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A NEW HIGH ORDER TWO LEVEL IMPLICIT DISCRETIZATION FOR THE SOLUTION OF 3D NON-LINEAR PARABOLIC EQUATIONS

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Abstract. We present a new two-level implicit difference method of $O(k^2 + kh^2 + h^4)$ for approximating the three space dimensional non-linear parabolic differential equation $u_{xx}+u_{yy}+u_{zz} = f(x, y, z, t, u, u_x, u_y, u_z, u_t), 0 < x, y, z < 1, t > 0$ subject to appropriate initial and Dirichlet boundary conditions, where $h > 0$ and $k > 0$ are mesh sizes in space and time directions, respectively. In addition, we also propose some new two-level explicit stable methods of $O(kh^2+h^4)$ for the estimates of $(\partial u/\partial n)$. When grid lines are parallel to $x-, y-$ and $z-$ coordinate axes, then $(\partial u/\partial n)$ at an internal grid point becomes $(\partial u/\partial x)$, $(\partial u/\partial y)$ and $(\partial u/\partial z)$, respectively. In all cases, we require only 19-spatial grid points and a single computational cell. The proposed methods are directly applicable to singular problems and we do not require any special technique to handle singular problems. We also discuss operator splitting method for solving linear parabolic equation. This method permits multiple use of the one-dimensional tri-diagonal solver. It is shown that the operator splitting method is unconditionally stable. Numerical tests are conducted which demonstrate the accuracy and effectiveness of the methods developed.

Key Words. non-linear parabolic equation, implicit scheme, high order method, normal derivatives, singular problem, operator splitting, Burgers' equation.

1. Introduction

Three space dimensional non-linear parabolic partial differential equations represent mathematical models of physical problems of great interest in physics and applied mathematics. Numerical solution of three space dimensional parabolic problems tends to be computationally intensive and may be prohibitive on conventional computers due to the requirements on the memory and the CPU time to obtain solutions of required accuracy. Traditional numerical methods are of lower order and require extremely smaller grid lengths. The size of the resulting linear or non-linear systems for 3-space dimensional problem is usually so large that even present day computers may not be able to handle them. One approach to alleviate these difficulties is to use higher-order methods, which yield approximates solutions with comparable accuracy using much coarser discretization, resulting in linear or non-linear systems of smaller size. It has been repeatedly demonstrated on model problems that even the simplest types of high-order methods should provide tremendous practical advantages in terms of diminishing the required number of storages and also the overall computing time for a desired solution (see Ciment et al [1]). Several authors have discussed high order finite difference methods for the

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solution of three space dimensional linear parabolic equations (see Ciment et al [1], Iyengar and Manohar [2] , Zhang and Zhao [3]). The solution requires the inversion of a block banded matrix. Alternating direction implicit (ADI) methods originally developed for a two-space dimensional diffusion equation have been extended to three-space dimension by Douglas and Rochford [4] , Brian [5] and Fairweather and Mitchell [6, 7]. Two-level implicit difference methods of order 2 in time and 4 in space for the numerical solution of three-space dimensional non-linear parabolic equations have been discussed by Jain et al [8], Mohanty and Jain [9], Mohanty [10] and Mohanty et al [11]. However, their methods are not directly applicable to singular parabolic problems. A special technique is required to handle singular parabolic problems.

In this paper, we consider the numerical solution of the non-linear parabolic partial differential equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=f(x,y,z,t,u,u_x,u_y,u_z,u_t), 0<x,y,z<1, t>0 \tag{1}$$

where $u=u(x,y,z,t)$. Let $\Omega=\{(x,y,z,t)|0<x,y,z<1,t>0\}$ be our solution domain with boundary $\partial\Omega$.

The initial condition is given by

$$u(x,y,z,0)=u_0(x,y,z) \qquad 0\le x,y,z\le 1 \tag{2}$$

and the boundary conditions are given by

$$u(0,y,z,t)=g_0(y,z,t), u(1,y,z,t)=g_1(y,z,t), 0\le y,z\le 1, t\ge 0 \tag{3}$$

$$u(x,0,z,t)=h_0(x,z,t), u(x,1,z,t)=h_1(x,z,t), 0\le x,z\le 1, t\ge 0 \tag{4}$$

$$u(x,y,0,t)=i_0(x,y,t), u(x,y,1,t)=i_1(x,y,t), 0\le x,y\le 1, t\ge 0 \tag{5}$$

where $u_0, g_0, g_1, h_0, h_1, i_0, i_1$ are given functions of sufficient smoothness.

In this paper, using 19-spatial grid points and a single computational cell (see Figure 1) we propose new formulas of order 2 in time and 4 in space coordinates for the solution of non-linear parabolic equation (1) and the estimates of $(\partial u/\partial n)$. When grid lines are parallel to coordinate axes, $(\partial u/\partial n)$ represents $(\partial u/\partial x)$, $(\partial u/\partial y)$ and $(\partial u/\partial z)$, respectively. The proposed methods are directly applicable to singular parabolic equations. We do not require any special technique or modification to handle the singular problem. Recently, Mohanty and Singh [12] have proposed a new fourth order finite difference method for the solution of three dimensional singularly perturbed non-linear elliptic partial differential equation. In next section, we give the description of new algorithms. The complete derivation of numerical methods is given in Section 3. In section 4, we discuss operator splitting method for the solution of a linear three space dimensional parabolic equation and its stability analysis. The operator splitting method requires the solution of tri-diagonal system of equations parallel to coordinates axes, at each time step, independent of the order of the method. In section 5, computational results of some test problems are provided to demonstrate the accuracy of the proposed numerical methods and compared with the corresponding second order methods. It is shown here that for a fixed mesh ratio parameter, the proposed methods are of fourth order in space. Concluding remarks are given in section 6.

2. Description of numerical algorithms

As usual, let us assume that the solution domain Ω is covered by a set of cubic grid with spacing $h>0$ and $k>0$ in space and time coordinates, respectively. The grid points (x_l, y_m, z_n, t_j) are given by $x_l=lh, y_m=mh, z_n=nh, t_j=jk$,

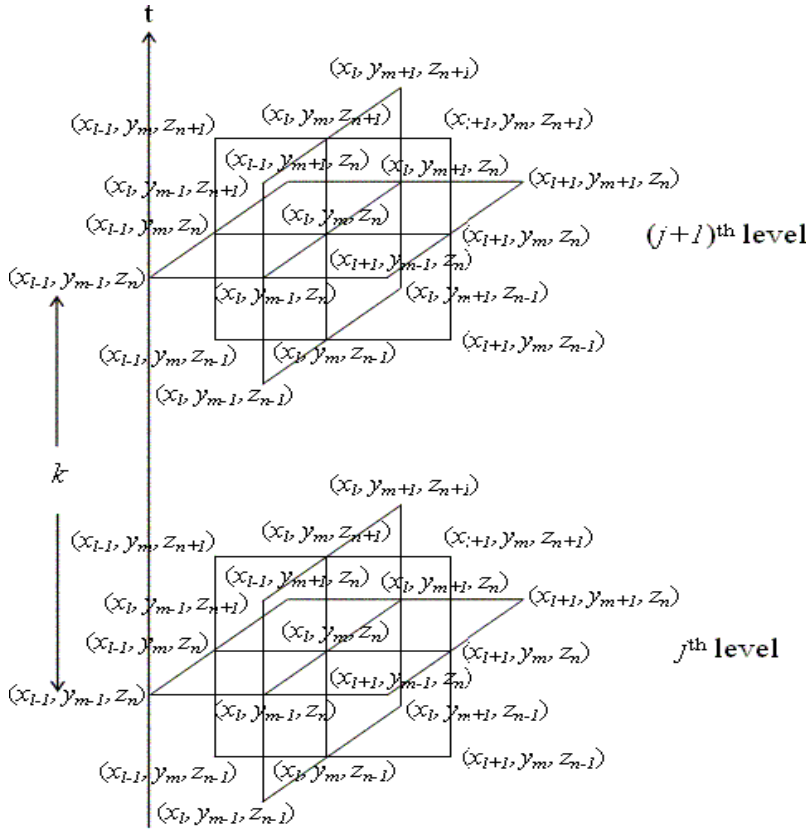

FIGURE 1. Nineteen Spatial Grid Points

with $l, m, n = 0(1)N+1$, and $j = 0, 1, 2, \ldots$, where N is a positive integer and $(N+1)h = 1$. Let $U^j_{l,m,n}$ and $u^j_{l,m,n}$ be the exact and approximate solution values of $u(x, y, z, t)$ at the grid point (x_l, y_m, z_n, t_j) respectively and $\lambda = (k/h^2)$ be the mesh ratio parameter.

We require the following approximations

$$\bar{t}_j = t_j + \theta k, 0 \le \theta \le 1 \tag{6}$$

$$\overline{U}^j_{l,m,n} = \theta U^{j+1}_{l,m,n} + (1-\theta) U^j_{l,m,n} \tag{7}$$

$$\overline{U}^j_{l\pm1,m,n} = \theta U^{j+1}_{l\pm1,m,n} + (1-\theta) U^j_{l\pm1,m,n} \tag{8}$$

$$\overline{U}^j_{l,m\pm1,n} = \theta U^{j+1}_{l,m\pm1,n} + (1-\theta) U^j_{l,m\pm1,n} \tag{9}$$

$$\overline{U}^j_{l,m,n\pm1} = \theta U^{j+1}_{l,m,n\pm1} + (1-\theta) U^j_{l,m,n\pm1} \tag{10}$$

$$\overline{U}^j_{l+1,m\pm1,n} = \theta U^{j+1}_{l+1,m\pm1,n} + (1-\theta) U^j_{l+1,m\pm1,n} \tag{11}$$

$$\overline{U}^j_{l-1,m\pm1,n} = \theta U^{j+1}_{l-1,m\pm1,n} + (1-\theta) U^j_{l-1,m\pm1,n} \tag{12}$$

$$\overline{U}^j_{l+1,m,n\pm1} = \theta U^{j+1}_{l+1,m,n\pm1} + (1-\theta) U^j_{l+1,m,n\pm1} \tag{13}$$

$$\overline{U}^j_{l-1,m,n\pm1} = \theta U^{j+1}_{l-1,m,n\pm1} + (1-\theta) U^j_{l-1,m,n\pm1} \tag{14}$$

$$\overline{U}^j_{l,m+1,n\pm1} = \theta U^{j+1}_{l,m+1,n\pm1} + (1-\theta) U^j_{l,m+1,n\pm1} \tag{15}$$

$$\overline{U}^j_{l,m-1,n\pm1} = \theta U^{j+1}_{l,m-1,n\pm1} + (1-\theta)U^j_{l,m-1,n\pm1} \tag{16}$$

$$\overline{U}^j_{l\pm\frac{1}{2},m,n} = (\overline{U}^j_{l\pm1,m,n} + \overline{U}^j_{l,m,n})/2 \tag{17}$$

$$\overline{U}^j_{l,m\pm\frac{1}{2},n} = (\overline{U}^j_{l,m\pm1,n} + \overline{U}^j_{l,m,n})/2 \tag{18}$$

$$\overline{U}^j_{l,m,n\pm\frac{1}{2}} = (\overline{U}^j_{l,m,n\pm1} + \overline{U}^j_{l,m,n})/2 \tag{19}$$

$$\overline{U}^j_{x_{l,m,n}} = (\overline{U}^j_{l+1,m,n} - \overline{U}^j_{l-1,m,n})/(2h) \tag{20}$$

$$\overline{U}^j_{x_{l\pm\frac{1}{2},m,n}} = \pm(\overline{U}^j_{l\pm1,m,n} - \overline{U}^j_{l,m,n})/h \tag{21}$$

$$\overline{U}^j_{x_{l,m\pm\frac{1}{2},n}} = (\overline{U}^j_{l+1,m\pm1,n} - \overline{U}^j_{l-1,m\pm1,n} + \overline{U}^j_{l+1,m,n} - \overline{U}^j_{l-1,m,n})/(4h) \tag{22}$$

$$\overline{U}^j_{x_{l,m,n\pm\frac{1}{2}}} = (\overline{U}^j_{l+1,m,n\pm1} - \overline{U}^j_{l-1,m,n\pm1} + \overline{U}^j_{l+1,m,n} - \overline{U}^j_{l-1,m,n})/(4h) \tag{23}$$

$$\overline{U}^j_{y_{l,m,n}} = (\overline{U}^j_{l,m+1,n} - \overline{U}^j_{l,m-1,n})/(2h) \tag{24}$$

$$\overline{U}^j_{y_{l\pm\frac{1}{2},m,n}} = (\overline{U}^j_{l\pm1,m+1,n} - \overline{U}^j_{l\pm1,m-1,n} + \overline{U}^j_{l,m+1,n} - \overline{U}^j_{l,m-1,n})/(4h) \tag{25}$$

$$\overline{U}^j_{y_{l,m\pm\frac{1}{2},n}} = \pm(\overline{U}^j_{l,m\pm1,n} - \overline{U}^j_{l,m,n})/h \tag{26}$$

$$\overline{U}^j_{y_{l,m,n\pm\frac{1}{2}}} = (\overline{U}^j_{l,m+1,n\pm1} - \overline{U}^j_{l,m-1,n\pm1} + \overline{U}^j_{l,m+1,n} - \overline{U}^j_{l,m-1,n})/(4h) \tag{27}$$

$$\overline{U}^j_{z_{l,m,n}} = (\overline{U}^j_{l,m,n+1} - \overline{U}^j_{l,m,n-1})/(2h) \tag{28}$$

$$\overline{U}^j_{z_{l\pm\frac{1}{2},m,n}} = (\overline{U}^j_{l\pm1,m,n+1} - \overline{U}^j_{l\pm1,m,n-1} + \overline{U}^j_{l,m,n+1} - \overline{U}^j_{l,m,n-1})/(4h) \tag{29}$$

$$\overline{U}^j_{z_{l,m\pm\frac{1}{2},n}} = (\overline{U}^j_{l,m\pm1,n+1} - \overline{U}^j_{l,m\pm1,n-1} + \overline{U}^j_{l,m,n+1} - \overline{U}^j_{l,m,n-1})/(4h) \tag{30}$$

$$\overline{U}^j_{z_{l,m,n\pm\frac{1}{2}}} = \pm(\overline{U}^j_{l,m,n\pm1} - \overline{U}^j_{l,m,n})/h \tag{31}$$

$$\overline{U}^j_{t_{l,m,n}} = (U^{j+1}_{l,m,n} - U^j_{l,m,n})/k \tag{32}$$

$$\overline{U}^j_{t_{l\pm1,m,n}} = (U^{j+1}_{l\pm1,m,n} - U^j_{l\pm1,m,n})/k \tag{33}$$

$$\overline{U}^j_{t_{l,m\pm1,n}} = (U^{j+1}_{l,m\pm1,n} - U^j_{l,m\pm1,n})/k \tag{34}$$

$$\overline{U}^j_{t_{l,m,n\pm1}} = (U^{j+1}_{l,m,n\pm1} - U^j_{l,m,n\pm1})/k \tag{35}$$

$$\overline{U}^j_{t_{l\pm\frac{1}{2},m,n}} = (U^{j+1}_{l\pm1,m,n} + U^{j+1}_{l,m,n} - U^j_{l\pm1,m,n} - U^j_{l,m,n})/(2k) \tag{36}$$

$$\overline{U}^j_{t_{l,m\pm\frac{1}{2},n}} = (U^{j+1}_{l,m\pm1,n} + U^{j+1}_{l,m,n} - U^j_{l,m\pm1,n} - U^j_{l,m,n})/(2k) \tag{37}$$

$$\overline{U}^j_{t_{l,m,n\pm\frac{1}{2}}} = (U^{j+1}_{l,m,n\pm1} + U^{j+1}_{l,m,n} - U^j_{l,m,n\pm1} - U^j_{l,m,n})/(2k) \tag{38}$$

Next we define

$$\begin{aligned}\overline{F}^j_{l\pm\frac{1}{2},m,n} = f(&x_{l\pm\frac{1}{2}}, y_m, z_n, \bar{t}_j, \overline{U}^j_{l\pm\frac{1}{2},m,n}, \overline{U}^j_{x_{l\pm\frac{1}{2},m,n}}, \overline{U}^j_{y_{l\pm\frac{1}{2},m,n}},\\ &\overline{U}^j_{z_{l\pm\frac{1}{2},m,n}}, \overline{U}^j_{t_{l\pm\frac{1}{2},m,n}})\end{aligned} \tag{39}$$

$$\begin{aligned}\overline{F}^j_{l,m\pm\frac{1}{2},n} = f(&x_l, y_{m\pm\frac{1}{2}}, z_n, \bar{t}_j, \overline{U}^j_{l,m\pm\frac{1}{2},n}, \overline{U}^j_{x_{l,m\pm\frac{1}{2},n}}, \overline{U}^j_{y_{l,m\pm\frac{1}{2},n}},\\ &\overline{U}^j_{z_{l,m\pm\frac{1}{2},n}}, \overline{U}^j_{t_{l,m\pm\frac{1}{2},n}})\end{aligned} \tag{40}$$

$$\begin{aligned}\overline{F}^j_{l,m,n\pm\frac{1}{2}} = f(&x_l, y_m, z_{n\pm\frac{1}{2}}, \bar{t}_j, \overline{U}^j_{l,m,n\pm\frac{1}{2}}, \overline{U}^j_{x_{l,m,n\pm\frac{1}{2}}}, \overline{U}^j_{y_{l,m,n\pm\frac{1}{2}}},\\ &\overline{U}^j_{z_{l,m,n\pm\frac{1}{2}}}, \overline{U}^j_{t_{l,m,n\pm\frac{1}{2}}})\end{aligned} \tag{41}$$

Further, we consider the following approximations:

$$\overline{\overline{U}}^j_{l,m,n} = \overline{U}^j_{l,m,n} + a_1 h^2 (\overline{F}^j_{l+\frac{1}{2},m,n} + \overline{F}^j_{l-\frac{1}{2},m,n} + \overline{F}^j_{l,m+\frac{1}{2},n} + \overline{F}^j_{l,m-\frac{1}{2},n} + \overline{F}^j_{l,m,n+\frac{1}{2}} + \overline{F}^j_{l,m,n-\frac{1}{2}}) \tag{42}$$

$$\overline{\overline{U}}^j_{x_{l,m,n}} = \overline{U}^j_{x_{l,m,n}} + a_2 h (\overline{F}^j_{l+\frac{1}{2},m,n} - \overline{F}^j_{l-\frac{1}{2},m,n}) \tag{43}$$

$$\overline{\overline{U}}^j_{y_{l,m,n}} = \overline{U}^j_{y_{l,m,n}} + a_3 h (\overline{F}^j_{l,m+\frac{1}{2},n} - \overline{F}^j_{l,m-\frac{1}{2},n}) \tag{44}$$

$$\overline{\overline{U}}^j_{z_{l,m,n}} = \overline{U}^j_{z_{l,m,n}} + a_4 h (\overline{F}^j_{l,m,n+\frac{1}{2}} - \overline{F}^j_{l,m,n-\frac{1}{2}}) \tag{45}$$

$$\overline{\overline{U}}^j_{t_{l,m,n}} = \overline{U}^j_{t_{l,m,n}} + a_5 (\overline{U}^j_{t_{l+1,m,n}} + \overline{U}^j_{t_{l-1,m,n}} + \overline{U}^j_{t_{l,m+1,n}} + \overline{U}^j_{t_{l,m-1,n}} + \overline{U}^j_{t_{l,m,n+1}} + \overline{U}^j_{t_{l,m,n-1}} - 6\overline{U}^j_{t_{l,m,n}}) \tag{46}$$

where $\theta = \frac{1}{2}, a_1 = \frac{1}{72}, a_2 = a_3 = a_4 = a_5 = \frac{1}{12}$.

Next we define

$$\overline{\overline{F}}^j_{l,m,n} = f(x_l, y_m, z_n, \bar{t}_j, \overline{\overline{U}}^j_{l,m,n}, \overline{\overline{U}}^j_{x_{l,m,n}}, \overline{\overline{U}}^j_{y_{l,m,n}}, \overline{\overline{U}}^j_{z_{l,m,n}}, \overline{\overline{U}}^j_{t_{l,m,n}},) \tag{47}$$

Then at each internal grid point (x_l, y_m, z_n, t_j), the proposed parabolic differential equation (1) is discretized by

$$\left[\delta_x^2 + \delta_y^2 + \delta_z^2 + \frac{1}{6}(\delta_x^2\delta_y^2 + \delta_y^2\delta_z^2 + \delta_z^2\delta_x^2)\right]\overline{U}^j_{l,m,n} = \frac{h^2}{3}\left[\overline{F}^j_{l+\frac{1}{2},m,n} + \overline{F}^j_{l-\frac{1}{2},m,n} + \overline{F}^j_{l,m+\frac{1}{2},n} + \overline{F}^j_{l,m-\frac{1}{2},n} + \overline{F}^j_{l,m,n+\frac{1}{2}} + \overline{F}^j_{l,m,n-\frac{1}{2}} - 3\overline{\overline{F}}^j_{l,m,n}\right] + \overline{T}^j_{l,m,n} \tag{48}$$

where $\delta_x U_l = (U_{l+\frac{1}{2}} - U_{l-\frac{1}{2}})$ and $\mu_x U_l = \frac{1}{2}(U_{l+\frac{1}{2}} + U_{l-\frac{1}{2}})$ are central and average difference operators with respect to x-direction etc. and $\overline{T}^j_{l,m,n} = O(k^2h^2 + kh^4 + h^6)$.

For the estimates of $(\partial u/\partial x)$, $(\partial u/\partial y)$ and $(\partial u/\partial z)$, we need the following approximations. Let

$$\widehat{U}^{j+1}_{l\pm\frac{1}{2},m,n} = (U^{j+1}_{l\pm1,m,n} + U^{j+1}_{l,m,n})/2 \tag{49}$$

$$\widehat{U}^{j+1}_{l,m\pm\frac{1}{2},n} = (U^{j+1}_{l,m\pm1,n} + U^{j+1}_{l,m,n})/2 \tag{50}$$

$$\widehat{U}^{j+1}_{l,m,n\pm\frac{1}{2}} = (U^{j+1}_{l,m,n\pm1} + U^{j+1}_{l,m,n})/2 \tag{51}$$

$$\widehat{U}^{j+1}_{x_{l\pm\frac{1}{2},m,n}} = \pm(U^{j+1}_{l\pm1,m,n} - U^{j+1}_{l,m,n})/h \tag{52}$$

$$\widehat{U}^{j+1}_{x_{l,m\pm\frac{1}{2},n}} = (U^{j+1}_{l+1,m\pm1,n} - U^{j+1}_{l-1,m\pm1,n} + U^{j+1}_{l+1,m,n} - U^{j+1}_{l-1,m,n})/(4h) \tag{53}$$

$$\widehat{U}^{j+1}_{x_{l,m,n\pm\frac{1}{2}}} = (U^{j+1}_{l+1,m,n\pm1} - U^{j+1}_{l-1,m,n\pm1} + U^{j+1}_{l+1,m,n} - U^{j+1}_{l-1,m,n})/(4h) \tag{54}$$

$$\widehat{U}^{j+1}_{y_{l\pm\frac{1}{2},m,n}} = (U^{j+1}_{l\pm1,m+1,n} - U^{j+1}_{l\pm1,m-1,n} + U^{j+1}_{l,m+1,n} - U^{j+1}_{l,m-1,n})/(4h) \tag{55}$$

$$\widehat{U}^{j+1}_{y_{l,m\pm\frac{1}{2},n}} = \pm(U^{j+1}_{l,m\pm1,n} - U^{j+1}_{l,m,n})/h \tag{56}$$

$$\widehat{U}^{j+1}_{y_{l,m,n\pm\frac{1}{2}}} = (U^{j+1}_{l,m+1,n\pm1} - U^{j+1}_{l,m-1,n\pm1} + U^{j+1}_{l,m+1,n} - U^{j+1}_{l,m-1,n})/(4h) \tag{57}$$

$$\widehat{U}^{j+1}_{z_{l\pm\frac{1}{2},m,n}} = (U^{j+1}_{l\pm1,m,n+1} - U^{j+1}_{l\pm1,m,n-1} + U^{j+1}_{l,m,n+1} - U^{j+1}_{l,m,n-1})/(4h) \tag{58}$$

$$\widehat{U}^{j+1}_{z_{l,m\pm\frac{1}{2},n}} = (U^{j+1}_{l,m\pm1,n+1} - U^{j+1}_{l,m\pm1,n-1} + U^{j+1}_{l,m,n+1} - U^{j+1}_{l,m,n-1})/(4h) \tag{59}$$

$$\widehat{U}^{j+1}_{z_{l,m,n\pm\frac{1}{2}}} = \pm(U^{j+1}_{l,m,n\pm1} - U^{j+1}_{l,m,n})/h \tag{60}$$

$$\widehat{U}^{j+1}_{t_{l\pm\frac{1}{2},m,n}} = (U^{j+1}_{l\pm1,m,n} + U^{j+1}_{l,m,n} - U^{j}_{l\pm1,m,n} - U^{j}_{l,m,n})/(2k) \tag{61}$$

$$\widehat{U}^{j+1}_{t_{l,m\pm\frac{1}{2},n}} = (U^{j+1}_{l,m\pm1,n} + U^{j+1}_{l,m,n} - U^{j}_{l,m\pm1,n} - U^{j}_{l,m,n})/(2k) \tag{62}$$

$$\widehat{U}^{j+1}_{t_{l,m,n\pm\frac{1}{2}}} = (U^{j+1}_{l,m,n\pm1} + U^{j+1}_{l,m,n} - U^{j}_{l,m,n\pm1} - U^{j}_{l,m,n})/(2k) \tag{63}$$

Then we define

$$\begin{aligned}\widehat{F}^{j+1}_{l\pm\frac{1}{2},m,n} = f(&x_{l\pm\frac{1}{2}}, y_m, z_n, t_{j+1}, \widehat{U}^{j+1}_{l\pm\frac{1}{2},m,n}, \widehat{U}^{j+1}_{x_{l\pm\frac{1}{2},m,n}}, \widehat{U}^{j+1}_{y_{l\pm\frac{1}{2},m,n}},\\ &\widehat{U}^{j+1}_{z_{l\pm\frac{1}{2},m,n}}, \widehat{U}^{j+1}_{t_{l\pm\frac{1}{2},m,n}})\end{aligned} \tag{64}$$

$$\begin{aligned}\widehat{F}^{j+1}_{l,m\pm\frac{1}{2},n} = f(&x_l, y_{m\pm\frac{1}{2}}, z_n, t_{j+1}, \widehat{U}^{j+1}_{l,m\pm\frac{1}{2},n}, \widehat{U}^{j+1}_{x_{l,m\pm\frac{1}{2},n}}, \widehat{U}^{j+1}_{y_{l,m\pm\frac{1}{2},n}},\\ &\widehat{U}^{j+1}_{z_{l,m\pm\frac{1}{2},n}}, \widehat{U}^{j+1}_{t_{l,m\pm\frac{1}{2},n}})\end{aligned} \tag{65}$$

$$\begin{aligned}\widehat{F}^{j+1}_{l,m,n\pm\frac{1}{2}} = f(&x_l, y_m, z_{n\pm\frac{1}{2}}, t_{j+1}, \widehat{U}^{j+1}_{l,m,n\pm\frac{1}{2}}, \widehat{U}^{j+1}_{x_{l,m,n\pm\frac{1}{2}}}, \widehat{U}^{j+1}_{y_{l,m,n\pm\frac{1}{2}}},\\ &\widehat{U}^{j+1}_{z_{l,m,n\pm\frac{1}{2}}}, \widehat{U}^{j+1}_{t_{l,m,n\pm\frac{1}{2}}})\end{aligned} \tag{66}$$

Following the techniques given by Stephenson [14], the estimates of $(\partial u/\partial x)$, $(\partial u/\partial y)$ and $(\partial u/\partial z)$ for the differential equation (1) are given by

$$\begin{aligned}U^{j+1}_{x_{l,m,n}} &= \frac{1}{12h}[U^{j+1}_{l+1,m+1,n} + U^{j+1}_{l+1,m-1,n} - U^{j+1}_{l-1,m+1,n} - U^{j+1}_{l-1,m-1,n}\\ &+ U^{j+1}_{l+1,m,n+1} + U^{j+1}_{l+1,m,n-1} - U^{j+1}_{l-1,m,n+1} - U^{j+1}_{l-1,m,n-1}\\ &+ 2(U^{j+1}_{l+1,m,n} - U^{j+1}_{l-1,m,n})] - \frac{h}{6}[\widehat{F}^{j+1}_{l+\frac{1}{2},m,n} - \widehat{F}^{j+1}_{l-\frac{1}{2},m,n}] + \widehat{T}^{j+1}_{x_{l,m,n}}\end{aligned} \tag{67}$$

$$\begin{aligned}U^{j+1}_{y_{l,m,n}} &= \frac{1}{12h}[U^{j+1}_{l+1,m+1,n} - U^{j+1}_{l+1,m-1,n} + U^{j+1}_{l-1,m+1,n} - U^{j+1}_{l-1,m-1,n}\\ &+ U^{j+1}_{l,m+1,n+1} - U^{j+1}_{l,m-1,n+1} + U^{j+1}_{l,m+1,n-1} - U^{j+1}_{l,m-1,n-1}\\ &+ 2(U^{j+1}_{l,m+1,n} - U^{j+1}_{l,m-1,n})] - \frac{h}{6}[\widehat{F}^{j+1}_{l,m+\frac{1}{2},n} - \widehat{F}^{j+1}_{l,m-\frac{1}{2},n}] + \widehat{T}^{j+1}_{y_{l,m,n}}\end{aligned} \tag{68}$$

$$\begin{aligned}U^{j+1}_{z_{l,m,n}} &= \frac{1}{12h}[U^{j+1}_{l+1,m,n+1} - U^{j+1}_{l+1,m,n-1} + U^{j+1}_{l-1,m,n+1} - U^{j+1}_{l-1,m,n-1}\\ &+ U^{j+1}_{l,m+1,n+1} + U^{j+1}_{l,m-1,n+1} - U^{j+1}_{l,m+1,n-1} - U^{j+1}_{l,m-1,n-1}\\ &+ 2(U^{j+1}_{l,m,n+1} - U^{j+1}_{l,m,n-1})] - \frac{h}{6}[\widehat{F}^{j+1}_{l,m,n+\frac{1}{2}} - \widehat{F}^{j+1}_{l,m,n-\frac{1}{2}}] + \widehat{T}^{j+1}_{z_{l,m,n}}\end{aligned} \tag{69}$$

where $\widehat{T}^{j+1}_{x_{l,m,n}} = O(kh^2+h^4)$, $\widehat{T}^{j+1}_{y_{l,m,n}} = O(kh^2+h^4)$ and $\widehat{T}^{j+1}_{z_{l,m,n}} = O(kh^2+h^4)$.

Note that, the matrices represented by the new formula (48) and (67)-(69) are tri-block-block diagonal and diagonal respectively. The formulas are of $O(k^2+kh^2+h^4)$ accuracy and free from the terms $(1/x_{l\pm1})$, $(1/y_{m\pm1})$ and $(1/z_{n\pm1})$, hence very easily solved for $l, m, n = 1(1)N$ in the region $0 < x, y, z < 1, t > 0$. If the differential equation is linear, we can solve the linear system by using operator splitting method, whereas for non-linear case, we can use Newton-Raphson method. The proposed numerical methods are directly applicable to singular parabolic problems in the region $0 < x, y, z < 1, t > 0$. It is mentioned here that in order to get $O(kh^2+h^4)$ numerical solution of $(\partial u/\partial x)$, $(\partial u/\partial y)$ and $(\partial u/\partial z)$ from (67)-(69), it is essential to know the corresponding accurate difference solution of u, which can be obtained using the formula (48).

3. Derivation of numerical methods

For the derivation of the new methods, we simply follow the techniques given by Mohanty and Singh [12] and Chawla and Shivakumar [13].

At the grid point (x_l, y_m, z_n, t_j), we denote

$$U_{abcd} = \frac{\partial^{a+b+c+d}U}{(\partial x)^a(\partial y)^b(\partial z)^c(\partial t)^d} \tag{70}$$

$$G = \frac{\partial f}{\partial t}, H = \frac{\partial f}{\partial u}, I = \frac{\partial f}{\partial u_x}, J = \frac{\partial f}{\partial u_y}, K = \frac{\partial f}{\partial u_z}, L = \frac{\partial f}{\partial u_t} \tag{71}$$

At the grid point (x_l, y_m, z_n, t_j), we denote

$$\begin{aligned} &\frac{\partial^2 U^j_{l,m,n}}{\partial x^2} + \frac{\partial^2 U^j_{l,m,n}}{\partial y^2} + \frac{\partial^2 U^j_{l,m,n}}{\partial z^2} \\ &\quad = f(x_l.y_m, z_n, t_j, U^j_{l,m,n}, U^j_{x_{l,m,n}}, U^j_{y_{l,m,n}}, U^j_{z_{l,m,n}} U^j_{t_{l,m,n}}) \\ &\quad \equiv F^j_{l,m,n} \end{aligned} \tag{72}$$

By the help of the Taylor expansion, we obtain

$$\begin{aligned} &[\delta_x^2 + \delta_y^2 + \delta_z^2 + \frac{1}{6}(\delta_x^2\delta_y^2 + \delta_y^2\delta_z^2 + \delta_z^2\delta_x^2)]U^j_{l,m,n} \\ &= \frac{h^2}{3}[F^j_{l+\frac{1}{2},m,n} + F^j_{l-\frac{1}{2},m,n} + F^j_{l,m+\frac{1}{2},n} + F^j_{l,m-\frac{1}{2},n} + F^j_{l,m,n+\frac{1}{2}} \\ &+ F^j_{l,m,n-\frac{1}{2}} - 3F^j_{l,m,n}] + O(h^6) \end{aligned} \tag{73}$$

Now differentiating the differential equation (1) with respect to t at the grid point (x_l, y_m, z_n, t_j), we obtain a relation of the form

$$\begin{aligned} -LU_{0002} = G + HU_{0001} + IU_{1001} + JU_{0101} + KU_{0011} \\ - (U_{2001} + U_{0201} + U_{0021}) \end{aligned} \tag{74}$$

By the help of the approximations (6)-(38), simplifying (39)-(41), we get

$$\overline{F}^j_{l\pm\frac{1}{2},m,n} = F^j_{l\pm\frac{1}{2},m,n} + \frac{k}{2}T_1 + \frac{h^2}{24}T_2 \pm O(kh + h^3) \tag{75}$$

$$\overline{F}^j_{l,m\pm\frac{1}{2},n} = F^j_{l,m\pm\frac{1}{2},n} + \frac{k}{2}T_1 + \frac{h^2}{24}T_3 \pm O(kh + h^3) \tag{76}$$

$$\overline{F}^j_{l,m,n\pm\frac{1}{2}} = F^j_{l,m,n\pm\frac{1}{2}} + \frac{k}{2}T_1 + \frac{h^2}{24}T_4 \pm O(kh + h^3) \tag{77}$$

where

$$\begin{aligned} T_1 &= 2\theta(G + U_{0001}H + U_{1001}I + U_{0101}J + U_{0011}K) + U_{0002}L, \\ T_2 &= 3U_{2000}H + U_{3000}I + (3U_{2100} + 4U_{0300})J + (3U_{2010} + 4U_{0030})K + 3U_{2001}L, \\ T_3 &= 3U_{0200}H + (4U_{3000} + 3U_{1200})I + U_{0300}J + (3U_{0210} + 4U_{0030})K + 3U_{0201}L, \\ T_4 &= 3U_{0020}H + (4U_{3000} + 3U_{1020})I + (3U_{0120} + 4U_{0300})J + U_{0030}K + 3U_{0021}L \end{aligned}$$

With the help of (75)-(77), from (42)-(46), we obtain

$$\overline{\overline{U}}^j_{l,m,n} = U^j_{l,m,n} + \theta k U_{0001} + 6a_1 h^2 (U_{2000} + U_{0200} + U_{0020}) + O(k^2 + kh^2 + h^4) \tag{78}$$

$$\overline{\overline{U}}^j_{x_{l,m,n}} = U^j_{x_{l,m,n}} + \theta k U_{1001} + \frac{h^2}{6}[(1+6a_2)U_{3000} + 6a_2(U_{1200} + U_{1020})] + O(kh^2 + h^4) \tag{79}$$

$$\overline{\overline{U}}^j_{y_{l,m,n}} = U^j_{y_{l,m,n}} + \theta k U_{0101} + \frac{h^2}{6}[(1+6a_3)U_{0300} + 6a_3(U_{2100} + U_{0120})] + O(kh^2 + h^4) \tag{80}$$

$$\overline{\overline{U}}^j_{z_{l,m,n}} = U^j_{z_{l,m,n}} + \theta k U_{0011} + \frac{h^2}{6}[(1+6a_4)U_{0030} + 6a_4(U_{2010} + U_{0210})] + O(kh^2 + h^4) \tag{81}$$

$$\overline{\overline{U}}^j_{t_{l,m,n}} = U^j_{t_{l,m,n}} + \frac{k}{2}U_{0002} + a_5 h^2 (U_{2001} + U_{0201} + U_{0021}) + O(kh^2 + h^4) \tag{82}$$

By the help of the approximations (78)-(82), from (47), we get

$$\overline{\overline{F}}^j_{l,m,n} = F^j_{l,m,n} + \frac{k}{2}T_1 + \frac{h^2}{6}T_5 + O(k^2 + kh^2 + h^4) \tag{83}$$

where

$$\begin{aligned} T_5 = {} & 36a_1(U_{2000} + U_{0200} + U_{0020})H + [(1+6a_2)U_{3000} + 6a_2(U_{1200} + U_{1020})]I \\ & + [(1+6a_3)U_{0300} + 6a_3(U_{2100} + U_{0120})]J \\ & + [(1+6a_4)U_{0030} + 6a_4(U_{2010} + U_{0210})]K \\ & + 6a_5(U_{2001} + U_{0201} + U_{0021})L \end{aligned}$$

Further, we may re-write

$$\begin{aligned} & \left(\delta_x^2 + \delta_y^2 + \delta_z^2 + \frac{1}{6}(\delta_x^2\delta_y^2 + \delta_y^2\delta_z^2 + \delta_z^2\delta_x^2)\right)\overline{U}^j_{l,m,n} \\ & = \left(\delta_x^2 + \delta_y^2 + \delta_z^2 + \frac{1}{6}(\delta_x^2\delta_y^2 + \delta_y^2\delta_z^2 + \delta_z^2\delta_x^2)\right)U^j_{l,m,n} \\ & + \theta k h^2 (U_{2001} + U_{0201} + U_{0021}) + O(k^2h^2 + kh^4 + h^6) \end{aligned} \tag{84}$$

Finally, by the help of the relations (74), (75)-(77), (83) and (84), from (48) and (73) we obtain the local truncation error

(85)

$$\overline{T}^j_{l,m,n} = -kh^2\left(\frac{1}{2} - \theta\right)U_{0002}L - \frac{h^4}{36}(T_2 + T_3 + T_4 - 6T_5) + O(k^2h^2 + kh^4 + h^6)$$

The proposed difference method (48) to be of $O(k^2 + kh^2 + h^4)$, the coefficients of kh^2 and h^4 in (85) must be zero, hence

$$\frac{1}{2} - \theta = 0 \tag{86}$$

and $T_2 + T_3 + T_4 - 6T_5 = 0$ or,

$$\begin{aligned} & (1 - 72a_1)(U_{2000} + U_{0200} + U_{0020})H + (1 - 12a_2)(U_{3000} + U_{1200} + U_{1020})I \\ & +(1 - 12a_3)(U_{0300} + U_{2100} + U_{0120})J + (1 - 12a_4)(U_{0030} + U_{2010} + U_{0210})K \\ & +(1 - 12a_5)(U_{2001} + U_{0201} + U_{0021})L = 0 \end{aligned} \tag{87}$$

Thus we obtain the values of parameters $\theta = \frac{1}{2}, a_1 = \frac{1}{72}, a_2 = a_3 = a_4 = a_5 = \frac{1}{12}$ for which the proposed method (48) becomes $O(k^2 + kh^2 + h^4)$ and $\overline{T}^j_{l,m,n} = O(k^2h^2 + kh^4 + h^6)$.

Next we discuss the methods of $O(kh^2+h^4)$ for the estimates of $(\partial u/\partial x)$, $(\partial u/\partial y)$ and $(\partial u/\partial z)$. Once the solution u has been obtained at $(j+1)^{th}$ level, one may compute these values using the central difference approximations

$$u^{j+1}_{x_{l,m,n}} = (u^{j+1}_{l+1,m,n} - u^{j+1}_{l-1,m,n})/(2h) \tag{88}$$

$$u^{j+1}_{y_{l,m,n}} = (u^{j+1}_{l,m+1,n} - u^{j+1}_{l,m-1,n})/(2h) \tag{89}$$

$$u^{j+1}_{z_{l,m,n}} = (u^{j+1}_{l,m,n+1} - u^{j+1}_{l,m,n-1})/(2h) \tag{90}$$

It has been verified that the standard central difference approximations (88)-(90) yield $O(h^2)$ accurate results irrespective of whether difference method (48), which is of $O(k^2 + kh^2 + h^4)$ or difference method of $O(k^2 + h^2)$ is used to solve the parabolic equation (1). New difference formulas of $O(kh^2 + h^4)$ for computing the numerical values of u_x, u_y and u_z are proposed. These new formulas are found to yield $O(h^4)$-accuracy for a fixed mesh ratio parameter λ, when used in conjunction with the method (48).

By the help of Taylor series expansion, we obtain

$$\begin{aligned} U^{j+1}_{x_{l,m,n}} = \frac{1}{12h}[&U^{j+1}_{l+1,m+1,n} + U^{j+1}_{l+1,m-1,n} - U^{j+1}_{l-1,m+1,n} - U^{j+1}_{l-1,m-1,n} \\ &+ U^{j+1}_{l+1,m,n+1} + U^{j+1}_{l+1,m,n-1} - U^{j+1}_{l-1,m,n+1} - U^{j+1}_{l-1,m,n-1} \\ &+ 2(U^{j+1}_{l+1,m,n} - U^{j+1}_{l-1,m,n})] - \frac{h}{6}[F^{j+1}_{l+\frac{1}{2},m,n} - F^{j+1}_{l-\frac{1}{2},m,n}] + O(h^4) \end{aligned} \tag{91}$$

$$\begin{aligned} U^{j+1}_{y_{l,m,n}} = \frac{1}{12h}[&U^{j+1}_{l+1,m+1,n} - U^{j+1}_{l+1,m-1,n} + U^{j+1}_{l-1,m+1,n} - U^{j+1}_{l-1,m-1,n} \\ &+ U^{j+1}_{l,m+1,n+1} - U^{j+1}_{l,m-1,n+1} + U^{j+1}_{l,m+1,n-1} - U^{j+1}_{l,m-1,n-1} \\ &+ 2(U^{j+1}_{l,m+1,n} - U^{j+1}_{l,m-1,n})] - \frac{h}{6}[F^{j+1}_{l,m+\frac{1}{2},n} - F^{j+1}_{l,m-\frac{1}{2},n}] + O(h^4) \end{aligned} \tag{92}$$

$$\begin{aligned} U^{j+1}_{z_{l,m,n}} = \frac{1}{12h}[&U^{j+1}_{l+1,m,n+1} - U^{j+1}_{l+1,m,n-1} + U^{j+1}_{l-1,m,n+1} - U^{j+1}_{l-1,m,n-1} \\ &+ U^{j+1}_{l,m+1,n+1} + U^{j+1}_{l,m-1,n+1} - U^{j+1}_{l,m+1,n-1} - U^{j+1}_{l,m-1,n-1} \\ &+ 2(U^{j+1}_{l,m,n+1} - U^{j+1}_{l,m,n-1})] - \frac{h}{6}[F^{j+1}_{l,m,n+\frac{1}{2}} - F^{j+1}_{l,m,n-\frac{1}{2}}] + O(h^4) \end{aligned} \tag{93}$$

where

$$\begin{aligned} F^{j+1}_{l\pm\frac{1}{2},m,n} =& f(x_{l\pm\frac{1}{2}}, y_m, z_n, t_{j+1}, U^{j+1}_{l\pm\frac{1}{2},m,n}, U^{j+1}_{x_{l\pm\frac{1}{2},m,n}}, U^{j+1}_{y_{l\pm\frac{1}{2},m,n}}, \\ & U^{j+1}_{z_{l\pm\frac{1}{2},m,n}}, U^{j+1}_{t_{l\pm\frac{1}{2},m,n}}) \\ F^{j+1}_{l,m\pm\frac{1}{2},n} =& f(x_l, y_{m\pm\frac{1}{2}}, z_n, t_{j+1}, U^{j+1}_{l,m\pm\frac{1}{2},n}, U^{j+1}_{x_{l,m\pm\frac{1}{2},n}}, U^{j+1}_{y_{l,m\pm\frac{1}{2},n}}, \\ & U^{j+1}_{z_{l,m\pm\frac{1}{2},n}}, U^{j+1}_{t_{l,m\pm\frac{1}{2},n}}) \\ F^{j+1}_{l,m,n\pm\frac{1}{2}} =& f(x_l, y_m, z_{n\pm\frac{1}{2}}, t_{j+1}, U^{j+1}_{l,m,n\pm\frac{1}{2}}, U^{j+1}_{x_{l,m,n\pm\frac{1}{2}}}, U^{j+1}_{y_{l,m,n\pm\frac{1}{2}}}, \\ & U^{j+1}_{z_{l,m,n\pm\frac{1}{2}}}, U^{j+1}_{t_{l,m,n\pm\frac{1}{2}}}) \end{aligned}$$

By the help of the approximations (49)-(63), from (64)-(66), we obtain

$$\widehat{F}^{j+1}_{l\pm\frac{1}{2},m,n} = F^{j+1}_{l\pm\frac{1}{2},m,n} + O(k+h^2) \tag{94}$$

$$\widehat{F}^{j+1}_{l,m\pm\frac{1}{2},n} = F^{j+1}_{l,m\pm\frac{1}{2},n} + O(k+h^2) \tag{95}$$

$$\widehat{F}^{j+1}_{l,m,n\pm\frac{1}{2}} = F^{j+1}_{l,m,n\pm\frac{1}{2}} + O(k+h^2) \tag{96}$$

With the help of the approximations (94)-(96) and using the relations (91)-(93), from (67)-(69), it is easy to verify that $\widehat{T}^{j+1}_{x_{l,m,n}} = O(kh^2+h^4)$, $\widehat{T}^{j+1}_{y_{l,m,n}} = O(kh^2+h^4)$ and $\widehat{T}^{j+1}_{z_{l,m,n}} = O(kh^2+h^4)$.

4. Operator Splitting method

Consider the three-space dimensional linear singular diffusion equation

$$\nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{1}{x}\frac{\partial u}{\partial x}\right) = \frac{\partial u}{\partial t} + g(x,y,z,t), \quad 0 < x,y,z < 1, t > 0 \tag{97}$$

subject to appropriate initial and Dirichlet boundary conditions of type (2)-(5) are prescribed, where $\nu > 0$ is called the diffusivity and $g(x,y,z,t)$ is the forcing function.

An application of the new difference method (48) to the differential equation (97) leads to a linear difference scheme

$$\begin{aligned}
&\Big[1 + \frac{1}{12}(1 - 6\nu\lambda + \lambda P_1)\delta_x^2 + \frac{1}{12}(1-6\nu\lambda)(\delta_y^2+\delta_z^2) \\
&\qquad + \frac{1}{12}\left(\frac{h}{2x_l} + \lambda P_2\right)(2\mu_x\delta_x) - \frac{\nu\lambda h}{24x_l}(\delta_y^2+\delta_z^2)(2\mu_x\delta_x) \\
&\qquad - \frac{\lambda\nu}{12}(\delta_x^2\delta_y^2 + \delta_y^2\delta_z^2 + \delta_z^2\delta_x^2)\Big] u^{j+1}_{l,m,n} \\
&= \Big[1 + \frac{1}{12}(1 + 6\nu\lambda - \lambda P_1)\delta_x^2 + \frac{1}{12}(1+6\nu\lambda)(\delta_y^2+\delta_z^2) \\
&\qquad + \frac{1}{12}\left(\frac{h}{2x_l} - \lambda P_2\right)(2\mu_x\delta_x) + \frac{\nu\lambda h}{24x_l}(\delta_y^2+\delta_z^2)(2\mu_x\delta_x) \\
&\qquad + \frac{\lambda\nu}{12}(\delta_x^2\delta_y^2 + \delta_y^2\delta_z^2 + \delta_z^2\delta_x^2)\Big] u^{j}_{l,m,n} - \frac{k}{12}\Sigma g
\end{aligned} \tag{98}$$

where

$\overline{g}^j_{l,m,n} = g(x_i, y_m, z_n, \overline{t}_j), \overline{g}^j_{l\pm\frac{1}{2},m,n} = g(x_{l\pm\frac{1}{2}}, y_m, z_n, \overline{t}_j)$ etc. and

$$P_1 = -\nu h\left(\frac{1}{x_{l+\frac{1}{2}}} - \frac{1}{x_{l-\frac{1}{2}}}\right) - \frac{\nu h^2}{2x_l^2},$$

$$P_2 = -\nu h\left(\frac{1}{x_{l+\frac{1}{2}}} + \frac{1}{x_l} + \frac{1}{x_{l-\frac{1}{2}}}\right) - \frac{\nu h^2}{4x_l}\left(\frac{1}{x_{l+\frac{1}{2}}} - \frac{1}{x_{l-\frac{1}{2}}}\right),$$

$$\begin{aligned}
\Sigma g = 4\Big(&\overline{g}^j_{l+\frac{1}{2},m,n} + \overline{g}^j_{l-\frac{1}{2},m,n} + \overline{g}^j_{l,m+\frac{1}{2},n} + \overline{g}^j_{l,m-\frac{1}{2},n} + \overline{g}^j_{l,m,n+\frac{1}{2}} + \overline{g}^j_{l,m,n-\frac{1}{2}} - 3\overline{g}^j_{l,m,n}\Big) \\
&+ \frac{h}{x_l}\left(\overline{g}^j_{l+\frac{1}{2},m,n} - \overline{g}^j_{l-\frac{1}{2},m,n}\right).
\end{aligned}$$

Note that, the linear difference scheme (98) is of $O(k^2+h^4)$ accuracy and requires solution of a system of equations with a large band width (19-diagonal) matrix at each time level.

We can rewrite (98) in product form as:

$$\begin{aligned}&[1+L_x][1+L_y][1+L_z]u_{l,m,n}^{j+1}\\ &\qquad\qquad = [1+M_x][1+M_y][1+M_z]u_{l,m,n}^{j} - \frac{k}{12}\Sigma g \equiv [R_u]\end{aligned} \tag{99}$$

where

$$\begin{aligned}
L_x &= \frac{1}{12}(1-6\nu\lambda+\lambda P_1)\delta_x^2 + \frac{1}{12}\left(\frac{h}{2x_l}+\lambda P_2\right)(2\mu_x\delta_x),\\
L_y &= \frac{1}{12}(1-6\nu\lambda)\delta_y^2, \qquad L_z = \frac{1}{12}(1-6\nu\lambda)\delta_z^2,\\
M_x &= \frac{1}{12}(1+6\nu\lambda-\lambda P_1)\delta_x^2 + \frac{1}{12}\left(\frac{h}{2x_l}-\lambda P_2\right)(2\mu_x\delta_x),\\
M_y &= \frac{1}{12}(1+6\nu\lambda)\delta_y^2, \qquad M_z = \frac{1}{12}(1+6\nu\lambda)\delta_z^2
\end{aligned}$$

The additional terms added in (99) are of high orders and do not affect the accuracy of the scheme but enables a factorization of the operators of the scheme (98) which is of $O(k^2+h^4)$.

Now we study the Von-Neumann linear stability of the method (99). Assume that the solution of (99) at each grid point (x_l, y_m, z_n, t_j) is of the form

$$u_{l,m,n}^{j} = \xi^j e^{i(\alpha l+\beta m+\gamma n)} \tag{100}$$

where $i=\sqrt{-1}$, ξ is the amplification factor and may be complex and α,β,γ are phase angles. For stability, the amplification factor ξ has to satisfy the inequality $|\xi| \le 1$ for $-\pi \le \alpha,\beta,\gamma \le \pi$. Substituting (100) into the homogeneous part of equation (99), the amplification factor ξ can be written as

$$\xi = \frac{A_1A_2A_3 + iA_2A_3A_4}{B_1B_2B_3 + iB_2B_3B_4} \tag{101}$$

where

$$\begin{aligned}
A_1 &= 1-\frac{1}{3}(1+6\nu\lambda-\lambda P_1)\sin^2\frac{\alpha}{2}, & A_2 &= 1-\frac{1}{3}(1+6\nu\lambda)\sin^2\frac{\beta}{2},\\
A_3 &= 1-\frac{1}{3}(1+6\nu\lambda)\sin^2\frac{\gamma}{2}, & A_4 &= \frac{1}{6}\left(\frac{h}{2x_l}-\lambda P_2\right)\sin\alpha,\\
B_1 &= 1-\frac{1}{3}(1-6\nu\lambda+\lambda P_1)\sin^2\frac{\alpha}{2}, & B_2 &= 1-\frac{1}{3}(1-6\nu\lambda)\sin^2\frac{\beta}{2},\\
B_3 &= 1-\frac{1}{3}(1-6\nu\lambda)\sin^2\frac{\gamma}{2}, & B_4 &= \frac{1}{6}\left(\frac{h}{2x_l}+\lambda P_2\right)\sin\alpha.
\end{aligned}$$

Since $\max\left(\sin^2\frac{\alpha}{2}\right) = \max\left(\sin^2\frac{\beta}{2}\right) = \max\left(\sin^2\frac{\gamma}{2}\right) = 1$ and imposing this condition directly on (101), we found that the inequality $|\xi|^2 \le 1$ is satisfied for all phase angles $\alpha,\beta,\gamma \in [-\pi,\pi]$. Thus the scheme (99) is stable for all choices of $h>0$ and $k>0$.

In order to facilitate the computation, we may write the scheme (99) in three-step operator split form as (see [4-7])

$$[1+L_z]u_{l,m,n}^{**} = [R_u] \tag{102}$$

$$[1+L_y]u_{l,m,n}^{*} = u_{l,m,n}^{**} \tag{103}$$

$$[1+L_x]u_{l,m,n}^{j+1} = u_{l,m,n}^{*} \tag{104}$$

Note that left hand sides of (102)-(104) are factorizations into $z-, y-$ and $x-$ differences which allows us to solve (102)-(104) by sweeping first in the $z-$, second in the $y-$ and then in $x-$ direction by the help of a tri-diagonal solver. $u^{*}_{l,m,n}$ and $u^{**}_{l,m,n}$ are intermediate values and the intermediate boundary conditions required for sweeping can be computed from (104) and (103), respectively.

Combining (103) and (104), we obtain

$$[1 + L_y][1 + L_x]u^{j+1}_{l,m,n} = u^{**}_{l,m,n} \tag{105}$$

Thus the intermediate boundary conditions from (104) and (105) are obtained as follows:

$$u^{*}_{l,0,n} = [1 + L_x]u^{j+1}_{l,0,n}, \quad u^{*}_{l,N+1,n} = [1 + L_x]u^{j+1}_{l,N+1,n} \tag{106}$$

$$u^{**}_{l,m,0} = [1 + L_y][1 + L_x]u^{j+1}_{l,m,0}, \quad u^{**}_{l,m,N+1} = [1 + L_y][1 + L_x]u^{j+1}_{l,m,N+1} \tag{107}$$

Further note that, the variable coefficients associated with (102)-(104) are free from the terms $(1/x_{l\pm1})$, $(1/y_{m\pm1})$ and $(1/z_{n\pm1})$, thus very easily solved for $l, m, n = 1(1)N$ in the solution region Ω and no fictitious points are required to calculate the intermediate boundary conditions.

5. Computational Results

If we substitute the approximations (6), (7), (20), (24), (28) and (32) with $\theta = \frac{1}{2}$ into the differential equation (1) we get the difference scheme

$$\begin{aligned}(\delta_x^2 + \delta_y^2 + \delta_z^2)\overline{U}^{j}_{l,m,n} &\\ &= h^2 f(x_l, y_m, z_n, \bar{t}_j, \overline{U}^{j}_{l,m,n}, \overline{U}^{j}_{x_{l,m,n}}, \overline{U}^{j}_{y_{l,m,n}}, \overline{U}^{j}_{z_{l,m,n}}, \overline{U}^{j}_{t_{l,m,n}}) \\ &\quad + O(k^2h^2 + h^4)\end{aligned} \tag{108}$$

To provide some indication of the accuracy of the proposed numerical methods, we have solved the following three problems, whose exact solutions are known. The right hand side functions, initial and boundary conditions may be obtained using the exact solution as a test procedure. The operator splitting method has solved the linear equation, whereas the generalized Newton-Raphson method has solved the non-linear equations (see Hageman and Young [15]). We have also compared the proposed method, with the method (108) which is of $O(k^2 + h^2)$. All computations were carried out using the double precision arithmetic.

Example 1: The problem is to solve (97) with the exact solution given by $u = e^{-\nu t}\cosh x \cosh y \cosh z$. The root mean square (RMS) errors for u, u_x, u_y, u_z are tabulated in Table 1 at $t = 1.0$ for $\nu = 01, 001, 0001$ and for a fixed mesh ratio parameter $\lambda = 1.6$.

Example 2: (Burgers' Equation)

$$\nu(u_{xx} + u_{yy} + u_{zz}) = u_t + u(u_x + u_y + u_z), 0 < x, y, z < 1, t > 0 \tag{109}$$

The exact solution is given by $u(x, y, z, t) = \dfrac{2\nu\pi \sin(\pi(x + y + z))e^{-3\nu\pi^2 t}}{2 + \cos(\pi(x + y + z))e^{-3\nu\pi^2 t}}$.
where $\nu = R_e^{-1} > 0$. The RMS errors for u, u_x, u_y, u_z are tabulated in Table 2 at $t = 1.0$ for various values of R_e and for a fixed mesh ratio parameter $\lambda = 1.6$.

Example 3:

$$u_{xx} + u_{yy} + u_{zz} = u_t + \alpha u(u_x + u_y + u_z) + f(x, y, z, t), \quad 0 < x, y, z < 1, t > 0 \tag{110}$$

The exact solution is given by $u = e^{-t} \sin(\pi x) \cos(\pi y) \sin(\pi z)$. The RMS errors for u, u_x, u_y, u_z are tabulated in Table 3 at $t = 1.0$ for various values of α and for a fixed mesh ratio parameter $\lambda = 1.6$.

6. Concluding Remarks

In this article, we have developed a new three-level 19-spatial grid point implicit finite difference method of $O(k^2 + kh^2 + h^4)$ based on grid points x_l, y_m, z_n, t_j, $x_{l\pm1/2}, y_{m\pm1/2}, z_{n\pm1/2}, t_{j+1}$ for the solution of three space dimensional non-linear parabolic partial differential equations and the estimates of first order derivatives $(\partial u/\partial x)$, $(\partial u/\partial y)$ and $(\partial u/\partial z)$. Although the proposed methods involve more algebra, the developments of new methods yield direct application to singular problems without any modification in the original scheme, which is an added advantage. The operator splitting method, which is unconditionally stable, is really impressive, for which we need only a tri-diagonal solver to solve the linear problem. Computational results indicate that the proposed high order methods are computationally more efficient than the corresponding difference methods of $O(k^2 + h^2)$. The numerical results confirm that the proposed methods produce oscillation free solutions for large values of R_e.

TABLE - 1 Example 1: The RMS errors

		$O(k^2 + kh^2 + h^4)$ - ADI method			$O(k^2 + h^2)$ - ADI method		
h		$\nu = .1$	$\nu = .01$	$\nu = .001$	$\nu = .1$	$\nu = .01$	$\nu = .001$
$\frac{1}{4}$	u	.6086(-5)	.4884(-6)	.1619(-7)	.3618(-4)	.4502(-5)	.4684(-6)
	u_x	.2965(-3)	.2878(-3)	.2868(-3)	.7341(-2)	.8032(-2)	.8105(-2)
	u_y	.4124(-3)	.4147(-3)	.4148(-3)	.7341(-2)	.8032(-2)	.8105(-2)
	u_z	.4124(-3)	.4147(-3)	.4148(-3)	.7341(-2)	.8032(-2)	.8105(-2)
$\frac{1}{8}$	u	.3860(-6)	.2221(-7)	.8688(-8)	.7520(-5)	.8615(-6)	.8881(-7)
	u_x	.1984(-4)	.1927(-4)	.1920(-4)	.1971(-2)	.2157(-2)	.2176(-2)
	u_y	.2751(-4)	.2767(-4)	.2767(-4)	.1971(-2)	.2157(-2)	.2176(-2)
	u_z	.2751(-4)	.2767(-4)	.2767(-4)	.1971(-2)	.2157(-2)	.2176(-2)
$\frac{1}{16}$	u	.2007(-7)	.1080(-8)	.5550(-9)	.1618(-5)	.2424(-6)	.2819(-7)
	u_x	.1133(-5)	.1121(-5)	.1119(-5)	.4188(-3)	.4280(-3)	.4284(-3)
	u_y	.2002(-5)	.2088(-5)	.2088(-5)	.4188(-3)	.4280(-3)	.4284(-3)
	u_z	.2002(-5)	.2088(-5)	.2088(-5)	.4188(-3)	.4280(-3)	.4284(-3)

TABLE -2 Example 2: The RMS errors

		$O(k^2+kh^2+h^4)$ method			$O(k^2+h^2)$ method		
h		$R_e=10$	$R_e=10^2$	$R_e=10^3$	$R_e=10$	$R_e=10^2$	$R_e=10^3$
$\frac{1}{4}$	u	.3168(-4)	.5551(-4)	.6116(-5)	.1733(-2)	.9990(-3)	.2020(-4)
	u_x	.1704(-2)	.2005(-2)	.3420(-3)	.2958(-2)	.8159(-2)	.1996(-2)
	u_y	.1704(-2)	.2005(-2)	.3420(-3)	.2958(-2)	.8159(-2)	.1996(-2)
	u_z	.1704(-2)	.2005(-2)	.3420(-3)	.2958(-2)	.8159(-2)	.1996(-2)
$\frac{1}{8}$	u	.1244(-5)	.2664(-5)	.3636(-6)	.3486(-3)	.2289(-3)	.6424(-5)
	u_x	.1108(-3)	.1074(-3)	.2468(-4)	.7957(-3)	.1933(-2)	.5249(-3)
	u_y	.1108(-3)	.1074(-3)	.2468(-4)	.7957(-3)	.1933(-2)	.5249(-3)
	u_z	.1108(-3)	.1074(-3)	.2468(-4)	.7957(-3)	.1933(-2)	.5249(-3)

TABLE- 3 Example 3: The RMS errors

		$O(k^2+kh^2+h^4)$ method			$O(k^2+h^2)$ method	
h		$\alpha=10$	$\alpha=20$	$\alpha=50$	$\alpha=10$	$\alpha=20$ and 50
$\frac{1}{4}$	u	.8163(-3)	.4181(-2)	.6318(-2)	.2892(-2)	Over Flow
	u_x	.4729(-2)	.1082(-1)	.3466(-1)	.2594(-1)	
	u_y	.1650(-1)	.2268(-1)	.5018(-1)	.5740(-1)	
	u_z	.4729(-2)	.1082(-1)	.3466(-1)	.2594(-1)	
$\frac{1}{8}$	u	.5115(-4)	.2020(-3)	.3615(-3)	.6543(-3)	Over Flow
	u_x	.3043(-3)	.8583(-3)	.2409(-2)	.7981(-2)	
	u_y	.9204(-3)	.1332(-2)	.3007(-2)	.1136(-1)	
	u_z	.3043(-3)	.8583(-3)	.2409(-2)	.7981(-2)	

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