

A Family of Relaxation Schemes for Nonlinear Convection Diffusion Problems

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Abstract. In this work we present a family of relaxation schemes for nonlinear convection diffusion problems, which can tackle also the cases of degenerate diffusion and of convection dominated regimes. The schemes proposed can achieve any order of accuracy, give non-oscillatory solutions even in the presence of singularities and their structure depends only weakly on the particular PDE being integrated. One and two dimensional results are shown, and a nonlinear stability estimate is given.

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1 Introduction

Relaxation approximations to non-linear PDE's are based on the replacement of the original PDE with a semi-linear hyperbolic system of equations, with a stiff source term, tuned by a relaxation parameter ε . When $\varepsilon \rightarrow 0$, the system relaxes onto the original PDE. A consistent discretization of the relaxation system for $\varepsilon = 0$ yields a consistent discretization of the original PDE, see for instance [2, 4, 9]. The advantage of this procedure is that the numerical scheme obtained in this fashion does not need computationally expensive Riemann solvers for the non-linear convective term, but enjoys the robustness of upwind

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discretizations, since the convective term is replaced by a constant coefficient linear hyperbolic system to which upwinding can be easily applied.

Moreover the complexity introduced replacing the original PDE with a stiff system of equations is only apparent, because when the discretization is carried out in the relaxed case, one recovers a scheme where only the original unknown is actually updated.

In this work, we consider convection-diffusion equations of the form:

$$\frac{\partial u}{\partial t} + \operatorname{div} f(u) = \Delta(p(u)), \quad (1.1)$$

where f is a differentiable function of the form $f(u) = [f_1, \dots, f_d]$, with d denoting the number of space dimensions.

We assume that all derivatives f'_j , $1 \leq j \leq d$, in absolute value are bounded by a constant α ; thus α is an upper bound of the maximum characteristic speed of the equation, measured in a suitable norm.

The function $p(u)$ appearing in the diffusion term is a non-negative, non-decreasing, Lipschitz continuous function with Lipschitz constant δ . The equation is called degenerate when (i) the (weak) derivative; (ii) $p'(u)$ vanishes for some nonnegative values of u : in this case, even the parabolic term generates travelling fronts. Here we illustrate the case of a scalar equation, but the construction can be easily generalized to systems.

Note also that analogies can be found between the relaxation approach and the LDG method (Local Discontinuous Galerkin) for convection diffusion equations [7]. In both cases the order of the PDE is reduced introducing an auxiliary equation. Furthermore, the relaxation system, unlike LDG, also linearizes the system with the help of an auxiliary variable. The linearized system provides a natural way to construct numerical fluxes (through upwinding) which results in a scheme which is stable under a CFL-like condition. On the other hand, LDG requires the construction of numerical fluxes and stabilization terms, so that stability is not built into the system being discretized. In the purely convective case the relaxation system utilizes the full technology of hyperbolic nonoscillatory reconstructions to prevent the onset of spurious oscillations.

Several strategies are possible to introduce relaxation systems for equation (1.1), see also [2]. We will compare the numerical properties of a few classes of relaxation approaches in a forthcoming paper. Here we propose a simple strategy which results in a particularly efficient scheme. It is interesting to note that the resulting scheme cannot be cast in the BGK framework studied in [2].

The present paper starts with a detailed description of the one-dimensional case, computing a stability estimate for the first order scheme, which will guide the selection of the time step even for higher order schemes. Next we discuss the implementation of boundary conditions and the extension of the scheme to the multidimensional case. We end with a few test problems carried out for schemes of order from 2 to 5, demonstrating the high resolution and non oscillatory properties of the family of schemes we propose.

2 Construction of the numerical schemes

We start describing in detail the numerical scheme in the one-dimensional case. Later we will generalize the schemes to the case of $x \in \mathbb{R}^d$.

2.1 Relaxation system

In order to construct a relaxation system for equation (1.1) we start relaxing the Laplacian operator as in [4]. We introduce the auxiliary variables v and w , and a parameter Φ , obtaining the system:

$$\begin{cases} u_t + v_x + (f(u))_x = 0, \\ v_t + \Phi^2 w_x = -\frac{1}{\varepsilon} (v - (\varepsilon \Phi^2 - 1) w_x), \\ w_t + v_x = -\frac{1}{\varepsilon} (w - p(u)). \end{cases} \quad (2.1)$$

When the relaxation parameter ε tends to 0, the solutions of (2.1) formally converge to solutions of (1.1). Next, to relax the convection operator, moving the non linearity of the flux f to the source term, we introduce the additional variable z and write

$$\begin{cases} u_t + z_x = 0, \\ v_t + \Phi^2 w_x = -\frac{1}{\varepsilon} (v - (\varepsilon \Phi^2 - 1) w_x), \\ w_t + v_x = -\frac{1}{\varepsilon} (w - p(u)), \\ z_t + A^2 u_x = -\frac{1}{\varepsilon} (z - v - f(u)), \end{cases} \quad (2.2)$$

where A is a scalar subject to a stability restriction, see Eq. (2.15). In the purely convective case, A must satisfy the Jin-Xin or Chen-Levermore-Liu subcharacteristic condition [6,9]:

$$A \geq \alpha \geq |f'(u)|.$$

We obtain the so-called relaxed scheme for (1.1), discretizing in time (2.2) and then letting $\varepsilon \rightarrow 0$ in the resulting semi-discrete equations.

2.2 IMEX semidiscretization in time

We first discretize system (2.2) in time. Due to the presence of both stiff and non-stiff terms, we exploit an IMEX time-integration procedure [1, 11], treating implicitly the stiff right hand side and integrating explicitly the non-stiff convective terms. In particular, we will consider Runge-Kutta IMEX schemes as in [1].

System (2.2) can be written in the form:

$$s_t + \operatorname{div} g(s) = \frac{1}{\varepsilon} h(s), \quad (2.3)$$

where

$$s = \begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix}, \quad g(s) = \begin{bmatrix} z \\ \Phi^2 w \\ v \\ A^2 u \end{bmatrix}, \quad h(s) = \begin{pmatrix} 0 \\ -v + (\Phi^2 \varepsilon - 1)w_x \\ p(u) - w \\ v + f(u) - z \end{pmatrix}. \quad (2.4)$$

We describe, for simplicity, the case of a uniform time step Δt . Let $s^n(x) = s(x, t^n)$, with $t^n = n\Delta t$. We obtain the following semidiscrete formulation

$$s^{n+1} = s^n - \Delta t \sum_{i=1}^v \tilde{b}_i \operatorname{div} g(s^{(i)}) + \frac{\Delta t}{\varepsilon} \sum_{i=1}^v b_i h(s^{(i)}), \quad (2.5)$$

where the $s^{(i)}$'s are the stage values of the Runge-Kutta scheme which are given by

$$s^{(i)} = s^n - \Delta t \sum_{k=1}^{i-1} \tilde{a}_{i,k} \operatorname{div} g(s^{(k)}) + \frac{\Delta t}{\varepsilon} \sum_{k=1}^i a_{i,k} h(s^{(k)}). \quad (2.6)$$

Here \tilde{b}_i , \tilde{a}_{ij} and b_i , a_{ij} denote the coefficients appearing in the Butcher's tableaux of the explicit and implicit RK schemes, respectively. We assume that the implicit scheme is diagonally implicit. To find the $s^{(i)}$'s it is necessary in principle to solve a nonlinear system of equations which however can be easily decoupled. The system for the first stage $s^{(1)}$ at time t^n is

$$\begin{pmatrix} u^{(1)} \\ v^{(1)} \\ w^{(1)} \\ z^{(1)} \end{pmatrix} = \begin{pmatrix} u^n \\ v^n \\ w^n \\ z^n \end{pmatrix} + \frac{\Delta t}{\varepsilon} a_{11} \begin{pmatrix} 0 \\ -v^{(1)} + (\Phi^2 \varepsilon - 1)\nabla w^{(1)} \\ p(u^{(1)}) - w^{(1)} \\ v^{(1)} + f(u^{(1)}) - z^{(1)} \end{pmatrix}. \quad (2.7)$$

The first equation yields $u^{(1)} = u^n$. Substituting in the third equation we immediately find $w^{(1)}$, then $v^{(1)}$ and finally we compute $z^{(1)}$ from the fourth equation. In other words the system can be solved simply by back-substitution. Similarly all stage values $s^{(i)}$ are computed solving an analogous system. Thus the nonlinearity of the source term is easily dealt with.

Following [4, 9] we set $\varepsilon = 0$, obtaining the so called *relaxed scheme*. Thus only the leading terms of Eq. (2.7) remain and the evaluation of the first stage reduces to:

$$\begin{aligned} u^{(1)} &= u^n, \\ w^{(1)} &= p(u^{(1)}), \\ v^{(1)} &= -\nabla w^{(1)}, \\ z^{(1)} &= v^{(1)} + f(u^{(1)}). \end{aligned} \quad (2.8)$$

For the following stages the first equation is

$$u^{(i)} = u^n - \Delta t \sum_{k=1}^{i-1} \tilde{a}_{i,k} \operatorname{div} z^{(k)}. \quad (2.9)$$

In the remaining equations the convective terms are dominated by the source terms and thus $v^{(i)}$ and $w^{(i)}$ are simply given by

$$\begin{aligned} v^{(i)} &= -\nabla w^{(i)}, \\ w^{(i)} &= p(u^{(i)}), \\ z^{(i)} &= v^{(i)} + f(u^{(i)}). \end{aligned} \quad (2.10)$$

The scheme reduces to an alternation of relaxation steps of the form (2.8) and (2.10) and of transport steps of the form (2.9), analogously to the lower order case of [9].

Note that only the explicit part of the Runge-Kutta method is actually involved in updating the solution. Therefore, in relaxed schemes, we use only the explicit part of the Runge-Kutta tableaux. In particular we consider second and third order strong stability preserving (SSP) Runge-Kutta schemes [8], namely

IMEX1 (1st order)

$$\begin{array}{c|c} & 0 \\ \hline & 1 \end{array}$$

IMEX2 (2nd order)

$$\begin{array}{c|cc} & 0 & 0 \\ \hline & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

IMEX3 (3rd order)

$$\begin{array}{c|ccc} & 0 & 0 & 0 \\ \hline & 1 & 0 & 0 \\ \hline & \frac{1}{4} & \frac{1}{4} & 0 \\ \hline & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{array}$$

2.3 Fully discrete

In order to complete the numerical scheme, we specify the spatial discretization employed to compute (2.5) and (2.6). We will use discretizations based on finite differences, in order to avoid cell coupling due to the implicit treatment of source terms.

Let us introduce a uniform grid on $[a, b] \subset \mathbb{R}$, $x_j = a - \frac{h}{2} + jh$ for $j = 1, \dots, N$, where $h = (b - a) / N$ is the grid spacing and N the number of cells.

In the relaxation steps (2.8) and (2.10), we compute the gradient of w with centered finite difference formulae of accuracy matching the order of the space discretization of the convective terms. Close to the boundaries, where the centered stencil would spill outside the domain, we employ asymmetric discretizations with matching accuracy.

In the transport steps (2.9), only the variable u needs to be advected. Let u_j^n denote the numerical solution at the grid point x_j at time t^n . Using a conservative finite difference discretization, (2.9) becomes

$$u_j^{(i)} = u_j^n - \frac{\Delta t}{h} \sum_{l=1}^{i-1} \left(z_{j+1/2}^{(l)} - z_{j-1/2}^{(l)} \right), \quad (2.11)$$

where $z_{j+1/2}^{(l)}$ is the reconstructed approximation to the flux z at the right cell border.

In order to compute $z_{j+1/2}^{(l)}$, we use the characteristic variables of the linear hyperbolic system

$$s_t + \operatorname{div} g(s) = 0.$$

The characteristic velocities of the system are $A, -A, \Phi, -\Phi$ and the corresponding characteristic variables are U, V, W, Z , but, due to the sparsity pattern of the Jacobian g' , only U and V are needed to compute the flux, with

$$U = \frac{Au+z}{2}, \quad V = \frac{-Au+z}{2}.$$

To minimize numerical viscosity we choose the Godunov flux, which in the case of a linear system reduces to upwinding along characteristic variables. Namely, since U has positive velocity and V negative velocity, we find

$$z_{j+1/2} = U_{j+1/2}^- + V_{j+1/2}^+, \tag{2.12}$$

where we suppressed the stage index (l) for clarity. The values $U_{j+1/2}^-$ and $V_{j+1/2}^+$ are the boundary extrapolated data of the non-oscillatory piecewise polynomial reconstructions $U(x)$ and $V(x)$ obtained from the point values $U_j = (Au+z)_j/2$ and $V_j = (-Au+z)_j/2$ at the centre of the computational cells. In this paper we use ENO reconstructions of order up to 6 and WENO reconstructions of order up to 5. See the review in [12] for all necessary details.

2.4 Boundary conditions

The numerical flux z appearing in (2.12) can be written as

$$z_{j+1/2} = \frac{A}{2} \left(u_{j+1/2}^- - u_{j+1/2}^+ \right) + \frac{1}{2} \left(z_{j+1/2}^- + z_{j+1/2}^+ \right),$$

with $z_{j+1/2}^\pm = [-\nabla p(u) + f(u)]_{j+1/2}^\pm$.

In order to treat the case of Neumann or Dirichlet boundary conditions, we employ one ghost point on each side of the computational domain. To fix ideas, suppose the left boundary is set at $x = 0$. At each Runge-Kutta stage, we choose a polynomial $P(x)$ of degree L , such that

$$P(x_j) = u_j^{(i)} \quad \text{for } j = 1, \dots, L,$$

while the last interpolation constraint is obtained from the boundary condition for u at $x = 0$. Then set

$$u_0^{(i)} = P(x_0) = P(-h/2).$$

Next, we compute $f(u)$ and $p(u)$ on the enlarged domain (interior domain plus ghost point). We also compute $\nabla p(u)$ on the enlarged domain using finite difference formulae of appropriate order. Having thus computed all characteristic variables on the enlarged domain, we then use the ENO/WENO procedures to compute $U_{j+1/2}^-$ and $V_{j+1/2}^+$ for $j = 0, \dots, N+1$. Close to the boundary, the ENO/WENO procedures are set to take into account only those stencils that are internal to the extended domain. This is sufficient to compute the fluxes (2.11) inside the physical domain and to perform the time integration.

2.5 CFL condition: parabolic/hyperbolic regimes

We now wish to find an estimate of the stability condition for the scheme. Working directly on a high order scheme is prohibitively complex, thus we estimate the growth of the Total Variation for the first order scheme of our relaxation methods. That is, we look for conditions which guarantee that:

$$TV(u^{n+1}) \leq TV(u^n),$$

where u^n denotes the set of the grid values of the numerical solution u , computed at the n -th time step. In the following, u^{n+1} is computed with the first order relaxed scheme, which reduces to:

$$u_j^{n+1} = u_j^n + \frac{\lambda A}{2} (u_{j+1}^n - 2u_j^n + u_{j-1}^n) + \frac{\lambda}{4h} (p_{j+2}^n - 2p_j^n + p_{j-2}^n) - \frac{\lambda}{2} (f_{j+1}^n - f_{j-1}^n), \quad (2.13)$$

where $\lambda = \Delta t/h$. Here p_j^n and f_j^n denote, respectively $p(u_j^n)$ and $f(u_j^n)$. To estimate the Total Variation of u^{n+1} , we apply the mean value theorem to f and p , with the following notation:

$$f(u_{j+1}^n) - f(u_j^n) = \alpha_{j+1/2}^n (u_{j+1}^n - u_j^n), \quad p(u_{j+1}^n) - p(u_j^n) = \delta_{j+1/2}^n (u_{j+1}^n - u_j^n),$$

where the Lipschitz continuity of f and p imply that

$$|\alpha_{j+1/2}^n| \leq \alpha, \quad 0 \leq \delta_{j+1/2}^n \leq \delta.$$

With these notations we can estimate the total variation of u^{n+1} as follows:

$$\begin{aligned} TV(u^{n+1}) &= \sum |u_j^{n+1} - u_{j-1}^{n+1}| \\ &= \sum \left| \frac{\lambda}{4h} \delta_{j+3/2}^n (u_{j+2}^n - u_{j+1}^n) + \frac{\lambda}{2} (A - \alpha_{j+1/2}^n) (u_{j+1}^n - u_j^n) \right. \\ &\quad \left. + \left(1 - \lambda A - \frac{\lambda}{2h} \delta_{j-1/2}^n \right) (u_j^n - u_{j-1}^n) \right. \\ &\quad \left. + \frac{\lambda}{2} (A + \alpha_{j-3/2}^n) (u_{j-1}^n - u_{j-2}^n) + \frac{\lambda}{4h} \delta_{j-5/2}^n (u_{j-2}^n - u_{j-3}^n) \right|, \end{aligned}$$

where on the right it is understood that all quantities are evaluated at the time level t^n . Assume that:

$$\delta_{j+1/2}^n \geq 0 \quad \forall j, n, \quad (2.14)$$

$$|\alpha_{j+1/2}^n| \leq \alpha \leq A \quad \forall j, n, \quad (2.15)$$

$$\lambda \leq \frac{1}{A + \frac{1}{2h} \delta_{j-1/2}^n} \quad \forall j, n. \quad (2.16)$$

Now we consider compactly supported initial data or periodic boundary conditions. Applying the triangle inequality, shifting indices in all sums, and applying the inequalities above, it is easy to see that the scheme is total variation non-increasing, provided the estimates above hold in all cells at each time step.

Inequality (2.14) is enforced by the hypothesis on $p(u)$ that guarantees the well posedness of the problem. The second condition (2.15) is analogous to the subcharacteristic condition of [6, 9]. Interestingly, the inequality (2.15) requires to link the relaxation velocity A with the characteristic speed of the convection term, irrespective of the diffusion term.

On the other hand, Eq. (2.16), which is the only one in the three conditions above restricting the time step, contains contributions from both the diffusion and the convective terms. For convection dominated and/or degenerate problems, the usual hyperbolic stability restriction results, with Δt proportional to h . For diffusion dominated problems, the second term in the denominator of (2.16) soon dominates, and the stability condition becomes parabolic, with Δt proportional to h^2 .

In our tests, we note that, when diffusion is present, the subcharacteristic condition (2.15) is too pessimistic and can be violated, so that A can be chosen much smaller than the characteristic speed of the hyperbolic term. As a matter of fact, the estimates (2.15) and (2.16) are obtained treating independently the diffusion and the convective term, so that the inequalities are strict when either of them is zero, but the stabilizing effect of the diffusion term on convection is disregarded. Sharper results on this effect can be obtained with linear stability analysis and will appear in [3].

Although the freedom in the choice of the subcharacteristic velocity does not result necessarily in a larger time step, it is nonetheless important, because the artificial diffusion of the scheme is considerably reduced if A can be chosen smaller. Note that the TVD condition is only sufficient for convergence. When diffusion is present, one sees numerically that wiggles developing in the first time steps are soon vanquished by the action of the diffusion term, suggesting that the total variation might be bounded but not necessarily diminishing.

Thus, in all tests, the time step will be chosen according to the CFL-like condition:

$$\Delta t \leq C \frac{h}{A + D/2h}, \quad (2.17)$$

where A will in general be smaller than α , when diffusion is present, $D \geq \delta$ is an upper bound for $\delta_{j+1/2}$, and C is a suitable constant, which depends on the scheme. Typically, C must be reduced for high order schemes, see for instance [5] for the tuning of the constant C on relaxation schemes for diffusion problems.

2.6 Dimensional splitting

An appropriate numerical approximation of (1.1) in \mathbb{R}^d that generalizes the scheme described in the previous sections can be obtained by additive dimensional splitting. We

again consider the relaxed scheme, i.e. $\varepsilon = 0$, and for the sake of simplicity, we focus on the square domain $[a, b] \times [a, b] \subset \mathbb{R}^2$. Here we shall generalize the 1D scheme described earlier to the case of two space dimensions.

Without loss of generality, we consider a uniform grid in $[a, b] \times [a, b] \subset \mathbb{R}^2$ such that $\vec{x}_{i,j} = (x_i, y_j) = (a - h/2, a - h/2) + i(h, 0) + j(0, h)$ for $i, j = 1, 2, \dots, N$ and $h = (b - a) / N$.

In the present case, u and w are one-dimensional variables, while $\vec{v} = (v_{(1)}, v_{(2)})$ and $\vec{z} = (z_{(1)}, z_{(2)})$ are now fields in \mathbb{R}^2 . First we observe that the relaxation steps (2.8) and (2.10) are easily generalized for $d > 1$. For the transport steps, one has to evolve in time the system

$$\begin{aligned} \frac{\partial}{\partial t} \begin{pmatrix} u \\ v_{(1)} \\ v_{(2)} \\ w \\ z_{(1)} \\ z_{(2)} \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \Phi^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ A^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{pmatrix} u \\ v_{(1)} \\ v_{(2)} \\ w \\ z_{(1)} \\ z_{(2)} \end{pmatrix} \\ + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Phi^2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ A^2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial y} \begin{pmatrix} u \\ v_{(1)} \\ v_{(2)} \\ w \\ z_{(1)} \\ z_{(2)} \end{pmatrix} = 0. \end{aligned} \tag{2.18}$$

Note that it is the sparsity of these flux matrices that ensures the success of the one-dimensional splitting. For the numerical integration we employ a two-dimensional generalization of (2.11), namely

$$\begin{aligned} \operatorname{div} \vec{z}_{i,j} &= \frac{1}{h} \left(z_{(1)i+1/2,j} - z_{(1)i-1/2,j} \right) + \frac{1}{h} \left(z_{(2)i,j+1/2} - z_{(2)i,j-1/2} \right) \\ &= \frac{1}{h} \left(U_{(1)i+1/2,j}^- + V_{(1)i+1/2,j}^+ - U_{(1)i-1/2,j}^- - V_{(1)i-1/2,j}^+ \right) \\ &\quad + \frac{1}{h} \left(U_{(2)i,j+1/2}^- + V_{(2)i,j+1/2}^+ - U_{(2)i,j-1/2}^- - V_{(2)i,j-1/2}^+ \right). \end{aligned} \tag{2.19}$$

Here $U_{(1)i\pm 1/2,j}^-$ and $V_{(1)i\pm 1/2,j}^+$ are reconstructions of the characteristic variables

$$U_{(1)} = \frac{Au + z_{(1)}}{2}, \quad V_{(1)} = \frac{-Au + z_{(1)}}{2},$$

along the x direction, while $U_{(2)i,j\pm 1/2}^-$ and $V_{(2)i,j\pm 1/2}^+$ are reconstructions of the characteristic variables

$$U_{(2)} = \frac{Au + z_{(2)}}{2}, \quad V_{(2)} = \frac{-Au + z_{(2)}}{2},$$

Table 1: The behaviour of errors and convergence rate for problem (3.1).

	$D = 10^{-1}$						rate
	N=20	N=40	N=80	N=160	N=320	N=640	
ENO2, RK1	2.04e-3	4.48e-4	1.35e-4	3.39e-5	8.52e-6	2.11e-6	1.9637
ENO4, RK2	3.26e-3	2.32e-4	3.07e-7	1.89e-8	1.17e-9	7.34e-11	5.2525
ENO6, RK3	8.34e-6	1.51e-7	2.44e-9	3.83e-11	5.98e-13	9.03e-15	5.9641
WENO3, RK2	9.55e-4	1.15e-4	1.25e-5	1.08e-6	7.75e-8	4.97e-9	3.5114
WENO5, RK3	7.80e-5	2.72e-6	8.91e-8	2.87e-9	9.04e-11	2.81e-12	4.9491
	$D = 10^{-6}$						
ENO2, RK2	1.98e-2	5.66e-3	1.57e-3	4.13e-4	1.10e-4	2.95e-5	1.8839
ENO3, RK3	1.08e-3	1.36e-4	1.71e-5	2.14e-6	2.68e-7	3.35e-8	2.9955
WENO3, RK3	1.75e-2	4.47e-3	1.16e-3	2.36e-4	2.62e-5	1.61e-6	2.6166

along the y direction.

The generalization to $d > 2$ and rectangular domains is now trivial. We stress once again that no two-dimensional reconstruction is used, but only d one-dimensional reconstructions are needed. Finally, boundary conditions can be implemented direction-wise with the same techniques used in the one-dimensional case.

3 Numerical tests

We consider some test problems in one and two space dimensions, which demonstrate the flexibility and robustness of the scheme we propose. Most tests are drawn from [10], which contains a large selection of benchmark problems, computed with a semidiscrete second order scheme (the KT scheme) which is also an explicit scheme for hyperbolic/parabolic problems, capable of dealing with the degenerate case.

3.1 Convergence tests

We first show a convergence test for a few schemes based on the relaxation system presented in this work. We consider the following one-dimensional test:

$$\begin{cases} u_t + u_x = Du_{xx} \text{ on } [0,1], \\ u_0(x) = \sin(2\pi x), \\ \text{periodic boundary conditions.} \end{cases} \quad (3.1)$$

We show the results of a diffusion dominated problem ($D = 0.1$, parabolic CFL) and a convection dominated problem ($D = 10^{-6}$, hyperbolic CFL).

The errors and convergence rates appear in Table 1. Note that the errors and the rate of convergence have the expected behaviour.

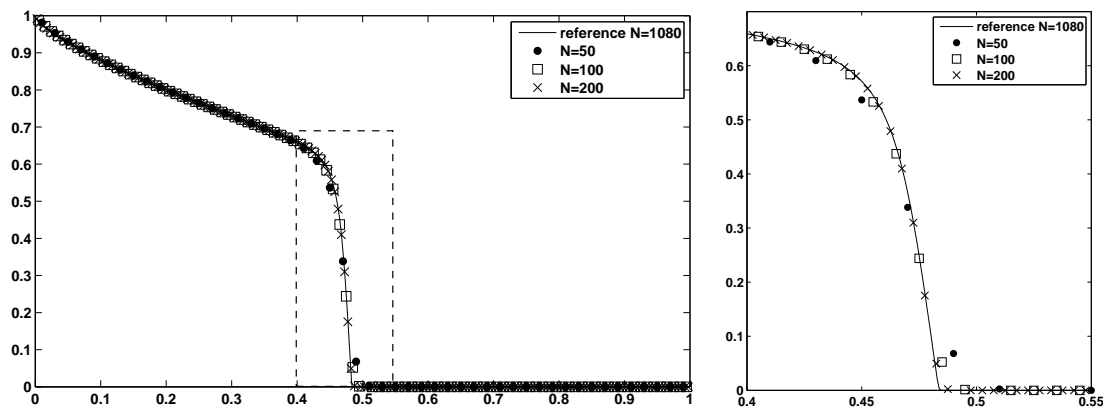


Figure 1: Convergence to the entropic solution for Test 1. ENO4 reconstruction, CFL condition $C=0.25$, $A=2$. The right-hand-side graph is a blow-up of the area inside the dashed line of the left-hand-side graph.

3.2 Buckley-Leverett equation (Test 1)

This is Example 7 in [10]. Here the flux and the diffusive term are given by

$$f(u) = \frac{u^2}{u^2 + (1-u)^2}, \quad p(u) = 2Du^2\left(1 - \frac{2}{3}u\right),$$

where we pick $D=0.01$, and with the initial condition:

$$u(x,0) = \begin{cases} 1-3x & 0 \leq x \leq 1/3, \\ 0 & 1/3 < x \leq 1. \end{cases}$$

This problem models a two phase flow, and, due to the nonconvexity of the flux, non entropic solutions may develop. Our results can be found in Fig. 1. The figure shows the numerical convergence of the relaxation scheme with the ENO4 reconstruction. The constant C in the CFL condition (2.17) is $C = 0.25$. We note that the numerical solution converges to the entropic solution, the profile is non-oscillatory, and the resolution of the angular point is very good. The relaxation speed is $A = 2$, which satisfies (2.15).

3.3 Glacier growth (Test 2)

This is Test 9 in [10]. The equation models the growth of a glacier, under the action of a seasonal forcing term. The equation is:

$$u_t + \left(\frac{u+3u^6}{4} \right)_x = D(3u^6u_x)_x + S(x,t)\chi_{u \geq 0},$$

where $D=0.01$ and $S(x,t)$ is the forcing term, which is active only for $u \geq 0$ and is given by

$$S(x,t) = \begin{cases} 0 & x \leq -5, \\ -0.01x + 0.05 \sin(2\pi t) & x > -5. \end{cases}$$

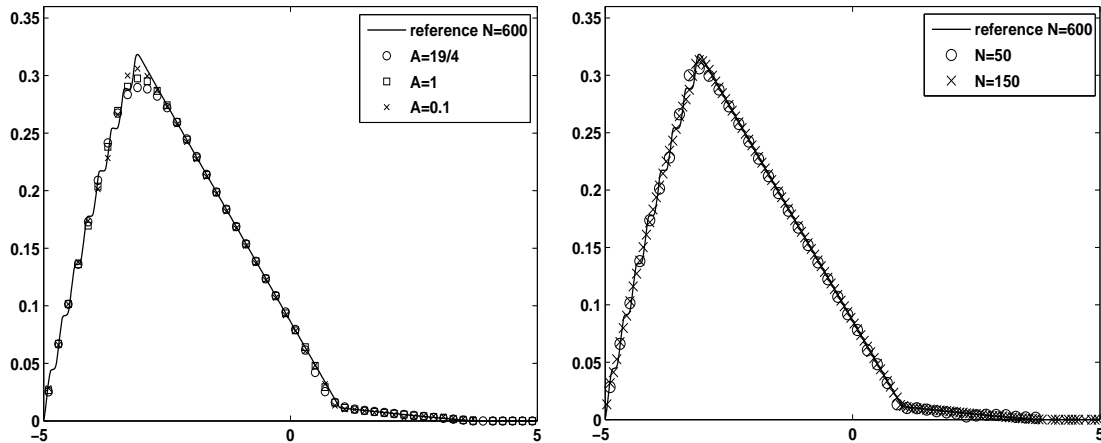


Figure 2: Solution for Test 2. WENO5 reconstruction with RK3, CFL condition $C=0.15$. Left: dependence on A , $N=50$. Right, convergence under grid refinement, with $A=0.1$.

The initial condition is $u(x, t=0) \equiv 0$. The computational domain is $(-5, 5)$ with homogeneous boundary conditions. Thus the model starts out with no ice, and the height u of the layer grows in time, while slowly sliding and diffusing downstream. Note that the solution is completely degenerate at the beginning, and it remains degenerate at the end points of the computational interval. We show the results at time $T=7.5$ in Fig. 2.

The left of the figure shows that the solution remains stable even when the subcharacteristic condition (2.15) is violated. Here (2.15) would prescribe $A \geq 19/4$, while the solution appears still stable for $A=0.01$. Note also that resolution increases as A is decreased. The zigzagging pattern exhibited by the reference solution is due to the sinusoidal forcing term. It is barely visible in Fig. 6.28 of [10] because the KT scheme is only second order accurate, and cannot match the high resolution given by the 5th order accurate computation shown here.

3.4 A 2D problem (Test 3)

We end with a 2D problem, again from [10], a convection diffusion equation, with a Buckley-Leverett flux:

$$u_t + \left(\frac{u^2}{u^2 + (1-u)^2} \right)_x + \left(\frac{u^2(1-5(1-u)^2)}{u^2 + (1-u)^2} \right)_y = D(u_{xx} + u_{yy}),$$

with $D=0.01$ and initial data $u(x, y, t=0) = 1$ for $x^2 + y^2 < 0.5$ and zero otherwise. The solution at time $T=0.5$ is shown in Fig. 3.

Here the characteristic velocities are $A_x=2$ and $A_y=3.5$, with $A = (A_x^2 + A_y^2)^{1/2}$.

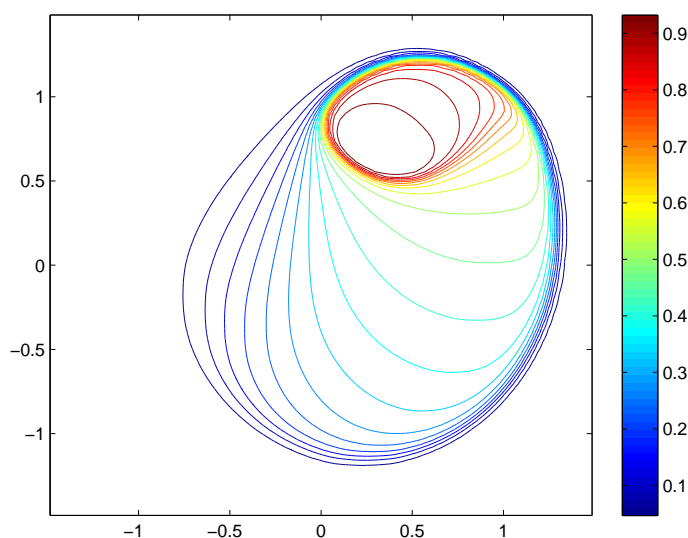


Figure 3: Solution of a 2D problem (Test 3) Solution with $N=100 \times 100$ points, with CFL $C=0.15$.

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