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PPA BASED PREDICTION-CORRECTION METHODS FOR MONOTONE VARIATIONAL INEQUALITIES*

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Abstract *In this paper we study the proximal point algorithm (PPA) based prediction-correction (PC) methods for monotone variational inequalities. Each iteration of these methods consists of a prediction and a correction. The predictors are produced by inexact PPA steps. The new iterates are then updated by a correction using the PPA formula. We present two profit functions which serve two purposes: First we show that the profit functions are tight lower bounds of the improvements obtained in each iteration. Based on this conclusion we obtain the convergence inexactness restrictions for the prediction step. Second we show that the profit functions are quadratically dependent upon the step lengths, thus the optimal step lengths are obtained in the correction step. In the last part of the paper we compare the strengths of different methods based on their inexactness restrictions.*

Key words *Monotone variational inequality, proximal point algorithm, prediction-correction method.*

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1 Introduction

Let Ω be a nonempty closed convex subset of R^n and F be a continuous monotone mapping from R^n into itself. The variational inequality problem is to determine a vector $u^* \in \Omega$ such that

$$\mathrm{VI}(\Omega, F) \qquad (u-u^*)^T F(u^*) \geq 0, \qquad \forall u \in \Omega. \tag{1.1}$$

$\mathrm{VI}(\Omega, F)$ problems include nonlinear complementarity problems (when $\Omega = R^n_+$) and systems of

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nonlinear equations (when $\Omega = R^n$), and thus have many important applications [7,8,9].

A classical method for solving variational inequality is the proximal point algorithm (abbreviated as PPA) [17,18]. Given $u^k \in \Omega$ and $\beta_k > 0$, the new iterate u^{k+1} of PPA is obtained by solving the following variational inequality:

$$\text{(PPA)} \qquad u \in \Omega, \qquad (u' - u)^T F_k(u) \geq 0, \qquad \forall u' \in \Omega, \tag{1.2}$$

where

$$F_k(u) = (u - u^k) + \beta_k F(u). \tag{1.3}$$

An equivalent recursion form of PPA is

$$u^{k+1} = P_\Omega[u^{k+1} - F_k(u^{k+1})], \tag{1.4}$$

where P_Ω denotes the projection on Ω. The above projection equation can be written as

$$u^{k+1} = P_\Omega[u^k - \beta_k F(u^{k+1})]. \tag{1.5}$$

Since u^{k+1} occurs on both sides of equation (1.5), we call the method an *implicit* method [11].

The ideal form (1.5) of the method is often impractical since in many cases solving problem (1.2) exactly is either impossible or expensive. In 1976 Rockafellar set up the fundamental convergence analysis for the approximate proximal point algorithm (abbreviated as APPA) to a general maximal monotone operator [17]. Extensive developments on APPA followed, focusing on different fields such as convex programming, mini-max problems, and variational inequality problems. To mention a few, see [1, 3, 4, 5, 6, 16]. The major challenges of such methods include setting the restrictions of the approximation which are both easy to implement and tight for convergence, and accelerating the convergence.

In this paper, we study a particular group of methods which share the flavor of APPA. We call the methods proximal point algorithm based prediction-correction methods (abbreviated as PPA-PC methods). Given $u^k \in \Omega$ and $\beta_k > 0$, let v^k be an approximate solution of (1.2) in the sense that

$$v^k \approx P_\Omega[v^k - F_k(v^k)] \tag{1.6}$$

and define

$$\tilde{v}^k := P_\Omega[v^k - F_k(v^k)]. \tag{1.7}$$

The new iterate of these methods is given by either

$$\text{(PPA-PC1)} \qquad u^{k+1}(\alpha, v^k) = P_\Omega[u^k - \alpha\beta_k F(v^k)] \tag{1.8}$$

or

$$\text{(PPA-PC2)} \qquad u^{k+1}(\alpha, \tilde{v}^k) = P_\Omega[u^k - \alpha\beta_k F(\tilde{v}^k)]. \tag{1.9}$$

In such methods, v^k and $\tilde{v}^k$ can be viewed as predictors generated by inexactly solving the variational inequality (1.2). Ignoring the step length α, the new iterate u^{k+1} in (1.8) (resp. in (1.9)) can be viewed as the corrector obtained from equation (1.5) via substituting the u^{k+1} in the right hand side by the predictor v^k (resp. $\tilde{v}^k$). Therefore, we refer (1.8) and (1.9) as PPA based prediction-correction methods (PPA-PC1) and (PPA-PC2), respectively. The method presented in [15] uses v^k as the predictor and belongs to PPA-PC1. The modified inexact PPA in [13] takes $\tilde{v}^k$ as the predictor and thus belongs to PPA-PC2.

Remark 1.1 At first glance it seems that PPA-PC2 is more complicated than PPA-PC1, since it requires one extra projection in order to obtain $\tilde{v}^k$ at the prediction step. However, the following discussion and the analysis in section 3 indicate that evaluating $\tilde{v}^k$ is also necessary in PPA-PC1.

According to equation (1.4), v^k is the exact solution in the k-th iteration of PPA if and only if $v^k = \tilde{v}^k$. Hence it is natural to view

$$\zeta^k := v^k - \tilde{v}^k \tag{1.10}$$

as an inexactness indicator of the predictors. Another natural choice for such an indicator is

$$\tilde{v}^k - P_\Omega[u^k - \beta_k F(\tilde{v}^k)].$$

However this quantity requires further projection. Note that from equation (1.3) we have

$$\tilde{v}^k = P_\Omega[u^k - \beta_k F(v^k)]. \tag{1.11}$$

Recalling that the projection is a non-expansive mapping we obtain

$$\|\tilde{v}^k - P_\Omega[u^k - \beta_k F(\tilde{v}^k)]\| = \|P_\Omega[u^k - \beta_k F(v^k)] - P_\Omega[u^k - \beta_k F(\tilde{v}^k)]\| \le \|\beta_k[F(\tilde{v}^k) - F(v^k)]\|.$$

Hence the second natural choice of an inexactness indicator is

$$\xi^k := \beta_k[F(\tilde{v}^k) - F(v^k)]. \tag{1.12}$$

On the other hand, the functions

$$\Theta_k(\alpha, v^k) := \|u^k - u^*\|^2 - \|u^{k+1}(\alpha, v^k) - u^*\|^2 \tag{1.13}$$

and

$$\Theta_k(\alpha, \tilde{v}^k) := \|u^k - u^*\|^2 - \|u^{k+1}(\alpha, \tilde{v}^k) - u^*\|^2 \tag{1.14}$$

can be viewed as the improvements in the k-th iteration of PPA-PC1 and PPA-PC2, respectively.

The major objective of this paper is to answer the following questions:

- (A). Given a predictor, how should one choose the step length α in order to gain more progress in each iteration?
- (B). In order to guarantee convergence, what restrictions on ζ^k in (1.10) (resp. ξ^k in (1.12)) should be applied when v^k (resp. $\tilde{v}^k$) is taken as the predictor?
- (C). Which method is more efficient to be implemented PPA-PC1 or PPA-PC2 ?

We will prove that the following α-dependent functions

$$\Psi_k(\alpha) := 2\alpha\{\|u^k - \tilde{v}^k\|^2 - \beta_k(\zeta^k)^T F(v^k)\} - \alpha^2\|u^k - \tilde{v}^k\|^2 \tag{1.15}$$

and

$$\Phi_k(\alpha) := 2\alpha\{\|u^k - \tilde{v}^k\|^2 + (u^k - \tilde{v}^k)^T\xi^k\} - \alpha^2\|(u^k - \tilde{v}^k) + \xi^k\|^2 \tag{1.16}$$

are tight lower bounds of the improvements (1.13) in PPA-PC1 and (1.14) in PPA-PC2, respectively. These differentiable functions offer us the bases to answer the above mentioned questions.

This paper is organized as follows: In Section 2, we summarize some basic concepts about variational inequalities and prove a lower bound which is shared by both progress functions $\Theta_k(\alpha, v^k)$ and $\Theta_k(\alpha, \tilde{v}^k)$. In Section 3, we answer questions (A) and (B) when v^k is taken as the predictor. In Section 4 we answer questions (A) and (B) when $\tilde{v}^k$ is taken as the predictor. In Section 5 we show that compared with PPA-PC1, PPA-PC2 has significant advantage in implementation.

Throughout this paper we assume that the operator F is continuous and monotone on Ω, i.e.,

$$(u - v)^T(F(u) - F(v)) \geq 0, \quad \forall u, v \in \Omega.$$

In addition, we assume that the solution set of $\mathrm{VI}(\Omega, F)$, denoted by Ω^*, is nonempty, and $\{\beta_k\}$ is a positive sequence such that $0 < \beta_{\min} = \inf_{k=0}^{\infty} \beta_k \leq \sup_{k=0}^{\infty} \beta_k = \beta_{\max} < +\infty$.

2 Preliminaries

This section summarizes some basic properties of variational inequalities and proves a common proposition of the PPA based correction methods.

2.1 Projection operator and variational inequality

We use the concept of projection under the Euclidean norm, which will be denoted by $P_\Omega(\cdot)$, i.e.,

$$P_\Omega(w) = \mathrm{argmin}\{\|w - u\| \mid u \in \Omega\}.$$

From the above definition, it follows that

$$\{w - P_\Omega(w)\}^T\{v - P_\Omega(w)\} \leq 0, \qquad \forall w \in R^n, \quad \forall v \in \Omega \tag{2.1}$$

and

$$(v - w)^T\{P_\Omega(v) - P_\Omega(w)\} \geq \|P_\Omega(v) - P_\Omega(w)\|^2, \quad \forall v, w \in R^n. \tag{2.2}$$

Consequently, we have

$$\|P_\Omega(v) - P_\Omega(w)\| \leq \|v - w\|, \quad \forall v, w \in R^n \tag{2.3}$$

and

$$\|P_\Omega(v) - u\|^2 \leq \|v - u\|^2 - \|v - P_\Omega(v)\|^2, \quad \forall u \in \Omega, \quad \forall v \in \mathbb{R}^n.. \tag{2.4}$$

Lemma 2.1([2]p.267) Let $\beta > 0$, then u^* solves $\mathrm{VI}(\Omega, F)$ if and only if

$$u^* = P_\Omega[u^* - \beta F(u^*)].$$

Denote

$$e(u, \beta) := u - P_\Omega[u - \beta F(u)]. \tag{2.5}$$

Then solving VI(Ω, F) is equivalent to finding a zero point of $e(u,\beta)$. The next lemma states that $\|e(u,\beta)\|$ is a non-decreasing function for $\beta > 0$.

Lemma 2.2([12]Lemma2) For all $u \in R^n$ and $\tilde{\beta} \geq \beta > 0$, it holds that

$$\|e(u,\tilde{\beta})\| \geq \|e(u,\beta)\|. \tag{2.6}$$

2.2 A common lower bound for the progress functions

The PPA based prediction-correction methods use v^k or $\tilde{v}^k$ as predictors. As a preparation we present a common lower bound held by both progress functions $\Theta_k(\alpha, v^k)$ and $\Theta_k(\alpha, \tilde{v}^k)$ defined by equations (1.13) and (1.14) respectively. This common lower bound does not depend on the unknown solution u^*.

Assume $v \in \Omega$ and consider the following general correction step, which is shared by all PPA-PC methods:

$$u^{k+1}(\alpha, v) = P_\Omega[u^k - \alpha\beta_k F(v)]. \tag{2.7}$$

Consider the progress function

$$\Theta_k(\alpha, v) := \|u^k - u^*\|^2 - \|u^{k+1}(\alpha, v) - u^*\|^2. \tag{2.8}$$

Given the predictor v, the progress $\Theta_k(\alpha, v)$ is a function of the step length α. It is natural to consider maximizing this function by choosing an optimal parameter α. However, since u^* is the solution point and thus is unknown, we can not maximize $\Theta_k(\alpha, v)$ directly. The following proposition introduces a tight lower bound of $\Theta_k(\alpha, v)$, namely the function $\Upsilon_k(\alpha, v)$, which does not include the unknown solution u^*. The proposition hereby converts the task of maximizing the function $\Theta_k(\alpha, v)$ to that of maximizing the function $\Upsilon_k(\alpha, v)$.

Proposition 2.1 For given $u^k \in \Omega$ and $\beta_k > 0$, let $v \in \Omega$ be any point in Ω and the new iterate be produced by (2.7). Then we have

$$\Theta_k(\alpha, v) \geq \Upsilon_k(\alpha, v), \tag{2.9}$$

where $\Theta_k(\alpha, v)$ is defined in (2.8) and

$$\Upsilon_k(\alpha, v) := \|u^k - u^{k+1}(\alpha, v)\|^2 + 2\alpha\beta_k\{u^{k+1}(\alpha, v) - v\}^T F(v). \tag{2.10}$$

Proof Since $u^* \in \Omega$ and $u^{k+1}(\alpha, v) = P_\Omega[u^k - \alpha\beta_k F(v)]$, it follows from (2.4) that

$$\|u^{k+1}(\alpha, v) - u^*\|^2 \leq \|u^k - \alpha\beta_k F(v) - u^*\|^2 - \|u^k - \alpha\beta_k F(v) - u^{k+1}(\alpha, v)\|^2. \tag{2.11}$$

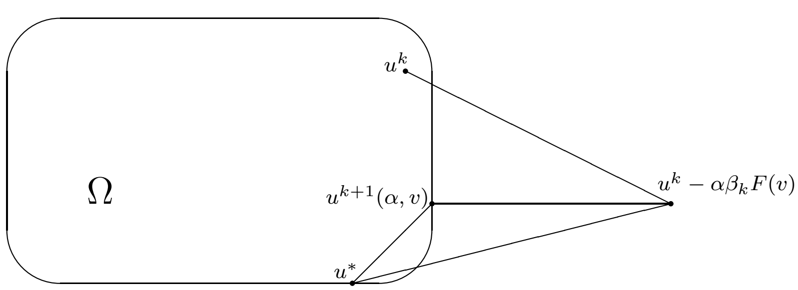

Fig. 2.1 Geometric interpretation of Inequality (2.11)

Consequently, using the definition of $\Theta_k(\alpha, v)$, we get

$$\begin{aligned}\Theta_k(\alpha, v) &\geq \|u^k - u^*\|^2 + \|u^k - u^{k+1}(\alpha, v) - \alpha\beta_k F(v)\|^2 - \|u^k - u^* - \alpha\beta_k F(v)\|^2 \\ &= \|u^k - u^{k+1}(\alpha, v)\|^2 + 2\alpha\beta_k\{u^{k+1}(\alpha, v) - u^k\}^T F(v) + 2\alpha\beta_k(u^k - u^*)^T F(v).\end{aligned} \quad (2.12)$$

Since $v \in \Omega$, using the monotonicity of F we have

$$(v - u^*)^T F(v) \geq (v - u^*)^T F(u^*) \geq 0$$

and consequently

$$(u^k - u^*)^T F(v) \geq (u^k - v)^T F(v). \quad (2.13)$$

Applying (2.13) to the last term in the right side of (2.12), we obtain

$$\Theta_k(\alpha, v) \geq \|u^k - u^{k+1}(\alpha, v)\|^2 + 2\alpha\beta_k\{u^{k+1}(\alpha, v) - v\}^T F(v) \quad (2.14)$$

and the assertion of this proposition is proved.

Remark 2.1 The inequality $\Theta_k(\alpha, v) \geq \Upsilon_k(\alpha, v)$ is tight in general. To see this we observe a special case in which $\Omega = R^n$, $F(u) = Mu + q$ and M is skew-symmetric. In this case, it follows from (2.7) that

$$u^{k+1}(\alpha, v) = u^k - \alpha\beta_k F(v)$$

and thus (2.11) is reduced to an equality. Because $\Omega = R^n$, we have $F(u^*) = 0$. In addition, since $F(u) = Mu + q$ and M is skew-symmetric, it follows that

$$(v - u^*)^T F(v) = (v - u^*)^T F(u^*)$$

and consequently

$$(u^k - u^*)^T F(v) = (u^k - v)^T F(v).$$

Inequality (2.13) is reduced to an equality. Therefore, we have $\Theta_k(\alpha, v) = \Upsilon_k(\alpha, v)$ in this special case.

The key technique applied in the proof of Proposition 2.1 is inequality (2.11). This technique was first used in [10] and later used in [12] for convergence analysis. Note that Proposition 2.1 is true for any $v \in \Omega$ and it does not guarantee that $\Upsilon_k(\alpha, v) > 0$ for all $\alpha > 0$ sufficiently small. In the following sections, we will convert $\Upsilon_k(\alpha, v)$ to some quadratic functions of α for $v = v^k$ and $v = \tilde{v}^k$, respectively.

3 Convergence Properties of Method PPA-PC1

In this section, we explore the convergence properties of the PPA-PC1 method. We investigate the choice of the optimal step length in the correction step and the inexactness restriction in the prediction step. The PPA-PC1 method uses v^k as the predictor, and the step length dependent correction formula is

$$u^{k+1}(\alpha, v^k) = P_\Omega[u^k - \alpha\beta_k F(v^k)]. \quad (3.1)$$

It follows from Proposition 2.1 that

$$\Theta_k(\alpha, v^k) = \|u^k - u^*\|^2 - \|u^{k+1}(\alpha, v^k) - u^*\|^2 \geq \Upsilon_k(\alpha, v^k)$$

and

$$\Upsilon_k(\alpha, v^k) = \|u^k - u^{k+1}(\alpha, v^k)\|^2 + 2\alpha\beta_k\{u^{k+1}(\alpha, v^k) - v^k\}^T F(v^k). \tag{3.2}$$

Notice that $u^{k+1}(\alpha, v^k)$ is obtained from (3.1) which includes a mapping of projection. Hence $\Upsilon_k(\alpha, v^k)$ is a non-differentiable function of α. Obtaining an optimal step length α for $\Upsilon_k(\alpha, v^k)$ directly is not straightforward. The following proposition offers us a tight lower bound of $\Upsilon_k(\alpha, v^k)$ which is a quadratic function of α.

Proposition 3.1 Given $u^k \in \Omega$ and $\beta_k > 0$, let $v^k \in \Omega$ be an approximate solution of (1.2) in the sense of (1.6) and the new iterate $u^{k+1}(\alpha, v^k)$ be given by (3.1). Then for any $\alpha > 0$ we have

$$\|u^k - u^*\|^2 - \|u^{k+1}(\alpha, v^k) - u^*\|^2 \geq \Upsilon_k(\alpha, v^k) \geq \Psi_k(\alpha), \tag{3.3}$$

where

$$\Psi_k(\alpha) := 2\alpha\{\|u^k - \tilde{v}^k\|^2 - \beta_k(\zeta^k)^T F(v^k)\} - \alpha^2\|u^k - \tilde{v}^k\|^2 \tag{3.4}$$

and ζ^k is as defined in (1.10).

Proof Note that Proposition 2.1 is true for any $v \in \Omega$ and

$$\begin{aligned}\Upsilon_k(\alpha, v^k) &= \|u^k - u^{k+1}(\alpha, v^k)\|^2 + 2\alpha\beta_k\{u^{k+1}(\alpha, v^k) - v^k\}^T F(v^k) \\ &= \|u^k - u^{k+1}(\alpha, v^k)\|^2 + 2\alpha\beta_k\{u^{k+1}(\alpha, v^k) - \tilde{v}^k\}^T F(v^k) \\ &\quad -2\alpha\beta_k(\zeta^k)^T F(v^k).\end{aligned} \tag{3.5}$$

In inequality (2.1) set $w := u^k - \beta_k F(v^k)$. From equation (1.11) we have $\tilde{v}^k = P_\Omega[u^k - \beta_k F(v^k)] = P_\Omega(w)$. Note that $u^{k+1}(\alpha, v^k) \in \Omega$, then it follows that for any $\alpha > 0$

$$0 \geq 2\alpha\{u^{k+1}(\alpha, v^k) - \tilde{v}^k\}^T\{[u^k - \beta_k F(v^k)] - \tilde{v}^k\}. \tag{3.6}$$

Adding (3.5) and (3.6), we obtain

$$\begin{aligned}\Upsilon_k(\alpha, v^k) \geq\ & \|u^k - u^{k+1}(\alpha, v^k)\|^2 + 2\alpha\{u^{k+1}(\alpha, v^k) - \tilde{v}^k\}^T(u^k - \tilde{v}^k) \\ & -2\alpha\beta_k(\zeta^k)^T F(v^k).\end{aligned} \tag{3.7}$$

Observe the first two terms of the right hand side of (3.7). We have

$$\begin{aligned}&\|u^k - u^{k+1}(\alpha, v^k)\|^2 + 2\alpha\{u^{k+1}(\alpha, v^k) - \tilde{v}^k\}^T(u^k - \tilde{v}^k) \\ &= \|u^k - u^{k+1}(\alpha, v^k)\|^2 + 2\alpha\{(u^{k+1}(\alpha, v^k) - u^k) + (u^k - \tilde{v}^k)\}^T(u^k - \tilde{v}^k) \\ &= \|u^k - u^{k+1}(\alpha, v^k)\|^2 + 2\alpha\{u^{k+1}(\alpha, v^k) - u^k\}^T(u^k - \tilde{v}^k) + 2\alpha\|u^k - \tilde{v}^k\|^2 \\ &= \|(u^k - u^{k+1}(\alpha, v^k)) - \alpha(u^k - \tilde{v}^k)\|^2 + (2\alpha - \alpha^2)\|u^k - \tilde{v}^k\|^2 \\ &\geq (2\alpha - \alpha^2)\|u^k - \tilde{v}^k\|^2.\end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.7), we obtain

$$\Upsilon_k(\alpha, v^k) \geq \Psi_k(\alpha)$$

and the assertion of this proposition is proved.

Remark 3.1 The inequality $\Upsilon_k(\alpha, v^k) \geq \Psi_k(\alpha)$ is tight for PPA-PC1. To see this we observe a special case $\Omega = R^n$. In this case, it follows from (1.11) that

$$\tilde{v}^k = v^k - F_k(v^k) = u^k - \beta_k F(v^k)$$

and thus (3.6) is reduced to an equality. In addition, equation (3.1) becomes

$$u^{k+1}(\alpha, v^k) = u^k - \alpha\beta_k F(v^k)$$

and thus

$$(u^k - u^{k+1}(\alpha, v^k)) - \alpha(u^k - \tilde{v}^k) = \alpha\beta_k F(v^k) - \alpha\beta_k F(v^k) = 0.$$

Hence inequality (3.8) is also reduced to an equality. Therefore $\Upsilon_k(\alpha, v^k) = \Psi_k(\alpha)$ in PPA-PC1 when $\Omega = R^n$.

The result of Proposition 3.1 is the foundation for investigating the convergence properties of the PPA-PC1 method. Based on the inequality

$$\|u^{k+1}(\alpha, v^k) - u^*\|^2 \leq \|u^k - u^*\|^2 - \Psi_k(\alpha), \tag{3.9}$$

where

$$\Psi_k(\alpha) = 2\alpha\{\|u^k - \tilde{v}^k\|^2 - \beta_k(\zeta^k)^T F(v^k)\} - \alpha^2\|u^k - \tilde{v}^k\|^2,$$

we explore the inexactness criteria in the prediction step as well as the step length in the correction step of method PPA-PC1. We call $\Psi_k(\alpha)$ a profit-function of method PPA-PC1 since it is the tight lower bound of the improvement obtained in the $k-$th iteration of the PPA-PC1 method. Let us consider the choice of the optimal step length α_k^* which maximizes the profit function $\Psi_k(\alpha)$ in the k-th iteration. Note that $\Psi_k(\alpha)$ is a quadratic function of α and it reaches its maximum at

$$\alpha_k^* = \frac{\|u^k - \tilde{v}^k\|^2 - \beta_k(\zeta^k)^T F(v^k)}{\|u^k - \tilde{v}^k\|^2} \tag{3.10}$$

with

$$\Psi_k(\alpha_k^*) = \alpha_k^*\Big(\|u^k - \tilde{v}^k\|^2 - \beta_k(\zeta^k)^T F(v^k)\Big). \tag{3.11}$$

As in the SOR methods for linear systems, for fast convergence, we propose a relaxation factor $\gamma_k \in [1, 2)$ and set the step-size α_k in (3.1) by $\alpha_k = \gamma_k\alpha_k^*$. The recommended correction formula of method PPA-PC1 is

$$\text{(PPA-PC1}^*) \qquad u^{k+1} = P_\Omega[u^k - \gamma_k\alpha_k^*\beta_k F(v^k)]. \tag{3.12}$$

By simple manipulations we obtain

$$\begin{aligned} \Psi_k(\gamma_k\alpha_k^*) &\overset{(3.4)}{=} 2\gamma_k\alpha_k^*\Big(\|u^k - \tilde{v}^k\|^2 - \beta_k(\zeta^k)^T F(v^k)\Big) - (\gamma_k^2\alpha_k^*)(\alpha_k^*\|u^k - \tilde{v}^k\|^2) \\ &\overset{(3.10)}{=} (2\gamma_k\alpha_k^* - \gamma_k^2\alpha_k^*)\Big(\|u^k - \tilde{v}^k\|^2 - \beta_k(\zeta^k)^T F(v^k)\Big) \\ &\overset{(3.11)}{=} \gamma_k(2 - \gamma_k)\Psi_k(\alpha_k^*). \end{aligned} \tag{3.13}$$

It follows from Proposition 3.1 that

$$\|u^{k+1} - u^*\|^2 \le \|u^k - u^*\|^2 - \gamma_k(2-\gamma_k)\Psi_k(\alpha_k^*). \tag{3.14}$$

In [10,12], the relaxation factor was recommended to be $\gamma_k \in [\gamma_l, \gamma_u] \subset [1.5, 1.8]$.

We are now in a position to consider the inexactness restriction in the prediction step of PPA-PC1. Notice that the result in Proposition 3.1 is valid only for $\alpha > 0$. In order to guarantee convergence of PPA-PC1 one must have $\alpha_k^* > 0$. Observe the check numerator in equation (3.10). Using $u^k - \tilde{v}^k = (u^k - v^k) + \zeta^k$ and $F_k(v) = (v - u^k) + \beta_k F(v)$, we have

$$\begin{aligned}
&\|u^k - \tilde{v}^k\|^2 - \beta_k(\zeta^k)^T F(v^k)\\
&= \frac{1}{2}\{\|(u^k - v^k) + \zeta^k\|^2 + \|u^k - \tilde{v}^k\|^2 - (\zeta^k)^T\beta_k F(v^k)\}\\
&= \frac{1}{2}\{\|u^k - v^k\|^2 + \|\zeta^k\|^2 + \|u^k - \tilde{v}^k\|^2\} - (\zeta^k)^T\{(v^k - u^k) + \beta_k F(v^k)\}\\
&= \frac{1}{2}\{\|u^k - v^k\|^2 + \|u^k - \tilde{v}^k\|^2\} - \{(\zeta^k)^T F_k(v^k) - \tfrac{1}{2}\|\zeta^k\|^2\}.
\end{aligned} \tag{3.15}$$

Therefore, we can consider the following inexactness restriction in the prediction step of PPA-PC1:

$$\boxed{(\zeta^k)^T F_k(v^k) - \tfrac{1}{2}\|\zeta^k\|^2 \le \tfrac{\nu}{2}\Big(\|u^k - v^k\|^2 + \|u^k - \tilde{v}^k\|^2\Big), \quad \nu < 1.} \tag{3.16}$$

Substituting (3.16) in (3.15) we obtain

$$\|u^k - \tilde{v}^k\|^2 - \beta_k(\zeta^k)^T F(v^k) \ge \frac{1-\nu}{2}\Big(\|u^k - v^k\|^2 + \|u^k - \tilde{v}^k\|^2\Big).$$

Consequently from (3.10) and (3.11) we have

$$\alpha_k^* > \frac{1-\nu}{2}$$

and

$$\Psi_k(\alpha_k^*) \ge \frac{(1-\nu)^2}{4}\Big(\|u^k - v^k\|^2 + \|u^k - \tilde{v}^k\|^2\Big). \tag{3.17}$$

Theorem 3.1 Given $u^k \in \Omega$ and $\beta_k > 0$, let $v^k \in \Omega$ be an approximate solution of (1.2) in the sense of (1.6) and the new iterate u^{k+1} be generated by (3.12). If the inexactness criterion (3.16) holds, then $\{u^k\}$ converges to some $u^\infty \in \Omega^*$.

Proof First, according to the analysis above, we have

$$\begin{aligned}
\|u^{k+1} - u^*\|^2 &\overset{(3.14)}{\le} \|u^k - u^*\|^2 - \gamma_k(2-\gamma_k)\Psi_k(\alpha_k^*)\\
&\overset{(3.17)}{\le} \|u^k - u^*\|^2 - \frac{\gamma_k(2-\gamma_k)(1-\nu)^2}{4}\Big(\|u^k - v^k\|^2 + \|u^k - \tilde{v}^k\|^2\Big).
\end{aligned}$$

Since $\gamma_k \in [\gamma_l, \gamma_u] \subset (0, 2)$, there is a constant $c_0 > 0$ such that

$$\|u^{k+1} - u^*\|^2 \le \|u^k - u^*\|^2 - c_0(\|u^k - v^k\|^2 + \|u^k - \tilde{v}^k\|^2), \qquad \forall u^* \in \Omega^*. \tag{3.18}$$

This means that the sequence $\{u^k\}$ is bounded. Next, we have

$$\sum_{k=1}^{\infty} c_0\Big(\|u^k - v^k\|^2 + \|u^k - \tilde{v}^k\|^2\Big) \le \|u^0 - u^*\|^2.$$

Therefore,

$$\lim_{k\to\infty} \|u^k - v^k\| = 0, \qquad \lim_{k\to\infty} \|u^k - \tilde{v}^k\| = 0,$$

and consequently $\{v^k\}$ is also bounded. Moreover, since $\zeta^k = (u^k - \tilde{v}^k) - (u^k - v^k)$, we have

$$\lim_{k\to\infty} \|\zeta^k\| = 0.$$

Since $\beta_k \geq \beta_{\min}$, it follows from Lemma 2.2 that

$$\begin{aligned}
\|e(v^k, \beta_{\min})\| &\leq \|v^k - P_\Omega[v^k - \beta_k F(v^k)]\| \\
&\overset{(1.10)}{=} \|\zeta^k + \tilde{v}^k - P_\Omega[v^k - \beta_k F(v^k)]\| \\
&\overset{(1.11)}{\leq} \|\zeta^k\| + \|P_\Omega[u^k - \beta_k F(v^k)] - P_\Omega[v^k - \beta_k F(v^k)]\| \\
&\overset{(2.3)}{\leq} \|\zeta^k\| + \|u^k - v^k\|
\end{aligned}$$

and thus

$$\lim_{k\to\infty} e(v^k, \beta_{\min}) = 0. \tag{3.19}$$

Let u^∞ be a cluster point of $\{v^k\}$ and the subsequence $\{v^{k_j}\}$ converges to u^∞. Since $e(u,\beta)$ is a continuous function of u, it follows from (3.19) that

$$e(u^\infty, \beta_{\min}) = \lim_{j\to\infty} e(v^{k_j}, \beta_{\min}) = 0.$$

According to Lemma 2.1, u^∞ is a solution point of $\mathrm{VI}(\Omega, F)$. Note that inequality (3.18) is true for all solution points of $\mathrm{VI}(\Omega, F)$, hence we have

$$\|u^{k+1} - u^\infty\|^2 \leq \|u^k - u^\infty\|^2, \quad \forall k \geq 0 \tag{3.20}$$

and it follows that the sequence $\{u^k\}$ converges to u^∞.

Remark 3.2 The method proposed in [15] is a specific implementation of the PPA-PC1 method (1.8). In [15] the inexactness restriction for prediction is set as

$$(\zeta^k)^T F_k(v^k) - \frac{1}{2}\|\zeta^k\|^2 \leq \frac{\nu}{2}\|u^k - v^k\|^2, \quad \nu < 1, \tag{3.21}$$

and the correction step is set as

$$u^{k+1} = P_\Omega[u^k - \beta_k F(v^k)]. \tag{3.22}$$

It is clear that condition (3.21) is more restrictive than condition (3.16). Under restriction (3.21) in the prediction step, it follows that

$$\begin{aligned}
\frac{1}{2}\|u^k - \tilde{v}^k\|^2 - \beta_k(\zeta^k)^T F(v^k) &= \frac{1}{2}\|u^k - v^k\|^2 - \Big((\zeta^k)^T[(v^k - u^k) + \beta_k F(v^k)] - \frac{1}{2}\|\zeta^k\|^2\Big) \\
&\geq \frac{1-\nu}{2}\|u^k - v^k\|^2.
\end{aligned}$$

Consequently from (3.10) we have

$$\alpha_k^* = \frac{\|u^k - \tilde{v}^k\|^2 - \beta_k(\zeta^k)^T F(v^k)}{\|u^k - \tilde{v}^k\|^2} \geq \frac{\frac{1}{2}\|u^k - \tilde{v}^k\|^2 + \frac{1-\nu}{2}\|u^k - v^k\|^2}{\|u^k - \tilde{v}^k\|^2} > 0.5.$$

The correction step (3.22) can be viewed as a special form of (3.12) by setting $\alpha_k = \gamma_k^* \alpha_k^*$ where

$$\gamma_k^* := \frac{1}{\alpha_k^*} \in (0, 2). \tag{3.23}$$

From equation (3.4) we have

$$\Psi_k(1) = \|u^k - \tilde{v}^k\|^2 - 2\beta_k(\zeta^k)^T F(v^k).$$

setting $u^k - \tilde{v}^k = u^k - v^k + \zeta^k$, we have

$$\Psi_k(1) = \|u^k - v^k\|^2 - \Big(2(\zeta^k)^T\{(v^k - u^k) + \beta_k F(v^k)\} - \|\zeta^k\|^2\Big).$$

Applying the inexactness restriction (3.21) we obtain

$$\Psi_k(1) \geq (1 - \nu)\|u^k - v^k\|^2. \tag{3.24}$$

It follows from Proposition 3.1 that the sequence $\{u^k\}$ generated by the algorithm proposed in [15] satisfies the inequality

$$\|u^{k+1} - u^*\|^2 \leq \|u^k - u^*\|^2 - (1 - \nu)\|u^k - v^k\|^2. \tag{3.25}$$

The convergence of the algorithm follows from (3.25) directly.

4 Convergence Properties of Method PPA-PC2

In this section, we investigate the similar properties of PPA-PC2, namely the optimal step length in the correction step and the inexactness restriction in the prediction step. The method PPA-PC2 uses $\tilde{v}^k$ (defined in (1.7)) as the predictor, and its correction formula is given by

$$u^{k+1}(\alpha, \tilde{v}^k) = P_\Omega[u^k - \alpha\beta_k F(\tilde{v}^k)], \tag{4.1}$$

which is dependent on the parameter α. It follows from Proposition 2.1 that

$$\Theta_k(\alpha, \tilde{v}^k) = \|u^k - u^*\|^2 - \|u^{k+1}(\alpha, \tilde{v}^k) - u^*\|^2 \geq \Upsilon_k(\alpha, \tilde{v}^k)$$

and

$$\Upsilon_k(\alpha, \tilde{v}^k) = \|u^k - u^{k+1}(\alpha, \tilde{v}^k)\|^2 + 2\alpha\beta_k(u^{k+1}(\alpha, \tilde{v}^k) - \tilde{v}^k)^T F(\tilde{v}^k). \tag{4.2}$$

Since $u^{k+1}(\alpha, \tilde{v}^k)$ is obtained from (4.1), $\Upsilon_k(\alpha, \tilde{v}^k)$ is again a non-differentiable function of α. Following the similar discussion as in the previous section, we seek a tight lower bound of $\Upsilon_k(\alpha, \tilde{v}^k)$ which is a smooth function of α.

Since $\tilde{v}^k = P_\Omega[u^k - \beta_k F(v^k)]$ and $\xi^k = \beta_k(F(\tilde{v}^k) - F(v^k))$ ((1.7) and (1.12)), the predictor $\tilde{v}^k$ in PPA-PC2 satisfies the following equation:

$$\tilde{v}^k = P_\Omega[u^k - \beta_k F(\tilde{v}^k) + \xi^k]. \tag{4.3}$$

According to Lemma 2.1, the pair $\tilde{v}^k$ and ξ^k satisfies the inequality

$$\tilde{v}^k \in \Omega, \quad (u' - \tilde{v}^k)^T\{(\tilde{v}^k - u^k) + \beta_k F(\tilde{v}^k) - \xi^k\} \geq 0, \quad \forall u' \in \Omega. \tag{4.4}$$

A method of type (4.1) which uses $\tilde{v}^k$ from (4.4) as predictor was proposed by the first author in [13].

Proposition 4.1 Given $u^k \in \Omega$ and $\beta_k > 0$, let $v^k \in \Omega$ be an approximate solution of (1.2) in the sense of (1.6), $\tilde{v}^k$ be defined in (1.7) and the new iterate $u^{k+1}(\alpha, \tilde{v}^k)$ be given by (4.1). Then for any $\alpha > 0$ we have

$$\|u^k - u^*\|^2 - \|u^{k+1}(\alpha, \tilde{v}^k) - u^*\|^2 \geq \Upsilon_k(\alpha, \tilde{v}^k) \geq \Phi_k(\alpha), \tag{4.5}$$

where

$$\Phi_k(\alpha) := 2\alpha(u^k - \tilde{v}^k)^T d^k - \alpha^2\|d^k\|^2, \tag{4.6}$$

$$d^k := u^k - \tilde{v}^k + \xi^k, \tag{4.7}$$

and ξ^k is defined as in (1.12).

Proof The proof is parallel to the proof of Proposition 3.1. Based on (4.3), it follows from (2.1) that

$$0 \geq 2\alpha\{u^{k+1}(\alpha, \tilde{v}^k) - \tilde{v}^k\}^T\{[u^k - \beta_k F(\tilde{v}^k) + \xi^k] - \tilde{v}^k\}.$$

Adding the inequality above and (4.2) similar to inequality (3.7), for any $\alpha \geq 0$, we obtain

$$\Upsilon_k(\alpha, \tilde{v}^k) \geq \|u^k - u^{k+1}(\alpha, \tilde{v}^k)\|^2 + 2\alpha(u^{k+1}(\alpha, \tilde{v}^k) - \tilde{v}^k)^T(u^k - \tilde{v}^k + \xi^k). \tag{4.8}$$

Using the notation of d^k, we obtain

$$\begin{array}{rcl} \Upsilon_k(\alpha, \tilde{v}^k) & \geq & \|u^k - u^{k+1}(\alpha, \tilde{v}^k)\|^2 + 2\alpha\{(u^{k+1}(\alpha, \tilde{v}^k) - u^k) + (u^k - \tilde{v}^k)\}^T d^k \\ & = & \|(u^k - u^{k+1}(\alpha, \tilde{v}^k)) - \alpha d^k\|^2 + 2\alpha(u^k - \tilde{v}^k)^T d^k - \alpha^2\|d^k\|^2 \\ & \geq & 2\alpha(u^k - \tilde{v}^k)^T d^k - \alpha^2\|d^k\|^2 \\ & = & \Phi_k(\alpha) \end{array} \tag{4.9}$$

and the assertion of this proposition is proved.

Remark 4.1 For the PPA-PC2 method, inequality $\Upsilon_k(\alpha, \tilde{v}^k) \geq \Phi_k(\alpha)$ is tight. Applying similar argument as in Remark 3.1 we obtain that $\Upsilon_k(\alpha, \tilde{v}^k) = \Phi_k(\alpha)$ when $\Omega = R^n$.

The result in Proposition 4.1 is the foundation for investigating the convergence properties of the PPA-PC2 method. Based on the inequality

$$\|u^{k+1}(\alpha, \tilde{v}^k) - u^*\|^2 \leq \|u^k - u^*\|^2 - \Phi_k(\alpha), \tag{4.10}$$

and

$$\Phi_k(\alpha) = 2\alpha(u^k - \tilde{v}^k)^T d^k - \alpha^2\|d^k\|^2,$$

we explore the inexactness criteria in the prediction step as well as the step length in the correction step of method PPA-PC2. We refer to $\Phi_k(\alpha)$ as a profit-function of method PPA-PC2, since it measures the improvement obtained in the k-th iteration of method PPA-PC2. It is natural to maximize the profit function $\Phi_k(\alpha)$ in each iteration. Note that $\Phi_k(\alpha)$ is a quadratic function of α and it reaches its maximum at

$$\alpha_k^* = \frac{(u^k - \tilde{v}^k)^T d^k}{\|d^k\|^2} \tag{4.11}$$

with

$$\Phi_k(\alpha_k^*) = \alpha_k^*(u^k - \tilde{v}^k)^T d^k. \tag{4.12}$$

Following the discussion in the previous section, for fast convergence we propose a relaxation factor $\gamma_k \in [1, 2)$ and set the step-size α_k in (4.1) by $\alpha_k = \gamma_k \alpha_k^*$. The recommended correction formula is

$$u^{k+1} = P_\Omega[u^k - \gamma_k \alpha_k^* \beta_k F(\tilde{v}^k)]. \tag{4.13}$$

By simple manipulations we obtain

$$\begin{aligned} \Phi_k(\gamma_k \alpha_k^*) &\overset{(4.6)}{=} 2\gamma_k \alpha_k^*(u^k - \tilde{v}^k)^T d^k - (\gamma_k^2 \alpha_k^*)(\alpha_k^* \|d^k\|^2) \\ &\overset{(4.11)}{=} (2\gamma_k \alpha_k^* - \gamma_k^2 \alpha_k^*)(u^k - \tilde{v}^k)^T d^k \\ &\overset{(4.12)}{=} \gamma_k(2 - \gamma_k)\Phi_k(\alpha_k^*). \end{aligned} \tag{4.14}$$

It follows from Proposition 4.1 that

$$\|u^{k+1} - u^*\|^2 \le \|u^k - u^*\|^2 - \gamma_k(2 - \gamma_k)\Phi_k(\alpha_k^*). \tag{4.15}$$

In [10,12], the relaxation factor was recommended to be $\gamma_k \in [\gamma_l, \gamma_u] \subset [1.5, 1.8]$.

In order to guarantee $\alpha_k^* > 0$, we only need

$$|(u^k - \tilde{v}^k)^T \xi^k| \le \nu \|u^k - \tilde{v}^k\|^2, \quad \nu < 1.$$

However, to obtain numerical stability we propose

$$\boxed{|(u^k - \tilde{v}^k)^T \xi^k| \le \nu \|u^k - \tilde{v}^k\|^2 \quad \text{and} \quad \|\xi^k\| \le \mu \|u^k - \tilde{v}^k\|, \quad 0 < \nu < 1 \le \mu} \tag{4.16}$$

as a recommended inexactness restriction in the prediction step of PPA-PC2.

We now proceed to obtain a lower bound for the profit function $\Phi_k(\alpha_k^*)$. From the first part of (4.16), we have

$$(u^k - \tilde{v}^k)^T d^k = \|u^k - \tilde{v}^k\|^2 + (u^k - \tilde{v}^k)^T \xi^k \ge (1 - \nu)\|u^k - \tilde{v}^k\|^2. \tag{4.17}$$

In the case of $(u^k - \tilde{v}^k)^T \xi^k \le 0$, it follows from (4.7) that

$$\begin{aligned} \|d^k\|^2 &\le \|u^k - \tilde{u}^k\|^2 + \|\xi^k\|^2 \\ &\overset{(4.16)}{\le} (1 + \mu^2)\|u^k - \tilde{v}^k\|^2 \\ &\overset{(4.17)}{\le} \tfrac{(1+\mu^2)}{1-\nu}(u^k - \tilde{v}^k)^T d^k. \end{aligned} \tag{4.18}$$

Otherwise, if $(u^k - \tilde{v}^k)^T \xi^k \ge 0$, noticing that $\mu \ge 1$ we have

$$\begin{aligned} \|d^k\|^2 &\overset{(4.16)}{\le} (1 + \mu^2)\|u^k - \tilde{v}^k\|^2 + 2(u^k - \tilde{v}^k)^T \xi^k \\ &\le (1 + \mu^2)\|u^k - \tilde{v}^k\|^2 + (1 + \mu^2)(u^k - \tilde{v}^k)^T \xi^k \\ &\overset{(4.7)}{=} (1 + \mu^2)(u^k - \tilde{v}^k)^T d^k. \end{aligned} \tag{4.19}$$

Therefore, from (4.11), (4.18) and (4.19), we have

$$\alpha_k^* \geq \frac{1-\nu}{1+\mu^2}.$$

and consequently from (4.12) and (4.17),

$$\Phi_k(\alpha_k^*) \geq \frac{(1-\nu)^2}{1+\mu^2}\|u^k - \tilde{v}^k\|^2. \tag{4.20}$$

If the inexactness restriction (4.16) holds, then from (4.15) and (4.20) we have

$$\|u^{k+1} - u^*\|^2 \leq \|u^k - u^*\|^2 - \frac{\gamma_k(2-\gamma_k)(1-\nu)^2}{1+\mu^2}\|u^k - \tilde{v}^k\|^2. \tag{4.21}$$

Theorem 4.1 Given $u^k \in \Omega$ and $\beta_k > 0$, let $v^k \in \Omega$ be an approximate solution of (1.2) in the sense of (1.6) and the new iterate u^{k+1} be generated by (4.13). If the inexactness criterion (4.16) holds, then $\{u^k\}$ converges to some $u^\infty \in \Omega^*$

Proof The proof is based on inequality (4.21) and is similar to the proof of Theorem 3.1. We omit the details.

5 Comparison of PPA-PC1 and PPA-PC2

In this section we compare the inexactness restrictions of the two methods. Both restrictions are tight for the convergence of the methods. We indicate that the restriction on PPA-PC2 is much more relaxed compared to the restriction on PPA-PC1, as illustrated in the particular case where the variational inequality is linear.

First we summarize the main features of each method. The PPA-PC1 method takes v^k as the predictor and the main result is

$$\Theta_k(\alpha, v^k) = \|u^k - u^*\|^2 - \|u^{k+1}(\alpha, v^k) - u^*\|^2 \geq \Upsilon_k(\alpha, v^k) \geq \Psi_k(\alpha), \tag{5.1}$$

where

$$\Psi_k(\alpha) = 2\alpha\{\|u^k - \tilde{v}^k\|^2 - \beta_k(\zeta^k)^T F(v^k)\} - \alpha^2\|u^k - \tilde{v}^k\|^2. \tag{5.2}$$

On the other hand, the method PPA-PC2 uses $\tilde{v}^k$ as the predictor and the relevant result is

$$\Theta_k(\alpha, \tilde{v}^k) = \|u^k - u^*\|^2 - \|u^{k+1}(\alpha, \tilde{v}^k) - u^*\|^2 \geq \Upsilon_k(\alpha, \tilde{v}^k) \geq \Phi_k(\alpha), \tag{5.3}$$

where

$$\Phi_k(\alpha) = 2\alpha\{\|u^k - \tilde{v}^k\|^2 + (u^k - \tilde{v}^k)^T\xi^k\} - \alpha^2\|(u^k - \tilde{v}^k) + \xi^k\|^2. \tag{5.4}$$

Functions $\Psi_k(\alpha)$ (5.2) and $\Phi_k(\alpha)$ (5.4) are the tight differentiable lower bounds of $\Theta_k(\alpha, v^k)$ and $\Theta_k(\alpha, \tilde{v}^k)$, respectively. Since Inequalities (5.1) and (5.3) can not be improved in the general case, it is reasonable to take functions $\Psi_k(\alpha)$ and $\Phi_k(\alpha)$ as the foundations for analysis of methods PPA-PC1 and PPA-PC2, respectively.

The inexactness restriction in PPA-PC1 is based on function $\Psi_k(\alpha)$. Since $v^k \in \Omega$ and $\tilde{v}^k = P_\Omega[v^k - F_k(v^k)]$, it follows from (2.1) that

$$\{(v^k - F_k(v^k)) - \tilde{v}^k\}^T\{v^k - \tilde{v}^k\} \leq 0.$$

From (1.10) and the above inequality we have

$$(\zeta^k)^T F_k(v^k) \geq \|\zeta^k\|^2. \tag{5.5}$$

Therefore, in order to satisfy the inexactness restriction (3.16) in the prediction step of PPA-PC1, namely,

$$(\zeta^k)^T F_k(v^k) - \frac{1}{2}\|\zeta^k\|^2 \leq \frac{\nu}{2}\Big(\|v^k - u^k\|^2 + \|u^k - \tilde{v}^k\|^2\Big), \quad \nu < 1,$$

it is necessary to have

$$\boxed{(v^k - \tilde{v}^k)^T F_k(v^k) \leq \nu(\|u^k - v^k\|^2 + \|u^k - \tilde{v}^k\|^2), \quad \nu < 1.} \tag{5.6}$$

Usually $F(u^*) \neq 0$, as $u^k \to u^*$, if the direction $v^k - P_\Omega[v^k - F_k(v^k)]$ is almost parallel to $F_k(v^k)$, it follows from (5.6) that

$$\|v^k - \tilde{v}^k\| = O(\|u^k - v^k\|^2 + \|u^k - \tilde{v}^k\|^2). \tag{5.7}$$

Meanwhile, for the PPA-PC2 method, $\Phi_k(\alpha)$ is the base for considering the inexactness restriction in the prediction. Recall that the inexactness restriction (4.16) is

$$|(u^k - \tilde{v}^k)^T \xi^k| \leq \nu\|u^k - \tilde{v}^k\|^2 \quad \text{and} \quad \|\xi^k\| \leq \mu\|u^k - \tilde{v}^k\|, \quad 0 < \nu < 1 \leq \mu.$$

Since $\xi^k = \beta_k[F(\tilde{v}^k) - F(v^k)]$, the above inequalities are satisfied if

$$\boxed{\beta_k\|F(v^k) - F(\tilde{v}^k)\| \leq \nu\|u^k - \tilde{v}^k\|, \quad \nu < 1.} \tag{5.8}$$

If F is Lipschitz continuous, then it follows from (5.8) that

$$\|v^k - \tilde{v}^k\| = O(\|u^k - \tilde{v}^k\|). \tag{5.9}$$

Comparing (5.7) and (5.9), it seems that the inexactness restriction (4.16) in PPA-PC2 is much more relaxed than (3.16) in PPA-PC1 when u^k is near a solution point.

In order to illustrate the difference in the two inexactness restrictions, we consider the phenomenon where the two PPA-PC methods are applied to the following linear variational inequality

$$\text{LVI}(\Omega, M, q) \qquad u^* \in \Omega, \quad (u - u^*)^T(Mu^* + q) \geq 0, \quad \forall u \in \Omega, \tag{5.10}$$

where M is a skew-symmetric matrix.

By setting $v^k := u^k$, we have

$$\tilde{v}^k = P_\Omega[u^k - \beta_k(Mu^k + q)] \tag{5.11}$$

and

$$\xi^k = \beta_k M(\tilde{v}^k - u^k).$$

Since M is skew-symmetric, $M^T = -M$, it follows that $(u^k - \tilde{v}^k)^T \xi^k = 0$. Therefore, for any $\nu \in (0, 1)$ and $\mu = \max\{1, \beta_{\max}\|M\|\}$, we have

$$|(u^k - \tilde{v}^k)^T \xi^k| \leq \nu\|u^k - \tilde{v}^k\|^2 \qquad \text{and} \qquad \|\xi^k\| \leq \mu\|u^k - \tilde{v}^k\|. \tag{5.12}$$

The inexactness restriction (4.16) in PPA-PC2 is held.

For the PPA-PC1 method, when $v^k = u^k$, from the left and right hand sides of (5.6) we obtain

$$(v^k - \tilde{v}^k)^T F_k(v^k) \overset{(5.5)}{\geq} \|v^k - \tilde{v}^k\|^2 = \|u^k - \tilde{v}^k\|^2$$

and

$$\nu(\|u^k - v^k\|^2 + \|u^k - \tilde{v}^k\|^2) = \nu\|u^k - \tilde{v}^k\|^2,$$

respectively. Therefore, this predictor ($v^k = u^k$) does not satisfy (5.6) and thus does not satisfy the inexactness restriction (3.16) in the method PPA-PC1.

The implementation advantage of the PPA-PC2 method over the PPA-PC1 method becomes obvious in this case. Some numerical experiments in [14] showed how efficiently the PPA-PC2 method can be implemented when it is applied to a linear variational inequality raised from a constrained shortest network problem [19].

6 Conclusion

In this paper we study two methods (PPA-PC1 and PPA-PC2), which are the proximal point algorithm (PPA) based prediction-correction (PC) methods for monotone variational inequalities. For each method we present a profit function. We show that the profit functions are tight lower bounds of the improvement obtained in each iteration for the methods. Based on this conclusion we then obtain the convergence inexactness restrictions for the prediction step. By comparing the inexactness restrictions for the two methods we conclude that the PPA-PC2 method possesses much stronger computational efficiency, since its inexactness restriction is much relaxed compared to that of the PPA-PC1 method.

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