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Vector Addition Theorem and Its Diagonalization

W. C. Chew*

Faculty of Engineering, The University of Hong Kong, Hong Kong, and Department of Electrical and Computer Engineering, University of Illinois, Urbana-Champaign, Urbana, IL 61801-2918, USA.

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Abstract. The conventional vector addition theorem is written in a compact notation. Then a new and succinct derivation of the vector addition theorem is presented that is as close to the derivation of the scalar addition theorem. Newly derived expressions in this new derivation are used to diagonalize the vector addition theorem. The diagonal form of the vector addition theorem is important in the design of fast algorithms for computational wave physics such as computational electromagnetics and computational acoustics.

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1 Introduction

The development of fast algorithms for integral equation solvers opens up new realms for the applications of integral equation solvers [1–5]. One of these is their use in the arena of circuits, micro-circuits and nanotechnologies [6]. Often time, the use of fast solvers in this arena calls for the combined use of fast algorithms where both wave physics and circuit physics are captured well by the solvers [7].

In the mid-frequency regime, where the wavelength is on the order of the object, or not extremely small compared to the size of the object, the vector nature of electromagnetic waves and their phases cannot be ignored. Hence, full vector wave physics is needed to describe the wave interaction with objects in this regime. This is the regime often encountered in microwave engineering [4].

*Corresponding author. *Email addresses:* `wcchew@hku.hk, w-chew@uiuc.edu` (W. C. Chew)

In the low-frequency regime, where the size of the object is much smaller than the wavelength, electromagnetic wave physics morphs into circuit physics, often not in a seamless fashion. Circuit physics is captured by quasistatic electromagnetics, but in modern circuit design, vast lengthscales ranging from sub-micrometers to centimeters are encountered concurrently. Hence, there often is a strong mixture of wave physics together with circuit physics in modern circuit design. Therefore the design of fast solvers in this regime, which has been nicknamed the "twilight zone", remains a challenge [6,8]. Moreover, in order to capture the "inductance" physics and the "capacitance" physics correctly, the vector nature of electromagnetic physics cannot be ignored [5]. This regime is encountered in micro-circuits in chip design, as well as nanotechnologies.

In this paper, the vector addition theorem for solenoidal vector wave functions is discussed, and a succinct derivation of the vector addition theorem is also presented. The new forms of the vector addition theorem facilitate their diagonalization, which is essential for developing fast algorithms in computational electromagnetics [9,10]. Previously, only the diagonalization of the scalar addition theorem has been presented [9,10]. The vector addition theorem can also be used to factorize the dyadic Green's function, which preserves the vector nature of electromagnetic field down to very long wavelength. It can be used for the development of a mixed-form fast multipole algorithm for vector electromagnetics which is valid from very low frequency to mid frequency [7]. It can also potentially result in memory savings.

2 Some fun with the vector addition theorem

Before the diagonalization of the vector addition theorem can be described, one needs to present the vector addition theorem and its expressions in compact notation. Their expressions in compact notation facilitate insight into their further diagonalization.

The vector addition theorem has been of great interest to the mathematical physics community [11–22]. The vector addition theorem for $\mathbf{r}=\mathbf{r}''+\mathbf{r}'$, for which $|\mathbf{r}'|<|\mathbf{r}''|$, can be written as [11,17,20, Appendix D of [20]]

$$\mathbf{M}_L(\mathbf{r})=\sum_{L'}[\Re g\mathbf{M}_{L'}(\mathbf{r}')A_{L',L}(\mathbf{r}'')+\Re g\mathbf{N}_{L'}(\mathbf{r}')B_{L',L}(\mathbf{r}'')], \tag{2.1}$$

$$\mathbf{N}_L(\mathbf{r})=\sum_{L'}[\Re g\mathbf{M}_{L'}(\mathbf{r}')B_{L',L}(\mathbf{r}'')+\Re g\mathbf{N}_{L'}(\mathbf{r}')A_{L',L}(\mathbf{r}'')]. \tag{2.2}$$

In the above, $\mathbf{M}$ and $\mathbf{N}$ are vector spherical harmonics expressed in terms of spherical Hankel functions [23] and spherical harmonics [24]. The subscript $L=(l,m)$ represents an ordered pair of integers where $-l\le m\le l$, and $l=1,\cdots,\infty$ (no monopole or $l=0$ term). Similar definition holds for L'. If the summation is truncated at $l=l_{max}$, then the number of terms involved is $P=(l_{max}+1)^2-1$. The $\Re g$ operator implies taking the regular part of the function where a spherical Hankel function (which is singular) is replaced by a spherical Bessel function (which is regular).

Only the solenoidal vector wave functions are discussed here and the irrotational vector wave functions, which are for longitudinal waves, can be discussed separately, as they are similar to the scalar wave functions [24].

More compactly, the above can be written as

$$\overline{\boldsymbol{\Psi}}^t_L(\mathbf{r})_{3\times 2}=\sum_{L'}\Re g\overline{\boldsymbol{\Psi}}^t_{L'}(\mathbf{r}')_{3\times 2}\cdot\overline{\boldsymbol{\alpha}}_{L',L}(\mathbf{r}'')_{2\times 2}, \quad \forall L, \quad |\mathbf{r}'|<|\mathbf{r}''|, \tag{2.3}$$

where $\forall L$ means for $l>0$ such that $L=(l,m)$ with the rule for the ordered pair prescribed above. The above is the analogue of Equation (1) of [10], but the elemental components are matrices whose dimensions are indicated by the subscripts. In particular,

$$\overline{\boldsymbol{\Psi}}^t_L(\mathbf{r})_{3\times 2}=[\mathbf{M}_L(\mathbf{r}),\mathbf{N}_L(\mathbf{r})], \tag{2.4}$$

$$\overline{\boldsymbol{\alpha}}_{L',L}(\mathbf{r}'')_{2\times 2}=\begin{bmatrix} A_{L',L}(\mathbf{r}'') & B_{L',L}(\mathbf{r}'') \\ B_{L',L}(\mathbf{r}'') & A_{L',L}(\mathbf{r}'') \end{bmatrix}. \tag{2.5}$$

The details of $A_{L',L}$ and $B_{L',L}$ can be found in [20, Appendix D]. It is to be noted that here, we define

$$\mathbf{M}_L(\mathbf{r})=\frac{1}{\sqrt{l(l+1)}}\frac{1}{i}\mathbf{r}\times\nabla\Psi_L(\mathbf{r}), \quad \mathbf{N}_L(\mathbf{r})=\frac{1}{k}\nabla\times\mathbf{M}_L(\mathbf{r}), \quad \Psi_L(\mathbf{r})=h_l(kr)Y_{l,m}(\theta,\phi).$$

This is in contrast to [20] where $\mathbf{M}_L(\mathbf{r})=\nabla\times\mathbf{r}\Psi_L(\mathbf{r})$; hence the equations for $A_{L',L}$ and $B_{L',L}$ have to be properly modified.

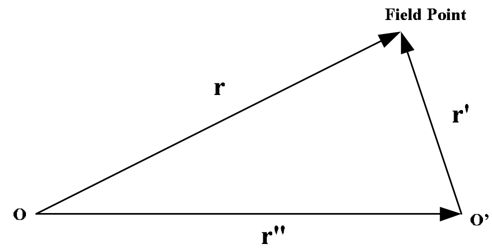

Figure 1: The addition theorem changes the coordinates of the vector wave functions defined with origin at O to origin at O'.

Another derivation of them is shown in the next section. Notice that the addition theorem changes the coordinates of the vector wave functions defined with origin at O to be expressed in terms of vector wave functions defined with origin at O' (see Fig. 1). This coordinate translation is achieved with the translator (or translation operator) $\overline{\boldsymbol{\alpha}}_{L,L'}$.

Similar forms of the theorem can be derived such that

$$\overline{\boldsymbol{\Psi}}^t_L(\mathbf{r})_{3\times 2}=\sum_{L'}\overline{\boldsymbol{\Psi}}^t_{L'}(\mathbf{r}')_{3\times 2}\cdot\overline{\boldsymbol{\beta}}_{L',L}(\mathbf{r}'')_{2\times 2}, \quad \forall L, \quad |\mathbf{r}'|>|\mathbf{r}''|, \tag{2.6}$$

$$\Re g\overline{\boldsymbol{\Psi}}^t_L(\mathbf{r})_{3\times 2}=\sum_{L'}\Re g\overline{\boldsymbol{\Psi}}^t_{L'}(\mathbf{r}')_{3\times 2}\cdot\overline{\boldsymbol{\beta}}_{L',L}(\mathbf{r}'')_{2\times 2}, \quad \forall L, \quad \forall|\mathbf{r}'|,\ |\mathbf{r}''|, \tag{2.7}$$

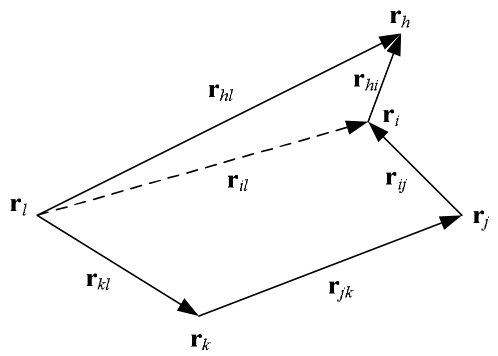

Figure 2: The translation of position vectors in the definition of the vector addition theorem.

where the translator $\overline{\boldsymbol{\beta}}_{L',L}(\mathbf{r}'') = \Re g \overline{\boldsymbol{\alpha}}_{L',L}(\mathbf{r}'')$ and again $\Re g$ means replacing Hankel functions with Bessel functions.

The above addition theorem can be expressed more compactly using a more compact notation: One can truncate the summation above with $l' \leq l_{max}$ so that the number of terms involved is $P = (l_{max}+1)^2 - 1$. The matrices can then be stacked into larger matrices, and an expanded matrix notation is used to express them. Eqs. (2.3), (2.6), and (2.7) can hence be compactly expressed as

$$\overline{\boldsymbol{\Psi}}^t(\mathbf{r})_{3\times 2P} = \Re g \overline{\boldsymbol{\Psi}}^t(\mathbf{r}')_{3\times 2P} \cdot \overline{\boldsymbol{\alpha}}(\mathbf{r}'')_{2P\times 2P}, \qquad |\mathbf{r}'| < |\mathbf{r}''|, \tag{2.8}$$

$$\overline{\boldsymbol{\Psi}}^t(\mathbf{r})_{3\times 2P} = \overline{\boldsymbol{\Psi}}^t(\mathbf{r}')_{3\times 2P} \cdot \overline{\boldsymbol{\beta}}(\mathbf{r}'')_{2P\times 2P}, \qquad |\mathbf{r}'| > |\mathbf{r}''|, \tag{2.9}$$

$$\Re g \overline{\boldsymbol{\Psi}}^t(\mathbf{r})_{3\times 2P} = \Re g \overline{\boldsymbol{\Psi}}^t(\mathbf{r}')_{3\times 2P} \cdot \overline{\boldsymbol{\beta}}(\mathbf{r}'')_{2P\times 2P}, \quad \forall |\mathbf{r}'|,\, |\mathbf{r}''|, \tag{2.10}$$

where $P \to \infty$ in the above. To ease the burden with notations, the dimensional subscripts in the following will be dropped.

Using the above addition theorem three times, then

$$\begin{aligned}
\overline{\boldsymbol{\Psi}}^t(\mathbf{r}_{hl}) &= \overline{\boldsymbol{\Psi}}^t(\mathbf{r}_{hk}) \cdot \overline{\boldsymbol{\beta}}(\mathbf{r}_{kl}) \\
&= \Re g \overline{\boldsymbol{\Psi}}^t(\mathbf{r}_{hj}) \cdot \overline{\boldsymbol{\alpha}}(\mathbf{r}_{jk}) \cdot \overline{\boldsymbol{\beta}}(\mathbf{r}_{kl}) \\
&= \Re g \overline{\boldsymbol{\Psi}}^t(\mathbf{r}_{hi}) \cdot \overline{\boldsymbol{\beta}}(\mathbf{r}_{ij}) \cdot \overline{\boldsymbol{\alpha}}(\mathbf{r}_{jk}) \cdot \overline{\boldsymbol{\beta}}(\mathbf{r}_{kl}),
\end{aligned} \tag{2.11}$$

where $\mathbf{r}_{ab} = \mathbf{r}_a - \mathbf{r}_b$. One can use (2.3) to write

$$\overline{\boldsymbol{\Psi}}^t(\mathbf{r}_{hl}) = \Re g \overline{\boldsymbol{\Psi}}^t(\mathbf{r}_{hi}) \cdot \overline{\boldsymbol{\alpha}}(\mathbf{r}_{il}). \tag{2.12}$$

Comparing (2.11) and (2.12), it implies that

$$\overline{\boldsymbol{\alpha}}(\mathbf{r}_{il}) = \overline{\boldsymbol{\beta}}(\mathbf{r}_{ij}) \cdot \overline{\boldsymbol{\alpha}}(\mathbf{r}_{jk}) \cdot \overline{\boldsymbol{\beta}}(\mathbf{r}_{kl}) \tag{2.13}$$

or more explicitly,

$$\overline{\boldsymbol{\alpha}}_{L,L'}(\mathbf{r}_{il}) = \sum_{L_1} \sum_{L_2} \overline{\boldsymbol{\beta}}_{L,L_1}(\mathbf{r}_{ij}) \cdot \overline{\boldsymbol{\alpha}}_{L_1,L_2}(\mathbf{r}_{jk}) \cdot \overline{\boldsymbol{\beta}}_{L_2,L'}(\mathbf{r}_{kl}). \tag{2.14}$$

One can trace through the translations in the above according to Fig. 2.

A dyadic Green's function in electromagnetics can be expressed as [20]

$$\begin{aligned}\overline{\mathbf{G}}(\mathbf{r}_h-\mathbf{r}_l) &= \left(\overline{\mathbf{I}}+\frac{\nabla_h\nabla_h}{k^2}\right)\cdot\frac{e^{ik|\mathbf{r}_h-\mathbf{r}_l|}}{4\pi|\mathbf{r}_h-\mathbf{r}_l|} \\ &= ik\sum_{nm}[\mathbf{M}_{nm}(\mathbf{r}_{hk})\Re g\mathbf{M}_{n,-m}(\mathbf{r}_{kl})+\mathbf{N}_{nm}(\mathbf{r}_{hk})\Re g\mathbf{N}_{n,-m}(\mathbf{r}_{kl})](-)^{m+1} \\ &= ik\overline{\boldsymbol{\Psi}}^t(\mathbf{r}_{hk})\cdot\Re g\hat{\overline{\boldsymbol{\Psi}}}(\mathbf{r}_{kl}),\quad |\mathbf{r}_{hk}|>|\mathbf{r}_{kl}|,\end{aligned} \tag{2.15}$$

where $\hat{\overline{\boldsymbol{\Psi}}}$ implies the conjugation of only the azimuthal angular part of $\overline{\boldsymbol{\Psi}}$ or m is replaced by $-m$. Applying the vector addition theorem once to $\overline{\boldsymbol{\Psi}}^t(\mathbf{r}_{hk})$, it implies that

$$\overline{\mathbf{G}}(\mathbf{r}_h-\mathbf{r}_l)_{3\times3}=ik\Re g\overline{\boldsymbol{\Psi}}^t(\mathbf{r}_{hj})_{3\times2P}\cdot\overline{\boldsymbol{\alpha}}(\mathbf{r}_{jk})_{2P\times2P}\cdot\Re g\hat{\overline{\boldsymbol{\Psi}}}(\mathbf{r}_{kl})_{2P\times3},\quad P\to\infty. \tag{2.16}$$

An interesting note about the vector addition theorem is that (2.16) is not a special case of (2.13) as in the scalar addition theorem. In the scalar addition case, the monopole to monopole translator is just the scalar Green's function, while the $\overline{\boldsymbol{\alpha}}$ translator cannot be simpler related to the dyadic Green's function. The important note is that the dyadic Green's function, the key function to the interaction between two point sources in electromagnetics, can be factorized in terms of the translator $\overline{\boldsymbol{\alpha}}$. It can be shown that

$$\Re g\mathbf{M}_{n,-m}(\mathbf{r})=(-)^{m+1}\Re g\mathbf{M}^*_{n,m}(\mathbf{r}), \tag{2.17}$$

$$\Re g\mathbf{N}_{n,-m}(\mathbf{r})=(-)^{m+1}\Re g\mathbf{N}^*_{n,m}(\mathbf{r}). \tag{2.18}$$

3 A succinct derivation of the vector addition theorem

The vector addition theorem has been derived by various means. The derivation here sets the stage for the diagonalization of the vector addition theorem in the next section. It parallels the derivation of scalar addition theorem using plane wave expansions [20, Appendix D]. But here, one needs the vector plane wave expansions. Special cases of the expansions of vector plane waves in terms of vector wave functions have been reported by Stratton [24] and Jackson [25]. More generally, one can show that a vector transverse plane wave (such as a transverse electromagnetic wave) can be expanded as [16, p. 442], [26,27, private communication with Wittmann]

$$\boldsymbol{\kappa}e^{i\mathbf{k}\cdot\mathbf{r}}=\sum_L 4\pi i^l\left[\Re g\mathbf{M}_L(\mathbf{r})\mathbf{X}_L^\dagger(\hat{k})+\Re g\mathbf{N}_L(\mathbf{r})\mathbf{Y}_L^\dagger(\hat{k})\right]\cdot\boldsymbol{\kappa}, \tag{3.1}$$

where $\boldsymbol{\kappa}\cdot\mathbf{k}=0$, $\mathbf{Y}_L=i\hat{k}\times\mathbf{X}_L$, and $\mathbf{X}_L^\dagger$ is defined in Jackson [25] as well as in Wittmann [26]. For coordinate space,

$$\mathbf{X}_L(\hat{r})=\mathbf{L}Y_{lm}(\hat{r})/\sqrt{l(l+1)},\quad \mathbf{L}=\frac{1}{i}\mathbf{r}\times\nabla. \tag{3.2}$$

It is clear that $\hat{r}\cdot\mathbf{X}_L(\hat{r})=0$, and hence, $\hat{k}\cdot\mathbf{X}_L(\hat{k})=0$. Hence, both $\mathbf{X}_L(\hat{k})$ and $\mathbf{Y}_L(\hat{k})$ are orthogonal to $\hat{k}$.

The above can be written more generally as

$$\begin{aligned}\overline{\mathbf{I}}_s e^{i\mathbf{k}\cdot\mathbf{r}} &= \sum_L 4\pi i^l \left[\underbrace{\Re g\mathbf{M}_L(\mathbf{r})}_{3\times 1}\underbrace{\mathbf{X}_L^\dagger(\hat{k})}_{1\times 3}+\underbrace{\Re g\mathbf{N}_L(\mathbf{r})}_{3\times 1}\underbrace{\mathbf{Y}_L^\dagger(\hat{k})}_{1\times 3}\right] \\ &= \sum_L 4\pi i^l \underbrace{\Re g\overline{\mathbf{\Psi}}_L^t(\mathbf{r})}_{3\times 2}\cdot\underbrace{\overline{\mathbf{W}}_L^*(\hat{k})}_{2\times 3},\end{aligned} \tag{3.3}$$

where

$$\overline{\mathbf{\Psi}}_L^t(\mathbf{r}) = [\mathbf{M}_L(\mathbf{r}),\mathbf{N}_L(\mathbf{r})] \in C_{3\times 2}, \tag{3.4}$$

$$\overline{\mathbf{W}}_L^t(\hat{k}) = \left[\mathbf{X}_L(\hat{k}),\mathbf{Y}_L(\hat{k})\right] \in C_{3\times 2}, \tag{3.5}$$

and $\overline{\mathbf{I}}_s$ is a symmetric 3×3 tensor with the property that $\overline{\mathbf{I}}_s\cdot\mathbf{k}=0$.

Converse to (3.1), the $\mathbf{M}$ and $\mathbf{N}$ functions can be expanded in terms of transverse vector plane waves [26]. Using these expansions, the following compact notation for plane-wave expansions of $\Re g\overline{\mathbf{\Psi}}_L^t(\mathbf{r})$ can be obtained

$$\Re g\overline{\mathbf{\Psi}}_L^t(\mathbf{r}) = \frac{1}{4\pi i^l}\int_{\bigcirc} d\hat{k}\,\overline{\mathbf{I}}_s\cdot\overline{\mathbf{W}}_L^t(\hat{k})e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{3.6}$$

where $\bigcirc$ indicates that the above integral is over a unit sphere. Since both $\mathbf{X}_L$ and $\mathbf{Y}_L$ are orthogonal to $\mathbf{k}$, the presence of $\overline{\mathbf{I}}_s$ is immaterial, but it is inserted here to facilitate the proof of an identity later on.

One lets

$$\overline{\mathbf{I}}_s e^{i\mathbf{k}\cdot\mathbf{r}} = \overline{\mathbf{I}}_s e^{i\mathbf{k}\cdot\mathbf{r}_1}e^{i\mathbf{k}\cdot\mathbf{r}_2}, \tag{3.7}$$

where $\mathbf{r}=\mathbf{r}_1+\mathbf{r}_2$. Then using (3.3) for $\overline{\mathbf{I}}_s e^{i\mathbf{k}\cdot\mathbf{r}_1}$ and that [20,24]

$$e^{i\mathbf{k}\cdot\mathbf{r}_2} = \sum_{L''} 4\pi i^{l''}\Re g\Psi_{L''}(k,\mathbf{r}_2)Y_{L''}^*(\hat{k}), \tag{3.8}$$

where

$$\Psi_L(kr) = h_l^{(1)}(kr)Y_{lm}(\theta,\phi), \tag{3.9}$$

$h_l^{(1)}(kr)$ is a spherical Hankel function of the first kind [23], and Y_{lm} is a spherical harmonic function, Eq. (3.6) becomes

$$\begin{aligned}\underbrace{\Re g\overline{\mathbf{\Psi}}_L^t(\mathbf{r})}_{3\times 2} &= \frac{1}{4\pi i^l}\sum_{L'} 4\pi i^{l'}\Re g\overline{\mathbf{\Psi}}_{L'}^t(\mathbf{r}_1)\cdot\sum_{L''}4\pi i^{l''}\Re g\Psi_{L''}(k,\mathbf{r}_2)\int_{\bigcirc} d\hat{k}\overbrace{\overline{\mathbf{W}}_{L'}^*(\hat{k})\cdot\overline{\mathbf{W}}_L^t(\hat{k})}^{2\times 2}Y_{L''}^*(\hat{k}) \\ &= \sum_{L'}\underbrace{\Re g\overline{\mathbf{\Psi}}_{L'}(\mathbf{r}_1)}_{3\times 2}\cdot\underbrace{\overline{\boldsymbol{\beta}}_{L',L}(\mathbf{r}_2)}_{2\times 2},\end{aligned} \tag{3.10}$$

where

$$\overline{\boldsymbol{\beta}}_{L',L}(\mathbf{r}_2)=\sum_{L''}4\pi i^{l'+l''-l}\Re g\Psi_{L''}(k,\mathbf{r}_2)\overline{\mathbf{A}}_{L,L',L''} \tag{3.11}$$

and

$$\overline{\mathbf{A}}_{L,L',L''}=\int_{\bigcirc}d\hat{k}\overline{\mathbf{W}}^*_{L'}(\hat{k})\cdot\overline{\mathbf{W}}^t_L(\hat{k})Y^*_{L''}(\hat{k}) \quad \in C_{2\times 2}. \tag{3.12}$$

It can be seen that

$$\overline{\mathbf{W}}^*_{L'}(\hat{k})\cdot\overline{\mathbf{W}}^t_L(\hat{k})=\begin{bmatrix}\mathbf{X}^\dagger_{L'}(\hat{k})\\ \mathbf{Y}^\dagger_{L'}(\hat{k})\end{bmatrix}\cdot\begin{bmatrix}\mathbf{X}_L(\hat{k})\,\mathbf{Y}_L(\hat{k})\end{bmatrix}. \tag{3.13}$$

Also, using the fact that both $\mathbf{Y}_L$ and $\mathbf{X}_L$ are orthogonal to $\hat{k}$, then

$$\mathbf{Y}^\dagger_{L'}(\hat{k})\cdot\mathbf{Y}_L(\hat{k})=\left(\hat{k}\times\mathbf{X}^\dagger_{L'}\right)\cdot\left(\hat{k}\times\mathbf{X}_L\right)=\mathbf{X}^\dagger_{L'}(\hat{k})\cdot\mathbf{X}_L(\hat{k}). \tag{3.14}$$

Consequently, (3.13) becomes

$$\overline{\mathbf{W}}^*_{L'}(\hat{k})\cdot\overline{\mathbf{W}}^t_L(\hat{k})=\begin{bmatrix}\mathbf{X}^\dagger_{L'}(\hat{k})\cdot\mathbf{X}_L(\hat{k}) & -i\hat{k}\times\mathbf{X}^\dagger_{L'}(\hat{k})\cdot\mathbf{X}_L(\hat{k})\\ -i\hat{k}\times\mathbf{X}^\dagger_{L'}(\hat{k})\cdot\mathbf{X}_L(\hat{k}) & \mathbf{X}^\dagger_{L'}(\hat{k})\cdot\mathbf{X}_L(\hat{k})\end{bmatrix}. \tag{3.15}$$

Eq. (3.12) is a generalization of the integral for Gaunt coefficients [20, Appendix D, Eq. (D9)]. It is a matrix that depends only on L, L', and L'' and is independent of k.

The factorized compact forms in (3.11) and (3.12) will facilitate the diagonalization of the vector addition theorem. Eqs. (3.11) and (3.12) combined is the vector analogue of Equation (4) of [10] for scalar addition theorem.

The vector addition theorem derived in Eq. (3.10) is valid for all $\mathbf{r}_1$ and $\mathbf{r}_2$, and all k. One can use the method of ref. [20, Appendix D] to derive different forms of the addition theorem in the following:

$$\overline{\boldsymbol{\Psi}}^t_L(\mathbf{r})=\sum_{L'}\Re g\overline{\boldsymbol{\Psi}}_{L'}(\mathbf{r}_1)\cdot\overline{\boldsymbol{\alpha}}_{L'L}(\mathbf{r}_2), \quad |\mathbf{r}_1|<|\mathbf{r}_2|, \tag{3.16}$$

$$\overline{\boldsymbol{\Psi}}^t_L(\mathbf{r})=\sum_{L'}\overline{\boldsymbol{\Psi}}_{L'}(\mathbf{r}_1)\cdot\overline{\boldsymbol{\beta}}_{L'L}(\mathbf{r}_2), \qquad |\mathbf{r}_1|>|\mathbf{r}_2|, \tag{3.17}$$

where

$$\overline{\boldsymbol{\alpha}}_{L',L}(\mathbf{r}_2) \;=\; \sum_{L''}4\pi i^{l'+l''-l}\Psi_{L''}(k,\mathbf{r}_2)\overline{\mathbf{A}}_{L,L',L''}. \tag{3.18}$$

Eq. (3.10) together with the above constitute the full set of addition theorem in different forms for the translation of the solenoidal vector wave functions.

The results above are similar to those derive in Wittmann [26] but the results here separate the $\overline{\mathbf{A}}_{L,L',L''}$ coefficients explicitly in the definition of the translators $\overline{\boldsymbol{\alpha}}$ and $\overline{\boldsymbol{\beta}}$. Moreover, the compact manner with which $\overline{\mathbf{A}}_{L,L',L''}$ are expressed is new. They are similar to those presented in [20]. Also, the above is in agreement with Eq. (2.5) which shows that the diagonal elements are the same, and the off-diagonal elements are equal to each other. The explicit expressions for $\overline{\mathbf{A}}_{L,L',L''}$ have been related to the Gaunt coefficients [11,12,18].

4 An important orthogonality identity

From (3.3),

$$\overline{\mathbf{I}}_s e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_L 4\pi i^l \Re g \overline{\mathbf{\Psi}}_L^t(\mathbf{r}) \cdot \overline{\mathbf{W}}_L^*(\hat{k}), \tag{4.1}$$

and from (3.6),

$$\Re g \overline{\mathbf{\Psi}}_L^t(\mathbf{r}) = \frac{1}{4\pi i^l} \int_{\bigcirc} d\hat{k}' \overline{\mathbf{I}}_s \cdot \overline{\mathbf{W}}_L^t(\hat{k}') e^{i\mathbf{k}'\cdot\mathbf{r}}. \tag{4.2}$$

Using (4.2) in (4.1), one arrives at

$$\overline{\mathbf{I}}_s e^{i\mathbf{k}\cdot\mathbf{r}} = \int_{\bigcirc} d\hat{k}' \overline{\mathbf{I}}_s e^{i\mathbf{k}'\cdot\mathbf{r}} \sum_L \overline{\mathbf{W}}_L^t(\hat{k}') \cdot \overline{\mathbf{W}}_L^*(\hat{k}). \tag{4.3}$$

From the above, one concludes that

$$\sum_L \overline{\mathbf{W}}_L^t(\hat{k}') \cdot \overline{\mathbf{W}}_L^*(\hat{k}) = \delta(\hat{k}' - \hat{k}) \overline{\mathbf{I}}_s. \tag{4.4}$$

The above is an important identity necessary for the diagonalization of the vector addition theorem in the next section. To the author's best knowledge, this is the first exposition of this identity.

5 Diagonalization of the vector addition theorem

The factorization of the translator and the Green's function is important for the development of fast multipole algorithms and its variants [28] in computational electromagnetics. The diagonal form of the factorized form greatly expedites the efficiency and reduces the computational complexity of the fast algorithms for wave physics problems [1–3]. Also, since the translators have a reducible representation in the plane-wave basis according to group theory, they have a diagonal form in this basis [29].

As shown in the previous section, the translators for vector wave functions can be factorized by the repeated use of the vector addition theorem. Namely,

$$\overline{\boldsymbol{\alpha}}(\mathbf{r}_{ij}) = \overline{\boldsymbol{\beta}}(\mathbf{r}_{i\lambda}) \cdot \overline{\boldsymbol{\alpha}}(\mathbf{r}_{\lambda\lambda'}) \cdot \overline{\boldsymbol{\beta}}(\mathbf{r}_{\lambda' j}), \tag{5.1}$$

or more explicitly,

$$\overline{\boldsymbol{\alpha}}_{L,L'}(\mathbf{r}_{ij}) = \sum_{L_1,L_2} \overline{\boldsymbol{\beta}}_{L,L_1}(\mathbf{r}_{i\lambda}) \cdot \overline{\boldsymbol{\alpha}}_{L_1,L_2}(\mathbf{r}_{\lambda\lambda'}) \cdot \overline{\boldsymbol{\beta}}_{L_2,L'}(\mathbf{r}_{\lambda' j}). \tag{5.2}$$

From the previous section, one can find an alternative way of writing the translators. Consequently,

$$\overline{\boldsymbol{\beta}}_{L',L}(\mathbf{r}) = \sum_{L''} 4\pi i^{l'+l''-l} \Re g \Psi_{L''}(k,\mathbf{r}) \overline{\mathbf{A}}_{L,L',L''}, \tag{5.3}$$

$$\overline{\mathbf{A}}_{L,L'L''} = \int_{\bigcirc} d\hat{k} \overline{\mathbf{W}}_{L'}^*(\hat{k}) \cdot \overline{\mathbf{W}}_L^t(\hat{k}) Y_{L''}^*(\hat{k}). \tag{5.4}$$

By using

$$e^{i\mathbf{k}\cdot\mathbf{r}}=\sum_{L''}4\pi i^{l''}\Re g\Psi_{L''}(k,\mathbf{r})Y^*_{L''}(\hat{k}), \tag{5.5}$$

Eq. (5.3) can be re-expressed as

$$\overline{\boldsymbol{\beta}}_{L',L}(\mathbf{r})=i^{l'-l}\int_{\bigcirc}d\hat{k}\overline{\mathbf{W}}^*_{L'}(\hat{k})\cdot\overline{\mathbf{W}}^t_L(\hat{k})e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{5.6}$$

The above is a similarity transform that diagonalizes the translator $\overline{\boldsymbol{\beta}}$ with plane wave representation. Using (5.6) in (5.2), then

$$\begin{aligned}\overline{\boldsymbol{\alpha}}_{L,L'}(\mathbf{r}_{ij})=&\sum_{L_1,L_2}i^{l-l_1}\int_{\bigcirc}d\hat{k}\overline{\mathbf{W}}^*_L(\hat{k})\cdot\overline{\mathbf{W}}^t_{L_1}(\hat{k})e^{i\mathbf{k}\cdot\mathbf{r}_{i\lambda}}\cdot\overline{\boldsymbol{\alpha}}_{L_1,L_2}(\mathbf{r}_{\lambda\lambda'})\\&\cdot i^{l_2-l'}\int_{\bigcirc}d\hat{k}'\overline{\mathbf{W}}^*_{L_2}(\hat{k}')\cdot\overline{\mathbf{W}}^t_{L'}(\hat{k}')e^{i\mathbf{k}'\cdot\mathbf{r}_{\lambda'j}},\end{aligned} \tag{5.7}$$

or after exchanging the order of integrations and summations, then

$$\begin{aligned}\overline{\boldsymbol{\alpha}}_{L,L'}(\mathbf{r}_{ij})=&\iint_{\bigcirc}d\hat{k}d\hat{k}'\,i^l\overline{\mathbf{W}}^*_L(\hat{k})e^{i\mathbf{k}\cdot\mathbf{r}_{i\lambda}}\cdot\sum_{L_1,L_2}i^{l_2-l_1}\overline{\mathbf{W}}^t_{L_1}(\hat{k})\\&\cdot\overline{\boldsymbol{\alpha}}_{L_1,L_2}(\mathbf{r}_{\lambda\lambda'})\cdot\overline{\mathbf{W}}^*_{L_2}(\hat{k}')\cdot i^{l'}\overline{\mathbf{W}}^t_{L'}(\hat{k}')e^{i\mathbf{k}'\cdot\mathbf{r}_{\lambda'j}}.\end{aligned} \tag{5.8}$$

One can simplify the above by defining a new function to replace the inner summations. Namely, one defines

$$\hat{\overline{\boldsymbol{\alpha}}}(\mathbf{r}_{\lambda,\lambda'},\hat{k},\hat{k}')=\sum_{L_1,L_2}i^{l_2-l_1}\overline{\mathbf{W}}^t_{L_1}(\hat{k})\cdot\overline{\boldsymbol{\alpha}}_{L_1,L_2}(\mathbf{r}_{\lambda\lambda'})\cdot\overline{\mathbf{W}}^*_{L_2}(\hat{k}'). \tag{5.9}$$

The expression for the above function can be simplified. Since

$$\begin{aligned}\overline{\boldsymbol{\alpha}}_{L_1,L_2}(\mathbf{r}_{\lambda\lambda'})=&\sum_{L''}4\pi i^{l_1+l''-l_2}\Psi_{L''}(k,\mathbf{r}_{\lambda\lambda'})\overline{\mathbf{A}}_{L_2,L_1,L''}\\=&\sum_{L''}4\pi i^{l_1+l''-l_2}\Psi_{L''}(k,\mathbf{r}_{\lambda\lambda'})\int_{\bigcirc}d\hat{k}''\overline{\mathbf{W}}^*_{L_1}(\hat{k}'')\cdot\overline{\mathbf{W}}^t_{L_2}(\hat{k}'')Y^*_{L''}(\hat{k}''),\end{aligned} \tag{5.10}$$

using (5.10) in (5.9),

$$\begin{aligned}\hat{\overline{\boldsymbol{\alpha}}}(\mathbf{r}_{\lambda\lambda'},\hat{k},\hat{k}')\ =\ &\sum_{L''}\Psi_{L''}(k,\mathbf{r}_{\lambda\lambda'})\int_{\bigcirc}d\hat{k}''Y^*_{L''}(\hat{k}'')4\pi i^{l''}\\&\cdot\sum_{L_1,L_2}\overline{\mathbf{W}}^t_{L_1}(\hat{k})\cdot\overline{\mathbf{W}}^*_{L_1}(\hat{k}'')\cdot\overline{\mathbf{W}}^t_{L_2}(\hat{k}'')\cdot\overline{\mathbf{W}}^*_{L_2}(\hat{k}').\end{aligned} \tag{5.11}$$

Using the results from the previous section that

$$\sum_L\overline{\mathbf{W}}^t_L(\hat{k}')\overline{\mathbf{W}}^*_L(\hat{k})=\delta(\hat{k}'-\hat{k})\overline{\mathbf{I}}_s, \tag{5.12}$$

Eq. (5.11) becomes

$$\hat{\tilde{\boldsymbol{\alpha}}}(\mathbf{r}_{\lambda\lambda'},\hat{k},\hat{k}') = \overline{\mathbf{I}}_s \sum_{L''} \Psi_{L''}(k,\mathbf{r}_{\lambda\lambda'}) 4\pi i^{l''} \delta(\hat{k}'-\hat{k}) Y^*_{L''}(\hat{k}'). \tag{5.13}$$

Using (5.13) in (5.8), one arrives at the diagonalized equivalence of the factorized translator in (5.1), that is

$$\overline{\boldsymbol{\alpha}}_{L,L'}(\mathbf{r}_{ij}) = \int_{\bigcirc} d\hat{k} i^{l} \overline{\mathbf{W}}^*_L(\hat{k}) \cdot e^{i\mathbf{k}\cdot\mathbf{r}_{i\lambda}} \tilde{\overline{\boldsymbol{\alpha}}}(\mathbf{r}_{\lambda\lambda'},\mathbf{k}) i^{-l'} \cdot e^{i\mathbf{k}\cdot\mathbf{r}_{\lambda' j}} \overline{\mathbf{W}}^t_{L'}(\hat{k}), \tag{5.14}$$

$$\tilde{\overline{\boldsymbol{\alpha}}}(\mathbf{r}_{\lambda\lambda'},\mathbf{k}) = \overline{\mathbf{I}}_s \sum_{L''} \Psi^*_{L''}(k,\mathbf{r}_{\lambda\lambda'}) 4\pi i^{l''} Y_{L''}(\hat{k}). \tag{5.15}$$

Reading the physical meaning of (5.14) from right to left, the first term $\overline{\mathbf{W}}^t_{L'}(\hat{k})$ transforms multipoles to plane waves. The exponential function $e^{i\mathbf{k}\cdot\mathbf{r}_{\lambda' j}}$ is the representation of the $\overline{\boldsymbol{\beta}}$ translator in the plane wave basis. The middle term is the representation of the $\overline{\boldsymbol{\alpha}}$ translator in the plane-wave basis, and the rest of the terms to the left mirror-image the operations to the right.

Since $L''=(l'',m'')$, the above involves a double summation, which can be removed by choosing the z axis to be in the $\mathbf{r}_{\lambda,\lambda'}$ direction [30]. Alternatively, one observes that since the integral in Eq. (5.14) is single fold, it is equivalent to having factorized the translator into diagonal forms [2].

The summation in (5.15) is a divergent series because the spherical Hankel function is increasingly large for large l''. The root of the reason can be traced to the invalid exchange of the order of integrations and summations in (5.8) [31]. This is also noted in the diagonalization of the scalar addition theorem [30]. This exchange is valid if our functional space includes distributions. Hence, the above should be thought of as a distribution which is undefined until it is integrated with other functions.

The exchanging of the order of integrations and summations is made valid in the classical sense by truncating the series in (5.10) beforehand. For practical computation however, a finite summation is used in the above, and various ways to truncate the summation have been discussed [4, Chapter 2 and references therein] .

6 Conclusion

Compact notations are introduced to simplify the expression of the vector addition theorem and the translation operators (translators). An alternative way to derive the vector addition theorem and the translation operators (translators) is illustrated so that it is as parallel as possible to the derivation of the scalar addition theorem. The new formulas and identity derived are used for the diagonalization of the translators in the vector addition theorem.

These factorized and diagonalized translators have applications in the development of fast algorithms where the vector nature of electromagnetic field cannot be ignored.

They can also be used to develop fast algorithms for mixed-form fast multipole algorithm [7] for vector electromagnetic fields all the way from static to electrodynamics. This is increasingly important in micro-circuits and nano-structures where the vector nature of electromagnetic field cannot be ignored, but meanwhile, the structures can be subwavelength or order of wavelength in size [6].

The above results can also be used to expedite other computational waves involving transverse waves such as shear waves in elastic solids [32]. The compressional waves in elastic solid, which propagate with a different phase velocity, can be treated as an additional scalar wave term [24].

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