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MULTILEVEL AUGMENTATION METHODS FOR SOLVING OPERATOR EQUATIONS*

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Abstract *We introduce multilevel augmentation methods for solving operator equations based on direct sum decompositions of the range space of the operator and the solution space of the operator equation and a matrix splitting scheme. We establish a general setting for the analysis of these methods, showing that the methods yield approximate solutions of the same convergence order as the best approximation from the subspace. These augmentation methods allow us to develop fast, accurate and stable nonconventional numerical algorithms for solving operator equations. In particular, for second kind equations, special splitting techniques are proposed to develop such algorithms. These algorithms are then applied to solve the linear systems resulting from matrix compression schemes using wavelet-like functions for solving Fredholm integral equations of the second kind. For this special case, a complete analysis for computational complexity and convergence order is presented. Numerical examples are included to demonstrate the efficiency and accuracy of the methods. In these examples we use the proposed augmentation method to solve large scale linear systems resulting from the recently developed wavelet Galerkin methods and fast collocation methods applied to integral equations of the second kind. Our numerical results confirm that this augmentation method is particularly efficient for solving large scale linear systems induced from wavelet compression schemes.*

Key words *Multilevel augmentation methods, operator equations, Fredholm integral equations of the second kind.*

AMS(2000)subject classifications 65J10, 65R20

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1 Introduction

Developing stable, efficient and fast numerical algorithms for solving operator equations including differential equations and integral equations is a main focus of research in numerical analysis and scientific computation. The development of such algorithms is particularly important for large scale computation. Solving an operator equation normally requires three steps of processing. The first step, at the level of approximation theory, is to choose appropriate subspaces and their suitable bases. The second step is to discretize the operator equations using these bases and to analyze convergence properties of the approximate solutions. This step of processing which results in a discrete linear system is a main task considered at the level of numerical solutions of operator equations. The third step, at the level of numerical linear algebra, is to design an efficient solver for the discrete linear system resulting from the second step. The ultimate goal is to efficiently solve the discrete linear system which gives an accurate approximate solution of the original operator equation. Theoretical consideration and practical implementation in the numerical solution of operator equations show that these three steps of processing are closely related. A good way of designing efficient algorithms for the discrete linear system should be taken into consideration choices of subspaces and their bases, methodologies of discretization of the operator equations and numerical solvers of the resulting discrete linear system. The well-known multigrid method (cf., [16, 17]) and the related two-grid method [3,4] are excellent examples of such algorithms.

A good algorithm for solving operator equations should be convenient for implementing adaptivity. When a computed approximate solution is confirmed to be not accurate enough, a local or global subdivision is often made aiming at a solution at a finer level. An efficient algorithm should be able to update the old computed approximate solution obtaining a new, more accurate approximate solution, at an additional expense proportional to the net gain in accuracy, avoiding solving the whole equation at the finer level. In other words, the additional computational costs to obtain the additional accuracy from a (local or global) subdivision should be proportional to the dimension of the difference space between the coarse level and the finer level, not at the expense in the order of the dimension of the finer level space. Presently, to our best knowledge, existing numerical methods in the literature are not yet able to meet this need. The research reported in this paper is an attempt to accomplish this goal.

We introduce a multilevel augmentation method for solving operator equations based on multilevel decompositions of the approximate subspaces, aiming at efficiently solving linear systems of large scale obtained from discretization of the operator equations. For this purpose, we require that both the range space of the operator and the solution space of the operator equation have direct sum decompositions. Our method consists of two steps. In the first step, we use the direct sum decompositions to discretize the operator equation and result in a linear system. Reflecting the direct sum decompositions of the subspaces, the coefficient matrix of the linear system has a special structure. Specifically, the matrix corresponding to a finer level of approximate spaces is obtained by augmenting the matrix corresponding to a coarser level with submatrices that correspond to the difference spaces between the spaces of the finer level and the coarser level. The second step is to split the matrix into a sum of two matrices, with one reflecting its lower frequency and the other reflecting its higher frequency. We are required to choose the splitting in a way that the inverse of the lower frequency matrix either has an explicit form or can be easily computed with a lower computational cost.

The general setting of the multilevel augmentation method presented in this paper is a new development closely related to the multilevel iteration method introduced in [9] (see, also [15]), where the idea of Gauss-Seidel is used to define a specific splitting of the matrix. In this paper we first establish an abstract setting of a general matrix splitting applicable to both first kind and second kind operator equations. We then specialize a splitting in the case of second kind equations (different from the Gauss-Seidel approach), which yields an efficient fast algorithm, aiming at overcoming the limitation of the multilevel iteration method. The limitation is that at every step we move from a coarse level to a finer level we are required to solve a linear system with a coefficient matrix of size related to the coarse level. The computational effort builds up when we repeat the iteration. This limitation is removed by the multilevel augmentation method introduced in this paper. Specifically, for equations of the second kind, we propose a matrix splitting scheme with which we need only to solve the linear system of a small scale corresponding to an initial coarse level k, when we move from a coarse level (say, $k+m$ for any positive integer m) to a finer level (say, $k+m+1$). Therefore, this method is particularly suitable in the context of adaptive solutions of operator equations in conjunction with an adaptive strategy based on a multiresolution decomposition.

Our multilevel augmentation method can be roughly described as follows. For a fixed integer k, we first solve a linear system of level k and obtain the k-th level approximate solution. Suppose that an adaptive strategy is available to guide the local or global subdivision which generates a finer level space. We then update this solution from the coarse level to the finer level by augmenting the coefficient matrix of the coarse level to one of the finer level with submatrices that reflect the difference between the coarse level and the finer level. This process is repeated until a satisfactory solution is obtained. Because of the special splitting scheme, at each level we need only to solve a linear system with the same coefficient matrix of the linear system of the initial level k but with a different right hand side vector. Noting that the computational cost for solving this linear system related to level k is constant, our method provides an efficient fast algorithm for adaptively solving operator equations.

A multiresolution analysis may provide a convenient decomposition for this development. However, the decomposition upon which this method is developed is not restricted to multiresolution analysis. To illustrate this point, we use both the wavelet Galerkin method which is based on a multiresolution analysis and the fast collocation method which is based on wavelet-like functions and collocation functionals (not exactly a multiresolution analysis) to demonstrate the efficiency of the method.

We organize this paper in five sections. Section 2 is set to describe a general setting of the multilevel augmentation method for solving operator equations including equations of the first kind and second kind. This setting covers both differential equations and integral equations. In terms of methodologies used in discretization of the equations, it is applicable to operator approximation methods including Galerkin methods, collocation methods and wavelet compression methods. We provide a general sufficient condition on the matrix splitting which ensures that the corresponding multilevel augmentation method yields an approximate solution having the same order of convergence as that of the best approximation from the subspace. Section 3 is devoted to the development of special multilevel augmentation methods for second kind equations. Using the special structure of second kind equations, we propose a particular matrix splitting scheme which results in a fast efficient algorithm for solving second kind equations. Stability analysis of the multilevel augmentation method for second kind equations is presented to show that such a method is stable. We estimate the computational complexity of this method. This augmentation method is particularly useful when it is applied to solve a linear system with a compressed coefficient matrix. We provide in Section 4 a complete analysis for the computational complexity and convergence order for this special case. In Section 5, we use the wavelet compressed Galerkin and collocation methods developed in [20] and [11] respectively as examples to illustrate the performance of the augmentation method. Numerical examples show that this method is indeed very efficient and accurate.

2 Multilevel Augmentation Methods

In this section, we describe a general setting of the multilevel augmentation method for solving operator equations. This method is based on a standard approximate method at a coarse level and updates the resulting approximate solutions by adding details corresponding to higher levels in a direct sum decomposition. We prove that this method provides the same order of convergence as the original approximate method.

We begin with a description of the general setup for the operator equations under consideration. Let $\mathbb{X}$ and $\mathbb{Y}$ be two Banach spaces, and $\mathcal{A} : \mathbb{X} \to \mathbb{Y}$ be a bounded linear operator. For a function $f \in \mathbb{Y}$, we consider the operator equation

$$\mathcal{A}u = f, \tag{2.1}$$

where $u \in \mathbb{X}$ is the solution to be determined. We assume that equation (2.1) has a unique solution in $\mathbb{X}$. To solve the equation, we choose two sequences of finite dimensional subspaces $\mathbb{X}_n, n \in \mathbb{N}_0 := \{0, 1, \ldots\}$ and $\mathbb{Y}_n, n \in \mathbb{N}_0$ of $\mathbb{X}$ and $\mathbb{Y}$, respectively, such that

$$\overline{\bigcup_{n\in\mathbb{N}_0} \mathbb{X}_n} = \mathbb{X}, \qquad \overline{\bigcup_{n\in\mathbb{N}_0} \mathbb{Y}_n} = \mathbb{Y},$$

and

$$\dim \mathbb{X}_n = \dim \mathbb{Y}_n, \quad n \in \mathbb{N}_0.$$

We suppose that equation (2.1) has an approximate operator equation

$$\mathcal{A}_n u_n = f_n, \tag{2.2}$$

where $\mathcal{A}_n : \mathbb{X}_n \to \mathbb{Y}_n$ is an approximate operator of $\mathcal{A}$, $u_n \in \mathbb{X}_n$ and $f_n \in \mathbb{Y}_n$ is an approximation of f. Examples of such equations include projection methods such as Galerkin methods and collocation methods. In particular, for solving integral equations they also include approximate operator equations obtained from quadrature methods and degenerate kernel methods. Wavelet compression schemes using both orthogonal projection (Galerkin methods) and interpolation projection (collocation methods) are also examples of this type.

Our method is based on an additional hypothesis that the subspaces are nested, i.e.,

$$\mathbb{X}_n \subset \mathbb{X}_{n+1}, \quad \mathbb{Y}_n \subset \mathbb{Y}_{n+1}, \quad n \in \mathbb{N}_0 \tag{2.3}$$

so that we can define two subspaces $\mathbb{W}_{n+1} \subset \mathbb{X}_{n+1}$ and $\mathbb{Z}_{n+1} \subset \mathbb{Y}_{n+1}$ such that $\mathbb{X}_{n+1}$ becomes a direct sum of $\mathbb{X}_n$ and $\mathbb{W}_{n+1}$ and likewise, $\mathbb{Y}_{n+1}$ is a direct sum of $\mathbb{Y}_n$ and $\mathbb{Z}_{n+1}$. Specifically, we assume that two direct sums $\oplus_1$ and $\oplus_2$ are defined so that we have the decompositions

$$\mathbb{X}_{n+1} = \mathbb{X}_n \oplus_1 \mathbb{W}_{n+1}, \quad \text{and} \quad \mathbb{Y}_{n+1} = \mathbb{Y}_n \oplus_2 \mathbb{Z}_{n+1}, \quad n \in \mathbb{N}_0. \tag{2.4}$$

In practice, the finer level subspaces $\mathbb{X}_{n+1}$ and $\mathbb{Y}_{n+1}$ are obtained respectively from the coarse level subspaces $\mathbb{X}_n$ and $\mathbb{Y}_n$ by local or global subdivisions. It follows from (2.4) for a fixed $k \in \mathbb{N}_0$ and any $m \in \mathbb{N}_0$ that

$$\mathbb{X}_{k+m} = \mathbb{X}_k \oplus_1 \mathbb{W}_{k+1} \oplus_1 \cdots \oplus_1 \mathbb{W}_{k+m}, \tag{2.5}$$

and

$$\mathbb{Y}_{k+m} = \mathbb{Y}_k \oplus_2 \mathbb{Z}_{k+1} \oplus_2 \cdots \oplus_2 \mathbb{Z}_{k+m}. \tag{2.6}$$

As in [9], for $g_0 \in \mathbb{X}_k$ and $g_i \in \mathbb{W}_{k+i}$, $i = 1, 2, \ldots, m$, we identify the vector $[g_0, g_1, \ldots, g_m]^T$ in $\mathbb{X}_k \times \mathbb{W}_{k+1} \times \cdots \times \mathbb{W}_{k+m}$ with the sum $g_0 + g_1 + \cdots + g_m$ in $\mathbb{X}_k \oplus_1 \mathbb{W}_{k+1} \oplus_1 \cdots \oplus_1 \mathbb{W}_{k+m}$. Similarly, for $g_0 \in \mathbb{Y}_k$ and $g_i \in \mathbb{Z}_{k+i}$, for $i = 1, 2, \ldots, m$, we also identify the vector $[g_0, g_1, \ldots, g_m]^T$ in $\mathbb{Y}_k \times \mathbb{Z}_{k+1} \times \cdots \times \mathbb{Z}_{k+m}$ with the sum $g_0 + g_1 + \cdots + g_m$ in $\mathbb{Y}_k \oplus_2 \mathbb{Z}_{k+1} \oplus_2 \cdots \oplus_2 \mathbb{Z}_{k+m}$. In this notation, we describe the multilevel method for solving equation (2.2) with $n := k + m$ which has the form

$$\mathcal{A}_{k+m} u_{k+m} = f_{k+m}. \tag{2.7}$$

According to decomposition (2.5), we write the solution $u_{k+m} \in \mathbb{X}_{k+m}$ as

$$u_{k+m} = u_{k,0} + \sum_{i=1}^{m} v_{k,i}, \tag{2.8}$$

where $u_{k,0} \in \mathbb{X}_k$ and $v_{k,i} \in \mathbb{W}_{k+i}$ for $i = 1, 2, \ldots, m$. Hence, u_{k+m} is identified as $u_k(m) := [u_{k,0}, v_{k,1}, \ldots, v_{k,m}]^T$. We will use both of these notations exchangeably.

We let $\mathcal{F}_{k,k+j} : \mathbb{W}_{k+j} \to \mathbb{Y}_k$, $\mathcal{G}_{k+i,k} : \mathbb{X}_k \to \mathbb{Z}_{k+i}$ and $\mathcal{H}_{k+i,k+j} : \mathbb{W}_{k+j} \to \mathbb{Z}_{k+i}$, $i, j = 1, 2, \ldots, m$, be given and assume that the operator $\mathcal{A}_{k+m}$ is identified as the matrix of operators

$$\mathcal{A}_{k,m} := \begin{bmatrix} \mathcal{A}_k & \mathcal{F}_{k,k+1} & \cdots & \mathcal{F}_{k,k+m} \\ \mathcal{G}_{k+1,k} & \mathcal{H}_{k+1,k+1} & \cdots & \mathcal{H}_{k+1,k+m} \\ \vdots & \vdots & & \vdots \\ \mathcal{G}_{k+m,k} & \mathcal{H}_{k+m,k+1} & \cdots & \mathcal{H}_{k+m,k+m} \end{bmatrix}. \tag{2.9}$$

Equation (2.7) is now equivalent to the equation

$$\mathcal{A}_{k,m} u_k(m) = f_{k+m}. \tag{2.10}$$

We remark that the nestedness of subspaces implies that the matrix $\mathcal{A}_{k,m}$ contains $\mathcal{A}_{k,m-1}$ as a submatrix. In other words, $\mathcal{A}_{k,m}$ is obtained by augmenting the matrix of the previous level $\mathcal{A}_{k,m-1}$.

With this setup, one can design various iteration schemes to solve equation (2.10) by splitting the matrix $\mathcal{A}_{k,m}$ defined by (2.9) into a sum of two matrices and applying matrix iteration algorithms to $\mathcal{A}_{k,m}$. A multilevel iteration method based on the Gauss-Seidel iteration was developed in [9]. In the present paper, we will first generalize the result in [9] and then specify a way of splitting the matrix so that the new algorithms obtained in this way improve the computational efficiency of the algorithm developed in [9].

For this purpose, we split operator $\mathcal{A}_{k,m}$ as the sum of two operators $\mathcal{B}_{k,m}, \mathcal{C}_{k,m} : \mathbb{X}_{k+m} \to \mathbb{Y}_{k+m}$, that is, $\mathcal{A}_{k,m} = \mathcal{B}_{k,m} + \mathcal{C}_{k,m}$, $m \in \mathbb{N}_0$. Note that matrices $\mathcal{B}_{k,m}$ and $\mathcal{C}_{k,m}$ are obtained from augmenting matrices $\mathcal{B}_{k,m-1}$ and $\mathcal{C}_{k,m-1}$, respectively. Hence, equation (2.10) becomes

$$\mathcal{B}_{k,m} u_k(m) = f_{k+m} - \mathcal{C}_{k,m} u_k(m), \quad m \in \mathbb{N}_0. \tag{2.11}$$

Instead of solving (2.11) directly, we solve (2.11) approximately by using the multilevel augmentation algorithm described below.

Algorithm 1 (Operator Form of the Augmentation Algorithm) Let $k > 0$ be a fixed integer.

Step 1 Solve equation (2.2) with $n := k$ for $u_k \in \mathbb{X}_k$ exactly.

Step 2 Set $u_{k,0} := u_k$ and compute the splitting matrices $\mathcal{B}_{k,0}$ and $\mathcal{C}_{k,0}$.

Step 3 For $m \in \mathbb{N}$, suppose that $u_{k,m-1} \in \mathbb{X}_{k+m-1}$ has been obtained and do the following.

· Augment the matrices $\mathcal{B}_{k,m-1}$ and $\mathcal{C}_{k,m-1}$ to form $\mathcal{B}_{k,m}$ and $\mathcal{C}_{k,m}$, respectively.

· Augment $u_{k,m-1}$ by setting $\overline{u}_{k,m} := \begin{bmatrix} u_{k,m-1} \\ 0 \end{bmatrix}$.

· Solve $u_{k,m} \in \mathbb{X}_{k+m}$ from equation

$$\mathcal{B}_{k,m} u_{k,m} = f_{k+m} - \mathcal{C}_{k,m} \overline{u}_{k,m}. \tag{2.12}$$

For a fixed positive integer k, if this algorithm can be carried out, it generates a sequence of approximate solutions $u_{k,m} \in \mathbb{X}_{k+m}$, $m \in \mathbb{N}_0$. Note that this algorithm is *not* an iteration method since for different m we are dealing with matrices of different order and for each m we only compute the approximate solution $u_{k,m}$ once. To ensure that the algorithm can be carried out, we have to guarantee that for $m \in \mathbb{N}_0$, the inverse of $\mathcal{B}_{k,m}$ exists and is uniformly bounded. Moreover, the approximate solutions generated by this augmentation algorithm neither necessarily have the same order of convergence as the approximation order of the subspaces $\mathbb{X}_n$ nor necessarily is more efficient to solve than solving equation (2.2) with $n := k+m$ directly, unless certain conditions on the splitting are satisfied. For this algorithm to be executable, accurate and efficient, we demand that the splitting of operator $\mathcal{A}_{k,m}$ fulfill three requirements. Firstly, $\mathcal{B}_{k,m}^{-1}$ is uniformly bounded. Secondly, the approximate solution $u_{k,m}$ preserves the convergence order of u_{k+m}. That is, $u_{k,m}$ converges to the exact solution u at the approximation order of the subspaces $\mathbb{X}_{k+m}$. Thirdly, the inverse of $\mathcal{B}_{k,m}$ is much easier to obtain than the inverse of $\mathcal{A}_{k,m}$.

We now address the first issue. To this end, we describe our hypotheses.

(I) There exist a positive integer N_0 and a positive constant α such that for $n \geq N_0$,

$$\|\mathcal{A}_n^{-1}\| \leq \alpha^{-1}. \tag{2.13}$$

(II) The limit

$$\lim_{n\to\infty} \|\mathcal{C}_{n,m}\| = 0 \tag{2.14}$$

holds uniformly for $m \in \mathbb{N}$.

Under these two assumptions, we have the following result.

Proposition 2.1 Suppose that hypotheses (I) and (II) hold. Then, there exists a positive integer $N > N_0$ such that for $k \geq N$ and $m \in \mathbb{N}$, equation (2.12) has a unique solution $u_{k,m} \in \mathbb{X}_{k+m}$.

Proof From hypothesis (I), whenever $k \geq N_0$ there holds that for $x \in \mathbb{X}_{k+m}$,

$$\|\mathcal{B}_{k,m}x\| = \|(\mathcal{A}_{k,m} - \mathcal{C}_{k,m})x\| \geq (\alpha - \|\mathcal{C}_{k,m}\|)\|x\|. \tag{2.15}$$

On the other hand, it follows from (2.14) that there exists a positive integer $N > N_0$ such that for $k \geq N$ and for $m \in \mathbb{N}_0$, $\|\mathcal{C}_{k,m}\| < \alpha/2$. Combining this inequality with (2.15), we find that for $k \geq N$ and for $m \in \mathbb{N}_0$ there holds the estimate

$$\|\mathcal{B}_{k,m}^{-1}\| \leq \frac{1}{\alpha - \|\mathcal{C}_{k,m}\|} \leq 2\alpha^{-1}. \tag{2.16}$$

This ensures that for all $k \geq N$ and $m \in \mathbb{N}_0$, equation (2.12) has a unique solution.

We next consider the second issue. For $n \in \mathbb{N}_0$, we let E_n denote the approximation error in space $\mathbb{X}_n$ for $u \in \mathbb{X}$, namely, $E_n = E_n(u) := \inf\{\|u - v\|_{\mathbb{X}} : v \in \mathbb{X}_n\}$. A sequence of nonnegative numbers γ_n, $n \in \mathbb{N}_0$ is called a majorization sequence of E_n if $\gamma_n \geq E_n$, $n \in \mathbb{N}_0$ and there exists a positive integer N_0 and a positive constant σ such that for $n \geq N_0$, $\dfrac{\gamma_{n+1}}{\gamma_n} \geq \sigma$.

We also need the following hypothesis.

(III) There exist a positive integer N_0 and a positive constant ρ such that for $n \geq N_0$ and for the solution $u_n \in \mathbb{X}_n$ of equation (2.2), $\|u - u_n\|_{\mathbb{X}} \leq \rho E_n$.

In the next theorem, we show that under the assumptions described above, $u_{k,m}$ approximates u at an order comparable to E_{k+m}.

Theorem 2.2 Suppose that hypotheses (I)-(III) hold. Let $u \in \mathbb{X}$ be the solution of equation (2.1), γ_n, $n \in \mathbb{N}_0$, be a majorization sequence of E_n and ρ be the constant appearing in hypothesis (III). Then, there exists a positive integer N such that for $k \geq N$ and for $m \in \mathbb{N}_0$

$$\|u - u_{k,m}\| \leq (\rho + 1)\gamma_{k+m},$$

where $u_{k,m}$ is the solution of equation (2.12).

Proof We prove this theorem by establishing an estimate on $\|u_{k,m} - u_{k+m}\|$. For this purpose, we subtract (2.11) from (2.12) to obtain $\mathcal{B}_{k,m}(u_{k,m} - u_{k+m}) = \mathcal{C}_{k,m}(u_{k+m} - \overline{u}_{k,m})$. The hypotheses of this theorem ensure that Proposition 2.1 holds. Hence, from the equation above and inequality (2.16) we have that

$$\|u_{k,m} - u_{k+m}\| \leq \frac{\|\mathcal{C}_{k,m}\|}{\alpha - \|\mathcal{C}_{k,m}\|}\|u_{k+m} - \overline{u}_{k,m}\|. \tag{2.17}$$

We next prove by induction on m that there exists a positive integer N such that for $k \geq N$ and for $m \in \mathbb{N}_0$,

$$\|u_{k,m} - u_{k+m}\| \leq \gamma_{k+m}. \tag{2.18}$$

When $m = 0$, since $u_{k,0} = u_k$, estimate (2.18) holds trivially. Suppose that the claim holds for $m = r - 1$ and we come to prove that it holds for $m = r$. To accomplish this, by using the definition of $\overline{u}_{k,r}$, hypothesis (III), the induction hypothesis and the definition of majorization sequences, we obtain that

$$\begin{aligned}\|u_{k+r} - \overline{u}_{k,r}\| &\leq \|u_{k+r} - u\| + \|u - u_{k+r-1}\| + \|u_{k+r-1} - u_{k,r-1}\| \\ &\leq \rho\gamma_{k+r} + (\rho+1)\gamma_{k+r-1} \leq \left(\rho + (\rho+1)\tfrac{1}{\sigma}\right)\gamma_{k+r}.\end{aligned}$$

Substituting this estimate into the right hand side of (2.17) with $m = r$ yields

$$\|u_{k,r} - u_{k+r}\| \leq \frac{\|\mathcal{C}_{k,r}\|}{\alpha - \|\mathcal{C}_{k,r}\|}\left(\rho + (\rho+1)\frac{1}{\sigma}\right)\gamma_{k+r}.$$

Again, employing hypothesis (II), there exists a positive integer N such that for $k \geq N$ and for $r \in \mathbb{N}_0$, $\|\mathcal{C}_{k,r}\| \leq \dfrac{\alpha}{(\rho+1)(1+\frac{1}{\sigma})}$. We then conclude that for $k \geq N$ and for $r \in \mathbb{N}_0$,

$\dfrac{\|\mathcal{C}_{k,r}\|}{\alpha - \|\mathcal{C}_{k,r}\|}\left(\rho + (\rho+1)\dfrac{1}{\sigma}\right) \leq 1$. Therefore, for $k \geq N$, estimate (2.18) holds for $m = r$. This advances the induction hypothesis and thus estimate (2.18) holds for all $m \in \mathbb{N}_0$.

Finally, the estimate of this theorem follows directly from estimate (2.18) and hypothesis (III).

We remark that when the exact solution u of equation (2.1) has certain Sobolev or Besov regularity and specific approximate subspaces $\mathbb{X}_n$ are chosen, we may choose the majorization sequence γ_n as the upper bound of E_n which gives the order of approximation of the subspaces $\mathbb{X}_n$ with respect to the regularity. For example, when $\mathbb{X}_n$ is chosen to be the usual finite element spaces with the mesh-size 2^{-n} and when the solution u of equation (2.1) belongs to the Sobolev space H^r, we may choose $\gamma_n := c2^{-rn}\|u\|_{H^r}$. In this case, the constant σ in the definition of majorization sequences can be taken as 2^{-r}. Therefore, Theorem 2.2 ensures that the approximate solution $u_{k,m}$ generated by the multilevel augmentation method has the same order of approximation of the subspaces $\mathbb{X}_n$.

3 Second Kind Equations

In this section, we present special results for projection methods for solving operator equations of the second kind. Consider equations

$$(\mathcal{I} - \mathcal{K})u = f, \tag{3.1}$$

where $\mathcal{K} : \mathbb{X} \to \mathbb{X}$ is a linear operator. We assume that equation (3.1) has a unique solution. In this special case, we identify that $\mathcal{A} := \mathcal{I} - \mathcal{K}$, $\mathbb{X} = \mathbb{Y}$, and $\mathbb{X}_n = \mathbb{Y}_n$. Suppose that $\mathcal{P}_n : \mathbb{X} \to \mathbb{X}_n$ are linear projections and we define the projection method for solving equation (3.1) by

$$\mathcal{P}_n(\mathcal{I} - \mathcal{K})u_n = \mathcal{P}_n f, \tag{3.2}$$

where $u_n \in \mathbb{X}_n$. To develop a multilevel augmentation method, we need another projection $\hat{\mathcal{P}}_n : \mathbb{X} \to \mathbb{X}_n$. We define operators $\mathcal{Q}_n := \mathcal{P}_n - \mathcal{P}_{n-1}$ and $\hat{\mathcal{Q}}_n := \hat{\mathcal{P}}_n - \hat{\mathcal{P}}_{n-1}$, and introduce subspaces $\mathbb{W}_n := \hat{\mathcal{Q}}_n \mathbb{X}_n$, $n \in \mathbb{N}$. We allow the projections $\mathcal{P}_n$ and $\hat{\mathcal{P}}_n$ to be different in order to have a wide range of applications. For example, for Galerkin methods, $\mathcal{P}_n$ and $\hat{\mathcal{P}}_n$ are both identical to the orthogonal projection and for the collocation method developed in [11], $\mathcal{P}_n$ is the interpolatory projection and $\hat{\mathcal{P}}_n$ is the orthogonal projection.

For $n \in \mathbb{N}$, we set $\mathcal{K}_n := \mathcal{P}_n \mathcal{K}|_{\mathbb{X}_n}$ and identify $\mathcal{A}_n := \mathcal{P}_n(\mathcal{I} - \mathcal{K})|_{\mathbb{X}_n}$. We further identify the operators in (2.9) with

$$\mathcal{F}_{k,k+j} := \mathcal{P}_k(\mathcal{I} - \mathcal{K})|_{\mathbb{W}_{k+j}}, \quad \mathcal{G}_{k+i,k} := \mathcal{Q}_{k+i}(\mathcal{I} - \mathcal{K})|_{\mathbb{X}_k}, \quad \mathcal{H}_{k+i,k+j} := \mathcal{Q}_{k+i}(\mathcal{I} - \mathcal{K})|_{\mathbb{W}_{k+j}}.$$

We split the operator $\mathcal{K}_{k+m}$ into the sum of two operators $\mathcal{K}_{k+m} = \mathcal{K}^L_{k,m} + \mathcal{K}^H_{k,m}$ where $\mathcal{K}^L_{k,m} := \mathcal{P}_k \mathcal{K}|_{\mathbb{X}_{k+m}}$, and $\mathcal{K}^H_{k,m} := (\mathcal{P}_{k+m} - \mathcal{P}_k)\mathcal{K}|_{\mathbb{X}_{k+m}}$. The operators $\mathcal{K}^L_{k,m}$ and $\mathcal{K}^H_{k,m}$ correspond to lower and higher frequency of the operator $\mathcal{K}_{k,m}$, respectively. According to the decomposition of $\mathcal{K}_{k+m}$, we write the operator $\mathcal{A}_{k,m} := \mathcal{I}_{k+m} - \mathcal{K}_{k,m}$ as a sum of lower and higher frequency $\mathcal{B}_{k,m} := \mathcal{I}_{k+m} - \mathcal{K}^L_{k,m}$ and $\mathcal{C}_{k,m} := -\mathcal{K}^H_{k,m}$. Using this specific splitting in formula (2.12) of Algorithm 1, we have that

$$(\mathcal{I}_{k+m} - \mathcal{K}^L_{k,m})u_{k,m} = f_{k,m} + \mathcal{K}^H_{k,m}\overline{u}_{k,m}. \tag{3.3}$$

The next theorem is concerned with convergence order for the multilevel augmentation method for second kind equations using projection methods.

Theorem 3.1 Suppose that $\mathcal{K}$ is a compact linear operator not having 1 as its eigenvalue and that there exists a positive constant p such that

$$\|\mathcal{P}_n\| \le p, \quad \|\hat{\mathcal{P}}_n\| \le p, \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.4}$$

Let $u \in \mathbb{X}$ be the solution of equation (3.1) and γ_n be a majorization sequence of E_n. Then, there exist a positive integer N and a positive constant c_0 such that for all $k \ge N$ and $m \in \mathbb{N}$,

$$\|u - u_{k,m}\| \le c_0\gamma_{k+m},$$

where $u_{k,m}$ is obtained from the augmentation algorithm with formula (3.3).

Proof We prove that the hypotheses of Theorem 2.2 hold for the special choice of the operators $\mathcal{B}_{k,m}$ and $\mathcal{C}_{k,m}$ for second kind equations. We first remark that the assumption on the operators $\mathcal{K}$ and $\mathcal{P}_n$ ensures that hypotheses (I) and (III) hold with $\mathcal{A}_n := \mathcal{I} - \mathcal{K}_n$. It remains to verify hypothesis (II). To this end, we recall the definition of $\mathcal{C}_{n,m}$ which has the form $\mathcal{C}_{n,m} = -(\mathcal{P}_{n+m} - \mathcal{P}_n)\mathcal{K}|_{\mathbb{X}_{n+m}}$. It follows from the second inequality of (3.4) that

$$\|\mathcal{C}_{n,m}\| = \|(\mathcal{P}_{n+m} - \mathcal{P}_n)\mathcal{K}|_{\mathbb{X}_{n+m}}\| \le p\|(\mathcal{P}_{n+m} - \mathcal{P}_n)\mathcal{K}\|.$$

By the first inequality of (3.4) and the nestedness of subspaces $\mathbb{X}_n$, we conclude that $\mathcal{P}_n$ pointwisely converges to the identity operator $\mathcal{I}$ of space $\mathbb{X}$. Hence, since $\mathcal{K}$ is compact, the last term of the inequality above converges to zero as $n \to \infty$ uniformly for $m \in \mathbb{N}$. Therefore, all hypotheses of Theorem 2.2 are satisfied and thus, we complete the proof of this theorem.

We next derive the matrix form of the multilevel augmentation method by choosing appropriate bases for the subspaces $\mathbb{X}_n$. For this purpose, we let $\mathbb{X}^*$ denote the dual space of $\mathbb{X}$ and for $\ell \in \mathbb{X}^*$, $x \in \mathbb{X}$ we let $\langle \ell, x\rangle$ denote the value of the linear functional ℓ at x. Suppose that $\mathbb{L}_n$, $n \in \mathbb{N}_0$ is a sequence of subspaces of $\mathbb{X}^*$ which has the property that $\mathbb{L}_n \subset \mathbb{L}_{n+1}$ and $\dim \mathbb{L}_n = \dim \mathbb{X}_n$, $n \in \mathbb{N}_0$. The operator $\mathcal{P}_n : \mathbb{X} \to \mathbb{X}_n$ is defined for $x \in \mathbb{X}$ by

$$\langle \ell, x - \mathcal{P}_n x\rangle = 0, \quad \text{for all} \quad \ell \in \mathbb{L}_n. \tag{3.5}$$

It is known (cf. [12]) that the operator $\mathcal{P}_n : \mathbb{X} \to \mathbb{X}_n$ is uniquely determined and is a projection if and only if $\mathbb{L}_n \cap \mathbb{X}_n^\perp = \{0\}$, $n \in \mathbb{N}_0$, where $\mathbb{X}_n^\perp$ denotes the annihilator of $\mathbb{X}_n$ in $\mathbb{X}^*$. Throughout the rest of this section we always assume that this condition is satisfied. We also assume that we have a decomposition of the space $\mathbb{L}_{n+1}$, namely,

$$\mathbb{L}_{n+1} = \mathbb{L}_n \oplus \mathbb{V}_{n+1}, \quad n \in \mathbb{N}_0. \tag{3.6}$$

Clearly, the spaces $\mathbb{W}_i$ and $\mathbb{V}_i$ have the same dimension. We will specify the direct sum in (3.6) later.

Set $w(0) := \dim \mathbb{X}_0$ and $w(i) := \dim \mathbb{W}_i$, for $i \in \mathbb{N}$ and $Z_n := \{0, 1, \ldots, n-1\}$. Suppose that

$$\mathbb{X}_0 = \operatorname{span}\{w_{0,j} : j \in Z_{w(0)}\}, \quad \mathbb{L}_0 = \operatorname{span}\{\ell_{0,j} : j \in Z_{w(0)}\},$$

$$\mathbb{W}_i = \operatorname{span}\{w_{i,j} : j \in Z_{w(i)}\}, \quad \mathbb{V}_i = \operatorname{span}\{\ell_{i,j} : j \in Z_{w(i)}\}, \quad i \in \mathbb{N}.$$

Introducing the index set $J_n := \{(i,j) : i \in Z_{n+1}, j \in Z_{w(i)}\}$, we have that for $n \in \mathbb{N}_0$

$$\mathbb{X}_n = \operatorname{span}\{w_{i,j} : (i,j) \in J_n\} \quad \text{and} \quad \mathbb{L}_n = \operatorname{span}\{\ell_{i,j} : (i,j) \in J_n\}.$$

We remark that the index set J_n has cardinality $d_n := \dim \mathbb{X}_n$ and we assume that the elements in J_n are ordered lexicographically.

We now present the matrix form of equation (3.2) using these bases. Note that for $v_n \in \mathbb{X}_n$, there exist unique constants $v_{i,j}$, $(i,j) \in J_n$, such that $v_n = \sum_{(i,j)\in J_n} v_{i,j} w_{i,j}$. It follows that the solution u_n, with $n := k+m$, of equation (3.2) has the vector representation $\mathbf{u}_n := [u_{i,j} : (i,j) \in J_n]^T$ under the basis $w_{i,j}$, $(i,j) \in J_n$. Using the bases for $\mathbb{X}_n$ and $\mathbb{L}_n$, we let $E_{i',j';i,j} := \langle \ell_{i',j'}, w_{i,j} \rangle$, and $K_{i',j';i,j} := \langle \ell_{i',j'}, \mathcal{K} w_{i,j} \rangle$, and introduce the matrices $\mathbf{E}_n := [E_{i',j';i,j} : (i',j'), (i,j) \in J_n]$, and $\mathbf{K}_n := [K_{i',j';i,j} : (i',j'), (i,j) \in J_n]$. We also introduce the column vectors $\mathbf{f}_n := [\langle \ell_{i',j'}, f \rangle : (i',j') \in J_n]^T$. In these notations, equation (3.2) is written in the matrix form as

$$(\mathbf{E}_{k+m} - \mathbf{K}_{k+m})\, \mathbf{u}_{k+m} = \mathbf{f}_{k+m}. \tag{3.7}$$

We partition matrices $\mathbf{K}_n$ and $\mathbf{E}_n$ into block matrices according to the decompositions of the spaces $\mathbb{X}_n$ and $\mathbb{L}_n$. Specifically, for $i', i \in Z_{n+1}$, we introduce the blocks $\mathbf{K}_{i',i} := [K_{i',j';i,j} : j' \in Z_{w(i')}, j \in Z_{w(i)}]$ and set $\mathbf{K}_n = [\mathbf{K}_{i',i} : i', i \in Z_{n+1}]$. Moreover, for a fixed $k \in \mathbb{N}$ we define the blocks $\mathbf{K}^k_{0,0} := \mathbf{K}_k$, and for $l', l \in \mathbb{N}$, $\mathbf{K}^k_{0,l} := [\mathbf{K}_{i',i} : i' \in Z_{k+1}, i = k+l]$, $\mathbf{K}^k_{l,0} := [\mathbf{K}_{i',i} : i' = k+l, i \in Z_{k+1}]$, and $\mathbf{K}^k_{l',l} := \mathbf{K}_{k+l',k+l}$. Using these block notations, for $n := k+m$ we write $\mathbf{K}_{k+m} = \left[\mathbf{K}^k_{i',i} : i', i \in Z_{m+1}\right]$. Likewise, we partition matrix $\mathbf{E}_n$ exactly in the same way.

The decomposition of operator $\mathcal{K}_{k+m}$ suggests the matrix decomposition $\mathbf{K}_{k+m} = \mathbf{K}^L_{k,m} + \mathbf{K}^H_{k,m}$, where

$$\mathbf{K}^L_{k,m} := \begin{bmatrix} \mathbf{K}^k_{0,0} & \mathbf{K}^k_{0,1} & \cdots & \mathbf{K}^k_{0,m} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{K}^H_{k,m} := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \mathbf{K}^k_{1,0} & \mathbf{K}^k_{1,1} & \cdots & \mathbf{K}^k_{1,m} \\ \vdots & \vdots & & \vdots \\ \mathbf{K}^k_{m,0} & \mathbf{K}^k_{m,1} & \cdots & \mathbf{K}^k_{m,m} \end{bmatrix}.$$

Note that matrices $\mathbf{K}_{k,m}^L$ and $\mathbf{K}_{k,m}^H$ correspond to lower and higher frequency of matrix $\mathbf{K}_{k,m}$. Moreover, we set $\mathbf{B}_{k,m} := \mathbf{E}_{k+m} - \mathbf{K}_{k,m}^L$ and $\mathbf{C}_{k,m} := -\mathbf{K}_{k,m}^H$. Next we describe the matrix form of the multilevel augmentation method for solving equation (3.7) using these two matrices.

Algorithm 2 (Matrix Form of the Augmentation Algorithm) Let $k > 0$ be a fixed integer.

Step 1 Solve $\mathbf{u}_k \in \mathbb{R}^{d_k}$ from the equation $(\mathbf{E}_k - \mathbf{K}_k)\,\mathbf{u}_k = \mathbf{f}_k$.

Step 2 Set $\mathbf{u}_{k,0} := \mathbf{u}_k$ and compute the splitting matrices $\mathbf{K}_{k,0}^L$ and $\mathbf{K}_{k,0}^H$.

Step 3 For $m \in \mathbb{N}$, suppose that $\mathbf{u}_{k,m-1} \in \mathbb{R}^{d_{k+m-1}}$ has been obtained and do the following.

· Augment the matrices $\mathbf{K}_{k,m-1}^L$ and $\mathbf{K}_{k,m-1}^H$ to form $\mathbf{K}_{k,m}^L$ and $\mathbf{K}_{k,m}^H$, respectively.

· Augment $\mathbf{u}_{k,m-1}$ by setting $\overline{\mathbf{u}}_{k,m} := \begin{bmatrix} \mathbf{u}_{k,m-1} \\ 0 \end{bmatrix}$.

· Solve $\mathbf{u}_{k,m} \in \mathbb{R}^{d_{k+m}}$ from the algebraic equations

$$(\mathbf{E}_{k,m} - \mathbf{K}_{k,m}^L)\mathbf{u}_{k,m} = \mathbf{f}_{k+m} + \mathbf{K}_{k,m}^H \overline{\mathbf{u}}_{k,m}. \tag{3.8}$$

It is important to know under what condition the matrix form (3.8) is equivalent to the operator form (3.3). This issue is addressed in the next theorem. To prepare a proof of this theorem, we consider an expression of the identity operator $\mathcal{I}$ in the subspace $\mathbb{X}_{k+m}$. Note that for any $x \in \mathbb{X}_{k+j}$, $j \in Z_m$, $\mathcal{Q}_{k+1+j}x = 0$. This is equivalent to the following equations

$$\mathcal{Q}_{k+1+j}\mathcal{I}|_{\mathbb{X}_k} = 0, \quad \text{and} \quad \mathcal{Q}_{k+1+j}\mathcal{I}|_{\mathbb{W}_{k+1+i}} = 0, \quad i \in Z_j. \tag{3.9}$$

Using this fact, we express the identity operator $\mathcal{I}$ in the subspace $\mathbb{X}_{k+m}$ as

$$\mathcal{I}_{k+m} := \mathcal{P}_{k+m}\mathcal{I}|_{\mathbb{X}_{k+m}} = \begin{bmatrix} \mathcal{P}_k\mathcal{I}|_{\mathbb{X}_k} & \mathcal{P}_k\mathcal{I}|_{\mathbb{W}_{k+1}} & \cdots & \mathcal{P}_k\mathcal{I}|_{\mathbb{W}_{k+m}} \\ 0 & \mathcal{Q}_{k+1}\mathcal{I}|_{\mathbb{W}_{k+1}} & \cdots & \mathcal{Q}_{k+1}\mathcal{I}|_{\mathbb{W}_{k+m}} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \mathcal{Q}_{k+m}\mathcal{I}|_{\mathbb{W}_{k+m}} \end{bmatrix}.$$

Taking this into consideration equation (3.3) becomes

$$\begin{aligned} &\begin{bmatrix} \mathcal{P}_k(\mathcal{I}-\mathcal{K})|_{\mathbb{X}_k} & \mathcal{P}_k(\mathcal{I}-\mathcal{K})|_{\mathbb{W}_{k+1}} & \cdots & \mathcal{P}_k(\mathcal{I}-\mathcal{K})|_{\mathbb{W}_{k+m}} \\ 0 & \mathcal{Q}_{k+1}\mathcal{I}|_{\mathbb{W}_{k+1}} & \cdots & \mathcal{Q}_{k+1}\mathcal{I}|_{\mathbb{W}_{k+m}} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \mathcal{Q}_{k+m}\mathcal{I}|_{\mathbb{W}_{k+m}} \end{bmatrix} u_{k,m} = \begin{bmatrix} \mathcal{P}_k f \\ \mathcal{Q}_{k+1} f \\ \vdots \\ \mathcal{Q}_{k+m} f \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \mathcal{Q}_{k+1}\mathcal{K}|_{\mathbb{X}_k} & \mathcal{Q}_{k+1}\mathcal{K}|_{\mathbb{W}_{k+1}} & \cdots & \mathcal{Q}_{k+1}\mathcal{K}|_{\mathbb{W}_{k+m}} \\ \vdots & \vdots & & \vdots \\ \mathcal{Q}_{k+m}\mathcal{K}|_{\mathbb{X}_k} & \mathcal{Q}_{k+m}\mathcal{K}|_{\mathbb{W}_{k+1}} & \cdots & \mathcal{Q}_{k+m}\mathcal{K}|_{\mathbb{W}_{k+m}} \end{bmatrix} \begin{bmatrix} u_{k,m-1} \\ 0 \end{bmatrix}. \end{aligned} \tag{3.10}$$

To state the next theorem, we let $\mathbb{N}_k := \{k, k+1, \ldots\}$ and introduce the following notion. For finite dimensional subspaces $A \subset \mathbb{X}^*$ and $B \subset \mathbb{X}$, we say $A \perp B$ if for any $\ell \in A$ and $x \in B$ there holds $\langle \ell, x \rangle = 0$.

Theorem 3.2 The following statements are equivalent.

(i) The matrix form (3.8) and the operator form (3.3) are equivalent for any $f \in \mathbb{X}$ and for any compact operator $\mathcal{K} : \mathbb{X} \to \mathbb{X}$.

(ii) For any $l \in \mathbb{N}_k$, $\mathbb{V}_{l+1} \perp \mathbb{X}_l$.

(iii) For any $l \in \mathbb{N}_k$ and any $j \in Z_{m+1} \setminus \{0\}$, $\mathbb{V}_{l+j} \perp \mathbb{X}_l$.

(iv) For $i', i \in Z_{m+1}$, $i' > i$, $\mathbf{E}^k_{i',i} = 0$.

(v) For $i', i \in Z_{m+1}$, $i' > i$, $\mathbf{B}^k_{i',i} = 0$.

Proof We first prove the equivalence of statements (i) and (ii). It is clear that for any $x \in \mathbb{X}$ and for any $\ell \in \mathbb{V}_n$, $\langle \ell, \mathcal{P}_{n-1}x \rangle = 0$ if and only if $\mathbb{V}_n \perp \mathbb{X}_{n-1}$. On the other hand, we observe from the definitions of $\mathcal{P}_n$ and $\mathcal{Q}_n$ for $n \in \mathbb{N}$ that for $x \in \mathbb{X}$ and for $\ell \in \mathbb{V}_n$

$$\langle \ell, \mathcal{Q}_n x \rangle = \langle \ell, \mathcal{P}_n x - \mathcal{P}_{n-1} x \rangle = \langle \ell, x \rangle - \langle \ell, \mathcal{P}_{n-1} x \rangle .$$

Hence, for $n \in \mathbb{N}$, the following equation holds $\langle \ell, \mathcal{Q}_n x \rangle = \langle \ell, x \rangle$, for all $x \in \mathbb{X}, \ell \in \mathbb{V}_n$ if and only if $\mathbb{V}_n \perp \mathbb{X}_{n-1}$. Therefore, statement (ii) is equivalent to saying that for any $j \in Z_m$,

$$\langle \ell, \mathcal{Q}_{k+j+1} x \rangle = \langle \ell, x \rangle , \quad \text{for all} \quad x \in \mathbb{X}, \ \ell \in \mathbb{V}_{k+j+1}. \tag{3.11}$$

Noting that for $x \in \mathbb{X}$ and $n \in \mathbb{N}$, $\mathcal{Q}_n x \in \mathbb{Z}_n := \mathcal{Q}_n \mathbb{X}_n \subset \mathbb{X}_n$, we conclude from (3.5) and (3.9) that equation (3.10) is equivalent to

$$\langle \ell, \mathcal{P}_k (\mathcal{I} - \mathcal{K}) u_{k,m} \rangle = \langle \ell, \mathcal{P}_k f \rangle , \quad \text{for all} \quad \ell \in \mathbb{L}_k, \tag{3.12}$$

and

$$\langle \ell, \mathcal{Q}_{k+j+1} u_{k,m} \rangle = \langle \ell, \mathcal{Q}_{k+j+1} f + \mathcal{Q}_{k+j+1} \mathcal{K} u_{k,m-1} \rangle , \text{ for all } \ell \in \mathbb{L}_{k+j+1}, \ j \in Z_m. \tag{3.13}$$

Using (3.5), equation (3.12) is written as

$$\langle \ell, (\mathcal{I} - \mathcal{K}) u_{k,m} \rangle = \langle \ell, f \rangle , \quad \text{for all} \quad \ell \in \mathbb{L}_k. \tag{3.14}$$

Again, from (3.5) we have that for $x \in \mathbb{X}$, and $\ell \in \mathbb{L}_{k+j}$, $j \in Z_m$,

$$\langle \ell, \mathcal{Q}_{k+j+1} x \rangle = \langle \ell, \mathcal{P}_{k+j+1} x - \mathcal{P}_{k+j} x \rangle = 0.$$

In particular, for all $\ell \in \mathbb{L}_{k+j}$, $j \in Z_m$, both sides of equation (3.13) are equal to zero. Noting that

$$\mathbb{L}_{k+j+1} = \mathbb{L}_{k+j} \oplus \mathbb{V}_{k+j+1},$$

equation (3.13) is equivalent to

$$\langle \ell, \mathcal{Q}_{k+j+1} u_{k,m} \rangle = \langle \ell, \mathcal{Q}_{k+j+1} f + \mathcal{Q}_{k+j+1} \mathcal{K} u_{k,m-1} \rangle , \quad \text{for all} \quad \ell \in \mathbb{V}_{k+j+1}, \ j \in Z_m.$$

Now suppose that statement (ii) holds. Using equation (3.11), the equation above is equivalent to

$$\langle \ell, u_{k,m} \rangle = \langle \ell, f + \mathcal{K}u_{k,m-1} \rangle, \quad \text{for all} \quad \ell \in \mathbb{V}_{k+j+1}, j \in Z_m. \tag{3.15}$$

In terms of the bases of the spaces $\mathbb{X}_k$, $\mathbb{L}_k$, $\mathbb{W}_{k+j+1}$ and $\mathbb{V}_{k+j+1}$, $j \in Z_m$, equations (3.14) and (3.15) are equivalent to the matrix equation (3.8). Conversely, if (i) holds, we can prove that equation (3.11) is satisfied and thus, (ii) holds.

The proof of (iii) implying (ii) is trivial. Statement (ii) and the nestedness assumption on $\mathbb{X}_n$ ensures the validity of (iii). Statement (iv) is the discrete version of (iii) and hence they are equivalent. Finally, the equivalence of (iv) and (v) follows from the definition of matrix $\mathbf{B}_{k,m}$.

Note that condition (ii) in Theorem 3.2 specifies the definition of the direct sum (3.6). In other words, the space $\mathbb{V}_{n+1}$ is uniquely determined by condition (ii). From now on, we will always assume that condition (ii) is satisfied to guarantee the equivalence of (3.3) and (3.8). Another way to write the equivalence conditions in Theorem 3.2 is

$$\langle \ell_{i',j'}, w_{i,j} \rangle = 0, \quad i, i' \in \mathbb{N}_k, i < i', j \in Z_{w(i)}, j' \in Z_{w(i')}. \tag{3.16}$$

When condition (3.16) is satisfied we call the bases $\ell_{i,j}$ and $w_{i,j}$ semi-biorthogonal. Under this condition, when the solution $\mathbf{u}_{k,m}$ of equation (3.8) is computed, we conclude from Theorem 3.2 that the function defined by $u_{k,m} := \mathbf{u}_{k,m}^T \mathbf{x}_{k+m}$ is the solution of the equation (3.3), where $\mathbf{x}_n = [w_{i,j} : (i,j) \in J_n]^T$. We remark that condition (3.16) is satisfied for wavelet Galerkin methods and wavelet collocation methods developed in [20] and [11], respectively. In fact, in the case of the Galerkin method using orthogonal piecewise polynomial wavelets constructed in [18] the matrix $\mathbf{E}_n$ is the identity and in the case of the collocation method using interpolating piecewise polynomial wavelets and multiscale functionals constructed in [8] the matrix $\mathbf{E}_n$ is upper triangular with the diagonal entries equal to one.

We now turn to a study of computational complexity of Algorithm 2. Specifically, we will estimate the number of multiplications used in the method. For this purpose, we rewrite equation (3.8) in a block form. Letting $n := k + m$ we partition the matrix $\mathbf{E}_{k+m}$ in the same way as we have done for the matrix $\mathbf{K}_{k+m}$ to obtain blocks $\mathbf{E}^k_{i,i'}$, $i, i' \in Z_{m+1}$. We also partition the vectors $\mathbf{u}_{k,m}$ and $\mathbf{f}_{k+m}$ accordingly as $\mathbf{u}_{k,m} := [\mathbf{u}^m_i : i \in Z_{m+1}]$ and $\mathbf{f}_{k+m} := [\mathbf{f}_{k,i} : i \in Z_{m+1}]$. Here and in what follows, we require that the appropriate bases are chosen so that $\mathbf{E}^k_{i',i} = 0$, for $0 \le i < i' \le m$, and $\mathbf{E}^k_{i,i} = \mathbf{I}$. With this assumption, we express the matrix $\mathbf{B}_{k,m}$ as

$$\mathbf{B}_{k,m} = \begin{bmatrix} \mathbf{I} - \mathbf{K}_k & \mathbf{E}^k_{0,1} - \mathbf{K}^k_{0,1} & \mathbf{E}^k_{0,2} - \mathbf{K}^k_{0,2} & \cdots & \mathbf{E}^k_{0,m-1} - \mathbf{K}^k_{0,m-1} & \mathbf{E}^k_{0,m} - \mathbf{K}^k_{0,m} \\ \mathbf{0} & \mathbf{I} & \mathbf{E}^k_{1,2} & \cdots & \mathbf{E}^k_{1,m-1} & \mathbf{E}^k_{1,m} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \cdots & \mathbf{E}^k_{2,m-1} & \mathbf{E}^k_{2,m} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{E}^k_{m-1,m} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{I} \end{bmatrix}.$$

It is clear from this matrix representation that inverting matrices $\mathbf{B}_{k,m}$, $m \in \mathbb{N}_0$, is basically equivalent to inverting $\mathbf{I} - \mathbf{K}_k$. The strength of Algorithm 2 is that it *only* requires computing the inverse $(\mathbf{I} - \mathbf{K}_k)^{-1}$. Using this block form of matrix $\mathbf{B}_{k,m}$, equation (3.8) becomes

$$\mathbf{u}_i^m = \mathbf{f}_{k,i} + \sum_{j=0}^{m-1} \mathbf{K}_{i,j}^k \mathbf{u}_j^{m-1} - \sum_{j=i+1}^{m} \mathbf{E}_{i,j}^k \mathbf{u}_j^m, \quad i = m, m-1, \ldots, 1, \tag{3.17}$$

$$\hat{\mathbf{f}}_{k,0} := \mathbf{f}_{k,0} - \sum_{j=1}^{m} (\mathbf{E}_{0,j}^k - \mathbf{K}_{0,j}^k)\mathbf{u}_j^m, \ \text{ and } \ \mathbf{u}_0^m = (\mathbf{I} - \mathbf{K}_k)^{-1}\hat{\mathbf{f}}_{k,0}. \tag{3.18}$$

For a matrix $\mathbf{A}$, we denote by $\mathcal{N}(\mathbf{A})$ the number of non-zero entries of $\mathbf{A}$. Note that we need $\mathcal{N}(\mathbf{K}_{k,m}^H) + \mathcal{N}(\mathbf{E}_{k+m})$ multiplications to obtain $\mathbf{u}_i^m$, $i = 1, 2, \ldots, m$ from equation (3.17). In addition, the computation of $\hat{\mathbf{f}}_{k,0}$ requires $\mathcal{N}(\mathbf{K}_{k,m}^L)$ number of multiplications. We assume that computing $\mathbf{u}_0^m$ from the second equation of (3.18) needs $\mathcal{M}(k)$ multiplications, which is constant independent of m. Hence, the number of multiplications for computing $\mathbf{u}_{k,m}$ from $\mathbf{u}_{k,m-1}$ is

$$N_{k,m} := \mathcal{N}(\mathbf{K}_{k+m}) + \mathcal{N}(\mathbf{E}_{k+m}) + \mathcal{M}(k). \tag{3.19}$$

Recall that to compute $\mathbf{u}_{k,m}$, we first compute $\mathbf{u}_k$ and then use algorithm (3.17)-(3.18) to compute $\mathbf{u}_{k,i}$, $i = 1, 2, \ldots, m$, successively. By formula (3.19), the total number of multiplications required to obtain $\mathbf{u}_{k,m}$ is given by

$$\mathcal{M}(k) + \sum_{i=1}^{m} N_{k,i} = (m+1)\mathcal{M}(k) + \sum_{i=1}^{m} \left[\mathcal{N}(\mathbf{K}_{k+i}) + \mathcal{N}(\mathbf{E}_{k+i})\right].$$

We now summarize the discussion above in a proposition.

Proposition 3.3 The total number of multiplications required for computing $\mathbf{u}_{k,m}$ from $\mathbf{u}_k$ is given by

$$(m+1)\mathcal{M}(k) + \sum_{i=1}^{m} [\mathcal{N}(\mathbf{K}_{k+i}) + \mathcal{N}(\mathbf{E}_{k+i})].$$

To close this section, we analyze stability of Algorithm 2. It can be shown that if the condition number $\kappa(\mathbf{B}_{k,m})$ of matrix $\mathbf{B}_{k,m}$ is small, then small perturbations of the matrices $\mathbf{B}_{k,m}$ and $\mathbf{C}_{k,m}$ and the vector $\overline{\mathbf{u}}_{k,m}$ only cause a small perturbation in the solution $\mathbf{u}_{k,m}$. For this reason, we study the condition number of matrix $\mathbf{B}_{k,m}$. Our theorem will confirm that the condition numbers $\kappa(\mathbf{B}_{k,m})$ and $\kappa(\mathbf{A}_{k+m})$ have the same order. In other words, the augmentation method will not ruin the well-condition property of the original multilevel method.

We first establish a result that stability of $\mathcal{B}_{k,m}$ is inherited from that of $\mathcal{A}_{k+m}$.

Lemma 3.4 Suppose that the family of operators $\mathcal{A}_n, n \in \mathbb{N}_0$ have the property that there exist positive constants c_1 and c_2 and a positive integer N_0 such that for $n \geq N_0$ $\|\mathcal{A}_n\| \leq c_1$, and $\|\mathcal{A}_n v\| \geq c_2 \|v\|$, for all $v \in \mathbb{X}_n$. Moreover, suppose that for any $k, m \in \mathbb{N}_0$, $\mathcal{A}_{k+m} = \mathcal{B}_{k,m} + \mathcal{C}_{k,m}$

where $\mathcal{C}_{k,m}$ satisfies hypothesis (II). Then, there exist positive constants c_1' and c_2' and a positive integer N_1 such that for $k > N_1$, $m \in \mathbb{N}_0$, $\|\mathcal{B}_{k,m}\| \le c_1'$, and $\|\mathcal{B}_{k,m}v\| \ge c_2'\|v\|$, for all $v \in \mathbb{X}_{k+m}$.

Proof By the triangular inequality, we have for any $k, m \in \mathbb{N}_0$ that

$$\|\mathcal{A}_{k+m}\| - \|\mathcal{C}_{k,m}\| \le \|\mathcal{B}_{k,m}\| \le \|\mathcal{A}_{k+m}\| + \|\mathcal{C}_{k,m}\|.$$

The hypotheses of this lemma ensure that there exists a positive integer N' such that for $k > N'$ and $m \in \mathbb{N}_0$, $\|\mathcal{C}_{k,m}\| \le c_2/2$. We let $N_1 := \max\{N_0, N'\}$ and observe that for $k > N_1$, $m \in \mathbb{N}_0$, $\|\mathcal{B}_{k,m}\| \le c_1 + c_2/2$, and $\|\mathcal{B}_{k,m}v\| \ge \|\mathcal{A}_{k+m}v\| - \|\mathcal{C}_{k,m}v\| \ge \frac{c_2}{2}\|v\|$, for all $v \in \mathbb{X}_{k+m}$. By choosing $c_1' := c_1 + \frac{c_2}{2}$ and $c_2' := \frac{c_2}{2}$, we complete the proof of this lemma.

We now return to the discussion of the condition number of matrix $\mathbf{B}_{k,m}$. To do this, we need auxiliary bases for $\mathbb{X}_0$ and $\mathbb{W}_i$, for $i \in \mathbb{N}$, which are bi-orthogonal to $\{\ell_{i,j} : j \in Z_{w(i)}, i \in \mathbb{N}_0\}$, that is, $\mathbb{X}_0 = \text{span}\ \{\zeta_{0,j} : j \in Z_{w(0)}\}$, $\mathbb{W}_i = \text{span}\ \{\zeta_{i,j} : j \in Z_{w(i)}\}$, with the bi-orthogonal property $\langle \ell_{i',j'}, \zeta_{i,j}\rangle = \delta_{i',i}\delta_{j',j}$, for $i, i' \in \mathbb{N}_0$, $j \in Z_{w(i)}$, $j' \in Z_{w(i')}$. For any $v \in \mathbb{X}_n$, we have two representations of v given by $v = \sum_{(i,j)\in J_n} v_{i,j}w_{i,j}$ and $v = \sum_{(i,j)\in J_n} v'_{i,j}\zeta_{i,j}$. We let $\mathbf{v}, \mathbf{v}' \in \mathbb{R}^{d_n}$ be the vectors of the coefficients in the two representations of v, respectively, i.e., $\mathbf{v} := [v_{i,j} : (i,j) \in J_n]^T$ and $\mathbf{v}' := [v'_{i,j} : (i,j) \in J_n]^T$.

Theorem 3.5 Let $n \in \mathbb{N}_0$ and suppose that there exist functions $\mu_i(n)$, $\nu_i(n)$, $i = 1, 2$, such that for any $v \in \mathbb{X}_n$,

$$\mu_1(n)\|\mathbf{v}\| \le \|v\| \le \mu_2(n)\|\mathbf{v}\|, \quad \nu_1(n)\|\mathbf{v}'\| \le \|v\| \le \nu_2(n)\|\mathbf{v}'\|. \tag{3.20}$$

Suppose that the hypothesis of Lemma 4.1 is satisfied. Then, there exists a positive integer N such that for any $k > N$ and any $m \in \mathbb{N}_0$,

$$\kappa(\mathbf{A}_{k+m}) \le \frac{c_1\mu_2(k+m)\nu_2(k+m)}{c_2\mu_1(k+m)\nu_1(k+m)} \quad \text{and} \quad \kappa(\mathbf{B}_{k,m}) \le \frac{c_1'\mu_2(k+m)\nu_2(k+m)}{c_2'\mu_1(k+m)\nu_1(k+m)}.$$

Proof We will prove the bound for both matrices at the same time. To this end, for $n := k+m$ we let $\mathbf{H}_{k,m}$ denote either $\mathbf{A}_{k+m}$ or $\mathbf{B}_{k,m}$ and $\mathcal{H}_{k,m}$ for the corresponding operator. For any $\mathbf{v} = [v_{i,j} : (i,j) \in J_{k+m}]^T \in \mathbb{R}^{d_{k+m}}$, we define a vector $\mathbf{g} = [g_{i,j} : (i,j) \in J_{k+m}]^T \in \mathbb{R}^{d_{k+m}}$ by letting $\mathbf{g} := \mathbf{H}_{k,m}\mathbf{v}$. Introducing $v = \sum_{(i,j)\in J_{k+m}} v_{i,j}w_{i,j}$, and $g = \sum_{(i,j)\in J_{k+m}} g_{i,j}\zeta_{i,j}$, we have the corresponding operator equation $\mathcal{H}_{k,m}v = g$. Since the hypotheses of Lemma 3.4 are satisfied, by using Lemma 3.4, there exist positive constants c_1' and c_2' and a positive integer N such that for $k > N$, $m \in \mathbb{N}_0$, $\|\mathcal{B}_{k,m}\| \le c_1'$, and $\|\mathcal{B}_{k,m}v\| \ge c_2'\|v\|$, for all $v \in \mathbb{X}_{k+m}$. Therefore, in either case, there exist positive constants c_1 and c_2 and a positive integer N such that for $k > N$, $m \in \mathbb{N}_0$, $\|\mathcal{H}_{k,m}\| \le c_1$, and $\|\mathcal{H}_{k,m}v\| \ge c_2\|v\|$, for all $v \in \mathbb{X}_{k+m}$. It follows from (3.20) that for any $k > N$ and $m \in \mathbb{N}_0$

$$\|\mathbf{H}_{k,m}\mathbf{v}\| = \|\mathbf{g}\| \le \frac{\|g\|}{\nu_1(k+m)} = \frac{\|\mathcal{H}_{k,m}v\|}{\nu_1(k+m)} \le \frac{c_1\|v\|}{\nu_1(k+m)} \le \frac{c_1\mu_2(k+m)}{\nu_1(k+m)}\|\mathbf{v}\|$$

which yields $\|\mathbf{H}_{k,m}\| \le \frac{c_1\mu_2(k+m)}{\nu_1(k+m)}$. Likewise, by (3.20) we have that for any $k > N$ and $m \in \mathbb{N}_0$

$$\mu_1(k+m)\|\mathbf{v}\| \le \|v\| \le \frac{\|\mathcal{H}_{k,m}v\|}{c_2} = \frac{\|g\|}{c_2} \le \frac{\nu_2(k+m)}{c_2}\|\mathbf{g}\| = \frac{\nu_2(k+m)}{c_2}\|\mathbf{H}_{k,m}\mathbf{v}\|,$$

which ensures that $\|\mathbf{H}_{k,m}^{-1}\| \le \dfrac{\nu_2(k+m)}{c_2\mu_1(k+m)}$. Combining the estimates for $\|\mathbf{H}_{k,m}\|$ and $\|\mathbf{H}_{k,m}^{-1}\|$, we confirm the bounds of the condition numbers of $\mathbf{A}_{k+m}$ and $\mathbf{B}_{k,m}$.

We next apply this theorem to two specific cases to obtain two useful special results.

Remark 3.6 For wavelet Galerkin methods developed in [20] and wavelet Petrov-Galerkin methods developed in [8], $\kappa(\mathbf{A}_{k+m}) = \mathcal{O}(1)$ and $\kappa(\mathbf{B}_{k,m}) = \mathcal{O}(1)$.

Proof In these wavelet methods, we use *orthogonal* wavelet bases. Thus, quantities $\mu_i(n)$ and $\nu_i(n)$, $i = 1, 2$, appearing in (3.20) are constant independent of n. By Theorem 3.5, in these cases condition numbers $\kappa(\mathbf{A}_{k+m})$ and $\kappa(\mathbf{B}_{k,m})$ are constant independent of n.

Remark 3.7 For wavelet collocation methods developed in [11], $\kappa(\mathbf{A}_{k+m}) = \mathcal{O}(\log^2 d_{k+m})$ and $\kappa(\mathbf{B}_{k,m}) = \mathcal{O}(\log^2 d_{k+m})$, where d_{k+m} denotes the order of matrices $\mathbf{A}_{k+m}$ and $\mathbf{B}_{k,m}$.

Proof In the wavelet collocation methods developed in [11], we have that $\mu_1(k+m) = \mathcal{O}(1)$, $\nu_1(k+m) = \mathcal{O}(1)$, $\mu_2(k+m) = \mathcal{O}(\log d_{k+m})$, and $\nu_2(k+m) = \mathcal{O}(\log d_{k+m})$. Therefore, the result of this remark follows from Theorem 3.5.

4 Wavelet Compression Schemes

This section is devoted to an application of the multilevel augmentation method for solving linear systems resulting from wavelet compression schemes.

We assume that a wavelet compression strategy has been applied to compress the full matrix $\mathbf{K}_n$ to obtain a sparse matrix $\tilde{\mathbf{K}}_n$, where the number of nonzero entries of $\tilde{\mathbf{K}}_n$ is of order $d_n \log^\alpha d_n$, with $\alpha = 0, 1$ or 2, that is,

$$\mathcal{N}(\tilde{\mathbf{K}}_n) = \mathcal{O}(d_n \log^\alpha d_n). \tag{4.1}$$

Methods of this type were studied in [1, 5, 7,10,11,13,14,20-24]. In particular, when orthogonal piecewise polynomial wavelets and interpolating piecewise polynomial wavelets constructed respectively in [18] and [8] are used to develop the wavelet Galerkin method [20], the wavelet Petrov-Galerkin methods [7,10] and the wavelet collocation method [11], we have that $\mathcal{N}(\tilde{\mathbf{K}}_n) = \mathcal{O}(n^\alpha \mu^n)$, where the corresponding wavelets are constructed with a μ-adic subdivision of the domain when the kernel $K(s,t)$, $s, t \in D \subseteq \mathbb{R}^d$, of the integral operator $\mathcal{K}$ satisfies the conditions described in these papers. The compression scheme for these methods has the form

$$(\mathbf{E}_n - \tilde{\mathbf{K}}_n)\tilde{\mathbf{u}}_n = \mathbf{f}_n. \tag{4.2}$$

Equation (4.2) has an equivalent operator equation. Let $\tilde{\mathcal{K}}_n : \mathbb{X}_n \to \mathbb{X}_n$ be the linear operator relative to the basis $\{w_{ij} : (i,j) \in J_n\}$ having the matrix representation $\mathbf{E}_n^{-1}\tilde{\mathbf{K}}_n$. We have that

$$(\mathcal{I}-\tilde{\mathcal{K}}_n)\tilde{u}_n=\mathcal{P}_n f, \tag{4.3}$$

where $\tilde{u}_n \in \mathbb{X}_n$ and it is related to the solution $\tilde{\mathbf{u}}_n$ of equation (4.2) by the formula $\tilde{u}_n=\tilde{\mathbf{u}}_n^T\mathbf{x}_n$, where $\mathbf{x}_n := [w_{ij} : (i,j)\in J_n]^T$. It is known that under certain conditions for Galerkin methods and collocation methods we have that if $u \in W_{r,p}(D)$,

$$\|u-\tilde{u}_n\|_p \le c\mu^{-rn/d} n^\alpha \|u\|_{r,p}, \tag{4.4}$$

where $p=2$ for the Galerkin method and $p=\infty$ for the collocation method and r denotes the order of piecewise polynomials used in these methods.

To develop the multilevel augmentation method for solving the operator equation (4.3), we note that $\tilde{\mathcal{K}}_n=\mathcal{P}_n\tilde{\mathcal{K}}_n$, from which equation (4.3) is rewritten as

$$(\mathcal{I}-\mathcal{P}_n\tilde{\mathcal{K}}_n)\tilde{u}_n=\mathcal{P}_n f. \tag{4.5}$$

Hence, for $n := k+m$, we have that $\tilde{\mathcal{K}}_{k+m}=\mathcal{P}_k\tilde{\mathcal{K}}_{k+m}+(\mathcal{P}_{k+m}-\mathcal{P}_k)\tilde{\mathcal{K}}_{k+m}$ and from this equation we define $\tilde{\mathcal{B}}_{k,m} := \mathcal{I}_{k+m}-\mathcal{P}_k\tilde{\mathcal{K}}_{k+m}$ and $\tilde{\mathcal{C}}_{k,m} := -(\mathcal{P}_{k+m}-\mathcal{P}_k)\tilde{\mathcal{K}}_{k+m}$. As done in Section 3, we can define the multilevel augmentation method for equation (4.3).

We have the following convergence result.

Theorem 4.1 Let $u\in W_{r,p}(D)$. Suppose that the estimate (4.4) holds and

$$\lim_{n\to\infty}\|\mathcal{K}_n-\tilde{\mathcal{K}}_n\|=0. \tag{4.6}$$

Then, there exist a positive integer N and a positive constant c such that for all $k>N$ and $m\in\mathbb{N}_0$,

$$\|u-\tilde{u}_{k,m}\| \le c\mu^{-r(k+m)/d}(k+m)^\alpha\|u\|_{r,p}.$$

Proof The proof is done by employing Theorem 2.2 with a majorization sequence $\gamma_n := c\mu^{-rn/d}n^\alpha\|u\|_{r,p}$. We conclude that $\dfrac{\gamma_{n+1}}{\gamma_n}\ge\mu^{-r/d}$. In other words, γ_n is a majorization sequence of E_n with $\sigma := \mu^{-r/d}$. It is readily shown that

$$\|\tilde{\mathcal{C}}_{k,m}\| \le 2p\|\tilde{\mathcal{K}}_{k+m}-\mathcal{K}_{k+m}\|+\|\mathcal{C}_{k,m}\|+2p^2\|(\mathcal{P}_{k+m}-\mathcal{I})\mathcal{K}\|.$$

By (4.6), we have that $\lim_{k\to\infty}\|\tilde{\mathcal{C}}_{k,m}\|=0$ uniformly for $m\in\mathbb{N}_0$. Also, it can be verified that other conditions of Theorem 2.2 are satisfied with both the wavelet Galerkin method and the collocation method using piecewise polynomial wavelets of order r. By applying Theorem 2.2 we complete the proof.

We now formulate the matrix form of the multilevel augmentation method directly from the compressed matrix. Because the compressed matrix $\tilde{\mathbf{K}}_n$ inherits the multilevel structure of matrix $\mathbf{K}_n$, we may use the multilevel augmentation method to solve equation (4.2) as described in Section 3. Specifically, we partition the compressed matrix $\tilde{\mathbf{K}}_n$ as it is done for the full matrix $\mathbf{K}_n$ in Section 3, let

$$\tilde{\mathbf{K}}^L_{k,m} := \begin{bmatrix} \tilde{\mathbf{K}}^k_{0,0} & \tilde{\mathbf{K}}^k_{0,1} & \cdots & \tilde{\mathbf{K}}^k_{0,m} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{K}}^H_{k,m} := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \tilde{\mathbf{K}}^k_{1,0} & \tilde{\mathbf{K}}^k_{1,1} & \cdots & \tilde{\mathbf{K}}^k_{1,m} \\ \vdots & \vdots & & \vdots \\ \tilde{\mathbf{K}}^k_{m,0} & \tilde{\mathbf{K}}^k_{m,1} & \cdots & \tilde{\mathbf{K}}^k_{m,m} \end{bmatrix},$$

and define $\tilde{\mathbf{B}}_{k,m} := \mathbf{E}_{k+m} - \tilde{\mathbf{K}}^L_{k,m}$ and $\tilde{\mathbf{C}}_{k,m} := -\tilde{\mathbf{K}}^H_{k,m}$.

Algorithm 3 (Matrix Form of the Augmentation Algorithm for Wavelet Compression Schemes) Let k be a fixed positive integer.

Step 1 Solve $\tilde{\mathbf{u}}_k \in \mathbb{R}^{d_k}$ from the equation $\left(\mathbf{E}_k - \tilde{\mathbf{K}}_k\right)\tilde{\mathbf{u}}_k = \mathbf{f}_k$.

Step 2 Set $\tilde{\mathbf{u}}_{k,0} := \tilde{\mathbf{u}}_k$ and compute the splitting matrices $\tilde{\mathbf{K}}^L_{k,0}$ and $\tilde{\mathbf{K}}^H_{k,0}$.

Step 3 For $m \in \mathbb{N}$, suppose that $\tilde{\mathbf{u}}_{k,m-1} \in \mathbb{R}^{d_{k+m-1}}$ has been obtained and do the following.

-Augment the matrices $\tilde{\mathbf{K}}^L_{k,m-1}$ and $\tilde{\mathbf{K}}^H_{k,m-1}$ to form $\tilde{\mathbf{K}}^L_{k,m}$ and $\tilde{\mathbf{K}}^H_{k,m}$, respectively.

-Augment $\tilde{\mathbf{u}}_{k,m-1}$ by setting $\overline{\tilde{\mathbf{u}}}_{k,m} := \begin{bmatrix} \tilde{\mathbf{u}}_{k,m-1} \\ 0 \end{bmatrix}$.

-Solve $\tilde{\mathbf{u}}_{k,m} \in \mathbb{R}^{d_{k+m}}$ from the algebraic equations

$$(\mathbf{E}_{k,m} - \tilde{\mathbf{K}}^L_{k,m})\tilde{\mathbf{u}}_{k,m} = \mathbf{f}_{k+m} + \tilde{\mathbf{K}}^H_{k,m}\overline{\tilde{\mathbf{u}}}_{k,m}.$$

Since $\mathbf{E}_n^{-1}\tilde{\mathbf{K}}_n$ is the matrix representation of the operator $\tilde{\mathcal{K}}_n$ relative to the basis $\{w_{ij} : (i,j) \in J_n\}$, we conclude that

$$\left\langle \ell_{i'j'}, \tilde{\mathcal{K}}_n w_{ij} \right\rangle = \sum_{(i'',j'') \in J_n} (\mathbf{E}_n^{-1}\tilde{\mathbf{K}}_n)_{i''j'',ij}\mathbf{E}_{i'j',i''j''} = (\tilde{\mathbf{K}}_n)_{i'j',ij}$$

and see that the matrix form of multilevel augmentation method derived above is equivalent to the corresponding operator form.

In the next result, we estimate the number of multiplications used in Algorithm 3.

Theorem 4.2 Let k be a fixed positive integer and $m \in \mathbb{N}_0$. Suppose that for some $\alpha \in \{0,1,2\}$ and some integer $\mu > 1$, $\mathcal{N}(\tilde{\mathbf{K}}_n) = \mathcal{O}(n^\alpha \mu^n)$, and suppose that for $n \in \mathbb{N}_0$, $\mathcal{N}(\mathbf{E}_n) \le \mathcal{N}(\tilde{\mathbf{K}}_n)$. Then, the total number of multiplications required for computing $\tilde{\mathbf{u}}_{k,m}$ from $\tilde{\mathbf{u}}_k$ is of $\mathcal{O}((m+k)^\alpha \mu^{m+k})$.

Proof According to Proposition 3.3, the total number of multiplications required for computing $\tilde{\mathbf{u}}_{k,m}$ from $\tilde{\mathbf{u}}_k$ is given by $\tilde{N}_{k,m} = \mathcal{O}(m) + \sum_{i=1}^{m}[\mathcal{N}(\tilde{\mathbf{K}}_{k+i}) + \mathcal{N}(\mathbf{E}_{k+i})]$. Since $\mathcal{N}(\mathbf{E}_n) \le \mathcal{N}(\tilde{\mathbf{K}}_n)$, it suffices to estimate the quantity $\sum_{i=1}^{m}\mathcal{N}(\tilde{\mathbf{K}}_{k+i})$. To this end, we may show the identity

$\sum_{i=1}^{m}(k+i)^\alpha \mu^{k+i} = \mathcal{O}((k+m)^\alpha \mu^{k+m})$. Using this formula and the hypotheses of this theorem, we complete the proof of this result.

It can be verified that for the wavelet compression schemes presented in [10,11,20] condition (4.6) and assumption on Theorem 4.2 are fulfilled. Therefore, the conclusions of Theorems 4.1 and 4.2 hold for the class of methods proposed in these papers.

5 Numerical Experiments

We present in this section numerical examples to demonstrate the performance of the multilevel augmentation method associated with the wavelet Galerkin method and the wavelet collocation method. To focus on the main issue of the multilevel augmentation method, we choose a second kind integral equation on the unit interval since the augmentation method is independent of the dimension of the domain of the integral equation. Consider equation (3.1) with the integral operator $\mathcal{K}$ defined by

$$(\mathcal{K}u)(s) := \int_0^1 \log|\cos(\pi s) - \cos(\pi t)| u(t) \mathrm{d}t, \qquad s \in I := [0,1].$$

In our numerical experiments, for convenience of comparison we choose the right hand side function in equation (3.1) as

$$f(s) = \sin(\pi s) + \frac{1}{\pi}[2 - (1-\cos(\pi s))\log(1-\cos(\pi s)) - (1+\cos(\pi s))\log(1+\cos(\pi s))]$$

so that $u(s) = \sin(\pi s)$, $s \in I$, is the exact solution of the equation. We choose $\mathbb{X}_n$ as the space of piecewise linear polynomials on I with knots at the dyadic points $j/2^n$, $j = 1, 2, \ldots, 2^n - 1$. Note that the theoretical convergence rate for piecewise linear approximation is 2. All our numerical algorithms are run on a personal computer with a 600MHz CPU and 256M memory.

Example 1: Wavelet Galerkin Methods. In our first experiment, we consider the wavelet Galerkin method for solving equation (3.1). Choose an orthonormal basis $w_{00}(t) := 1$ and $w_{01}(t) := \sqrt{3}(2t-1)$, for $t \in I$ for $\mathbb{X}_0$ and

$$w_{10}(t) := \begin{cases} 1 - 6t, & t \in [0, \frac{1}{2}], \\ 5 - 6t, & t \in [\frac{1}{2}, 1] \end{cases} \qquad w_{11}(t) := \begin{cases} \sqrt{3}(1-4t), & t \in [0, \frac{1}{2}], \\ \sqrt{3}(4t-3), & t \in [\frac{1}{2}, 1] \end{cases}$$

for $\mathbb{W}_1$. An orthonormal basis w_{ij}, $j = 0, 1, \ldots, 2^i$ for $\mathbb{W}_i$ is constructed according to the construction given in [18]. We choose $\mathcal{P}_n = \mathcal{R}_n$ as the orthogonal projection mapping $L^2(I)$ onto $\mathbb{X}_n$ and $\ell_{ij} = w_{ij}$.

In this case, $\mathbf{E}_n$ is an identity matrix because of the orthogonality of the basis and the matrix $\mathbf{K}_n$ is truncated to A54FFFFFFFFFFF54AA5454AA54FFFFFFFFFFF54AA5454AA54FFFFFFFFFFF54AA54

is smaller than that of $\tilde{\mathbf{A}}_{k+m}$.

Table 4 Condition numbers of the matrices from collocation method

m	$\kappa(\tilde{\mathbf{A}}_{5+m})$	Δ_1	$\kappa(\tilde{\mathbf{B}}_{5,m})$	Δ_2
0	7.742223			
1	9.446764	1.704541	9.383285	
2	11.096629	1.649865	11.003597	1.620312
3	12.695535	1.598926	12.589373	1.585776
4	14.256332	1.560797	14.151313	1.561940
5	15.789732	1.533380	15.690938	1.539625

Both theoretical analysis and numerical experiment show that the multilevel augmentation method is particularly suitable for solving large scale linear systems resulting from wavelet compression schemes applied to integral equations. It is a stable and fast algorithm which provides accurate numerical solutions of integral equations.

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