^T} U_m & 0 \end{bmatrix} \\ &= \begin{bmatrix} B^{(m)} & U_m^T c^{(m)} \\ -c^{(m)^T} U_m & 0 \end{bmatrix} \end{aligned} \tag{30}$$

is a real anti-symmetric $(m+1) \times (m+1)$ matrix with eigenvalues $\{\lambda_j^{(k)} i\}_{j=1}^k$ in the leading $k \times k$ submatrix of $B^{(m+1)}(k = n-p+1, \cdots, m, m+1)$. This process for $m = n-p+1, \cdots, n-1$ gives us a matrix B which satisfies all conditions in Theorem 3.1. This completes the proof of Theorem 3.1.

4 Construction of banded anti-symmetric matrices from spectral data

Theorem 4.1. *Let $\{\lambda_j^{(k)}\}_{j=1}^k (k = n-p+1, \cdots, n. 0 < p \leq n)$ be a set of real numbers satisfying (1), (2), then there exists a real anti-symmetric $n \times n$ matrix A with eigenvalues $\{\lambda_j^{(k)} i\}_{j=1}^k$ in the leading $k \times k$ submatrix of $A(k = n-p+1, \cdots, n)$ and $a_{st} = 0$ for $|s-t| \geq p$.*

Proof By Theorem 3.1, there exists a real anti-symmetric $n \times n$ matrix B with eigenvalues $\{\lambda_j^{(k)} i\}_{j=1}^k$ in the leading $k \times k$ submatrix of $B(k = n-p+1, \cdots, n)$. But B is not necessary a banded matrix. In order to transform B into $(2p-1)$-diagonal form we begin to zero the elements outside the band in the nth column(row) and continue with the $(n-1)$th, $(n-2)$th,$\cdots$, $(p+1)$th column(row), using Householder transformations. Working backward in this way, we do not destroy anti-symmetry and the eigenvalues of the p leading submatrices. We construct similar matrices $B = A_0, A_1, \cdots, A_{n-p} = A$, where the last j columns and rows of A_j are of $(2p-1)$-diagonal form. To be specific, for $j = 0, 1, \cdots, n-p-1$, let

$$\overline{a}_{n-j} = \begin{bmatrix} \overline{a}_{1,n-j} \\ \vdots \\ \overline{a}_{n-j-p+1,n-j} \end{bmatrix}$$

be the upper part of the $(n-j)$th column of A_j. Let $\overline{H}_j \in R^{n-j-p+1}$ be Householder matrix such that

$$\overline{H}_j \overline{a}_{n-j} = r e_{n-j-p+1},$$

where $r \in R$ and $e_{n-j-p+1} \in R^{n-j-p+1}$ the $(n-j-p+1)$th unit vector. Now let

$$H_j = \begin{bmatrix} \overline{H}_j & 0 \\ 0 & I_{j+p-1} \end{bmatrix}. \tag{31}$$

Then the $(n-j)$th column and row of

$$A_{j+1} = H_j A_j H_j^T \tag{32}$$

are of $(2p-1)$-diagonal form. This transformation reserves the $(2p-1)$-diagonal form of the last j columns and rows of A_j. It reserves the eigenvalues of the p greatest leading submatrices of A_j either. As consequence, the matrix $A = A_{n-p}$ has $(2p-1)$-diagonal form and same eigenvalues in the p greatest leading submatrices as those in B.

5 Numerical methods and examples

The process of the proof of Theorems 3.1 and 4.1 provide us with an algorithm to construct the required matrix as follows:

Algorithm 1 This algorithm construct a real anti-symmetric matrix from given spectrum data.

Step 1 Compute $B^{(n-p+1)}$ by (26).
Set $U^{(n-p+1)} = I_{n-p+1}$.

Step 2 For $m = n-p+2, \cdots, n$ do Step 3-5.

- **Step 3** Compute $c^{(m)}$ by (9), (25) when m is odd, or by (14),(15),(25) when m is even.
- **Step 4** Compute $B^{(m)}$ by (30).
- **Step 5** If $m < n$, compute $U^{(m+1)}$ by (10), (11).

Step 6 For $j = 0, 1, \cdots, n-p-1$ do Step 7-8.

Step 7 Compute A_{j+1} by (31), (32).

Step 8 Set $A_0 = B$.

Step 9 Output $A = A_{n-p}$.

Using the above algorithm for the construction of real anti-symmetric matrix from given spectrum data, we give some examples here to illustrate that the results obtained in this paper are correct. Numerical experiments have been performed implementing a MATLAB routine on an PC.

Example 1 ($p = 2, n = 7$) Given $\{\lambda_j^{(7)}\}_{j=1}^{7}$=$\{-7, -5, -3, 0, 3, 5, 7\}$ and $\{\lambda_j^{(6)}\}_{j=1}^{6} = \{-6, -4, -2, 2, 4, 6\}$. The computed real anti-symmetric tri-diagonal matrix is given below:

$$\begin{bmatrix} 0.000000 & 3.285052 & & & & & \\ -3.285052 & 0.000000 & -2.389232 & & & & \\ & 2.389232 & 0.000000 & -3.772709 & & & \\ & & 3.772709 & 0.000000 & -3.204164 & & \\ & & & 3.204164 & 0.000000 & -3.872983 & \\ & & & & 3.872983 & 0.000000 & -5.196152 \\ & & & & & 5.196152 & 0.000000 \end{bmatrix}.$$

Example 2 ($p = 2, n = 7$) Given $\{\lambda_j^{(7)}\}_{j=1}^{7}$=$\{-5, -5, -2, 0, 2, 5, 5\}$ and $\{\lambda_j^{(6)}\}_{j=1}^{6} = \{-5, -3, -1, 1, 3, 5\}$. This time (1), (2) hold with equality. The desired matrix is computed as follows:

$$\begin{bmatrix} 0.000000 & -5.000000 & & & & & \\ 5.000000 & 0.000000 & 0.000000 & & & & \\ & 0.000000 & 0.000000 & -1.314257 & & & \\ & & 1.314257 & 0.000000 & -1.749915 & & \\ & & & 1.749915 & 0.000000 & -2.282658 & \\ & & & & 2.282658 & 0.000000 & -4.358899 \\ & & & & & 4.358899 & 0.000000 \end{bmatrix}.$$

Example 3 ($p = 3, n = 8$) Given $\{\lambda_j^{(8)}\}_{j=1}^{8}$=$\{-7.5, -5.5, -3.5, -1.5, 1.5, 3.5, 5.5, 7.5\}$, $\{\lambda_j^{(7)}\}_{j=1}^{7}$ = $\{-7, -5, -3, 0, 3, 5, 7\}$ and $\{\lambda_j^{(6)}\}_{j=1}^{6}$=$\{-6, -4, -2, 2, 4, 6\}$. Because A is anti-symmetric penta-diagonal, we only list two upper sub-diagonal entries in Table 1. The eigenvalues of tree greatest leading principal submatrices of A are computed and list in the table to compare with the given data. The results are rather satisfying.

Table 1: Example 3: two upper sub-diagonal entries of matrix A.

j	$a_{j,j+1}$	$a_{j,j+2}$	computed $\lambda_j^{(8)}$	computed $\lambda_j^{(7)}$	computed $\lambda_j^{(6)}$
1	- 3.569251945735	+2.448044172744	+7.500000000000i	+7.000000000000i	+5.999999999999i
2	- 1.697317755236	- 1.070337761180	- 7.500000000000i	- 7.000000000000i	- 5.999999999999i
3	- 3.132965850439	- 3.376274833182	+5.500000000000i	+4.999999999999i	+4.000000000000i
4	+2.269296078105	- 1.633815326846	- 5.500000000000i	- 4.999999999999i	- 4.000000000000i
5	+2.051157187872	- 3.000355819338	+3.500000000000i	+3.000000000000i	+2.000000000000i
6	+4.242389062469	- 4.136546918404	- 3.500000000000i	- 3.000000000000i	- 2.000000000000i
7	- 0.942857142857		+1.500000000000i	+0.000000000000	
8			- 1.500000000000i		

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