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QUINTIC SPLINE SOLUTIONS OF FOURTH ORDER BOUNDARY-VALUE PROBLEMS

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Dedicated to the memory of Dr. M. Rafique

Abstract. In this paper quintic spline is used for the numerical solutions of the fourth order linear special case boundary value problems. End conditions for the definition of spline are derived, consistent with the fourth order boundary value problem. The algorithm developed approximates the solutions, and their higher order derivatives. It has also been proved that the method is a second order convergent. Numerical illustrations are tabulated to demonstrate the practical usefulness of method.

Key Words. Quintic spline; Boundary value problem; End conditions.

1. Introduction

Spline functions are used in many areas such as interpolation, data fitting, numerical solution of ordinary and partial differential equations. Spline functions are also used in curve and surface designing.
Usmani [7], considered the fourth order boundary value problem to be the problem of bending a rectangular clamped beam of length l resting on an elastic foundation. The vertical deflection w of the beam satisfies the system

$$\left[L \ + \ \left(\frac{K}{D}\right)\right] w = D^{-1}q(x), \text{ where } \ L \equiv \frac{d^4}{dx^4}\ , \tag{1. 1}$$

$$w(0) = w(l) = w'(0) = w'(l) = 0, \tag{1. 2}$$

where D is the flexural rigidity of the beam, and k is the spring constant of the elastic foundation and the load $q(x)$ acts vertically downwards per unit length of the beam. Mathematically, the system (1.1) belongs to a general class of boundary value problems of the form

$$\left.\begin{array}{c} \left(\frac{d^4}{dx^4} \ + \ f(x)\right) y(x) \ = \ g(x), \ x \ \in \ [a,b], \\ y(a) = \alpha_0, \ y(b) \ = \ \alpha_1, \\ y'(a) = \gamma_0, \ y'(b) \ = \ \gamma_1, \end{array}\right\} \tag{1. 3}$$

where $\alpha_i, \ \gamma_i$; i= 0,1 are finite real constants and the functions $f(x)$ and $g(x)$ are continuous on [a,b]. The analytic solution of (1.3) for special choices of $f(x)$ and $g(x)$ are easily obtained, but for arbitrary choices, the analytic solution cannot be determined.

Numerical methods for obtaining an approximation to $y(x)$ are introduced. Usmani [7] derived numerical techniques of order 2, 4 and 6 for solution of a fourth

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order linear boundary value problem. Usmani [8] derived cubic, quartic, quintic and sextic spline solution of nonlinear boundary value problems. Usmani and Sakai [9] developed a quartic spline for the approximation of the solution of third order linear (special case) two point boundary value problems involving third order linear differential equation.

Papamichael and Worsey [4] derived end conditions for cubic spline interpolation at equally spaced knots. Papamichael and Worsey [5] have developed a cubic spline method, similar to that proposed by Daniel and Swartz [3] for second order problems.

Siddiqi and Twizell [10-13] presented the solutions of 6, 8, 10 and 12 order linear boundary value problems, using the sixth, eighth, tenth and twelfth degree splines.

In this paper a quintic spline method is described for the solution of (1.3). The end conditions for quintic spline interpolation, at equally spaced knots are derived, which is discussed in the next section.

2. Quintic Spline

Let Q be a quintic spline defined on $[a,\ b]$ with equally spaced knots

$$x_i = a\ +\ ih, \quad i = 0, 1, 2, ..., k, \tag{2. 4}$$

where

$$h = \frac{b-a}{k}. \tag{2. 5}$$

Moreover, for $i = 0, 1, 2, \ldots,\ k$, taking

$$Q(x_i) = y_i\ ,\ Q^{(1)}(x_i) = m_i, \tag{2. 6}$$

$$Q^{(2)}(x_i) = M_i\ ,\ Q^{(3)}(x_i) = n_i \tag{2. 7}$$

and

$$Q^{(4)}(x_i) = N_i. \tag{2. 8}$$

Also, let $y(x)$ be the exact solution of the system (1.3) and y_i be an approximation to $y(x_i)$, obtained by the quintic spline $Q(x_i)$. It may be noted that the $Q_i(x),\ i = 1, 2, 3, \ldots, k$ can be defined on the interval $[x_{i-1}, x_i]$, integrating

$$Q_i^4(x) = \frac{1}{h}[N_{i-1}(x_i - x) + N_i(x - x_{i-1})], \tag{2. 9}$$

four times w.r.t. x, gives

$$Q_i(x) = \frac{1}{120h}[N_{i-1}(x_i - x)^5\ +\ N_i(x - x_{i-1})^5] + \frac{Ax^3}{6} + \frac{Bx^2}{2} + Cx\ +\ D. \tag{2. 10}$$

To calculate the constants of integrations, the following conditions are used:

$$\begin{aligned} Q_i(x_i) = y_i\ , &\quad Q^{(2)}{}_i(x_i) = M_i, \\ Q_i(x_{i-1}) = y_{i-1}\ , &\quad Q^{(2)}{}_i(x_{i-1}) = M_{i-1}. \end{aligned} \tag{2. 11}$$

The identities of quintic splines for the solution of (1.3) can be written as

$$\begin{aligned} & m_{i-2}\ +\ 26m_{i-1}\ +\ 66m_i\ +\ 26m_{i+1}\ +\ m_{i+2} \\ = \ & \frac{5}{h}\left[-y_{i-2} - 10y_{i-1} + 10y_{i+1} + y_{i+2}\right]; \\ & i = 2, 3, ..., k-2, \end{aligned} \tag{2. 12}$$

$$(2.\ 13)\qquad \begin{aligned} &M_{i-2}+26M_{i-1}+66M_i+26M_{i+1}+M_{i+2}\\ =\ &\frac{20}{h^2}[y_{i-2}+2y_{i-1}-6y_i+2y_{i+1}+y_{i+2}]; \end{aligned}$$

$$(2.\ 14)\qquad \begin{aligned} &n_{i-2}\ +\ 26n_{i-1}\ +\ 66n_i\ +\ 26n_{i+1}\ +\ n_{i+2}\\ =\ &\frac{60}{h^3}[-y_{i-2}+2y_{i-1}-2y_{i+1}+y_{i+2}]; \end{aligned}$$

$$(2.\ 15)\qquad \begin{aligned} &N_{i-2}\ +\ 26N_{i-1}\ +\ 66N_i\ +\ 26N_{i+1}\ +\ N_{i+2}\\ =\ &\frac{120}{h^4}[y_{i-2}-4y_{i-1}+6y_i-4y_{i+1}+y_{i+2}]; \end{aligned}$$

for $i=2,3,...,k-2$,

$$(2.\ 16)\qquad N_{i-1}\ +\ 4N_i\ +\ N_{i+1}-\frac{6}{h^2}[M_{i-1}-2M_i+M_{i+1}]=0;$$

$$(2.\ 17)\qquad \begin{aligned} &60h\{m_{i-1}\ +\ 2m_i\ +\ m_{i+1}\}-h^3\{3n_{i-1}\ +\ 14n_i\ +\ 3n_{i+1}\}\\ &\qquad =120\{-y_{i-1}+y_{i+1}\}; \end{aligned}$$

$$(2.\ 18)\qquad \begin{aligned} &8h\{m_{i+1}\ -\ m_{i-1}\}-h^2\{M_{i-1}\ -\ 6M_i\ +\ M_{i+1}\}\\ =\ &20\{y_{i-1}-2y_i+y_{i+1}\}; \end{aligned}$$

for $i=1,2,3,...,k-1$,

$$(2.\ 19)\qquad \begin{aligned} m_i\ =\ &\frac{h}{6}\{2M_i\ +\ M_{i-1}\}\ -\frac{h^3}{360}\{8N_i\ +\ 7N_{i-1}\}\ +\ \frac{1}{h}\{y_i-y_{i-1}\};\\ &i=1,2,3,...,k, \end{aligned}$$

$$(2.\ 20)\qquad \begin{aligned} m_i\ =\ &-\frac{h}{6}\{2M_i\ +\ M_{i+1}\}+\frac{h^3}{360}8N_i\ +\ 7N_{i+1}\ +\ \frac{1}{h}\{y_{i+1}-y_i\};\\ &i=0,1,,...,k-1, \end{aligned}$$

$$(2.\ 21)\qquad \begin{aligned} m_i\ =\ &-\frac{h^2}{120}\{n_{i-1}\ +\ 18n_i\ +\ n_{i+1}\}+\frac{1}{2h}\{y_{i+1}-y_{i-1}\};\\ &i=1,2,3,...,k-1, \end{aligned}$$

$$(2.\ 22)\qquad \begin{aligned} M_i\ =\ &-\frac{h^2}{120}\{N_{i-1}\ +\ 8N_i\ +\ N_{i+1}\}+\frac{1}{h^2}\{y_{i-1}-2y_i+y_{i+1}\};\\ &i=1,2,3,...,k-1, \end{aligned}$$

$$(2.\ 23)\qquad \begin{aligned} M_i\ =\ &\frac{1}{32h}\{m_{i-2}\ +\ 32m_{i-1}\ -\ 32m_{i+1}\ -\ m_{i+2}\}\\ &+\frac{5}{32h^2}\{y_{i-2}+16y_{i-1}-34y_i+16y_{i+1}+y_{i+2}\};\\ &i=2,3,...,k-2 \end{aligned}$$

and

$$(2.\ 24)\qquad \begin{aligned} N_i\ =\ &-\frac{3}{2h^2}\{M_{i-1}+18M_i\ +\ M_{i+1}\}+\frac{30}{h^4}\{y_{i-1}-2y_i+y_{i+1}\};\\ &i=1,2,3,...,k-1. \end{aligned}$$

The relations (2.12)–(2.24) can be derived from the results of Albasiny and Hoskins [1] and Ahlberg, Nilson and Walsh [2] discussed by Papamichael and Behforooz [6]. The uniqueness of Q can be established showing that any of the four $(k+1)\times(k+1)$ linear systems, obtained, using one of the relations (2.12), (2.13), (2.14) and (2.15) together with the four end conditions of Q, is non singular. In this paper the linear system corresponding to (2.15) has been chosen and has a unique solution

for $N_i s,\ i = 0, 1, ..., k$.
Equation (2.16) and (2.22) give the parameters $M_i s,\ i = 0, 1, ..., k$.
Consider the system (2.15)

$$\begin{aligned} & N_{i-2} \ + \ 26N_{i-1} \ + \ 66N_i \ + \ 26N_{i+1} \ + \ N_{i+2} \\ = \ & \frac{120}{h^4}\ [y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2}]; \\ & i = 2, 3, ..., k-2, \end{aligned} \tag{2. 25}$$

where

$$N_i = -f_i y_i + g_i, \qquad i = 0, 1, \ldots, k. \tag{2. 26}$$

The above system gives $(k-3)$ linear algebraic equations in the $(k-1)$ unknowns $(y_i,\ i = 1, 2, ..., k-1)$. Two more equations are needed to have complete solution of $y_i s$, which are derived in section 3.

It may be noted that *Papamichael and Worsey [5] needed two consistency systems for the solution of boundary value problem (1.3) but the method developed in this paper, needs only one such system.*

Papamichael and Behforooz [6] proved the following lemma to determine the end conditions in terms of first derivative of interpolatory quintic spline.

Lemma 1

Let $\lambda_i \ = \ m_i \ - \ y_i^{(1)}$. If $y \in C^7[a, b]$ then

$$\lambda_{i-2} + 26\lambda_{i-1} + 66\lambda_i + 26\lambda_{i+1} + \lambda_{i+2} = \beta_i\ ,\ i = 2, 3, ..., k-2, \tag{2. 27}$$

where

$$|\beta_i| \ \leq \ \frac{11}{21}\ h^6\ \|y^{(7)}\|, \quad i = 2, 3, ..., k-2.$$

Following the above lemma along with equation (2.15) and with Taylor series expansion about the point x_i, the following lemma can easily be proved, to determine the end conditions for the solution of boundary value problem (1.3).

Lemma 2

Let

$$\lambda_i = N_i - y_i^{(4)}\ , i = 0, 1, \ldots, k, \tag{2. 28}$$

then for $y \in C^6[a, b]$,

$$\lambda_{i-2} + 26\lambda_{i-1} + 66\lambda_i + 26\lambda_{i+1} + \lambda_{i+2} = \beta_i,\ i = 2, 3, ..., k-2, \tag{2. 29}$$

while

$$|\beta_i| \ \leq \ \frac{158}{3}\ h^2\ \|y^{(6)}\|, \quad i = 2, 3, ..., k-2.$$

Finally the required end conditions are derived in the following section.

3. End Conditions

Consider the end conditions of the form

$$N_0 \ + \ \alpha N_1 \ + \ \beta N_2 \ + \ \gamma N_3 \ + \ N_4 = \frac{1}{h^4}\left[\ \sum_{i=0}^{3} a_i y_i \ + \ bhy_0^{(1)} \ + \ h^4 c y_0^{(4)}\right] \tag{3. 30}$$

and

(3. 31)
$$N_k + \alpha N_{k-1} + \beta N_{k-2} + \gamma N_{k-3} + N_{k-4} = \frac{1}{h^4}\left[\sum_{i=0}^{3} a_i y_{k-i} - bhy_k^{(1)} + h^4 c y_k^{(4)}\right].$$

Recalling that $y \in C^6[a,b]$ and

$$\lambda_i = N_i - y_i^{(4)}, \quad i = 0, 1, ..., k,$$

the equations (3.30), (2.15) and (3.31) give

(3. 32)
$$\lambda_0 + \alpha\lambda_1 + \beta\lambda_2 + \gamma\lambda_3 + \lambda_4 = \beta_1,$$

(3. 33)
$$\lambda_{i-2} + 26\lambda_{i-1} + 66\lambda_i + 26\lambda_{i+1} + \lambda_{i+2} = \beta_i; \quad i = 2, 3, ..., k-2,$$

(3. 34)
$$\lambda_k + \alpha\lambda_{k-1} + \beta\lambda_{k-2} + \gamma\lambda_{k-3} + \lambda_{k-4} = \beta_{k-1},$$

where

(3. 35)
$$\beta_1 = \frac{1}{h^4}\left[\sum_{i=0}^{3} a_i y_i + bhy_0^{(1)} + h^4 c y_0^{(4)} - h^4\left(y_0^{(4)} + \alpha y_1^{(4)} + \beta y_2^{(4)} + \gamma y_3^{(4)} + y_4^{(4)}\right)\right]$$

and

(3. 36)
$$\beta_{k-1} = \frac{1}{h^4}\left[\sum_{i=0}^{3} a_i y_{k-i} - bhy_k^{(1)} + h^4 c y_k^{(4)} - h^4\left(y_k^{(4)} + \alpha y_{k-1}^{(4)} + \beta y_{k-2}^{(4)} + \gamma y_{k-3}^{(4)} + y_{k-4}^{(4)}\right)\right].$$

Moreover, from lemma 2

(3. 37)
$$|\beta_i| \le \frac{158}{3}\, h^2\, \|y^{(6)}\|, \quad i = 2, 3, ..., k-2.$$

The scalars b, c and a_i, $i = 0, 1, 2, 3$ are determined in terms of α, β and γ, such that β_1 and β_{k-1} are bounded by $O(h^2)$. Moreover, α, β and γ are chosen such that the system (3.32)–(3.34) has unique solution.

Following Papamichael and Behforooz [6], the required end conditions may be written as

(3. 38)
$$N_0 + N_4 = \frac{1}{h^4}\left[\frac{-220}{9}y_0 + 40y_1 - 20y_2 + \frac{40}{9}y_3 - \frac{40}{3}hy_0^{(1)} - \frac{4}{3}h^4 y_0^{(4)}\right]$$

and

(3. 39)
$$N_k + N_{k-4} = \frac{1}{h^4}\left[\frac{-220}{9}y_k + 40\,y_{k-1} - 20\,y_{k-2} + \frac{40}{9}y_{k-3} + \frac{40}{3}hy_k^{(1)} - \frac{4}{3}h^4 y_k^{(4)}\right].$$

The quintic spline solution of the system (1.3) is defined in the next section.

4. Quintic Spline Solution

Since the interpolatory quintic spline Q alongwith end conditions (3.38) and (3.39) satisfies

$$\|Q^{(r)} - y^{(r)}\| = O(h^{6-r}), \quad r = 0, 1, ..., 5,$$

therefore, it follows from (1.3) that

$$\left.\begin{array}{ll} N_i + f_i y_i = g_i + O(h^2); & i = 0, 1, \ldots, k, \\ y_0 = \alpha_0, & y_k = \alpha_1, \\ m_0 = \gamma_0, & m_k = \gamma_1 \end{array}\right\} \tag{4. 40}$$

The system (4.40) leads naturally to the method of collocation, where a quintic spline $\tilde{Q}$ approximating the solution of (1.3) is attained from (4.40) by simply dropping the $O(h^2)$ terms. That is, the quintic spline $\tilde{Q}$ is defined by the following $k + 5$ linear equations

$$\left.\begin{array}{ll} \tilde{N}_i + f_i \tilde{y}_i = g_i; & i = 0, 1, \ldots, k, \\ \tilde{y_0} = \alpha_0, & \tilde{y_k} = \alpha_1, \\ \tilde{m_0} = \gamma_0, & \tilde{m_k} = \gamma_1 \end{array}\right\} \tag{4. 41}$$

The quintic spline solution Q is based on the linear equations (3.38), (2.15) and (3.39) together with (4.40). The linear system obtained can, thus, be written in matrix form as

$$(\mathbf{A} + h^4 \mathbf{BF})\mathbf{Y} = \mathbf{C} + \mathbf{e}, \tag{4. 42}$$

where

$$e_i = O(h^6), \quad i = 1, 2, \ldots, k-1.$$

The parameters $\tilde{y}_i$ of the approximating spline $\tilde{Q}$ satisfy the linear system

$$(\mathbf{A} + h^4 \mathbf{BF})\tilde{Y} = \mathbf{C}, \tag{4. 43}$$

where $\tilde{Y} = [\tilde{y_1}, \tilde{y_2}, \ldots, \tilde{y_{k-1}}]^T$ and $\mathbf{C} = [c_1, c_2, \ldots, c_{k-1}]^T$.
Also $A = (a_{m,n})$, $m, n = 1, 2, \ldots, k-1$,
where $a_{1,1} = a_{k-1,k-1} = 9$, $a_{1,2} = a_{k-1,k-2} = -9/2$; otherwise,

$$a_{m,n} = \begin{cases} 6, & m = n, \\ -4, & |m-n| = 1, \\ 1, & |m-n| = 2, \\ 0, & |m-n| > 2, \end{cases} \tag{4. 44}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & \frac{9}{40} & & & & & \\ \frac{13}{60} & \frac{11}{20} & \frac{13}{60} & \frac{1}{120} & & & & & \\ \frac{1}{120} & \frac{13}{60} & \frac{11}{20} & \frac{13}{60} & \frac{1}{120} & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & \frac{1}{120} & \frac{13}{60} & \frac{11}{20} & \frac{13}{60} & \frac{1}{120} \\ & & & & \frac{1}{120} & \frac{13}{60} & \frac{11}{20} & \frac{13}{60} \\ & & & & \frac{9}{40} & 0 & 0 & 0 \end{bmatrix}$$

and
$\mathbf{F} = diag(f_i); \quad i = 1, 2, \ldots, k-1.$
Moreover,

$$c_1 = \frac{11}{2}\tilde{y_0} + 3h\tilde{m}_0 + h^4\left(\frac{21}{40}(f_0\tilde{y_0} - g_0) + \frac{9}{40}g_4\right), \tag{4. 45}$$

$$c_2 = \frac{h^4}{120}(g_0 - f_0\tilde{y_0} + 26g_1 + 66g_2 + 26g_3 + g_4) - \tilde{y_0}, \tag{4. 46}$$

$$\begin{aligned} c_i &= \frac{h^4}{120}(g_{i-2} + 26g_{i-1} + 66g_i + 26g_{i+1} + g_{i+2}); \\ &\quad i = 3, 4, \ldots, k-3, \end{aligned} \tag{4. 47}$$

$$\begin{aligned} c_{k-2} &= \frac{h^4}{120}(g_k - f_k\tilde{y_k} + 26g_{k-1} + 66g_{k-2} + 26g_{k-3} \\ &\quad + g_{k-4}) - \tilde{y_k} \end{aligned} \tag{4. 48}$$

and

$$c_{k-1} = \frac{11}{2}\tilde{y_k} - 3h\tilde{m}_k + h^4\left(\frac{21}{40}(f_k\tilde{y_k} - g_k) + \frac{9}{40}g_{k-4}\right). \tag{4. 49}$$

Extending the method for the solution of sixth, eighth and higher order boundary value problems, is in process.

Convergence analysis of the method is discussed in the next section.

5. Convergence Analysis

Using eq.(4.42) and eq.(4.43), it can be written as

$$(\mathbf{A} + h^4\mathbf{BF})(\mathbf{Y}\text{-}\tilde{Y}) = \mathbf{e}.$$

That is

$$\mathbf{A}(\mathbf{I} + h^4A^{-1}\mathbf{BF})(\mathbf{Y}\text{-}\tilde{Y}) = \mathbf{e}. \tag{5. 50}$$

To determine the bound on $\|\mathbf{Y}\text{-}\tilde{Y}\|$, the following lemma is needed [7].

Lemma 5.1
If G is a matrix of order N and $\|G\| < 1$, then $(I+G)^{-1}$ exists and $\|(I+G)^{-1}\| < \frac{1}{1-\|G\|}$.

Equation (5.50) can be expressed as

$$(\mathbf{Y}\text{-}\tilde{Y}) = A^{-1}(\mathbf{I} + h^4A^{-1}\mathbf{BF})^{-1}\mathbf{e} \tag{5. 51}$$

Using lemma 5.1,

$$\|\mathbf{Y}\text{-}\tilde{Y}\|_\infty \leq \frac{\|A^{-1}\|_\infty\|\mathbf{e}\|_\infty}{1 - h^4\|A^{-1}\|_\infty\|B\|_\infty\|F\|_\infty}, \tag{5. 52}$$

provided

$$h^4\|A^{-1}\|_\infty\|B\|_\infty\|F\|_\infty < 1. \tag{5. 53}$$

Now

$$\|B\|_\infty = 1, \tag{5. 54}$$

$$\|F\|_\infty \leq \|f\| = \max_{x\in[a,b]}|f(x)|. \tag{5. 55}$$

From Usmani[7],

$$\|A^{-1}\|_\infty \leq \frac{1}{384h^4}(b-a)^4\left(1+\frac{8h^3}{(b-a)^3}\right), \tag{5. 56}$$

so

$$h^4\|A^{-1}\|_\infty\|B\|_\infty\|F\|_\infty \leq h^4\frac{1}{384h^4}(b-a)^4\left(1+\frac{8h^3}{(b-a)^3}\right)\|f\|, \tag{5. 57}$$

which shows that eq.(5.53) holds only if,

$$\|f\| < \frac{384}{(b-a)^4+8h^3(b-a)}. \tag{5. 58}$$

Considering the restriction (5.58) on $\|f\|$, $\|\mathbf{Y}\text{-}\tilde{Y}\|_\infty$ can be determined as

$$\begin{aligned}\|\mathbf{Y}\text{-}\tilde{Y}\|_\infty &\leq \frac{1}{1-\frac{(b-a)^4}{384}\left(1+\frac{8h^3}{(b-a)^3}\right)\|f\|}\frac{1}{384h^4}(b-a)^4\left(1+\frac{8h^3}{(b-a)^3}\right)h^6 \\ &= (1-\|f\|G)^{-1}\, Gh^2, \end{aligned} \tag{5. 59}$$

where

$$G = \frac{(b-a)^4}{384}\left(1+\frac{8h^3}{(b-a)^3}\right) \tag{5. 60}$$

which shows that the method developed for the solution of fourth order boundary value problem (1.3), is second order convergent.
To implement the method for the quintic spline solution of the boundary value problem (1.3), three examples are discussed in the following section.

6. Numerical Examples

Example 1

Consider the following boundary value problem

$$\left.\begin{aligned} &y^{(iv)} + 4y = 1, \qquad -1 \leq x \leq 1, \\ &y(-1) = y(1) = 0, \quad y'(-1) = -y'(1) = \tfrac{\sinh(2)-\sin(2)}{4(\cosh(2)+\cos(2))} \end{aligned}\right\} \tag{6. 61}$$

The analytic solution of the given problem is

$$\begin{aligned} y(x) &= 0.25[1-2[\sin(1)\ \sinh(1)\ \sin(x)\ \sinh(x) \\ &\quad +\cos(1)\ \cosh(1)\ \cos(x)\ \cosh(x)]/(\cos(2)+\cosh(2))] . \end{aligned}$$

The observed maximum errors (in absolute values) associated with $y_i^{(\mu)}$, $\mu = 0,1,2,3,4,$ for the system (6.61) are summarized in Table 1, which confirms the method to be second order convergent.

Example 2

Consider the following boundary value problem

$$\left.\begin{aligned} &y^{(iv)} + xy = -(8+7x+x^3)e^x, \qquad 0 \leq x \leq 1, \\ &y(0) = y(1) = 0, \qquad y'(0) = 1,\ y'(1) = -e \end{aligned}\right\} \tag{6. 62}$$

The analytic solution of the above differential system is

$$y(x) = x(1-x)\ e^x .$$

TABLE 1. Maximum absolute errors for problem (6.61) in $y_i^{(\mu)}$, $\mu = 0, 1, 2, 3, 4$.

h	y_i	$y_i^{(1)}$	$y_i^{(2)}$	$y_i^{(3)}$	$y_i^{(4)}$
$\frac{1}{8}$	6.35×10^{-5}	9.42×10^{-5}	4.06×10^{-4}	1.00×10^{-3}	2.54×10^{-4}
$\frac{1}{16}$	1.33×10^{-5}	2.01×10^{-5}	9.61×10^{-5}	2.41×10^{-4}	5.33×10^{-5}
$\frac{1}{32}$	3.17×10^{-6}	4.85×10^{-6}	2.35×10^{-5}	5.96×10^{-5}	1.27×10^{-5}
$\frac{1}{64}$	7.84×10^{-7}	1.18×10^{-6}	5.85×10^{-6}	1.47×10^{-5}	3.21×10^{-6}
$\frac{1}{128}$	1.95×10^{-7}	2.76×10^{-7}	1.45×10^{-6}	3.60×10^{-6}	8.60×10^{-7}
$\frac{1}{256}$	4.14×10^{-8}	4.06×10^{-8}	3.10×10^{-7}	6.86×10^{-7}	2.45×10^{-7}
$\frac{1}{512}$	1.09×10^{-7}	1.94×10^{-7}	8.10×10^{-7}	2.12×10^{-6}	3.59×10^{-7}

The observed maximum errors (in absolute values) associated with $y_i^{(\mu)}$, $\mu = 0, 1, 2, 3, 4$, for the system (6.62) are summarized in Table 2.

TABLE 2. Maximum absolute errors for problem (6.62) in $y_i^{(\mu)}$, $\mu = 0, 1, 2, 3, 4$.

h	y_i	$y_i^{(1)}$	$y_i^{(2)}$	$y_i^{(3)}$	$y_i^{(4)}$
$\frac{1}{8}$	3.18×10^{-4}	1.00×10^{-3}	4.9×10^{-3}	4.12×10^{-2}	1.91×10^{-4}
$\frac{1}{16}$	4.71×10^{-5}	1.49×10^{-4}	1.5×10^{-3}	1.09×10^{-2}	2.71×10^{-5}
$\frac{1}{32}$	1.08×10^{-5}	3.46×10^{-5}	3.94×10^{-4}	2.7×10^{-3}	6.15×10^{-6}
$\frac{1}{64}$	2.68×10^{-6}	8.60×10^{-6}	9.91×10^{-5}	6.84×10^{-4}	1.52×10^{-6}
$\frac{1}{128}$	6.71×10^{-7}	2.15×10^{-6}	2.47×10^{-5}	1.71×10^{-4}	3.81×10^{-7}
$\frac{1}{256}$	1.67×10^{-7}	5.38×10^{-7}	6.19×10^{-6}	4.27×10^{-5}	9.53×10^{-8}
$\frac{1}{512}$	3.60×10^{-8}	1.15×10^{-7}	1.35×10^{-6}	9.64×10^{-6}	2.04×10^{-8}

Example 3

Consider the differential system
(6. 63)

$$\left.\begin{array}{ll} y^{(iv)} - y = -4(2x\ cos(x) + 3\ sin(x)), & -1 \le x \le 1, \\ y(-1) = y(1) = 0, & y^{'}(-1) = y^{'}(1) = 2\ sin(1) \end{array}\right\}$$

The analytic solution of the above system is

$$y(x) = (x^2 - 1)\ sin(x)\ .$$

The observed maximum errors (in absolute values) associated with $y_i^{(\mu)}$, $\mu = 0, 1, 2, 3, 4$, for the system (6.63) are summarized in Table 3.

TABLE 3. Maximum absolute errors for problem (6.63) in $y_i^{(\mu)}$, $\mu = 0, 1, 2, 3, 4$.

h	y_i	$y_i^{(1)}$	$y_i^{(2)}$	$y_i^{(3)}$	$y_i^{(4)}$
$\frac{1}{8}$	1.93×10^{-4}	6.01×10^{-4}	2.3×10^{-3}	1.8×10^{-2}	1.93×10^{-4}
$\frac{1}{16}$	3.40×10^{-5}	1.14×10^{-4}	7.90×10^{-4}	4.7×10^{-3}	3.40×10^{-5}
$\frac{1}{32}$	7.83×10^{-6}	2.72×10^{-5}	2.05×10^{-4}	1.2×10^{-3}	7.83×10^{-6}
$\frac{1}{64}$	1.92×10^{-6}	6.76×10^{-6}	5.16×10^{-5}	2.96×10^{-4}	1.92×10^{-6}
$\frac{1}{128}$	4.79×10^{-7}	1.68×10^{-6}	1.29×10^{-5}	7.42×10^{-5}	4.79×10^{-7}
$\frac{1}{256}$	1.19×10^{-7}	4.22×10^{-7}	3.23×10^{-6}	1.85×10^{-5}	1.19×10^{-7}
$\frac{1}{512}$	3.33×10^{-8}	1.02×10^{-7}	8.32×10^{-7}	4.62×10^{-6}	3.33×10^{-8}

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