English Series | NUMERICAL MATHEMATICS A Journal of Chinese Universities | Vol.14, No.1 Feb. 2005

NUMERICAL SOLUTIONS OF AN EIGENVALUE PROBLEM IN UNBOUNDED DOMAINS*

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Abstract *A coupling method of finite element and infinite large element is proposed for the numerical solution of an eigenvalue problem in unbounded domains in this paper. With some conditions satisfied, the considered problem is proved to have discrete spectra. Several numerical experiments are presented. The results demonstrate the feasibility of the proposed method.*

Key words *eigenvalue problem, unbounded domain, infinite large element method.*

AMS(2000)subject classifications 35P05

1 Introduction

A great deal of work has been done on the numerical solution of eigenvalue problems, which have widespread applications in physics and engineering. In this paper, we will consider an eigenvalue problem in unbounded domains and introduce a numerical approach for the proposed problem.

This paper is inspired by the successful application of infinite large element to Helmholtz equation in exterior domains by K.Gerds[7,8,9] and L.Demkowicz[9] and various concepts of large element and infinite large elements developed by Han Hou-de and Ying Lung-an[2], P.Bettess[4,5], and D.S.Burnett[6]. Here we will introduce a coupling method of finite element(FE) and infinite large element(ILE) to overcome the essential difficulty for obtaining the numerical solution of the given problem which originates from the unboundedness of physical domain.

We now consider the following eigenvalue problem in the unbounded domain Ω^e

* Supported partial by the Natinal Science Foundation of China under Grant No. 10401020 and Grant No. 10471073.
Received: Jun. 10, 2004.

Find $\lambda \in \mathbb{C}, u \neq 0$ such that

$$-\triangle u - \lambda\rho(x)u = 0, \qquad \text{in } \Omega^e, \tag{1.1}$$

$$u|_\Gamma = 0, \qquad \text{on } \Gamma, \tag{1.2}$$

$$\int_{\Omega^e} |\nabla u|^2 \mathrm{d}x + \int_{\Omega^e} \rho|u|^2 \mathrm{d}x < \infty. \tag{1.3}$$

Here $\Omega = \mathbb{R}^2 \backslash \bar{\Omega}^e$ is assumed to be a bounded domain with smooth boundary Γ. $\rho(x) > 0$ is continuous in $\mathbb{R}^2$. Besides, we assume

$$B = \{x : |x| < 1\} \subset\subset \Omega.$$

Problem (1.1)-(1.3) can be deduced from the following initial-boundary of heat equation on the unbounded domain $\Omega^e \times (0, T]$:

$$\rho(x)\frac{\partial w}{\partial t} = \triangle w, \quad (x, t) \in \Omega^e \times (0, T], \tag{1.4}$$

$$w|_\Gamma = 0, \qquad 0 < t \leq T, \tag{1.5}$$

$$w|_{t=0} = w_0(x), \tag{1.6}$$

$$w(x, t) \ \text{ is bounded}, \tag{1.7}$$

where $w_0(x)$ is a given function, $w|_\Gamma = 0$ and support $\{w_0(x)\}$ is compact. Consider the solution of the problem (1.4)-(1.7) in the following form

$$w(x, t) = e^{-\lambda t} u(x), \tag{1.8}$$

where $(\lambda, u(x))$ is to be determined. Substituting (1.8) into problem (1.4)-(1.7) we know that $\{\lambda, u(x)\}$ is determined by eigenvalue problem (1.1)-(1.3).

The organization is as the following. In section 2, we introduce the variational formulation of the eigenvalue problem and analyze some of its properties. In section 3, we present the discretization of exterior domain. Some numerical examples are given in section 4.

2 Some properties of the eigenvalue problem

We suppose $\rho(x)$ satisfies:

$$0 < \delta(R) \leq \min_{1 \leq |x| \leq R} \rho(x), \quad \text{for} \quad R \geq 1, \tag{2.1}$$

$$\rho(x) \leq \rho_0(|x|), \quad 1 \leq |x| < +\infty, \tag{2.2}$$

$$M = \int_1^\infty r \ln r \rho_0(r) \mathrm{d}r < +\infty,$$

where $\delta(R)$ is a given function. Denote

$$C_*^\infty(\Omega^e)=\Big\{v: v\in C^\infty(\Omega^e), \frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}\in C_0^\infty(\overline{\Omega}^e), v|_\Gamma=0\Big\}$$

and introduce an inner product

$$(u,v)_{1,\rho,\Omega^e}=\int_{\Omega^e}\nabla u\cdot\nabla\bar{v}\mathrm{d}x+\int_{\Omega^e}\rho u\bar{v}\mathrm{d}x \tag{2.3}$$

and a semi-inner product

$$[u,v]_{1,\rho,\Omega^e}=\int_{\Omega^e}\nabla u\cdot\nabla\bar{v}\mathrm{d}x$$

in this space. The induced norm and seminorm are defined by

$$\|u\|_{1,\rho,\Omega^e}=\sqrt{(u,u)_{1,\rho,\Omega^e}},\ |u|_{1,\rho,\Omega^e}=\sqrt{[u,u]_{1,\rho,\Omega^e}}.$$

Denote $H_0^{1,\rho}(\Omega^e)$ as the completed space of $C_*^\infty(\Omega^e)$ under inner product (2.3), and it is easy to prove

$$H_0^{1,\rho}(\Omega^e)\subset\Big\{u:\int_{\Omega^e}|\nabla u|^2\mathrm{d}x+\int_{\Omega^e}\rho|u|^2dx<\infty, u|_\Gamma=0\Big\}.$$

Furthermore, we denote

$$H^{0,\rho}(\Omega^e)=\Big\{u:\int_{\Omega^e}\rho|u|^2\mathrm{d}x<\infty\Big\}.$$

$H^{0,\rho}(\Omega^e)$ has a natural inner product:

$$(u,v)_{0,\rho,\Omega^e}=\int_{\Omega^e}\rho u\bar{v}\mathrm{d}x,$$

and the corresponding norm is denoted as

$$\|u\|_{0,\rho,\Omega^e}=\sqrt{(u,u)_{0,\rho,\Omega^e}}.$$

It is obvious that $H^{0,\rho}(\Omega^e)$ is a Hilbert space. Let R be a positive number and satisfy

$$\Gamma_R\overset{\text{def}}{=}\{x:|x|=R\}\subset\Omega^e.$$

Denote:

$$H_0^{1,\rho}(\Omega_R)=\Big\{u|_{\Omega_R}: u\in H_0^{1,\rho}(\Omega^e)\Big\},$$

$$H^{0,\rho}(\Omega_R)=\Big\{u|_{\Omega_R}: u\in H^{0,\rho}(\Omega^e)\Big\},$$

where $\Omega_R=\{x: x\in\Omega^e\ , |x|<R\}$. Let $\Omega_R^*=\{x:|x|>R\}$.

Lemma 2.1 (Poincare inequality) Suppose that $\rho(x)$ satisfies the condition (2.1)-(2.2), then there exists a constant $C>0$ such that

$$\|u\|_{1,\rho,\Omega^e}\le C|u|_{1,\rho,\Omega^e}, \qquad \forall u\in H_0^{1,\rho}(\Omega^e).$$

Proof Since $C_*^\infty(\Omega^e)$ is dense in $\in H_0^{1,\rho}(\Omega^e)$, we need only to prove $\forall u \in C_*^\infty(\Omega^e)$, there exists a constant $C > 0$ such that

$$\|u\|_{1,\rho,\Omega^e} \leq C|u|_{1,\rho,\Omega^e}, \qquad \forall u \in H_0^{1,\rho}(\Omega^e).$$

Let

$$\hat{u} = \begin{cases} u(x) & x \in \Omega^e, \\ 0 & x \in \bar{\Omega}, \end{cases}$$

then, we have

$$|\hat{u}^2(r,\theta)| = \Big|\int_1^r \frac{\partial \hat{u}}{\partial r}\mathrm{d}r\Big|^2 \leq \int_1^r r\Big|\frac{\partial \hat{u}}{\partial r}\Big|^2 \mathrm{d}r \cdot \ln r \leq \int_1^\infty r\Big|\frac{\partial \hat{u}}{\partial r}\Big|^2 \mathrm{d}r \cdot \ln r, \quad r \geq 1.$$

Multiplying $r\rho_0(r)$ on the above inequality and integrating with respect θ and r we obtain

$$\int_0^{2\pi}\int_1^\infty \rho_0(r)|\hat{u}(r,\theta)|^2 r\mathrm{d}r\mathrm{d}\theta \leq M\int_0^{2\pi}\int_1^\infty \Big|\frac{\partial \hat{u}(r,\theta)}{\partial r}\Big|^2 r\mathrm{d}r\mathrm{d}\theta.$$

It follows immediately that

$$\int_{\mathbb{R}^2\setminus B} \rho|\hat{u}|^2\mathrm{d}x \leq M\int_0^{2\pi}\int_1^\infty \Big|\frac{\partial \hat{u}(r,\theta)}{\partial r}\Big|^2 r\mathrm{d}r\mathrm{d}\theta.$$

Namely,

$$\|u\|_{0,\rho,\Omega^e} \leq \sqrt{M}|u|_{1,\rho,\Omega^e}.$$

The proof is complete.

Lemma 2.2 Suppose that $\rho(x)$ satisfies the conditions (2.1)-(2.2), then the imbedding from $H_0^{1,\rho}(\Omega^e)$ to $H^{0,\rho}(\Omega^e)$ is compact.

Proof Let S be a bounded subset of $H_0^{1,\rho}(\Omega^e)$, and for any $u \in S$,

$$\|u\|_{1,\rho,\Omega^e} \leq M_S.$$

We only need to prove that S is a compact set in $H^{0,\rho}(\Omega^e)$.

For any $u \in S$, by the Lemma 2.1, we have

$$\begin{aligned}\int_{\Omega_\eta^*} \rho|u|^2\mathrm{d}x \leq \int_{\Omega_\eta^*} \rho_0|u|^2\mathrm{d}x &= \int_0^{2\pi}\int_\eta^\infty \rho_0|u|^2 r\mathrm{d}r\mathrm{d}\theta \leq \int_{\Omega^e}\Big|\frac{\partial u}{\partial r}\Big|^2\mathrm{d}x\int_\eta^\infty r\ln r\rho_0(r)\mathrm{d}r, \\ &\leq M_S^2\int_\eta^\infty r\ln r\rho_0(r)\mathrm{d}r.\end{aligned}$$

Therefore we obtain: for any $\varepsilon > 0, \exists R$, such that

$$\int_{\Omega_R^*} \rho|u|^2\mathrm{d}x \leq \varepsilon/4, \qquad \forall u \in S. \tag{2.4}$$

By conditions (2.1)-(2.2),the imbedding $H_0^{1,\rho}(\Omega_R) \hookrightarrow H^{0,\rho}(\Omega_R)$ is compact.

Then the restriction of S on Ω_R is a compact set in $H^{0,\rho}(\Omega_R)$, thus for any $\varepsilon > 0$, there exists $\{u_1, u_2, \dots, u_M\} \in S$ such that $\forall u \in S, \ \ \exists i, \ \ 1 \le i \le M$, there is

$$\|u - u_i\|_{0,\rho,\Omega_R} < \varepsilon/2$$

by (2.4), we know that $\forall \varepsilon > 0$, there exists $\{u_1, u_2, \dots, u_M\} \in S$ such that $\forall u \in S, \ \ \exists i, \ \ 1 \le i \le M$, there is

$$\|u - u_i\|_{0,\rho,\Omega^e} < \varepsilon.$$

Hence S is a compact set in $H^{0,\rho}(\Omega^e)$. Therefore the imbedding from $H_0^{1,\rho}(\Omega^e)$ to $H^{0,\rho}(\Omega^e)$ is compact.

The variational formulation of eigenvalue problem (1.1)-(1.3) is

$$\begin{cases} \text{Find } \lambda \in \mathbb{C}, u \in H_0^{1,\rho}(\Omega^e), \text{ and } u \neq 0 \text{ such that} \\ \displaystyle\int_{\Omega^e} \nabla u \cdot \nabla \bar{v} \mathrm{d}x - \lambda \int_{\Omega^e} \rho u \bar{v} \mathrm{d}x = 0, \quad \forall v \in H_0^{1,\rho}(\Omega^e). \end{cases} \tag{2.5}$$

We can see that if (λ, u) satisfies (2.5), and $u \in C^2(\Omega^e)$, then (λ, u) satisfies (1.1)-(1.3).

Theorem 2.1 Assume that $\rho(x)$ satisfies conditions (2.1)-(2.2), then eigenvalue problem (2.5) has real discrete eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \cdots.$$

Proof Let

$$a(u, v) = \int_{\Omega^e} \nabla u \cdot \nabla \bar{v} \mathrm{d}x, \quad b(u, v) = \int_{\Omega^e} \rho u \bar{v} \mathrm{d}x.$$

The eigenvalue problem (2.5) can be written in the following form:

$$\begin{cases} \text{Find } \lambda \in \mathbb{C}, \ u \in H_0^{1,\rho}(\Omega^e) \text{ and } u \neq 0 \text{ such that} \\ a(u, v) - \lambda b(u, v) = 0, \qquad \forall v \in H_0^{1,\rho}(\Omega^e). \end{cases}$$

The bilinear form $a(u, v)$ is bounded and coercive in $H_0^{1,\rho}(\Omega^e)$, namely there is a positive constant μ, such that

$$|a(u, v)| \le \|u\|_{1,\rho,\Omega^e} \|v\|_{1,\rho,\Omega^e}, \qquad \forall u, v \in H_0^{1,\rho}(\Omega^e),$$

$$a(u, u) \ge \mu \|u\|_{1,\rho,\Omega^e}^2, \qquad \forall u \in H_0^{1,\rho}(\Omega^e). \tag{2.6}$$

It is obvious that

$$b(u, u) > 0, \forall u \in H^{0,\rho}(\Omega^e), \quad u \neq 0. \tag{2.7}$$

Introduce the operator $T : H_0^{1,\rho}(\Omega^e) \to H_0^{1,\rho}(\Omega^e)$ defined by $Tu \in H_0^{1,\rho}(\Omega^e)$, Tu is the unique solution of the following variational problem for given $u \in H_0^{1,\rho}(\Omega^e)$

$$a(Tu, v) = b(u, v), \forall v \in H_0^{1,\rho}(\Omega^e), \tag{2.8}$$

Taking $v = Tu$ in (2.8), we obtain

$$\mu\|Tu\|^2_{1,\rho,\Omega^e} \leq a(Tu, Tu) = b(u, Tu) \leq \|u\|_{0,\rho,\Omega^e}\|Tu\|_{0,\rho,\Omega^e}.$$

Therefore we have

$$\|Tu\|_{1,\rho,\Omega^e} \leq \frac{1}{\mu}\|u\|_{0,\rho,\Omega^e}, \qquad \forall u \in H^{0,\rho}(\Omega^e). \tag{2.9}$$

If $\{u_j\}$ is a bounded sequence in $H_0^{1,\rho}(\Omega^e)$, then, since $H_0^{1,\rho}(\Omega^e)$ imbeds in $H^{0,\rho}(\Omega^e)$ compactly, we know there is a subsequence $\{u_{j_l}\}$ that is Cauchy in $H^{0,\rho}(\Omega^e)$. It then follows immediately from (2.9), applied to $u_{j_l} - u_{j_k}$, that $\{Tu_{j_l}\}$ is Cauchy, and hence convergent in $H_0^{1,\rho}(\Omega^e)$, Thus $T : H_0^{1,\rho}(\Omega^e) \to H_0^{1,\rho}(\Omega^e)$ is compact. Suppose (λ, u) is an eigenvalue of (2.5) if and only if

$$Tu = \frac{1}{\lambda}u, u \neq 0.$$

Finally from the spectral theory of compact operators we know that the eigenvalue problem (2.5) has discrete eigenvalues. For the symmetry of bilinear form $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and (2.6)-(2.7), we know that the eigenvalues of problem (2.5) are real and positive.

3 The numerical solution of eigenvalue problem (2.5)

In this section, we discuss the numerical approximation of eigenvalue problem (2.5). We introduce a coupling method of finite element and infinite large element to overcome the difficulty, which originates from the unboundedness of physical domain Ω^e.

Suppose R_1 is large enough such that $\Gamma_{R_1} = \{x : |x| = R_1\} \subset \Omega^e$. Thus Γ_{R_1} divides domain Ω^e into two parts: the unbounded part $\Omega^*_{R_1} = \{x : |x| > R_1\}$ and the bounded part $\Omega_{R_1} = \{x : x \in \Omega^e, |x| < R_1\}$.

On circle Γ_{R_1}, we take N nodes $\{x_i = (R_1\cos\theta_i, R_1\sin\theta_i) : \theta_i = \dfrac{2\pi i}{N}, i = 1, ..., N\}$. The rays $\{\overrightarrow{Ox_i}\}_{i=1}^N$ divide the domain $\Omega^*_{R_1}$ into N subsets:

$$K_j^e = \{x : |x| \geq R_1, \theta_{j-1} \leq \theta \leq \theta_j\}, j = 1, ...N,\ \theta_0 = 0.$$

Each subset $K_j^e(j = 1, ...N)$ is an infinite large element. Let $\daleth_e^h = \{K_j^e\}_{1\leq j\leq N}$.

Then we divide domain Ω_{R_1} into finite number of triangles $\{K_j^i\ , j = 1, ..., M\}$, which may have one curve side, and $\{x_i, i = 1, ..., N\}$ belong to the set of vertexes of the triangles. Let $\daleth_i^h = \{K_j^i\}_{1\leq j\leq M}$ and $\daleth^h = \daleth_e^h \cup \daleth_i^h$. Finally we obtain the partition $\daleth^h$ of domain Ω^e, which include the triangle elements $\{K_j^i\ , j = 1, ..., M\}$ and infinite large elements $\{K_j^e, j = 1, ..., N\}$. On triangle element K_j^i we use the linear shape function in planimetric rectangular coordinates. On infinite large element $K_j^e = \{x : r \geq R_1, \theta_{j-1} \leq \theta \leq \theta_j\}, j = 1, ..., N$ we take another $2K - 1$ nodes $x_{i,j-1} = \{R_i\cos\theta_{j-1}, R_i sin\theta_{j-1}\}$, $x_{i,j} = \{R_i\cos\theta_j, R_i\sin\theta_j\}$, $i = 2, ..., K$, with

$R_1 < R_2 < ... < R_K$, and node x_∞ at infinity which is shared by all infinite large elements. Let

$$V = \left\{ a_0 + \sum_{i=1}^{K} \sum_{j=1}^{2} a_{i,j} \frac{\theta^{j-1}}{r^i} : a_0, a_{i,j} \in \mathbb{R}, i = 1, 2, ...K, j = 1, 2 \right\},$$

then the shape function $f_j(r, \theta)$ on K_j^e is taken to be

$$f_j(r, \theta) \in V, \tag{3.1}$$

define the value of $f_j(r, \theta)$ at x_∞ as

$$f_j(x_\infty) = \lim_{r \to \infty} f_j(r, \theta),$$

which is independent of j. On element K_j^e, the shape function (3.1) can be determined by the values at the nodes $\{x_{i,j-1}, x_{i,j}\}, i = 1, ..., K$ and node x_∞, where $(x_{1,j-1} = x_{j-1}, x_{1,j} = x_j)$. Here we present a specific example to give a further explanation.

For example Figure.1 shows an infinite large element K_j^e with node $x_{j-1}, x_j, x_{2,j-1}, x_{2,j}$,

Figure 1

and their corresponding coordinates are $(R_1, \theta_{j-1}), (R_1, \theta_j), (R_2, \theta_{j-1}), (R_2, \theta_j)$, x_∞ at infinity is a public node. Let

$$p_1(r, \theta) = \frac{(r - R_2)(\theta - \theta_j)R_1^2}{\delta R \delta\theta r^2}, p_2(r, \theta) = -\frac{(r - R_2)(\theta - \theta_{j-1})R_1^2}{\delta R \delta\theta r^2},$$

$$p_3(r, \theta) = -\frac{(r - R_1)(\theta - \theta_j)R_2^2}{\delta R \delta\theta r^2}, p_4(r, \theta) = \frac{(r - R_1)(\theta - \theta_{j-1})R_2^2}{\delta R \delta\theta r^2},$$

$$p_5(r, \theta) = \frac{(r - R_1)(r - R_2)}{r^2},$$

where $\delta R = R_2 - R_1$, $\delta\theta = \theta_j - \theta_{j-1}$, It is obvious that $\{p_i(r, \theta)\}_{1 \le i \le 5}$ are the basis of function space V, therefore for $f_j(r, \theta) \in V$, $f_j(r, \theta)$ can be written as

$$f_j(r, \theta) = f_j(x_{j-1})p_1(r, \theta) + f_j(x_j)p_2(r, \theta) + f_j(x_{2,j-1})p_3(r, \theta) +$$

$$f_j(x_{2,j})p_4(r, \theta) + f_j(x_\infty)p_5(r, \theta).$$

Now we introduce a finite dimensional function space to approximate $H_0^{1,\rho}(\Omega^e)$.

Let

$$V_h = \{v_h : v_h|_{\Omega_{R_1}} \in C^0(\Omega_{R_1}),\ \ v_h|_{\Omega^*_{R_1}} \in C^0(\Omega^*_{R_1}),\ v_h\ is \text{ continuous at the nodes}$$
$$x_i (i = 1, ..., N).\ v_h|_{K^i_j} \in P_1,\ \forall K^i_j \in \daleth^h_i;\ v_h|_{K^e_j} \in V(K^e_j), \forall K^e_j \in \daleth^h_e;\ v_h|_{y_i} = 0,$$
$$\text{when } y_i\ \in \Gamma(i = 1, ..., J)\},$$

where P_1 is a linear polynomial space, and $\{y_i, i = 1, ..., J\}$ denote the nodes on the boundary Γ.

Generally, V_h is not a subspace of $H^{1,\rho}_0(\Omega^e)$. Therefore the coupling method of FE\ILE is nonconforming.

Let $\{z_l\}_{1\le l\le L}$, including the node x_∞, be the all nodes except $\{y_i, i = 1, ..., J\}$ of the discretization, $\{\varphi_i\}_{1\le i\le L}$ be the basis of V_h, such that

$$\varphi_i(z_l) = \delta_{il}.$$

Then we obtain an approximate variational formulation of eigenvalue problem (2.5)

$$\begin{cases} \text{Find } \lambda_h \in \mathbb{R}, u_h \in V_h, \text{ and } u_h \neq 0 \text{ such that} \\ \sum\limits_{e\in\daleth^h} \int_e \nabla u_h \cdot \nabla v \mathrm{d}x - \lambda_h \sum\limits_{e\in\daleth^h} \int_e \rho u_h v \mathrm{d}x = 0, \quad \forall v \in V_h. \end{cases} \tag{3.2}$$

Suppose

$$u_h(x) = \sum_{j=1}^{L} \varphi_j u_j. \tag{3.3}$$

Substituting (3.3) into (3.2) and taking $v = \varphi_i$, for $i = 1, 2, ..., L$ lead to

$$\sum_{j=1}^{L} \sum_{e\in\daleth^h} \int_e \nabla\varphi_j \cdot \nabla\varphi_i u_j \mathrm{d}x - \lambda_h \sum_{j=1}^{L} \sum_{e\in\daleth^h} \int_e \rho\varphi_j\varphi_i u_j \mathrm{d}x = 0, i = 1, ...L.$$

Let

$$A = (A_{i,j})_{L\times L}, B = (B_{i,j})_{L\times L},$$
$$x = (u_1, u_2, ...u_L)^T,$$

where

$$A_{i,j} = \sum_{e\in\daleth^h} \int_e \nabla\varphi_j \nabla\varphi_i \mathrm{d}x, \quad B_{i,j} = \sum_{e\in\daleth^h} \int_e \rho\varphi_j\varphi_i \mathrm{d}x.$$

Then we have a linear generalized eigenvalue system:

$$\begin{cases} \text{Find } \lambda_h \in \mathbb{R}, x \in \mathbb{R}^L, \text{ and } x \neq 0 \text{ such that} \\ Ax = \lambda_h Bx \end{cases} \tag{3.4}$$

Solving (3.4), we can get a series of approximate eigenvalues

$$0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \dots,$$

and their corresponding approximate eigenfunctions of problem (2.5).

4 Numerical experiments

4.1 Example 1

First we consider the eigenvalue problem (1.1)-(1.3) in exterior domain

$$\Omega^e = \{x : |x| > 1\},$$

and $\rho(r,\theta) = \frac{1}{r^4}$.

By separation of variables and condition (1.3), we have the following form of eigenfunction $u(r,\theta)$ corresponding to eigenvalue λ:

$$u(r,\theta) = J_m(\frac{\sqrt{\lambda}}{r})(a_m \cos(m\theta) + b_m \sin(m\theta)), m = 0, 1, 2, ...,$$

where J_m is m-order Bessel function of the first kind, and by condition (1.2), we know λ satisfies

$$J_m(\sqrt{\lambda}) = 0, m = 0, 1, 2, ...$$

Solving the above equation for λ, we have a series of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots,$$

which are shown in table 1.

Table 1 Exact eigenvalue

eigenvalue	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
value	5.7840	14.6842	14.6842	26.3785	26.3785	30.4704

Let

$$0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \dots$$

be the approximate eigenvalues given by (3.2), then error can be defined as

$$\text{error}(\lambda_i) = \frac{|\mu_i - \lambda_i|}{\lambda_i}.$$

We divide domain Ω^e into two parts by $\Gamma_{1.2}=\{x:|x|=1.2\}$, one is the truncated domain $\Omega_{1.2}$, the other is an exterior domain $\Omega^*_{1.2}$. Let n be the numbers of infinite large elements and

$$R_i = 1.2 + (i-1)^2\frac{\pi}{32}, i = 1, 2, ...K,$$

where $R_i, i = 1, 2, ...K$ and K is defined in section 3.

On $\Omega_{1.2}$, interval $[1, 1.2]$ is partitioned into $\frac{n}{16}$ subintervals. Then we obtain a mesh structure with $n \times \frac{n}{16} \times 2$ triangle elements for domain $\Omega_{1.2}$ and n infinite large elements for domain $\Omega^*_{1.2}$. Figure 2 shows the disretization with $n = 16$.

Figure 2 Discretization of disc exterior domain

Firstly let $K = 3$, and n varies from 16 to 128, and we get Table 2. Table 2 indicates that at a certain point, it does not make any sense to simply refine the FE\ILE meshes.

Table 2 Numerical results with K=3

n \ error	error(λ_1)	error(λ_2)	error(λ_3)	error(λ_4)	error(λ_5)	error(λ_6)
16	0.0044	0.0184	0.0184	0.0598	0.0598	0.0153
32	0.0014	0.0079	0.0079	0.0210	0.0210	0.0140
64	0.0006	0.0053	0.0053	0.0115	0.0115	0.0137
128	0.0005	0.0046	0.0046	0.0091	0.0091	0.0136

Secondly let $K = 7$, and n varies from 16 to 128, and we get Table 3. Table 2 and Table 3 indicate that increasing K may improve not only the accuracy but also the order of convergence rate.

Table 3 Numerica results with $K = 7$

n \ error	error(λ_1)	error(λ_2)	error(λ_3)	error(λ_4)	error(λ_5)	error(λ_6)
16	0.00583	0.01842	0.01842	0.06044	0.06044	0.01017
32	0.00134	0.00457	0.00457	0.01524	0.01524	0.00279
64	0.00023	0.00103	0.00103	0.00372	0.00372	0.00073
128	0.00006	0.00014	0.00014	0.00082	0.00082	0.00021

Thirdly, the finite mesh is fixed as $64 \times \frac{64}{16} \times 2$, which means we have a 64 infinite large elements mesh.When K varies from 3 to 7, we have Table 4. From Table 4, we can see that with more degrees of freedom in the radial direction, we may obtain more accurate numerical results, but after a certain point, it does not make any sense to increase degrees of freedom in the radial direction of infinite large element. That is because the error caused by infinite large elements is not so significant as finite element approximation error in domain Ω^e.

Table 4 Numerical result when K varies from 3 to 7

n \ error	error(λ_1)	error(λ_2)	error(λ_3)	error(λ_4)	error(λ_5)	error(λ_6)
3	0.00064	0.00528	0.00528	0.01148	0.01148	0.01369
4	0.00023	0.00104	0.00104	0.00448	0.00448	0.00172
5	0.00023	0.00104	0.00104	0.00373	0.00373	0.00077
6	0.00023	0.00104	0.00104	0.00372	0.00372	0.00074
7	0.00023	0.00103	0.00103	0.00372	0.00372	0.00073

4.2 Example 2

In the following, we consider the eigenvalue problem (1.1)-(1.3) in exterior domain

$$\Omega^e = \mathbb{R}^2 \backslash \{x : |x_1| \le \sqrt{2}/2, |x_2| \le \sqrt{2}/2\}$$

to illustrate the effectiveness of our coupling method of FE\ILE.

In our first experiment of this example, We let $\rho(r, \theta) = 1/r^{2.1}$. We divide domain Ω^e into two parts by $\Gamma_{1.2}$, one is the truncated domain $\Omega_{1.2}$, the other is an exterior domain $\Omega^*_{1.2}$. Let $4n$ be the numbers of intervals in θ direction, which means we have

$4n$ infinite large elements. Let

$$R_i = 1.2 + (i-1)^2 \frac{\pi}{32}, i = 1, 2, ...K.$$

Let K be 5. Figue 3 shows the disretization when $n = 4$. Let $n = 4, 8, 16, 32$, we obtain Table 5.

Figure 3 Discretization of rectangle exterior domain

Table 5 Numerical results with $\rho(r, \theta) = 1/r^{2,1}$

$n\backslash$ eigenvalue	μ_1	μ_2	μ_3	μ_4
4	0.0276	0.6320	1.7987	1.7987
8	0.0274	0.6257	1.7788	1.7788
16	0.0274	0.6232	1.7728	1.7728
32	0.0274	0.6222	1.7710	1.7710

In second experiment of the example, let $\rho(r, \theta) = (1+\sin^2\theta)/r^{2.1}$. The disretization of domain Ω^e is the same as that in the first experiment. Let $n = 4, 8, 16, 32$, we obtain Table 6.

Table 6 Numerical results with $\rho(r, \theta) = (1 + sin^2\theta)/r^{2.1}$

$n\backslash$eigenvalue	μ_1	μ_2	μ_3	μ_4
4	0.0184	0.4182	1.0229	1.4310
8	0.0183	0.4142	1.0116	1.4145
16	0.0183	0.4127	1.0083	1.4096
32	0.0182	0.4121	1.0073	1.4081

From Table 5 and Table 6, we can see that the coupling method of finite and infinite large element for eigenvalue problem does converge on rectangle exterior domain.

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