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An Iterative Domain Decomposition Algorithm for the Grad(div) Operator

E. Ahusborde[1], M. Azaïez[1], M. O. Deville[2] and E. H. Mund[3,*]

[1] *TREFLE, (UMR CNRS 8505), Ecole Nationale Supérieure de Chimie et de Physique de Bordeaux, Pessac, France.*
[2] *Laboratory of Computational Engineering, Ecole Polytechnique Fédérale de Lausanne, 1015 Lausanne, Switzerland.*
[3] *Unité de Thermodynamique, UCL, 1348 Louvain-La-Neuve, Belgium.*

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Abstract. This paper describes an iterative solution technique for partial differential equations involving the **grad**(div) operator, based on a domain decomposition. Iterations are performed to solve the solution on the interface. We identify the transmission relationships through the interface. We relate the approach to a Steklov-Poincaré operator, and we illustrate the performance of technique through some numerical experiments.

AMS subject classifications: 65N55, 65F10

Key words: Domain decomposition, **grad**(div) operator, stable approximation, iterative substructuring method, Steklov-Poincaré operator.

1 Introduction

The purpose of this paper is to take benefit of recent advances in the use of spectral methods for the stable approximation of the **grad**(div) operator in tensorised Cartesian domains to solve a large class of problems involving this operator in more sophisticated domains that can be viewed as unions of tensorised Cartesian domains [2]. More precisely, we want to solve the symmetric linear elliptic boundary value problem : *for a given data* $\mathbf{f}$*, find* $\mathbf{u}$ *solution to*

$$-\nabla(\nabla\cdot\mathbf{u})+\alpha^2\mathbf{u}=\mathbf{f}, \quad \text{in } \Omega, \tag{1.1}$$

$$\mathbf{u}\cdot\mathbf{n}=0, \quad \text{on } \partial\Omega, \tag{1.2}$$

*Corresponding author. *Email addresses:* ahusborde@enscpb.fr (E. Ahusborde), azaiez@enscpb.fr (M. Azaïez), Michel.Deville@epfl.ch (M. O. Deville), emund@ulb.ac.be (E. H. Mund)

by a domain decomposition technique using an iterative algorithm between sub-domains in the spirit of the well-known Dirichlet-Neumann algorithm introduced for the Laplacian operator by Quarteroni (see [5] and the references therein). Here, and in the rest of the paper, $\Omega \subset \mathbb{R}^2$ is a bounded open domain with Lipschitzian border and $\mathbf{n}$ denotes the outer unit normal along the boundary. The constant α is given arbitrarily.

The first question we address in Section 2 is the identification of transmission conditions for the vector operator, on the 'skeleton' of the decomposition, that is on the interface between adjacent sub-domains. This is followed in Section 2.1 by the formulation of an iterative substructuring algorithm. In Sections 2.2 and 2.3 we relate the ensuing problem on the skeleton to a Steklov-Poincaré operator and we give some numerical results. Finally Section 3 concludes the paper.

2 A domain decomposition for the grad(div) operator

We assume that the domain Ω is partitioned into a set of non-overlapping and conforming sub-domains $\Omega_i, i=1\cdots,I$ (see [3]) and for simplicity we assume $I=2$. Let $\overline{\Gamma}:=\overline{\Omega_1}\cap\overline{\Omega_2}$ denote the interface between the two sub-domains considered in our analysis and shown on Fig. 1. Γ will be called the skeleton of the decomposition in the sequel of the paper. We shall also assume that Γ is a Lipschitz one-dimensional manifold.

We call $\mathbf{u}_i$ the restriction to sub-domain Ω_i, $i=1,2$, of the solution $\mathbf{u}$ to the problem (1.1)-(1.2), and by $\mathbf{n}_i$ the outward oriented normal vector on $\partial\Omega_i\cap\Gamma$. For convenience we will set $\mathbf{n}=\mathbf{n}_1$.

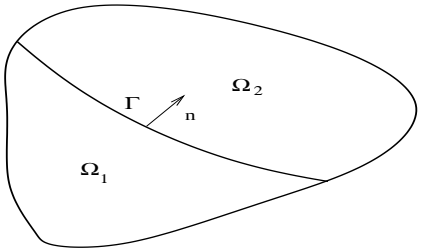

Figure 1: Non-overlapping partition of the domain Ω into two sub-domains.

One can easily prove that the problem (1.1)-(1.2) can be reformulated into the equivalent multi-domain set of local coupled problems (see [5]):

$$-\nabla(\nabla\cdot\mathbf{u}_1)+\alpha^2\mathbf{u}_1=\mathbf{f}, \quad \text{in } \Omega_1, \tag{2.1}$$

$$-\nabla(\nabla\cdot\mathbf{u}_2)+\alpha^2\mathbf{u}_2=\mathbf{f}, \quad \text{in } \Omega_2, \tag{2.2}$$

$$\mathbf{u}_1\cdot\mathbf{n}=0, \quad \text{on } \partial\Omega_1\cap\partial\Omega, \tag{2.3}$$

$$\mathbf{u}_2\cdot\mathbf{n}=0, \quad \text{on } \partial\Omega_2\cap\partial\Omega, \tag{2.4}$$

$$\mathbf{u}_1\cdot\mathbf{n}=\mathbf{u}_2\cdot\mathbf{n}, \quad \text{on } \Gamma, \tag{2.5}$$

$$\mathrm{div}\mathbf{u}_1=\mathrm{div}\mathbf{u}_2, \quad \text{on } \Gamma. \tag{2.6}$$

Eqs. (2.5) and (2.6) are the transmission conditions for $\mathbf{u}_1$ and $\mathbf{u}_2$ on Γ. These conditions are for the **grad**(div) operator the corresponding transmission conditions of the Dirichlet-Neumann conditions for the Laplacian operator (see [5]). Notice that (2.6) must be taken in weak sense.

Note that the convergence proof should be similar to that already performed for finite element discretization (see [5], Sections 5.5 and 4.1.2).

2.1 Iterative substructuring method

There are several ways to uncouple Eqs. (2.1)-(2.6). The one we choose consists in the introduction of a sequence of subproblems in Ω_1 and Ω_2 for which conditions (2.5) and (2.6) provide the missing condition on Γ. This can be performed in the following iterative way: *Starting from a given λ^0 on the skeleton, for $k \geq 0$ we solve*

$$\begin{aligned} &-\nabla(\nabla\cdot\mathbf{u}_1^{k+1})+\alpha^2\mathbf{u}_1^{k+1}=\mathbf{f}, && \text{in } \Omega_1,\\ &\mathbf{u}_1^{k+1}\cdot\mathbf{n}=0, && \text{on } \partial\Omega_1\cap\partial\Omega,\\ &\mathbf{u}_1^{k+1}\cdot\mathbf{n}=\lambda^k, && \text{on } \Gamma. \end{aligned} \tag{2.7}$$

This is followed by:

$$\begin{aligned} &-\nabla(\nabla\cdot\mathbf{u}_2^{k+1})+\alpha^2\mathbf{u}_2^{k+1}=\mathbf{f}, && \text{in } \Omega_2,\\ &\mathbf{u}_2^{k+1}\cdot\mathbf{n}=0, && \text{on } \partial\Omega_2\cap\partial\Omega,\\ &\mathrm{div}\mathbf{u}_2^{k+1}=\mathrm{div}\mathbf{u}_1^{k+1} && \text{on } \Gamma, \end{aligned} \tag{2.8}$$

and we update

$$\lambda^{k+1}=\mathbf{u}_2^{k+1}\cdot\mathbf{n} \qquad \text{on } \Gamma. \tag{2.9}$$

In order to accelerate convergence of the iterative process the transmission condition (2.9) will be replaced by a relaxed version

$$\lambda^{k+1}=\omega\mathbf{u}_2^{k+1}\cdot\mathbf{n}+(1-\omega)\lambda^k \qquad \text{on } \Gamma, \tag{2.10}$$

adopted for our numerical investigation [4,5]. The choice of some optimal value of ω will be discussed in Section 2.3.

Convergence of the numerical process obtains when, after completion of the step (2.8), $|\mathbf{u}_2^{k+1}\cdot\mathbf{n}-\mathbf{u}_1^{k+1}\cdot\mathbf{n}|\leq\epsilon$, with ϵ a given error level.

Now, we shall show that the transmission conditions (2.5) and (2.6) lead to a problem on the skeleton involving the Steklov-Poincaré operator (see [1]). The unknown of this interface problem is the trace of the normal component of the vector solution to (1.1)-(1.2).

2.2 Steklov-Poincaré interface equation

Let λ denote again the unknown value of the trace of $\mathbf{u}\cdot\mathbf{n}$ along Γ. We consider the two following problems : *Find* $\mathbf{w}_i$, $(i=1,2)$ *solution to*

$$\begin{aligned} &-\nabla(\nabla\cdot\mathbf{w}_i)+\alpha^2\mathbf{w}_i=\mathbf{f}, && \text{in } \Omega_i,\\ &\mathbf{w}_i\cdot\mathbf{n}=0, && \text{on } \partial\Omega_i\cap\partial\Omega,\\ &\mathbf{w}_i\cdot\mathbf{n}=\lambda, && \text{on } \Gamma. \end{aligned} \tag{2.11}$$

We can state that

$$\mathbf{w}_i=\mathbf{u}_i^0+\mathbf{u}_i^*,$$

where we have defined $\mathbf{u}_i^0$ and $\mathbf{u}_i^*$ to be solutions to

$$\begin{aligned} &-\nabla(\nabla\cdot\mathbf{u}_i^0)+\alpha^2\mathbf{u}_i^0=\mathbf{0}, && \text{in } \Omega_i,\\ &\mathbf{u}_i^0\cdot\mathbf{n}=0, && \text{on } \partial\Omega_i\cap\partial\Omega,\\ &\mathbf{u}_i^0\cdot\mathbf{n}=\lambda, && \text{on } \Gamma, \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} &-\nabla(\nabla\cdot\mathbf{u}_i^*)+\alpha^2\mathbf{u}_i^*=\mathbf{f}, && \text{in } \Omega_i,\\ &\mathbf{u}_i^*\cdot\mathbf{n}=0, && \text{on } \partial\Omega_i\cap\partial\Omega,\\ &\mathbf{u}_i^*\cdot\mathbf{n}=0, && \text{on } \Gamma. \end{aligned} \tag{2.13}$$

For $i=1,2$, $\mathbf{u}_i^0$ is the extension of λ from Γ into Ω_i. It will be denoted hereafter as $\mathbf{H}_i\lambda$. We will also denote $\mathbf{u}_i^*$ by $\mathcal{G}_i\mathbf{f}$ to recall that the quantities $\mathbf{u}_i^*$ relate to the RHS of (2.13) in the two sub-domains.

If we proceed formally and compare (1.1)-(1.2) with (2.11), it follows that

$$\mathbf{w}_i=\mathbf{u}_i, \quad \text{for } i=1,2 \qquad \text{if and only if} \qquad \mathrm{div}\mathbf{w}_1=\mathrm{div}\mathbf{w}_2 \text{ on } \Gamma. \tag{2.14}$$

Using this relationship one can show that λ is the trace of the normal component of the solution if and only if it satisfies the Steklov-Poincaré interface equation

$$S\check{}=\chi \quad \text{on } \Gamma, \tag{2.15}$$

where

$$\chi:=\mathrm{div}\mathcal{G}_1\mathbf{f}-\mathrm{div}\mathcal{G}_2\mathbf{f},$$

and S is the Steklov-Poincaré operator formally defined as

$$S\check{}:=S_1\check{}-S_2\check{}=\mathrm{div}\mathbf{H}_1\lambda-\mathrm{div}\mathbf{H}_2\lambda.$$

Notice that the Steklov-Poincaré operator S has been built starting from the transmission condition (2.5). A dual version of this algorithm would start from the transmission condition (2.6).

As demonstrated for the Laplacian operator (see [5]), the iterative algorithm (2.7)-(2.8) with the transmission condition (2.9) is nothing else than the S_1-preconditioned Richardson iterative algorithm applied to (2.15).

2.3 Numerical results

Up to this point we didn't specify the discretization technique used to solve the local problems in the domain decomposition approach. Any stable numerical technique could be applied like FEM, SEM, etc. Since, in a series of studies on the spectral properties of the -**grad**(div) operator, we exhibited the stability properties of the staggered grid Legendre spectral element $X_N := \mathbb{P}_N \otimes \mathbb{P}_{N-1} \times \mathbb{P}_{N-1} \otimes \mathbb{P}_N$ on the reference square $\widehat{\Omega} :=]-1,+1[^2$, it seemed quite natural to use this approximation tool to implement the method (see for details [2]). Here $\mathbb{P}_N$ denotes the set of polynomials of degree less or equal to N. This spectral element is used here to solve the local problems on the two sub-domains Ω_i mapped onto $\widehat{\Omega}$.

To test the efficiency of the iterative algorithm outlined above, we solved problem (1.1)-(1.2) on two different domains with data and boundary conditions such that the analytical solution is given by:

$$u_x(x,y) = 2\sin(x)\cos(y),$$
$$u_y(x,y) = \cos(x)\sin(y).$$

In both test cases the coefficient α in Eq. (1.1) was set equal to 1. The numerical experiments have been conducted with an error criterion $\epsilon = 10^{-12}$. Several values of the relaxation factor were tested that indicated an optimal value equal to 0.5, used in all subsequent computations.

A first numerical test was made on the rectangle $\Omega :=]-1,+3[\times]-1,+1[$, subdivided into two square sub-domains. The exact solution $\mathbf{u} = (u_x, u_y)$ is approximated by $\mathbf{u}_N \in X_N$. The $(L^2(\Omega))^2$–norm $||\mathbf{u} - \mathbf{u}_N||$ error behavior with respect to N is plotted on Fig. 2 with a linear-logarithmic scale. As long as we approximate analytical solutions by high degree polynomials, we expect an exponential error convergence to zero. The curve on Fig. 2 confirms this expectation.

Table 1 gives the number of iterations needed to reach convergence in the iterative process between sub-domains, as a function of the polynomial degree N. We remark that the iteration number is almost independent on the polynomial degree. This seems to imply that the S_1 preconditioner of the Steklov-Poincaré operator is nearly optimal, having a convergence radius independent of N. As seen later on the property is closely linked to the topology of the domain decomposition.

Table 1: Number of iterations between sub-domains v.s. the polynomial degree N.

N	4	8	12	16	20
Number of iterations	9	8	8	7	7

In our second numerical experiment, the domain Ω is L-shaped.

Fig. 3 exhibits the L-shaped domain partitioned into 3 sub-domains with some Gauss-Lobatto-Legendre (GLL) grid corresponding to $N = 20$.

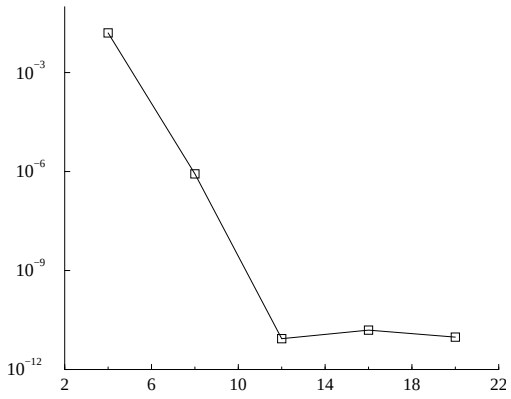

Figure 2: L^2 error norm $||\mathbf{u}-\mathbf{u}_N||_{(L^2(\Omega))^2}$ as a function of the polynomial degree on the rectangle domain.

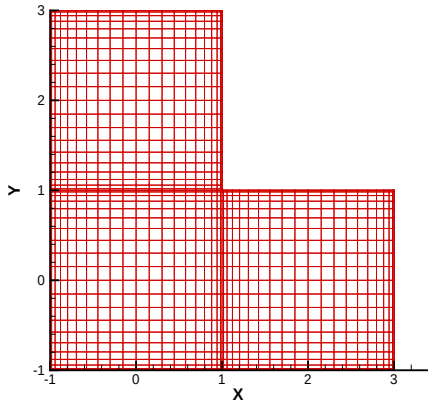

Figure 3: L-shaped domain: Anatomy of the decomposition.

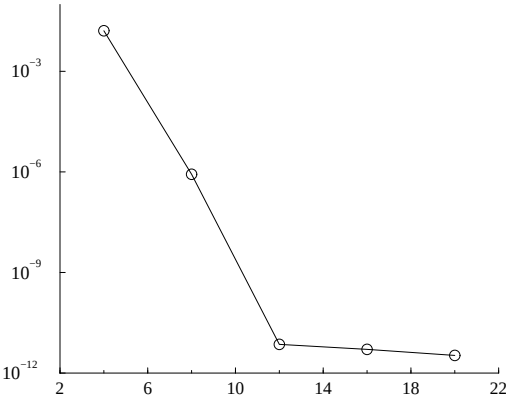

Figure 4: L^2 error norm $||\mathbf{u}-\mathbf{u}_N||_{(L^2(\Omega))^2}$ as a function of the polynomial degree N on the L-shaped domain.

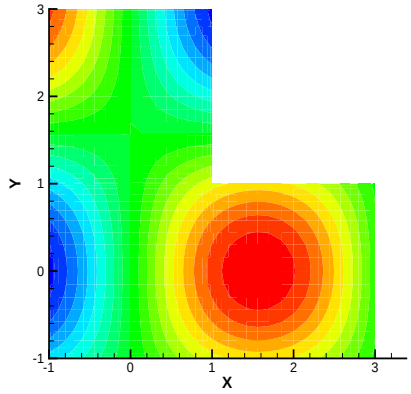

Figure 5: Isovalues of the approximation u_{xN} with $N=20$.

Fig. 4 displays the $(L^2(\Omega))^2$-norm $||\mathbf{u}-\mathbf{u}_N||$ error behavior with respect to N. Again the convergence is exponential as expected, the error reaching round-off for polynomial degrees larger or equal to 12.

Fig. 5 displays the isovalues of the approximation u_{xN} with $N=20$. One notes the continuity of this velocity component through the interfaces between the sub-domains.

Finally, Table 2 displays the number of iterations between the three sub-domains to reach the convergence, as a function of N. We remark that in this case the number of iterations is slightly dependent on the polynomial degree N. By slightly we mean something of the order $\mathcal{O}(N^\tau)$ with $0<\tau<1$. Note that this conclusion is not linked to the choice of ϵ. Relating this behavior to the Steklov-Poincaré preconditioner we conclude that the operator S_1 is optimal for domain decompositions in 'slices' but not for decompositions with skeleton cross points as demonstrated for the Laplace operator.

Table 2: Number of iterations between sub-domains v.s. the polynomial degree N.

N	4	8	12	16	20
Number of iterations	30	34	36	38	42

3 Conclusion

In this paper we generalized a classical domain decomposition approach to the vector equations involving the **grad**(div) operator. In particular we have identified the transmission conditions for the normal velocity component and for the divergence of the velocity along the decomposition skeleton. We presented an iterative substructuring algorithm based on these transmission conditions and we restated the problem as an interface problem involving a Steklov-Poincaré operator. Numerical results obtained using an efficient stable approximation closely related to the **grad**(div) operator illustrate the efficiency of the iterative approach for the domain decomposition.

As a final comment let us stress the fact that the work done so far is only a preliminary step towards a more sophisticated fully spectral element approach.

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