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INVALIDITY OF DECOUPLING A BIHARMONIC EQUATION TO TWO POISSON EQUATIONS ON NON-CONVEX POLYGONS

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Abstract. We clarify the validity of a method that decouples a boundary value problem of biharmonic equation to two Poisson equations on polygonal domains. The method provides a way of computing deflections of simply supported polygonal plates by using Poisson solvers. We show that such decoupling is not valid if the polygonal domain is not convex. It may fail even when the right hand side function is infinitely smooth and supported away from the reentrant corners.

Key Words. Simply supported plate, biharmonic, Poisson, decoupling.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a two-dimensional domain. The boundary value problem of biharmonic equation

$$\Delta^2 u = f \text{ in } \Omega; \quad u|_{\partial\Omega} = 0,\ \Delta u|_{\partial\Omega} = 0. \tag{1}$$

can be formally de-coupled to two Poisson equations by introducing an intermediate function v such that

$$-\Delta v = f \text{ in } \Omega,\ v|_{\partial\Omega} = 0 \quad \text{and} \quad -\Delta\tilde{u} = v \text{ in } \Omega,\ \tilde{u}|_{\partial\Omega} = 0. \tag{2}$$

When $\partial\Omega$ and f are smooth, (1) and (2) are equivalent in the sense that $u = \tilde{u}$. This is easily seen from the regularity of solutions of these equations up to the domain boundary [3, 5]. This decoupling gives rise to a numerical method of solving the biharmonic equation (1) by using Poisson solvers. Incidentally, for polygonal domains, the equation (1) also determines the transverse deflection of a simply supported plate loaded by a resultant transverse force f [1, 2, 4], and the decoupling has been used in numerical computation of the plate deflection. However, for polygonal domains, the validity of this decoupling is not obvious. Actually, we show that it is not valid when the polygon is not convex.

To see the well-posedness of these equations on polygonal domains, we write them in weak forms. In the following, even without explicit indication, a function space is composed of functions defined on Ω. For example, H^1 means $H^1(\Omega)$. The weak formulation of (1) seeks $u \in H^2 \cap H_0^1$ such that

$$(\Delta u, \Delta w)_{L^2} = \langle f, w\rangle \quad \forall\ w \in H^2 \cap H_0^1. \tag{3}$$

Here the parenthesis stands for the inner product in the Hilbert space indicated by the subscript. The right hand side is the dual product between f, viewed as an

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element of the duality of the $H^2 \cap H_0^1$, and w. The equations in (2) become seeking v and $\tilde{u}$ in H_0^1, respectively, such that

$$
\begin{aligned}
(\nabla v, \nabla w)_{[L^2]^2} &= \langle f, w \rangle \quad \forall\, w \in H_0^1, \\
(\nabla \tilde{u}, \nabla w)_{[L^2]^2} &= (v, w)_{L^2} \quad \forall\, w \in H_0^1.
\end{aligned} \tag{4}
$$

Of course, when $\partial\Omega$ and f are smooth, the equations (3) and (4) are well-posed and $u = \tilde{u}$, which are also solutions of the differential equations (1) and (2). However, the weak equations make broader sense than the differential equations: Their well-posedness requires less regularity on the domain and the loading function. The question is whether or not the equivalence between (3) and (4) remains when the domain boundary and loading function are not smooth. This is an important issue that is crucial to the validity of the numerical method for the simply supported plate model (3) obtained by combining the Poisson solvers for the two equations in (4).

The weak equation (3) is well-posed for polygonal domains as long as the function f defines a linear continuous functional on $H^2 \cap H_0^1$. This well-posedness relies on the inequality that for polygonal domains one has [4]

$$
\|\Delta w\|_{L^2} \geq C\|w\|_{H^2} \quad \forall\, w \in H^2 \cap H_0^1. \tag{5}
$$

Here C is a positive constant depending on Ω. The equations in (4) are well-posed if $f \in H^{-1}$. Thus, (3) requires less regularity on f than (4) does. We shall assume that $f \in H^{-1}$ such that both u and $\tilde{u}$ are uniquely determined, and consider the question that whether $u = \tilde{u}$. The answer is that if Ω is a convex polygon then $u = \tilde{u}$. However, if Ω is not convex, then (3) is not equivalent to (4) in the sense that there exists loading function f such that $u \neq \tilde{u}$. Our argument is based on the observation that the solution u of (3) lies in $H^2 \cap H_0^1$, while $\tilde{u}$ is only required to be in H_0^1. Thus if $\tilde{u} \notin H^2$, then $\tilde{u} \neq u$. In the next section, we construct an example to show that on non-convex polygons this may occur even when f is smooth and supported away from the reentrant corners. In the last section we prove that if $\tilde{u} \in H^2$ then $\tilde{u} = u$. Therefore, if the polygonal domain is convex, then (3) and (4) are equivalent as long as $f \in H^{-1}$.

2. An example for the invalidity of the decoupling

As an example, we consider a non-convex polygonal domain and put one of its reentrant corners at the origin of the Cartesian coordinate system. See Figure 1 in which the reentrant angle $\omega > \pi$. Let (r, θ) be the usual polar coordinates. With a slight abuse of notations, we use the same letter to denote a function expressed in both Cartesian and polar coordinates. Let the loading function f be defined by

$$
\begin{aligned}
&f(r,\theta) = \sin\alpha\theta \\
&\left(r^\alpha \frac{d^4\phi(r)}{dr^4} + (4\alpha+2) r^{\alpha-1} \frac{d^3\phi(r)}{dr^3} + (4\alpha^2-1) r^{\alpha-2} \frac{d^2\phi(r)}{dr^2} - (4\alpha^2-1) r^{\alpha-3} \frac{d\phi(r)}{dr} \right).
\end{aligned} \tag{6}
$$

Here $\alpha = \frac{\pi}{\omega} < 1$, and ϕ is a smooth function such that $\phi(r) = 1$ for $0 < r < r_1$ and $\phi(r) = 0$ for $r > r_2$. See Figure 1 for the meanings of r_1 and r_2. This is a smooth function whose support is the shaded region in the figure, which is away from the reentrant corner. For such an f, the solution of (4) is given by

$$
\begin{aligned}
v(r,\theta) &= -\left((2\alpha+1) r^{\alpha-1} \frac{d\phi(r)}{dr} + r^\alpha \frac{d^2\phi(r)}{dr^2} \right) \sin\alpha\theta, \\
\tilde{u}(r,\theta) &= r^\alpha \phi(r) \sin\alpha\theta.
\end{aligned} \tag{7}
$$

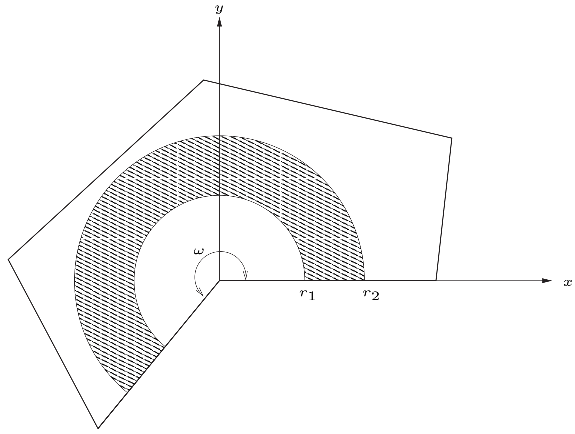

FIGURE 1. A polygonal domain with a concave corner

The function $\tilde{u}$ is in H_0^1 but is not in H^2 since its second derivatives are not square integrable. It is certainly different from the solution of the equation (3), since the latter must be in H^2. It may be worthwhile to note that the functions defined above are strong solutions of the equations in (2), and $\tilde{u}$ satisfies all the conditions in the equation (1), except the boundary condition $\Delta u = 0$ at the reentrant corner. It is the violation of the boundary condition at this single point that makes $\tilde{u}$ not a solution of (1).

3. A condition for the validity of the decoupling

It follows from (5) that the operator Δ maps $H^2 \cap H_0^1$ onto a closed subspace R of L^2 when Ω is a polygon. It is well known that $R = L^2$ when Ω is convex. But $R \subsetneqq L^2$ when Ω has reentrant corners. The co-dimension of R is finite and is equal to the number of reentrant corners [4]. Now let f be such a function that the $v \in H_0^1$ determined by the first equation in (4) does not belong to R. Then the second equation in (4) yields a $\tilde{u}$ that is not in $H^2 \cap H_0^1$. On the other hand, as a solution of (3), u must belong to $H^2 \cap H_0^1$. Generally, one should not expect $u = \tilde{u}$. The following theorem shows that if $\tilde{u}$ has the H^2 regularity, then $\tilde{u} = u$.

Theorem 1. *The solution u of* (3) *and the $\tilde{u}$ determined by* (4) *are equal if and only if the $\tilde{u} \in H^2$. This is equivalent to saying that the v determined by* (4) *belongs to the closed subspace R of L^2, which is the image of $H^2 \cap H_0^1$ under the mapping Δ.*

Proof. The necessity is trivial. If the solution of (4) is such that $\tilde{u} \in H^2 \cap H_0^1$, then from the second equation we see

$$-(\Delta\tilde{u}, w)_{L^2} = (v, w)_{L^2} \quad \forall\, w \in H_0^1. \tag{8}$$

Since $\Delta\tilde{u} \in L^2$, by a limiting argument we see

$$-(\Delta\tilde{u}, w)_{L^2} = (v, w)_{L^2} \quad \forall\, w \in L^2. \tag{9}$$

Therefore,

$$-(\Delta\tilde{u}, \Delta w)_{L^2} = (v, \Delta w)_{L^2} \quad \forall\, w \in H^2 \cap H_0^1. \tag{10}$$

Since v satisfies the first equation in (4), we have $(v, \Delta w)_{L^2} = -(\nabla v, \nabla w)_{[L^2]^2} = -\langle f, w\rangle$. This proves that $\tilde{u}$ is the solution of (3). $\square$

Since Δ is a one-to-one correspondence between $H^2 \cap H_0^1$ and L^2 when Ω is a convex polygon, so the $\tilde{u}$ yield by (4) must be in $H^2 \cap H_0^1$. Therefore, for convex polygonal domains, (3) and (4) is equivalent as long as $f \in H^{-1}(\Omega)$.

For non-convex polygonal domains, if one finds a $\tilde{u}$ by solving the two Poisson equations (4) that is not in H^2, then it must be different from the solution of (3). The example in the previous section indicates that even if thus found $\tilde{u}$ is in H^2, a perturbation of f on a arbitrarily thin ring section could push the $\tilde{u}$ out of H^2. Using the technique of [6], it can be shown that a small perturbation of the value of f on a subregion of Ω that is arbitrarily small in diameter could do the same damage.

It would be desirable to find some kind of post processing technique to modify the $\tilde{u}$ yielded by (4) to reproduce the solution of (3) for concave polygons.

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