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Majority-Vote on Undirected Barabási-Albert Networks

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Abstract. On Barabási-Albert networks with z neighbours selected by each added site, the Ising model was seen to show a spontaneous magnetisation. This spontaneous magnetisation was found below a critical temperature which increases logarithmically with system size. On these networks the majority-vote model with noise is now studied through Monte Carlo simulations. However, in this model, the order-disorder phase transition of the order parameter is well defined in this system and this was not found to increase logarithmically with system size. We calculate the value of the critical noise parameter q_c for several values of connectivity z of the undirected Barabási-Albert network. The critical exponentes β/ν, γ/ν and $1/\nu$ were also calculated for several values of z.

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Key words: Monte Carlo simulation, vote, networks, nonequilibrium.

1 Introduction

It has been argued that nonequilibrium stochastic spin systems on regular square lattice with up-down symmetry fall in the universality class of the equilibrium Ising model [1]. This conjecture was found in several models that do not obey detailed balance [2–4]. Campos *et. al.* [5] investigated the majority-vote model on small-world network by rewiring the two-dimensional square lattice. These small-world networks, aside from presenting quenched disorder, also possess long-range interactions. They found that the critical exponents γ/ν and β/ν are different from the Ising model and depend on the rewiring probability. However, it was not evident whether the exponent change was due to the disordered nature of the network or due to the presence of long-range interactions. Lima *et. al.* [6] studied the majority-vote model on Voronoi-Delaunay random lattices

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with periodic boundary conditions. These lattices posses natural quenched disorder in their conecctions. They showed that the presence of quenched connectivity disorder is enough to alter the exponents β/ν and γ/ν and therefore is a relevant term to such non-equilibrium phase-transition. Sumour and Shabat [7,8] investigated the Ising models on the directed Barabási-Albert networks [9] with the usual Glauber dynamics. No spontaneous magnetisation was found, in contrast to the case of undirected Barabási-Albert networks [10–12] where a spontaneous magnetisation was found lower a critical temperature which increases logarithmically with system size. Lima and Stauffer [13] simulated directed square, cubic and hypercubic lattices in two to five dimensions with heat bath dynamics in order to separate the network effects form the effects of directedness. They also compared different spin flip algorithms, including cluster flips [15], for the Ising-Barabási-Albert networks. They found a freezing-in of the magnetisation similar to [7,8], following an Arrhenius law at least in low dimensions. This lack of a spontaneous magnetisation (in the usual sense) is consistent with the fact that if on a directed lattice a spin S_j influences a spin S_i, then the spin S_i in turn does not influence S_j, and there may be no well-defined total energy. Thus, they showed that for the same scale-free networks, different algorithms give different results. More recently, Lima [14] investigated the majority-vote model on the directed Barabási-Albert network and calculated the β/ν, γ/ν and $1/\nu$ exponents. These results are different from those obtained using the Ising model and depend on the values of connectivity z in the directed Barabási-Albert network. In this work, we calculate the same β/ν, γ/ν and $1/\nu$ exponents for the majority-vote model on *undirected* Barabási-Albert network. Numerical results from the Monte Carlo simulations will be reported and discussed.

2 Model and simulation

We consider the majority-vote model, on directed Barabási-Albert Networks, defined [6, 16–18] by a set of "voters" or spins variables σ taking the values $+1$ or -1, situated on every site of an undirected Barabási-Albert Network with N sites, and evolving in time by single spin-flip like dynamics with a probability w_i given by

$$w_i(\sigma)=\frac{1}{2}\left[1-(1-2q)\sigma_i S\left(\sum_{\delta=1}^{k_i}\sigma_{i+\delta}\right)\right], \tag{2.1}$$

where $S(x)$ is the sign ±1 of x if $x\neq 0$, $S(x)=0$ if $x=0$, and the sum runs over all nearest neighbours of σ_i. In this network, each new site added to the network selects old sites as neighbours influencing it; and the newly added spin does not influence these neighbours. The control parameter q plays the role of the temperature in the equilibrium systems and measures the probability of aligning antiparallel to the majority of neighbours.

To study the critical behavior of the model we define the variable $m=\sum_{i=1}^{N}\sigma_i/N$. In particular, we are interested in the magnetisation, the susceptibility and the reduced fourth-order cumulant:

$$M(q)=[<|m|>]_{av}, \tag{2.2}$$

$$\chi(q)=N[<m^2>-<|m|>^2]_{av}, \tag{2.3}$$

$$U(q)=\left[1-\frac{<m^4>}{3<|m|>^2}\right]_{av}, \tag{2.4}$$

where $<...>$ stands for a thermodynamics average and $[...]_{av}$ square brackets for a average over 20 realizations.

These quantities are functions of the noise parameter q and obey the finite-size scaling relations

$$M=N^{-\beta/\nu}f_m(x)[1+\cdots], \tag{2.5}$$

$$\chi=N^{\gamma/\nu}f_\chi(x)[1+\cdots], \tag{2.6}$$

$$\frac{dU}{dq}=N^{1/\nu}f_U(x)[1+\cdots], \tag{2.7}$$

where ν, β, and γ are the usual critical exponents, $f_i(x)$ are the finite size scaling functions with

$$x=(q-q_c)N^{1/\nu} \tag{2.8}$$

being the scaling variable, and the brackets $[1+\cdots]$ indicate corretions-to-scaling terms. Therefore, from the size dependence of M and χ we obtained the exponents β/ν and γ/ν, respectively. The maximum value of susceptibility also scales as $N^{\gamma/\nu}$. Moreover, the value of q for which χ has a maximum, $q_c^{\chi_{max}}=q_c(N)$, is expected to scale with the system size as

$$q_c(N)=q_c+bN^{-1/\nu}, \tag{2.9}$$

were the constant b is close to the unity. Therefore, the relations (2.7) and (2.9) are used to determine the exponente $1/\nu$. We have checked also if the calculated exponents satisfy the hyperscaling hypothesis

$$2\beta/\nu+\gamma/\nu=D_{eff} \tag{2.10}$$

in order to get the effective dimensionality, D_{eff}, for various values of z.

We performed Monte Carlo simulation on the *undirected* Barabási-Albert networks with various values of connectivity z. For a given z, we used systems of size $N=1000$, 2000, 4000, 8000, and 16000. We waited 10000 Monte Carlo steps (MCS) to make the system reach the steady state, and the time averages were estimated from the next 10000 MCS. In our simulations, one MCS is accomplished after all the N spins are updated. For all sets of parameters, we have generated 20 distinct networks, and have simulated 20 independent runs for each distinct network.

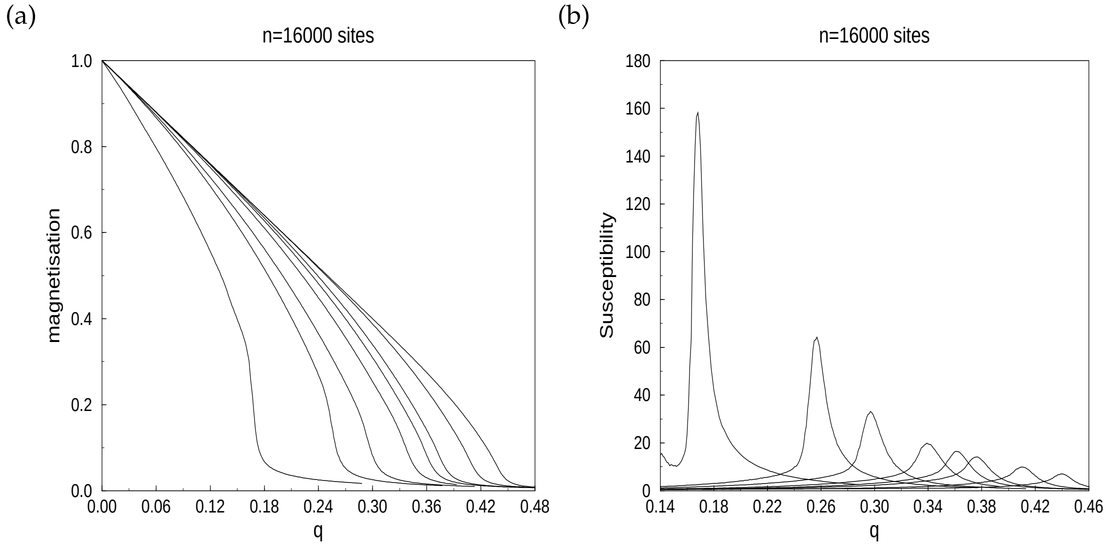

Figure 1: Magnetisation and susceptibility as a function of the noise parameter q, for $N=16000$ sites. Curves from left to right are for $z=2$, 3, 4, 6, 8, 10, 20, and 50.

3 Results and discussion

In Fig. 1 we show the dependence of the magnetisation M and the susceptiblity χ on the noise parameter, obtained from simulations on the *undirected* Barabási-Albert network with 16000 sites and several values of connectivity z. In Fig. 1(a) the curves for M, for given values of N and z, suggest that there is a phase transition from an ordered state to a disordered state. The phase transition occurs at a value of the critical noise parameter q_c, which is an increasing function of the connectivity z in the directed Barabási-Albert network. In Fig. 1(b) we show the corresponding behavior of the susceptibility χ against the value of q, where the maximum of χ is identified as q_c. In Fig. 2, we plot the Binder's fourth-order cumulant for different values of N and two different values of z. The critical noise parameter q_c, for a given value of z, is estimated as the point where the curves for different system sizes N intercept each other. In Fig. 3, the phase diagram is shown as a function of the critical noise parameter q_c on connectivity z obtained from the data of Fig. 2.

The phase diagram of the majority-vote model on the *undirected* Barabási-Albert network shows that for a given network (fixed z) the system becomes ordered for $q<q_c$, whereas it has zero magnetisation for $q \geq q_c$. We notice that the increasement of q_c as a function of z is slower than that in [18]. In Figs. 4 and 5, we plot the dependence of the magnetisation and susceptibility, respectively at $q=q_c$ with the system size. The slopes of the curves correspond to the exponent ratio β/ν and γ/ν according to Eqs. (2.5) and (2.6), respectively. The results show that the exponent ratio β/ν increase and γ/ν decrease at q_c when z increase, see Table 1.

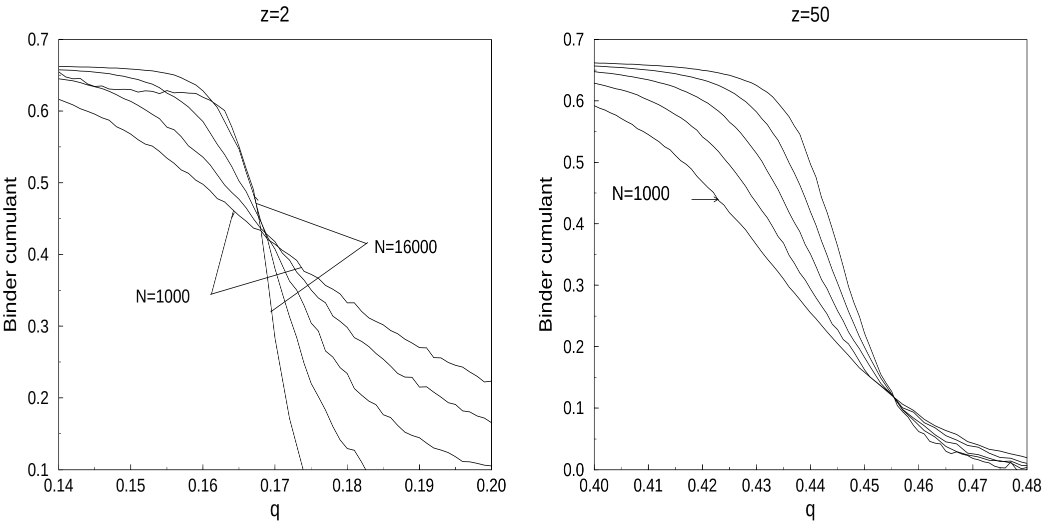

Figure 2: Binder's fourth-order cumulant as a function of q: $z=2$ (left) and $z=50$ (right) for $N=1000$, 2000, 4000, 8000 and 16000 sites.

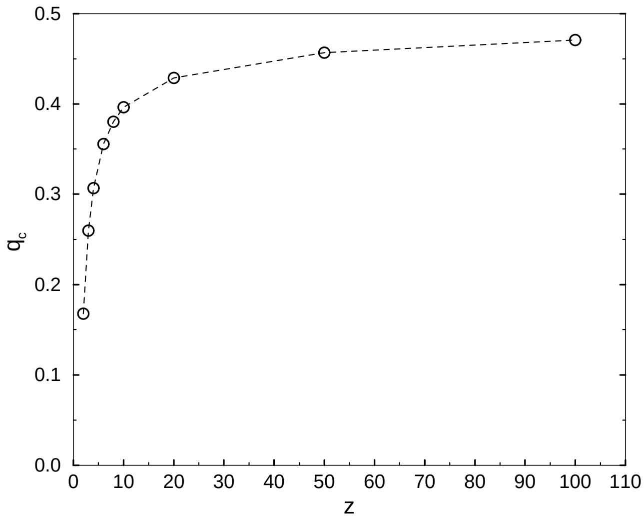

Figure 3: The dependence of the critical noise parameter q_c on the connectivity z.

In Fig. 6 we display the scalings for the susceptibility at $q=q_c$ i.e., $\chi(q_c(N))$, using its maximum amplitude, $\chi^{max}(N)$ (square), and the scalings for the susceptibility at $q=q_c$ obtained from the Binder's cumulant, $\chi(q_c)$ (circle), versus N for the connectivity $z=8$. The exponents ratio γ/ν are obtained from the slopes of the straight lines. For most values of z, the exponents γ/ν of the two estimates agree (along with errors), see Table 1. An increased z means a tendency to decrease the exponent ratio γ/ν, see Table 1. These results agree with those of Pereira and Moreira [18], but disagree with the results

Table 1: The critical noise q_c, the critical exponents, and the effective dimensionality D_{eff}, for undirected Barabási-Albert network with connectivity z. Error bars are statistical only.

z	q_c	β/ν	γ/ν^{q_c}	$\gamma/\nu^{q_c(N)}$	$1/\nu$	D_{eff}
2	0.167(3)	0.036(8)	0.828(6)	0.805(11)	0.76(3)	0.90(1)
3	0.259(2)	0.133(21)	0.713(18)	0.655(31)	0.83(7)	0.979(27)
4	0.306(3)	0.231(22)	0.537(8)	0.519(17)	0.43(2)	0.999(23)
6	0.355(2)	0.283(8)	0.445(15)	0.423(3)	0.35(5)	1.011(17)
8	0.380(6)	0.323(2)	0.358(7)	0.405(6)	0.39(3)	1.004(7)
10	0.396(3)	0.338(2)	0.324(2)	0.380(3)	0.324(5)	1.000(2)
20	0.428(2)	0.344(2)	0.305(2)	0.350(2)	0.307(5)	0.993(2)
50	0.456(3)	0.366(2)	0.255(2)	0.341(3)	0.30(1)	0.987(2)
100	0.471(3)	0.373(2)	0.218(5)	0.330(3)	0.308(4)	0.964(5)

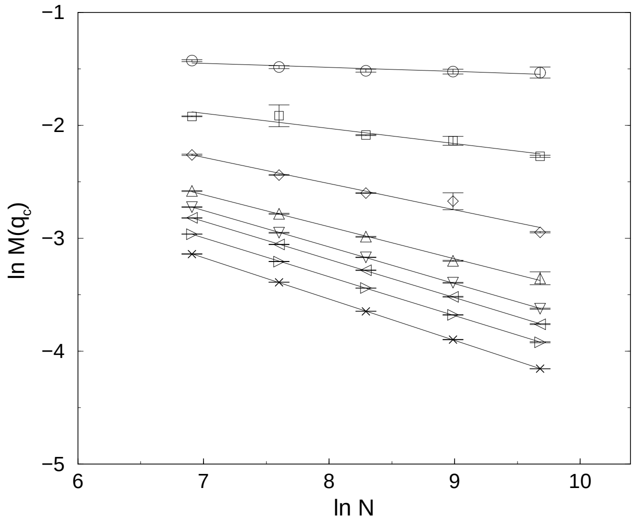

Figure 4: $\ln M(q_c)$ versus $\ln N$. From top to bottom, z=2, 3, 4, 6, 8, 10, 20, and 50.

of Lima [14] for the *directed* Barabási-Albert network, where the values of the exponents ratio γ/ν are all different and with a slight tendency to increase at q_c and decrease at $q_c(N)$. Therefore we can use the Eq. (2.9), for fixed z, to obtain the critical exponent $1/\nu$, see Fig. 7. In Fig. 8, we show the critical behaviors of the exponentes β/ν, γ/ν and $1/\nu$ as a function of the connectivity z.

To obtain the critical exponent $1/\nu$, we calculated numerically $U'(q)=dU(q)/dq$ at the critical point for each value of N at some fixed z. The results agree well with the scaling relation (2.7). Then, we can calculate the exponent $1/\nu$ through this relation. Therefore, we do not need the values of the exponents $1/\nu$ for each connectivity z.

Table 1 summarizes the values of q_c, the exponents β/ν, γ/ν, and the effective dimensionality of the systems. For all values of z, $D_{eff}=1$, which has been obtained using

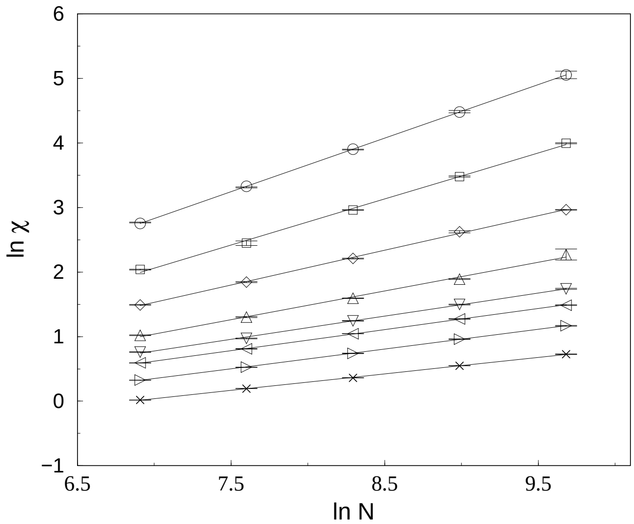

Figure 5: $\ln\chi$ versus $\ln N$. From top to bottom $z=2$, 3, 4, 6, 8, 10, 20, and 50.

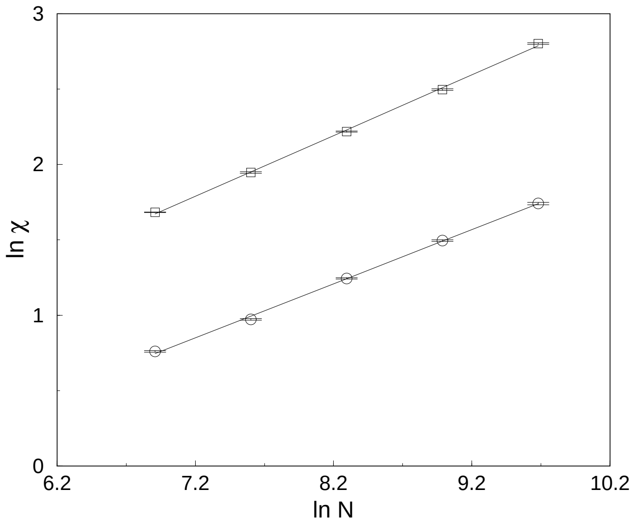

Figure 6: Plot of $\ln\chi^{max}(N)$ (square) and $\ln\chi(q_c)$ (circle) versus $\ln N$ for connectivity $z=8$.

Eq. (2.9). Therefore, when z increases, β/ν increases and γ/ν decreases at q_c, thus providing the value of $D_{eff}=1$ (along with errors). Therefore, the *undirected* Barabási-Albert network has the same effective dimensionality as that of Erdös-Rényi's random graphs [18] and the *directed* Barabási-Albert network [14]. Oliveira [16] showed that the majority-vote model defined on regular lattice has critical exponents that fall into the same class of universality as the corresponding equilibrium Ising model. Campos et. al [5] investigated the critical behavior of the majority-vote on small-world networks by rewiring the two-

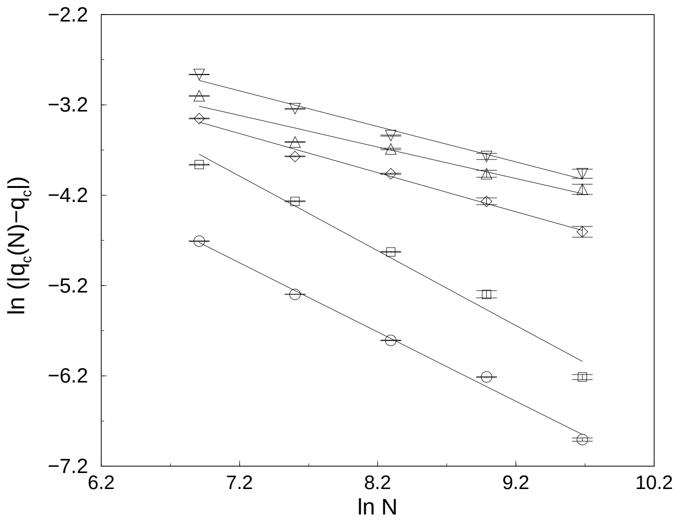

Figure 7: Plot of $\ln|q_c(N)-q_c|$ versus $\ln N$. From bottom to top $z=2$, 3, 4, 6, and 8.

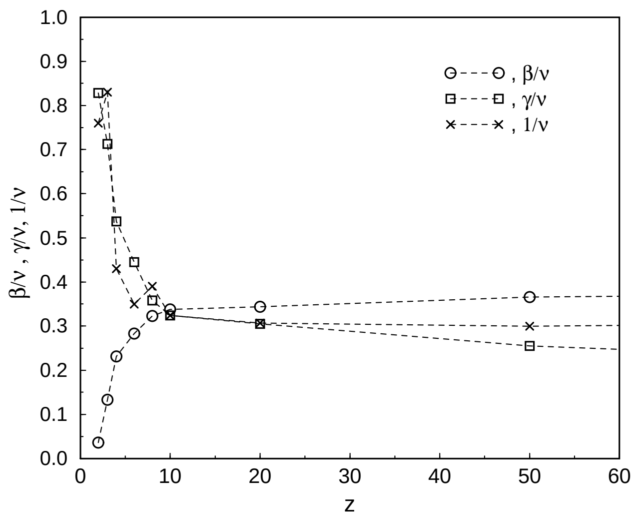

Figure 8: Critical behavior of β/ν, γ/ν and $1/\nu$ as a function of the connectivity z.

dimensional square lattice. Pereira and Moreira [18] studied this model on Erdös-Rényi's random graphs, and Lima et. al [6] also studied this model on Voronoy-Delaunay lattice. The results obtained by these authors show that the critical exponents of the majority-vote model belong to different universality classes.

Finally, we remark that our MC results obtained on the *undirected* Barabási-Albert network for the majority-vote model show that the critical exponents are different from the results of [16] for regular lattice, of Pereira and Moreira [18] for Erdös-Rényi's random graphs and Lima [14] for the *directed* Barabási-Albert network.

4 Conclusion

In conclusion, we have presented a very simple nonequilibrium model on *undirected* Barabási-Albert network [7, 8]. Different from the Ising model, in these networks, the majority-vote model presents a second-order phase transition which occurs with connectivity $z>1$. The exponents obtained are different from the other models. Nevertheless, our Monte Carlo simulations have demonstrated that the effective dimensionality D_{eff} equals unity, for all values of z, which agrees with the results of Pereira and Moreira [18]. However, when z grows, the exponent β/ν (in the critical point q_c) obtained through the Binder's cumulant grows and the exponent γ/ν decreases, which satisfy the hyperscaling relation $D_{eff}=1$.

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