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# Exponentially-Convergent Strategies for Defeating the Runge Phenomenon for the Approximation of Non-Periodic Functions, Part I: Single-Interval Schemes

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**Abstract.** Approximating a function from its values $f(x_i)$ at a set of evenly spaced points $x_i$ through $(N+1)$-point polynomial interpolation often fails because of divergence near the endpoints, the "Runge Phenomenon". Here we briefly describe seven strategies, each employing a single polynomial over the entire interval, to wholly or partially defeat the Runge Phenomenon such that the error decreases *exponentially* fast with $N$. Each is successful in obtaining high accuracy for Runge's original example. Unfortunately, each of these single-interval strategies also has liabilities including, depending on the method, various permutations of inefficiency, ill-conditioning and a lack of theory. Even so, the Fourier Extension and Gaussian RBF methods are worthy of further development.

**AMS subject classifications**: 42C10, 65D05

**Key words**: Interpolation, Runge Phenomenon, Fourier extension, radial basis functions.

## 1 Introduction

More than a century ago, Carl Runge, C. Meray and Emilie Borel independently made an astonishing discovery: polynomial interpolation on an equispaced grid was unreliable [11, 27–29, 33]. Borel gave an example of non-convergent interpolation at the Heidelberg Mathematical Congress in 1904, but apparently did not publish it. Even if $f(x)$ is analytic for all real $x$, its interpolants $f_N(x)$ will diverge as $N \rightarrow \infty$ near the endpoints $x = \pm 1$ if $f(x)$ has singularities within the "Runge Zone" in the complex $x$-plane illustrated in Fig. 1.

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Runge's own example was $f(x)=1/(1+25x^2)$, which is analytic for all real $x$, but has a divergent equispaced polynomial interpolant sequence because of poles at $x=\pm\pi i/5$. Exponential convergence can be recovered by using the highly nonuniform Chebyshev grid [3], but what is one to do with experimental data collected at evenly spaced levels?

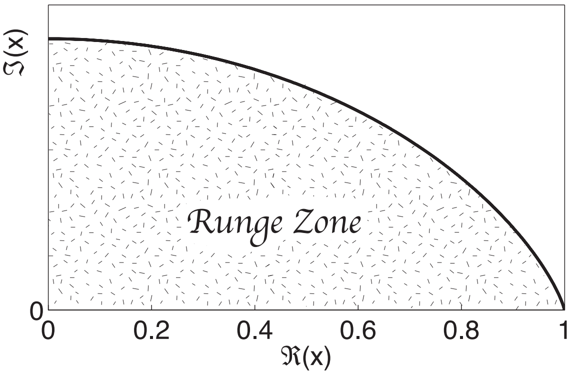


Figure 1: The Runge Zone in the complex $x$-plane for polynomial interpolation with a uniformly spaced grid on $x\in[-1,1]$. Because the boundary curve is symmetric under reflection with respect to the both the real and imaginary axes, only the portion in the upper right quadrant of the complex plane is illustrated. If $f(x)$ has any singularities in the sense of complex variable theory anywhere within the shaded region (or its reflections about either axis), then interpolation diverges as $N\to\infty$. If $f(x)$ has singularities only outside the shaded "Runge Zone", then interpolation will converge everywhere on $x\in[-1,1]$.

As explained in the reviews [5, 6, 16], defeating Gibbs Phenomenon in Fourier also requires reconstructing a function $f(x)$ everywhere on $x\in[-1,1]$ with exponential accuracy from knowledge only of its analyticity on the interval and its samples on an evenly spaced grid of $(N+1)$ points on the interval. Symbolically,

$$\text{anti}-\text{Gibbs}=\text{EdgeDetection}+\text{Anti}-\text{Runge}, \tag{1.1}$$

where edge detection identifies the boundaries of regions that are free of discontinuities ("edges", alias "shocks" and "fronts" in fluids), and then an anti-Runge procedure is applied on each smooth sub-interval to approximate $f(x)$.

There is a wide variety of *finite order* strategies to defeat the Runge Phenomenon whereby "finite order" denotes an approximation scheme whose error decreases as $\mathcal{O}(1/P^K)$ where $P$ is the number of sample points and $K>0$ is the "algebraic order of convergence". The simplest is piecewise polynomial interpolation: the "connect-the-dots" diagrams of coloring books allow a preschooler to draw a butterfly or a fish by drawing linear polynomials from dot to dot with a crayon. This is only first order, but cubic splines provide a higher order "Old Reliable".

Our goal is more ambitious, which is to develop schemes with an *exponential* rate of convergence as $P\to\infty$. Curiously, although this problem is over a century old, it is only in recent times that exponentially-convergent Runge-defeating methods have been developed. Now, there are so many strategies that it is impossible to describe them in a single article. In this work, we shall specialize to *single-interval* schemes:

- Tikhonov Regularization [2];
- Gegenbauer Regularization and its generalizations; [5, 6, 13–15, 17–24, 30, 34];
- Overdetermined Least-Squares [9];
- Mock-Chebyshev sub-sampling [9];
- Gaussian Radial Basis Functions, Slowly Flattening with $N$ [8];
- Fourier Extension [4];
- Platte's Undetermined $L_1$ Extension.

In contrast, three-subdomain methods use one algorithm for a large, central subdomain while applying different tactics to the two thin subintervals that include the endpoints $x=\pm 1$. Borrowing a term from fluid mechanics, it is helpful to label these two narrow subdomains the "boundary layers". A common theme is that the width $D$ of the boundary layers must shrink as the number of interpolation points $P$ increases in order to ensure convergence. The central layer of width $2(1-D)$ can be treated by Fourier interpolation in combination with either

1. series acceleration by any one of several sequence acceleration schemes or
2. windowing using either an analytic or a $C^\infty$ window.

The boundary layers can be treated by standard Lagrangian polynomial interpolation on an evenly-spaced grid; we prove in Part II that convergence is guaranteed provided that the width $D$ of the boundary layers contracts with increasing $N$. Alternatively, one may overlap the two domains and use a semi-infinite basis set such as the $TM_n(x)$ rational Chebyshev functions.

Another option is to use Lagrangian polynomial interpolation of degree $M$ on *many* subdomains, increasing $M$ with $N$ so that the number of subdomains increases and the width of each subdomain steadily decreases. Still another option is Martin Berzins' ENO-type polynomial interpolation scheme. This does not explicitly subdivide the interval into non-overlapping subdomains, but does use many different polynomial approximations of different orders to adapt to $f(x)$ [1].

These multidomain or multi-polynomial algorithms will be compared and contrasted in Part II. However, an accelerated Fourier/polynomial-boundary-layer scheme has already been analyzed in [7]. This note shows that at least some multidomain methods are very robust, well-conditioned, and backed by theory. Similarly, Platte and Gelb have presented an as yet unpublished three domain scheme using windowing at the SIAM Annual Meeting in 2006, and shown it works well.

It is, of course, easier to apply a single tactic over the whole domain than to use different algorithms on different subintervals or to use a whole lot of subintervals. It is therefore important to understand the virtues and flaws of the single-interval schemes as we shall try to catalogue here.

One property common to many anti-Runge methods needs explication here: A *subgeometric* rate of convergence.

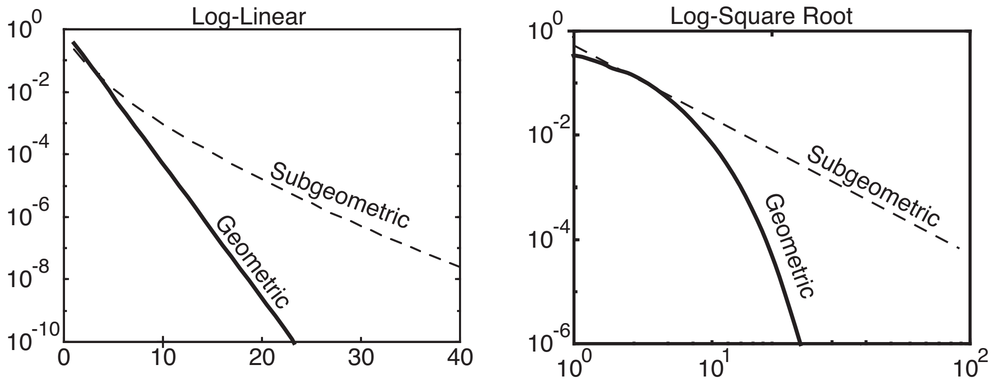


Figure 2: Left: Schematic of geometric and subgeometric convergence when either error or absolute value of the coefficients is plotted versus degree using a logarithmic scale for the error or coefficients, and a scale linear in $N$. Right: Same except the horizontal axis is a plot of $\sqrt{N}$.

The usual or generic rate of convergence for Chebyshev interpolation of an $f(x)$ that is analytic on $x\in[-1,1]$ is a *geometric* rate of convergence, that is, the error $E_N$ with a polynomial of degree $N$ is proportional to

$$\exp(-qN), \tag{1.2}$$

where $q>0$ is a constant that depends on $f(x)$; this is an exponential whose argument is *linear* in $N$. (The exponential may be multiplied by a more slowly-varying function of $N$ such as a power or logarithm, but the leading order approximation to the *logarithm* of the error is $-qN$.) The price that must be paid to guarantee the defeat of the Runge Phenomenon is that many "anti-Runge" methods yield only a *subgeometric* rate of convergence, that is, the error is proportional to

$$\exp(-qN^r) \tag{1.3}$$

for some exponent $r>0$ which is less than one so that the logarithm of the error grows slower-than-linearly with $N$. Fig. 2 compares how these rates of convergence appear on different plots; the right panel assumes that $r=1/2$, the most common case of subgeometric convergence seen.

## 2 Tikhonov regularization

Tikhonov regularization has been applied to many ill-conditioned problems including tomography: reconstructing three-dimensional reality from two-dimensional scattering behavior. For such "Tikhonov applicable" problems, the ill-conditioning arises because there are many solutions which are very close to matching the data. For example, to approximate a function like Runge's example, $f(x)=1/(1+25x^2)$, there are many polynomials that interpolate or almost interpolate $f(x)$ at $(N+1)$ equidistant points. One is the polynomial interpolant, which is bad because it oscillates wildly between the grid points. Another is the polynomial which is the truncation of the Chebyshev series of $f(x)$ after $(N+1)$ terms. The truncated Chebyshev series converges geometrically as $N\to\infty$, so its values at the points of the equispaced grid must differ from $f(x)$ only by exponentially small amounts; the Chebyshev approximation is much more desirable than the interpolant because the Chebyshev series is accurate *everywhere* on $x\in[-1,1]$. Tikhonov's brain storm, originally applied to problems such as solving ill-conditioned integral equations, was to pick the smoothest approximation from the large set of functions that almost fit the data.

The Tikhonov regularization of polynomial interpolation is the polynomial which *minimizes* a *cost function* that is the sum of the interpolation residual norm $R$ plus a smoothness norm $S$ where

$$R\equiv\sum_{k=0}^{N}\left(f(x_k)-f_N(x_k)\right)^2 \tag{2.1}$$

with

$$x_k=-1+2k/N, \qquad k=0,1,\cdots,N, \tag{2.2}$$

and one of many possible choices for $S$ is

$$S\equiv\sum_{k=0}^{N}\left(\frac{d^2}{dx^2}(y_k)\right)^2, \tag{2.3}$$

where

$$y_k\equiv\cos(\pi[2k+1]/[2N+2]). \tag{2.4}$$

The cost function is

$$\rho\equiv R+\chi S, \tag{2.5}$$

where $\chi$ is the "Tikhonov parameter". The parameter $\chi$ can be estimated by the "L-shaped curve method" — essentially minimizing the cost function for many different values of $\chi$. The approximation is insensitive to $\chi$ for an intermediate range of this parameter, and the optimum $\chi$ is the middle of this range.

It should be noted that similar ideas have been developed in statistics under the name of "smoothing splines" [36] and "ridge regression"; the method is also called "penalized least squares" and "regularization theory" [12, p. 167].

Boyd showed that Tikhonov Regularization triumphs for Runge's example [2]. Hurrah! But there are also, alas, some "howevers".

One is that it is expensive to solve the problem repeatedly for many different $\chi$. Another is that the interpolation matrix on an equispaced grid is very ill-conditioned [35]. The smoother $S$ improves the situation, but the matrix condition number is still high, and Boyd's best accuracy was limited to about $10^{-6}$.

This method is sufficiently promising to warrant another look, even though it does not seem as good as the radial basis function and Fourier Extension methods described below. It is unknown whether Tikhonov Regularization is successful, even in infinite precision arithmetic, for functions with singularities arbitrarily close to the expansion interval.

## 3 Gegebauer regularization and its generalizations

The Gegenbauer method was invented to defeat Gibbs' Phenomenon. If a fluid flow develops a shock wave where $f(x)$ is discontinuous, its Fourier series (computed using an evenly-spaced grid) will converge poorly, but still correctly encode information about the function. To retrieve an accurate approximation, Gottlieb and Shu proposed to employ a polynomial approximation, restricted to an interval with the discontinuity as an endpoint, derived from the Fourier series. This is also *implicitly* a solution to Runge's problem since a polynomial approximation is being generated from knowledge of $f(x)$ only on an equispaced grid.

This idea was sufficiently successful to spawn a large literature and continuing generalizations. However, there are drawbacks. One is that the degree of the polynomial approximation $N$ must be small compared to the number $P$ of samples of $f(x)$ — typically, $N \approx (P/4)$.[†] Another problem is severe numerical ill-conditioning, increasing rapidly with $N$.

Another difficulty is that to obtain a geometric rate of convergence, it is necessary to increase the Gegenbauer order parameter $m$ linearly with $N$, the Gegenbauer degree, so that $\beta \equiv m/N$ is a constant in the limit $N \to \infty$. Boyd showed [5] that this *shrinks* the Runge Zone, that is, the region of the complex $x$-plane where $f(x)$ must be free of singularities for convergence. Unfortunately, the Runge Zone contracts to the expansion interval only when $\beta \to 0$, sacrificing a geometric rate of convergence.

However, this idea has been successful in a variety of applications [14, 15, 25, 26, 30] with new generalizations such as inverse fits and substitution of Freud polynomials for Gegenbauer.

It may be possible to guarantee convergence if $\beta$ is allowed to decrease slowly with $N$, allowing a subgeometric but still exponential rate of convergence. However, this conjecture has not been tested.

## 4 Overdetermined least-squares

This option was investigated numerically by Boyd and Xu [9] and theoretically by Rakhmanov [32]. If an approximation of degree $N$ is determined by an unweighted least-squares fit to a uniform grid with $P$ points, then defining $\beta \equiv (N+1)/P$, the Runge Zone shrinks linearly with $\beta$. Unfortunately, it appears the only way to guarantee convergence

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[†]Note that the Gegenbauer literature employs $N$ for our $P$, $m$ for our $N$, and $\lambda$ for the Gegenbauer superscript $m$.

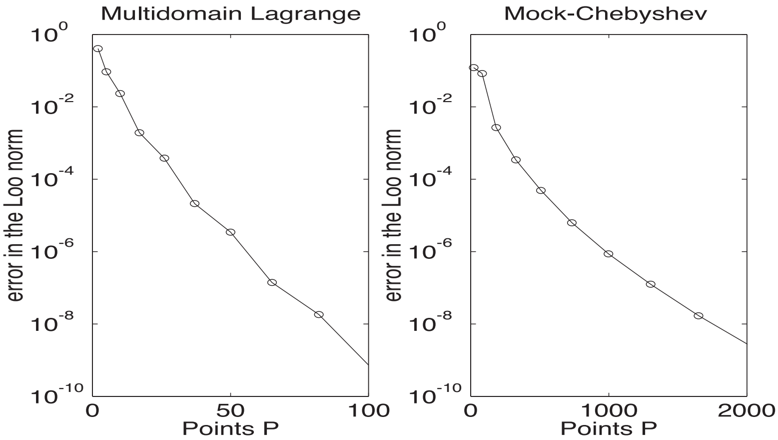


Figure 3: Left: Lagrange polynomial interpolation on subdomains with the polynomial degree equal to the number of subdomains, a multi-interval strategy discussed in Part II. Right: Interpolation of a single polynomial over the entire domain using mock-Chebyshev subsampling.

for $f(x)$ with singularities very close to $[-1,1]$ is to choose $\beta=0$, or more precisely, to allow $P$ to grow as the *square* of $N$. If $\beta$ is finite as $N,P$ simultaneously tend to infinity, then there is always a class of $f(x)$, analytic everywhere on the expansion interval, for which the overdetermined approximation will diverge.

## 5 Subsampling: The mock-Chebyshev grid

The Runge Phenomenon never occurs when the interpolation points are those of a Chebyshev grid:

$$x_j^{Cheb}=\cos(\pi j/N), \qquad j=0,1,\cdots N. \tag{5.1}$$

Chebyshev himself knew in the nineteenth century that as long as $f(x)$ was analytic on the interval $[-1,1]$, Chebyshev interpolation would converge exponentially fast. However, the Chebyshev grid is very nonuniform with neighboring points separated by only $\mathcal{O}(1/N^2)$ near the endpoints.

A simple way to recover the virtues of Chebyshev interpolation from samples of $f(x)$ on an evenly spaced grid is to sub-sample, that is, to select from an equispaced grid with $P$ points the $(N+1)$ points which are closest to the Chebyshev points:

$$x_j^{mock-Cheb}=x_k, \text{ such that } |x_k-x_j^{Cheb}|=\min_m|x_m-x_j^{Cheb}| \qquad \forall m=1,\cdots,P. \tag{5.2}$$

When the number of points on the equispaced grid is

$$P=\text{round}\left(\chi N^2\right)-1 \quad \text{with } \chi>2/\pi^2,$$

an exponential rate of convergence is indeed retrieved as illustrated in [9].

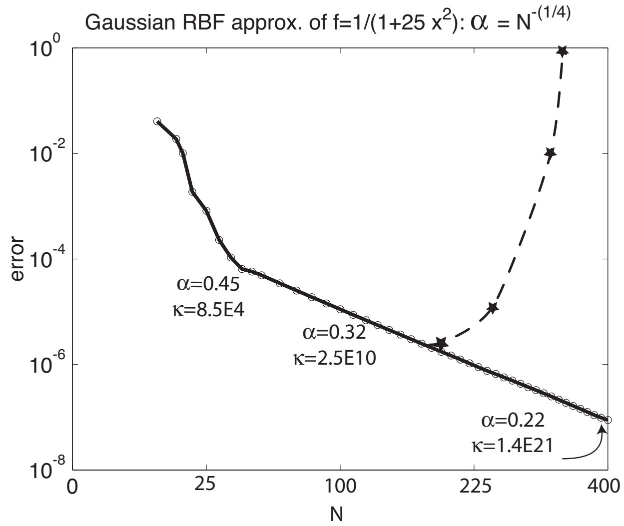


Figure 4: Errors in the $L_\infty$ norm for the Gaussian RBF approximation of $f(x)=1/(1+25x^2)$, plotted versus $\sqrt{N}$, as computed using multiple precision (solid/circles) and single precision (dashed/stars). The absolute width $\epsilon$ varies with $N$ as $(1/2)N^{3/4}$. The numerical labels give the values of $\alpha$, the inverse width parameter of the RBFs, and $\kappa$, the condition number of the interpolation matrix.

However, Fig. 3 shows performance is very poor compared to another exponentially convergent method that will be discussed in Part II. This is not surprising because almost all values of $f(x)$ on the equispaced grid are simply being *discarded*, utterly unused in the approximation. The fact that $P \sim \mathcal{O}(N^2)$ is completely consistent with the theory of Rakhmanov [32] mentioned in the previous subsection.

Convergence of the method is uniform in the location of the singularities of $f(x)$ in the sense that the mock-Chebyshev grid succeeds even when the function has a singularity an arbitrarily small distance $\epsilon$ from the expansion interval. However, to reach a given error tolerance, the degree $N$ of the Chebyshev series must grow proportionally to $1/\epsilon$. This implies the number of points $P$ required by the mock-Chebyshev method must grow as $\mathcal{O}(1/\epsilon^2)$, just as bad as for the overdetermined fit method.

It is noteworthy, though, that like the other schemes, the "mock-Chebyshev" grid is able to retrieve an exponential-but-subgeometric rate of convergence.

## 6 Gaussian RBFs

A one-dimensional function can be approximated by a Gaussian radial basis function (RBF) series as

$$f(x) \approx \sum_{j=1}^{N} \lambda_j \exp\left(-\frac{\alpha^2}{h^2}\left(x-c_j\right)^2\right), \tag{6.1}$$

where the $c_j, j=1,\cdots N$, are "centers", $\lambda_j$ are "coefficients", $h$ is the average grid spacing, and $\alpha$ is the inverse width relative to $h$. For our purposes, it is sufficient to specialize by choosing the $c_j$ to be the points $x_j$ of an evenly spaced grid. The coefficients $\lambda_j$ can be found by interpolation of $f(x)$ at the set $x_j$.

Platte & Driscoll [31] show that if $\alpha$ decreases as $\sqrt{\beta}/\sqrt{N}$ where $\beta$ is a constant, the RBF interpolant will *diverge* if $f(x)$ is singular within a certain domain in the complex $x$-plane, which is the Runge Zone for RBF interpolation. (The RBF Zone for RBFs is *different* from that for polynomial interpolation using the same interpolation points except in the limit $\alpha \rightarrow 0$.) They show that this domain contracts as $\beta$ decreases. If the interpolant converges for *fixed* $\beta$, the rate of convergence is geometric.

Fig. 4 shows that when the width parameter decreases more slowly as $\alpha = N^{-1/4}$, the rate of convergence slows to *subgeometric*: the error is proportional to $\exp(-p\sqrt{N})$ for some constant $p$. By using $\sqrt{N}$ instead of $N$ as the horizontal axis, the error on a log-linear scale asymptotes to a straight line as shown. The reward for the slower rate of convergence is that the Runge Phenomenon is *eliminated*. However, the condition number $\kappa$ of the interpolation matrix grows exponentially proportional to $\exp(2.4\sqrt{N})$. In single precision floating point arithmetic, there is thus a race as $N$ increases: Will adequate accuracy be achieved before the numerical singularity of the interpolation matrix spoils the RBF method?

## 7 Fourier extension

Fourier Extension is a strategy of approximating a function on a "physical" interval, here normalized to $x \in [-1,1]$, by means of a Fourier series which is periodic on a larger interval $x \in [-(1+D),(1+D)]$ where $D>0$. When $f(x)$ is unknown outside $[-1,1]$, this is "Fourier Extension of the Third Kind" in the terminology of [4]. Boyd employed collocation on an evenly-spaced grid on $x \in [-1,1]$ (only) with the Fourier coefficients as the unknowns and with typically twice as many collocation points as Fourier coefficients ("FPIC-SU" in his jargon). The overdetermined system was solved by the SVD factorization with typically three iterative corrections. He also employed SVD filtering, that is, discarded SVD modes of very small singular value, but this did not prove necessary in our experiments. To reduce cost by a factor of four, it is advantageous to split $f(x)$ into its symmetric and antisymmetric parts and fit each separately as explained in [4].

Fig. 5 shows the errors, plotted in the $P-D$ plane where $D$ is the width of the "extension" zone such that the antisymmetric function $f(x)$ is approximated by a sine series of period $2(1+D)$ and $P/4$ terms. This particular function is much more difficult than Runge's example because it has poles at $x = \pm i/40$, only one-eighth of the distance from the real axis to the poles of Runge's function, and located with real parts at $x = -1,0,1$ so that both the center of the interval and the endpoints have narrow peaks. Nevertheless, Fourier Extension yields approximations whose error is smaller than $10^{-10}$! (More limited experiments are reported in [10].)

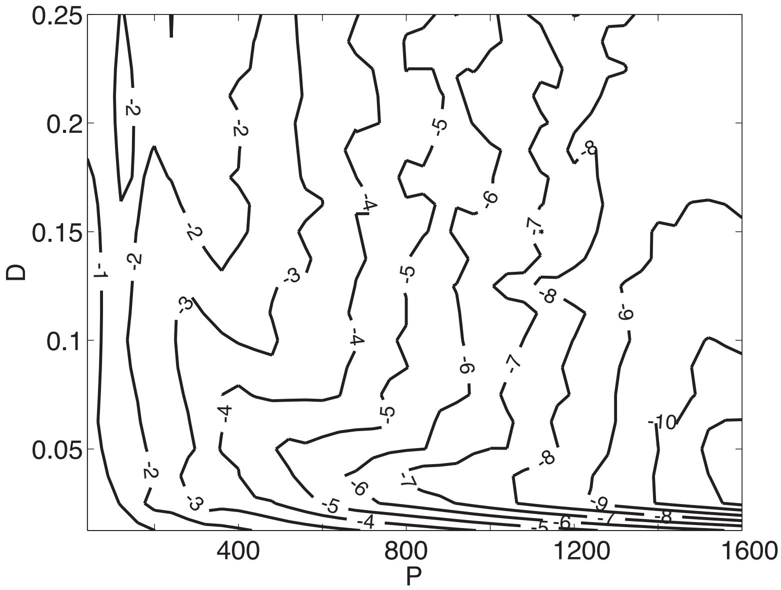


Figure 5: Fourier Extension. Base 10 logarithm of the errors in the $L_\infty$ norm for the approximation of $f(x)=x/(1+1600x^2)+1/(1+1600[x-1]^2-1/(1+1600[x+1]^2$ through Fourier Extension from $x\in[-1,1]$ to the extended interval $x\in[-(1+D),(1+D)]$ through a trigonometric polynomial of degree $N$ which has period $2(1+D)$. All collocation points are restricted to the "physical" interval, $[-1,1]$, and twice as many collocation points $P$ as coefficients of the trigonometric polynomial were employed, that is, $P=2N$. The overdetermined collocation equations, a matrix problem of dimension $2N\times N$, was solved by SVD factorization with three iterative corrections.

Numerical ill-conditioning is a problem, but not as much as for some of the competing single-interval methods. The need for SVD factorization of an overdetermined interpolation matrix makes the method relatively expensive, but on modern workstations, not very time-consuming. Programming is very easy.

All in all, Fourier Extension warrants further study.

## 8 Platte's undetermined $L_1/L_2$ approximation

Rodrigo Platte has experimented with an undetermined polynomial fit, that is, the number of free parameters $(N+1)$ is greater than the number of grid points $P$. An additional criterion is necessary to determine a unique approximation. Borrowing a successful idea from image processing, Platte chose the approximation of smallest $L_1$ norm. The undetermined $L_1$ fit approximated Runge's function to several decimal places. In later experiments, he obtained even better performance by minimizing the weighted $L_2$ norm instead. A full discussion must be deferred to Platte's own future publication. However, this intriguing idea is yet another illustration of the rich variety of methods that can be deployed against the Runge Phenomenon.

## 9 Summary

As noted in Table 1, even the best of the single domain methods are "good but not great", which motivates the more complicated multidomain schemes of Part 2.

One criterion of merit is that an anti-Runge method should converge over all of $x \in [-1,1]$ even when $f(x)$ has a singularity only an arbitrarily small distance $\epsilon$ away from the real axis with a number of degrees of freedom that grows only linearly with $1/\epsilon$. We may dub this the "Uniform Convergence" criterion. Unfortunately, the Gegenbauer, Overdetermined Polynomial Fit and the Mock-Chebyshev grid all fail, the last only because the number of points grows *quadratically* with $1/\epsilon$. For the Tikhonov and $L_1$-Minimizing schemes, uniform convergence is uncertain. For the Gaussian RBF and Fourier Extension algorithms, uniform convergence is plausible, based on numerical and theoretical evidence, but has not been rigorously proved.

Numerical ill-conditioning is a serious problem for many single-interval anti-Runge methods. The mock-Chebyshev subset scheme is the exception, being blessed with the same $\mathcal{O}(1)$ condition method as standard Chebyshev interpolation with $N$ points. There is a good theory for the Gaussian RBF method that shows that the condition grows proportional to $\exp(q\sqrt{N})$ for some constant $q$. For the others, there is strong numerical evidence that the condition number $\kappa$ grows rapidly with $N$ also, but there is as yet no precise characterization.

Another issue for many methods is the lack of a rigorous theory or in some cases, any theory. The mock-Chebyshev subsampling algorithm is again the exception as it falls under the umbrella of standard Chebyshev interpolation. It seems likely that the Gaussian RBF method can be rigorously justified by combining the theory of polynomial interpolation on a shrinking interval given in Part II with the Platte-Driscoll theory for Gaussian RBF interpolation. Rakhmanov has developed a theory for overdetermined least-squares that shows that this method shrinks the Runge Zone, but cannot eliminate it for any finite ratio of P/N. There is a limited theory for Fourier Extension that proves that such extensions are always possible and exponentially convergent when $f(x)$ is known analytically outside $[-1,1]$, an "Extension of the First Kind", but no theory for when $f(x)$ is known only from its samples on $[-1,1]$, an "Extension of the Third Kind". The $L_1$-minimizing method does not have a rigorous theory in the anti-Runge context though there has been some development of similar ideas in image processing.

Finally, many of the anti-Runge methods are very inefficient compared to standard interpolation with $P$ points. The mock-Chebyshev scheme, otherwise so desirable, generates only an approximation of degree $\sqrt{P}$ and discards most of the grid point values of $f(x)$. The Gegenbauer and overdetermined least squares both shrink the Runge Zone only when $N$ is much smaller than $P$. Fourier Extension in practice usually needs $P=2N$ and a costly SVD factorization. The Gaussian RBF has $P=N$, and thus is relatively attractive if sufficient accuracy can be obtained before the ill-conditioning, growing exponentially fast with $N$, ruins it all. Platte's intriguing $L_1$ method is still unpublished, and so its numerical efficiency is difficult to assess.

Table 1: Single-Domain Anti-Runge.

| Method | Runge Domain | $\kappa$ | Points $P$ & Degree $N$ |
|---|---|---|---|
| Tikhonov | ? | Large | $P=N$ |
| Gegenbauer | Shrinks | Large | $P>>N$ |
| Overdetermined | Shrinks | Large | $P>>N$ |
| Mock-Chebyshev | $[-1,1]$ | $\mathcal{O}(1)$ | Inefficient: $N\sim\mathcal{O}(\sqrt{P})$ |
| Gaussian RBF | $[-1,1]$ (?) | $\exp(q\sqrt{N})$ | $N=P$ |
| Fourier Extension | $[-1,1]$ (?) | Large | Typically $P=2N$ |
| Platte's $L_1/L_2$ Method | ? | Large | $P<N$ |

Note: $\kappa$ is the matrix condition number, $P$ is the number of samples of $f(x)$ and $N$ is the degree of the approximation.

Other than labeling some methods as "worth-further-development" and others "not-worth-development", it is difficult to make definitive judgments about the relative merits of the algorithms. The first reason is that most criteria of merit — accuracy, cost, theory, ill-conditioning and ease-of-programming — are all problem-dependent. (The importance of theory is perhaps "culturally-dependent" in the sense that a mathematician is much more perturbed by a lack of theory than an engineer.) When one only needs a few one-dimensional approximations, any method will yield an answer in seconds and ease-of-programming is paramount. When applying these ideas in tensor-product form to generate multi-dimensional approximations, cost is much more important. Similarly, when the approximations will used as subroutines, called millions of times, the cost of *evaluating* an approximation may be vastly more important than the cost of *obtaining* the coefficients of the polynomial. When $f(x)$ is smooth, ill-conditioning is a minor problem because $N$ is small; for complicated $f(x)$, the need for large $N$ makes ill-conditioning a much more serious and perhaps insuperable difficulty.

The second reason to suspend definitive judgments is that multi-interval methods await full accounting in Part II. Some interesting single-interval ideas have only been sketched and still await definitive treatment. This short article is necessarily an interim account written on foolscap rather than a magisterial tome laser-etched into copper.

Nonetheless, it is gratifying that there has been so much recent progress in a problem more than a century old. Single-interval methods are good but not great. The Gegenbauer Regularization method, though only a "Runge-Zone-shrinker" rather than a tactic that completely eliminates the Runge Phenomenon, has been used successfully in a wide variety of applications.

The Gaussian RBF and Fourier Extension methods are sufficiently promising to warrant further studies. However, the limitations of ill-conditioning and a lack of theory motivate the multidomain methods studied in Part II.

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### References

[1] M. Berzins, Adaptive polynomial interpolation on evenly spaced meshes, SIAM Rev., 49 (2007), 604–627.

[2] J. P. Boyd, Defeating the Runge phenomenon for equispaced polynomial interpolation via Tikhonov regularization, Appl. Math. Lett., 5 (1992), 57–59.

[3] J. P. Boyd, Chebyshev and Fourier Spectral Methods, Dover, Mineola, New York, 2d ed., 2001. pp. 665.

[4] J. P. Boyd, A comparison of numerical algorithms for Fourier Extension of the First, Second and Third Kinds, J. Comput. Phys., 178 (2002), 118–160.

[5] J. P. Boyd, Trouble with Gegenbauer reconstruction for defeating Gibbs' phenomenon: Runge phenomenon in the diagonal limit of Gegenbauer polynomial approximations, J. Comput. Phys., 204 (2005), 253–264.

[6] J. P. Boyd, Defeating Gibbs Phenomenon in Fourier and Chebyshev spectral methods for solving differential equations, in Gibbs Phenomenon, A. Jerri, ed., Sampling Publishing, Potsdam, New York, 2007.

[7] J. P. Boyd, Exponentially accurate Runge-Free approximation of non-periodic functions from samples on an evenly-spaced grid, Appl. Math. Lett., 20 (2007), 971–975.

[8] J. P. Boyd and S. Parhar, Gaussian radial basis functions on a finite interval, Comput. Math. Applic., (2008). submitted.

[9] J. P. Boyd and F. Xu, Divergence (Runge Phenomenon) for least-squares polynomial approximation on an equispaced grid, Appl. Math. Comput., (2007). submitted.

[10] O. Bruno, Y. Han, and M. M. Pohlman, Accurate, high-order representation of complex three-dimensional surfaces via Fourier-continuation analysis, J. Comput. Phys., (2007). in press.

[11] J. F. Epperson, On the Runge example, Amer. Math. Monthly, 94 (1987), 329–341.

[12] G. F. Fasshauer, Meshfree Approximation Methods with MATLAB, Interdisciplinary Mathematical Sciences, World Scientific Publishing Company, Singapore, 2007.

[13] A. Gelb, Parameter optimization and reduction of roundoff error for the Gegenbauer reconstruction method, J. Sci. Comput., 20 (2004), 433–459.

[14] A. Gelb and Z. Jackiewicz, Determing analyticity for parameter optimization of the Gegenbauer reconstruction method, SIAM J. Sci. Comput., 27 (2005), 1014–1031.

[15] A. Gelb and J. Tanner, Robust reprojection methods for the resolution of Gibbs' Phenomenon, Appl. Comput. Harmonic Anal., 20 (2006), 3–25.

[16] D. Gottlieb and S. Gottlieb, Spectral methods for compressible reactive flows, Comptes Rend. Mec., 333 (2005), 3–16.

[17] D. Gottlieb and C.-W. Shu, Resolution properties of the Fourier method for discontinuous waves, in: C. Bernardi and Y. Maday, (Eds.), Analysis, Algorithms and Applications of Spec-

tral and High Order Methods for Partial Differential Equations, Selected Papers from the International Conference on Spectral and High Order Methods (ICOSAHOM '92), Le Corum, Montpellier, France, 22-26 June 1992, North-Holland, Amsterdam, 1994, pp. 27–38. Also in Comput. Meths. Appl. Mech. Engrg., vol. 116.

[18] D. Gottlieb and C.-W. Shu, On the Gibbs phenomenon IV: Recovering exponential accuracy in a subinterval from a Gegenbauer partial sum of a piecewise analytic function, Math. Comput., 64 (1995), 1081–1095.

[19] D. Gottlieb and C.-W. Shu, On the Gibbs phenomenon V: Recovering exponential accuracy from collocation point values of a piecewise analytic function, Numerische Mathematik, 71 (1995), 511–526.

[20] D. Gottlieb and C.-W. Shu, On the Gibbs phenomenon 3: Recovering exponential accuracy in a subinterval from a spectral partial sum of a piecewise analytic function, SIAM J. Numer. Anal., 33 (1996), 280–290.

[21] D. Gottlieb and C.-W. Shu, On the Gibbs phenomenon and its resolution, SIAM Rev., 39 (1997), 644–668.

[22] D. Gottlieb, C.-W. Shu, A. Solomonoff, and H. Vandeven, On the Gibbs phenomenon I: Recovering exponential accuracy from the Fourier partial sum of a nonperiodic analytic function, J. Comput. Appl. Math., 43 (1992), 81–98.

[23] Z. Jackiewicz, Determination of optimal parameters for the Chebyshev-Gegenbauer reconstruction method, SIAM J. Sci. Comput., 25 (2003), 1187–1198.

[24] J. Jung and B. D. Shizgal, Generalization of the inverse polynomial reconstruction method in the resolution of the Gibbs phenomenon, J. Comput. Appl. Math., 172 (2004), 131–151.

[25] J. Jung and B. D. Shizgal, Inverse polynomial reconstruction method in the resolution of the Gibbs phenomenon, J. Sci. Comput., 25 (2005), 367–399.

[26] J. Jung and B. D. Shizgal, On the numerical convergence with the inverse polynomial reconstruction method for the resolution of the Gibbs phenomenon, J. Comput. Phys., 224 (2007), 477–488.

[27] C. Meray, Observations sur la légitime de l'interpolation, Ann. Sci. Ec. Norm. Super., 1 (1884), 165–176.

[28] C. Meray, Nouvelles examples d'interpolation illusoires, Bull. Sci. Math., 20 (1896), 266–270.

[29] T. M. Mills and S. J. Smith, Lagrange interpolation polynomials based equidistant nodes, Math. Scientist, 16 (1991), 107–117.

[30] R. Pasquetti, On inverse methods for the resolution of the Gibbs phenomenon, J. Comput. Appl. Math., 170 (2004), 303–315.

[31] R. B. Platte and T. A. Driscoll, Polynomials and potential theory for Gaussian radial basis function interpolation, SIAM J. Numer. Anal., 43 (2005), 750–766.

[32] E. A. Rakhmanov, Bounds for polynomials with a unit discrete norm, Ann. Math., 165 (2006), 55–88.

[33] C. Runge, Über empirische Funktionen und die Interpolation zwischen äquidistanten Ordinaten, Zeitschrift fur Mathematik und Physik, 246 (1901), 224–243.

[34] B. D. Shizgal and J. H. Jung, Towards the resolution of the Gibbs phenomena, J. Comput. Appl. Math., 161 (2003), 41–65.

[35] L. N. Trefethen and J. A. C. Weideman, Two results on polynomial interpolation in equally spaced points, J. Approx. Theor., 65 (1991), 247–260.

[36] G. Wahba, Spline Models for Observational Data, vol. 59 of CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, 1990. pp. 169.