Potts Model with q States on Directed Barabási-Albert Networks

F. W. S. Lima*

Departamento de Física, Universidade Federal do Piauí, 57072-970 Teresina PI, Brazil.

Abstract. On directed Barabási-Albert networks with two and seven neighbours selected by each added site, the Ising model with spin $S=1/2$ was seen not to show a spontaneous magnetisation. Instead, the decay time for flipping of the magnetisation followed an Arrhenius law for Metropolis and Glauber algorithms, but for Wolff cluster flipping the magnetisation decayed exponentially with time. However, on these networks the Ising model spin $S=1$ was seen to show a spontaneous magnetisation. In this case, a first-order phase transition for values of connectivity $z=2$ and $z=7$ is well defined. On these same networks the Potts model with $q=3$ and 8 states is now studied through Monte Carlo simulations. We also obtained for $q=3$ and 8 states a first-order phase transition for values of connectivity $z=2$ and $z=7$ for the directed Barabási-Albert network. Theses results are different from the results obtained for the same model on two-dimensional lattices, where for $q=3$ the phase transition is of second-order, while for $q=8$ the phase transition is of first-order.

1 Introduction

Sumour and Shabat [1,2] investigated the Ising models with spin $S=1/2$ on the directed Barabási-Albert networks [3] with the usual Glauber dynamics. No spontaneous magnetisation was found, in contrast to the case of undirected Barabási-Albert networks [4–6] where a spontaneous magnetisation was found lower a critical temperature which increases logarithmically with system size. In $S=1/2$ systems on undirected, scale-free hierarchical-lattice small-world networks [7], conventional and algebraic (Berezinskii-Kosterlitz-Thouless) ordering, with finite transition temperatures, have been found. Lima

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*Corresponding author. *Email address:* `wel@ufpi.br` (F. W. S. Lima)

and Stauffer [8] simulated directed square, cubic and hypercubic lattices in two to five dimensions with heat bath dynamics in order to separate the network effects form the effects of directedness. They also compared different spin flip algorithms, including cluster flips [9], for Ising-Barabási-Albert networks. They found a freezing-in of the magnetisation similar to [1, 2], following an Arrhenius law at least in low dimensions. This lack of a spontaneous magnetisation (in the usual sense) is consistent with the fact that if on a directed lattice a spin S_j influences spin S_i, then spin S_i in turn does not influence S_j, and there may be no well-defined total energy. Thus, they show that for the same scale-free networks, different algorithms give different results. More recently, Lima [10, 11] simulated the Ising model for spin $S=1$ on the directed Barabási-Albert network and different from the Ising model for spin $S=1/2$, the order-disorder phase transition of order parameter is well defined in this system. He obtained a first-order phase transition for values of connectivity $z=2$ and $z=7$ of the directed Barabási-Albert network. Now we study the Potts model for $q=3$ and 8 on the directed Barabási-Albert network for values of connectivity $z=2$ and $z=7$. Unlike the Ising model for spin $S=1/2$, the order-disorder phase transition of order parameter is well defined in this system. We obtained a first-order phase transition for values of connectivity $z=2$ and $z=7$ for the directed Barabási-Albert network.

2 Model and simulation

We consider the Potts model with $q=3$ and 8 states, on the directed Barabási-Albert Networks, defined by a set of spin variables σ taking the values 1, 2 and 3 for $q=3$, and $\sigma=1,\cdots,8$ for $q=8$ situated on every site of a directed Barabási-Albert Network with N sites.

The Potts interaction energy is given by

$$E=-J\sum_i\sum_k \delta_{\sigma_i\sigma_k} \tag{2.1}$$

where k-sum runs over all nearest neighbors of i. In this network, each new site added to the network selects z already existing sites as neighbours influencing it; the newly added spin does not influence these neighbours. To study the critical behavior of the model we define the variable $e=E/N$ and $m=(q\cdot\max_i[n_i]-N)/(q-1)$, where $n_i \leq N$ denotes the number of spins with "orientation" $i=1$, 2 and 3 for $q=3$ and $i=1,2,\cdots,8$ for $q=8$, in one network configuration. From the variable energy we can compute the average energy, the specific heat and the energetic fourth-order cumulant,

$$u(K)=[<E>]_{av}/N, \tag{2.2}$$

$$C(K)=K^2N[<e^2>-<e>^2]_{av}, \tag{2.3}$$

$$B_i(K)=\left[1-\frac{<e^4>}{3<e^2>^2}\right]_{av}, \tag{2.4}$$

where $K = J/k_B T$, with $J = 1$, and k_B is the Boltzmann constant. Similarly, we can derive from the magnetisation measurements the average magnetisation, the susceptibility and the magnetic cumulants,

$$m(K) = [<|m|>]_{av},$$
(2.5)

$$\chi(K) = KN[<m^2> - <|m|>^2]_{av},$$
(2.6)

$$U_4(K) = \left[1 - \frac{<m^4>}{3<|m|>^2}\right]_{av},$$
(2.7)

where $< \cdots >$ stands for a thermodynamics average and $[\cdots]_{av}$ square brackets for an average over 20 realizations.

In order to verify the order of transition, we apply finite-size scaling (FSS) [12]. Initially we search for the minima of the energetic fourth-order cumulant in Eq. (2.4). This quantity gives a qualitative as well as a quantitative description of the order of transition [13]. It is known [14] that this parameter takes a minimun value $B_{i,min}$ at the effective transition temperature $T_c(N)$. One can show [15] that for a second-order transition $\lim_{N \to \infty} (2/3 - B_{i,min}) = 0$, even at T_c, while at a first-order transition the same limit measures the latent heat $|e_+ - e_-|$:

$$\lim_{N \to \infty} (2/3 - B_{i,min}) = \frac{1}{3} \frac{(e_+ - e_-)^2(e_+ + e_-)^2}{(e_+^2 - e_-^2)^2}.$$
(2.8)

A more quantitative analysis can be carried out through the FSS of the specific heat C_{max}, the susceptibility maxima χ_{max} and the minima of the Binder parameter $B_{i,min}$. If the hypothesis of a first-order phase transition is correct, we should then expect, for large systems sizes, an asymptotic FSS behavior of the form [16,17],

$$C_{max} = a_C + b_C N + \cdots$$
(2.9)

$$\chi_{max} = a_\chi + b_\chi N + \cdots$$
(2.10)

$$B_{i,min} = a_{B_i} + b B_i N + \cdots.$$
(2.11)

We have performed Monte Carlo simulation on the directed Barabási-Albert networks with values of connectivity $z = 2$ and 7. For a given z, we used systems of size $N = 250, 500, 1000, 2000, 4000,$ and 8000 sites. We waited 10000 Monte Carlo steps (MCS) to make the system reach the steady state, and the time averages were estimated from the next 10000 MCS. In our simulations, one MCS is accomplished after all the N spins are updated. For all sets of parameters, we have generated 20 distinct networks, and have simulated 20 independent runs for each distinct network.

3 Results and discussion

Our simulations, using the HeatBath algorithm, indicate that the model displays a first order phase transition. In Fig. 1 we show the dependence of the magnetisation M on the
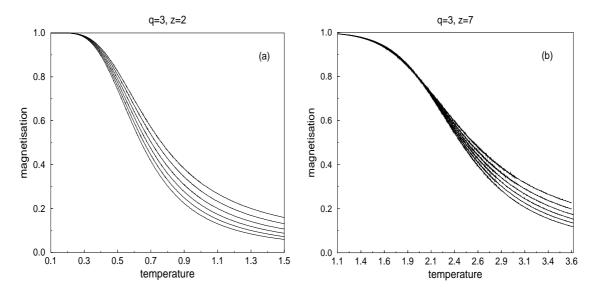
Figure 1: Plot of the magnetisation M versus K for $q=3$ and severals systems sizes (from top to bottom: $N=250$, 500, 1000, 2000, 4000, and 8000): (a) $z=2$ and (b) $z=7$.
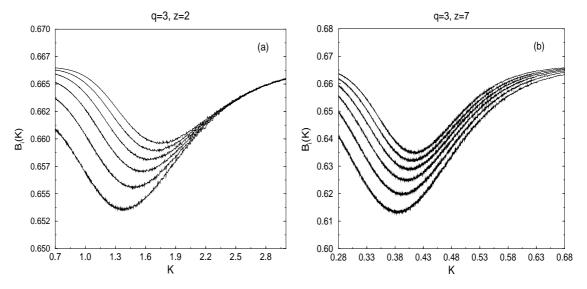

Figure 2: Plot of the Binder cumulant $B_i(K)$ versus K for $q=3$ and severals systems sizes (from top to bottom: $N=250$, 500, 1000, 2000, 4000, and 8000): (a) $z=2$ and (b) $z=7$.

temperature T for $q=3$ and several system sizes. Fig. 1(a) shows the curves for $z=2$ from $N=250$ (top) to 8000 (bottom), and Fig. 1(b) for $z=7$. In Fig. 2 we show the dependence of the Binder parameter $B_i(K)$ for connectivity $z=2$ and 7 and various systems size. Fig. 2(a) shows the curves for $z=2$ from $N=250$ (top) to 8000 (bottom), and Fig. 2(b) for $z=7$. The Binder parameter clearly goes to a value which is different from 2/3. This is
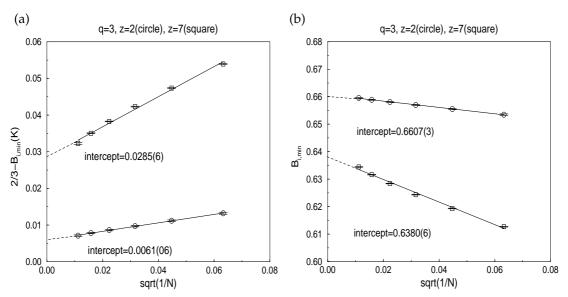
(a)

(b)

Figure 3: (a): Plot of the Binder parameter $2/3 - B_i(K)$ versus $1/N$; and (b): $B_{i,min}(K)$ versus $1/N$, for $q = 3$, $z = 2$ (circle) and $z = 7$ (square).
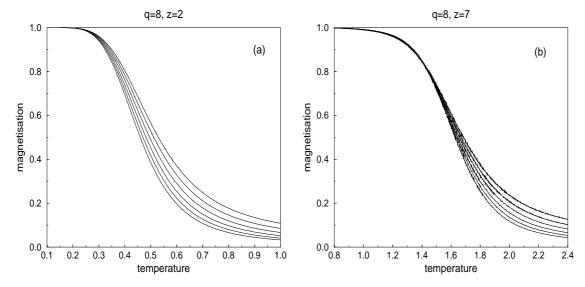

Figure 4: Same as Fig. 1, except $q = 8$.

a sufficient condition to characterize a first-order transition. In Fig. 3, we plot the Binder parameter B_i versus $1/N$ for $z = 2$ (circle) and $z = 7$ (square), and several systems sizes ($N = 250, 500, 1000, 2000, 4000$, and 8000). We show the scaling of the Binder parameter minima, and again the first order phase transition is verified. The order of the transitions can be confirmed by plotting the values of $2/3 - B_{i,min}$ again versus $1/N$. For a second-
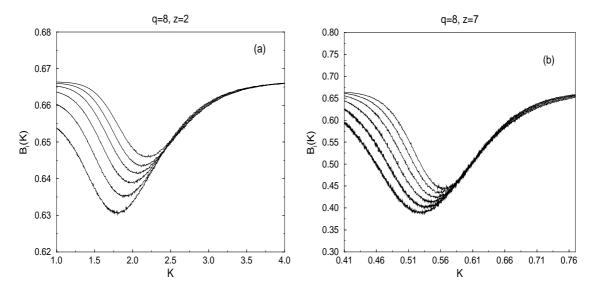
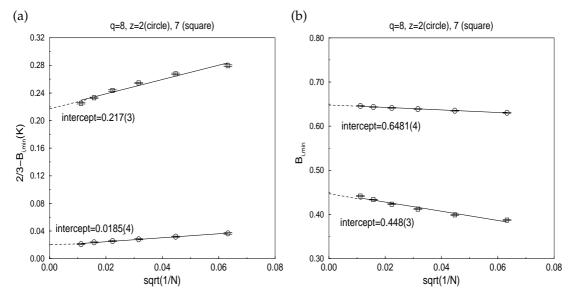Figure 5: Same as Fig. 2, except $q=8$.

Figure 6: Same as Fig. 3, except $q=8$.

order transition the curves go to zero as we increase the system size. Here, the quantity $2/3-B_{i,min}$ approaches a nonvanishing value in the limit of small $1/N$, for $z=2$ and $z=7$, see Fig. 3(a). In Figs. 4, 5, and 6 we show the same behaviors as those of Figs. 1, 2, and 3, but for $q=8$. As depicted in Figs. 7 and 8, our results for scaling of the specific heat and susceptibility are consistent with Eqs. (2.9)-(2.10). In Fig. 7(a) we show this behavior for $z=2$ and in Fig. 7(b) for $z=7$, with $q=3$. The same occurs with Fig. 8, but for $q=8$.
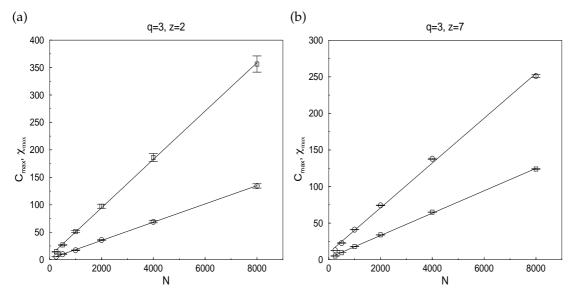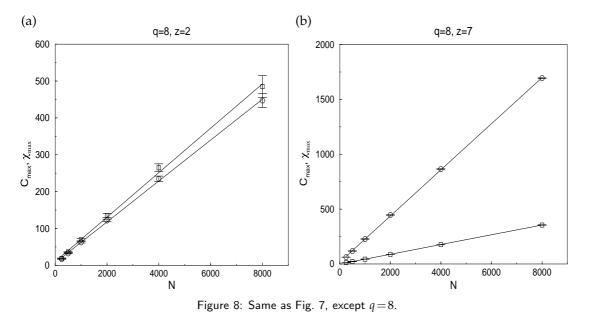
(a)

(b)

Figure 7: Plot of the specific heat C_{max} (circle) and susceptibility χ_{max} (square) versus N for $q=3$: (a) $z=2$, and (b) $z=7$.

(a)

(b)

Figure 8: Same as Fig. 7, except $q=8$.

4 Conclusion

In conclusion, we have presented a very simple equilibrium model on the directed Barabási-Albert network [1,2]. Different from the spin 1/2 Ising model, in these networks, the Potts model with $q=3$ and 8 presents a first-order phase transition which occurs with connec-

tivity $z=2$ and $z=7$ as studied here. These results disagree with the results for the Potts model on two-dimensional lattice [15], where for $q \leq 4$ the transitions are of second-order and are of first-order for $q>4$.

Acknowledgments

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