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Composite Laguerre-Legendre Pseudospectral Method for Exterior Problems

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Abstract. In this paper, we propose a composite Laguerre-Legendre pseudospectral method for exterior problems with a square obstacle. Some results on the composite Laguerre-Legendre interpolation, which is a set of piecewise mixed interpolations coupled with domain decomposition, are established. As examples of applications, the composite pseudospectral schemes are provided for two model problems. The convergence of proposed schemes are proved. Efficient algorithms are implemented. Numerical results demonstrate the spectral accuracy in space of this new approach.

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1 Introduction

Many practical problems require solving partial differential equations defined on exterior domains. Considerable progress has been made in spectral and pseudospectral methods for unbounded domains. A direct and commonly used approach is based on Hermite and Laguerre orthogonal approximations, see, e.g., [4–7,9,17,18,20]. Some authors also developed the mixed Laguerre-Legendre spectral and pseudospectral methods for an infinite strip, see [14, 19]. A challenging problem is how to design spectral and pseudospectral schemes for exterior problems. However, so far, there have been only few literatures concerning the spectral method for exterior problems, see [2, 10, 21].

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We now consider exterior problems with a square obstacle $\Omega_0=\{\ (x,y)\mid -1\le x,y\le 1\}$. Its boundary $\Gamma=\bigcup_{j=1}^{4}\Gamma_j$, where

$$\begin{aligned}
&\Gamma_1=\{\ (x,y)\mid x=1,\ |y|\le 1\}, &&\Gamma_2=\{\ (x,y)\mid |x|\le 1,\ y=1\ \},\\
&\Gamma_3=\{\ (x,y)\mid x=-1,\ |y|\le 1\}, &&\Gamma_4=\{\ (x,y)\mid |x|\le 1,\ y=-1\}.
\end{aligned}$$

In this case, we divide the unbounded domain $\Omega=R^2/\Omega_0$ into eight subdomains $\Omega_j,1\le j\le 8$. In other words, $\Omega=\bigcup_{j=1}^{8}\Omega_j$, $1\le j\le 8$, where

$$\begin{aligned}
&\Omega_1=\{(x,y)\mid x>1,\ |y|\le 1\}, &&\Omega_2=\{(x,y)\mid x>1,\ y>1\},\\
&\Omega_3=\{(x,y)\mid |x|\le 1,\ y>1\}, &&\Omega_4=\{(x,y)\mid x<-1,\ y>1\},\\
&\Omega_5=\{(x,y)\mid x<-1,\ |y|\le 1\}, &&\Omega_6=\{(x,y)\mid x<-1,\ y<-1\},\\
&\Omega_7=\{(x,y)\mid |x|\le 1,\ y<-1\}, &&\Omega_8=\{(x,y)\mid x>1,\ y<-1\}.
\end{aligned}$$

The common boundary of adjacent subdomains Ω_j and Ω_{j+1} are denoted by $\Gamma_{j,j+1}$. In particular, $\Gamma_{8,9}=\Gamma_{8,1}$. Namely,

$$\begin{aligned}
&\Gamma_{12}=\{(x,y)\in\Omega,\ x>1,\ y=1\}, &&\Gamma_{23}=\{(x,y)\in\Omega,\ x=1,\ y>1\},\\
&\Gamma_{34}=\{(x,y)\in\Omega,\ x=-1,\ y>1\}, &&\Gamma_{45}=\{(x,y)\in\Omega,\ x<-1,\ y=1\},\\
&\Gamma_{56}=\{(x,y)\in\Omega,\ x<-1,\ y=-1\}, &&\Gamma_{67}=\{(x,y)\in\Omega,\ x=-1,\ y<-1\},\\
&\Gamma_{78}=\{(x,y)\in\Omega,\ x=1,\ y<-1\}, &&\Gamma_{81}=\{(x,y)\in\Omega,\ x>1,\ y=-1\}.
\end{aligned}$$

Recently, the authors developed the composite Lagurre-Legendre approximation with its applications to exterior problems with the obstacle Ω_0, see [13]. But, the pseudospectral method is more preferable in actual computation. In particular, it is convenient to match numerical solutions on the common boundaries $\Gamma_{j,j+1},1\le j\le 8$, of adjacent subdomains, and is easy to deal with nonlinear terms. Moreover, we can use the pseudospectral method coupled with finite element methods, for various exterior problems with more complex geometry.

In this paper, we investigate the composite Laguerre-Legendre pseudospectral method coupled with domain decomposition for exterior problems with the obstacle Ω_0. We shall use the mixed Laguerre-Legendre interpolations on the subdomains $\Omega_j,j=1,3,5,7$, and the two-dimensional Laguerre interpolations on the subdomains $\Omega_j,j=2,4,6,8$. We also introduce certain specific basis functions, induced by the scaled Lagurre functions and the Legendre polynomials, so that the global numerical solutions belong to the space $H^1(\Omega)$ and possess high accuracy. In order to derive precise estimates of numerical solutions, we develop the composite Laguerre-Legendre interpolations on the whole domain Ω, which play important roles in the related pseudospectral methods for various exterior problems. As examples of applications, we consider two model exterior problems. The convergence of proposed pseudospectral schemes are proved. Efficient implementations are described. The corresponding linear discrete systems are symmetric and sparse, which can be resolved easily. Especially, they are suitable for parallel computation. Numerical results demonstrate the spectral accuracy in space of this new approach and confirm well the theoretical analysis. The approximation results and some techniques developed in this paper are also applicable to many other problems on unbounded domains, as well as exterior problems.

This paper is organized as follows. In the next section, we establish the basic results on the composite Laguerre-Legendre interpolation. In Section 3, we propose the composite pseudospectral schemes for model problems and derive sharp error estimates of numerical solutions. In Section 4, we describe the numerical implementations and present some numerical results. The final section is for concluding remarks.

2 Composite Laguerre-Legendre interpolation

In this section, we derive some basic results on the composite Laguerre-Legendre interpolation.

2.1 Laguerre-Gauss-Radau interpolation

Let $\Lambda = \{ x \mid 0 < x < \infty \}$ and $\chi(x)$ be a certain weight function. For integer $r \geq 0$,

$$H^r_\chi(\Lambda) = \{ u \mid u \text{ is measurable on } \Lambda \text{ and } ||u||_{r,\chi,\Lambda} < \infty \},$$

equipped with the following inner product, semi-norm and norm,

$$\begin{aligned}(u,v)_{r,\chi,\Lambda} &= \sum_{0\leq k\leq r} \int_{\Lambda_0} \partial_x^k u(x) \partial_x^k v(x) \chi(x) dx, \\ |u|_{r,\chi,\Lambda} &= \int_\Lambda (\partial_x^r u(x))^2 \chi(x) dx, \quad ||u||_{r,\chi,\Lambda} = (u,u)^{\frac{1}{2}}_{r,\chi,\Lambda}.\end{aligned}$$

In particular, $H^0_\chi(\Lambda) = L^2_\chi(\Lambda)$, with the inner product $(u,v)_{\chi,\Lambda}$ and the norm $||u||_{\chi,\Lambda}$. We omit the subscript χ in the notations whenever $\chi(x) \equiv 1$.

Let $\omega_{\alpha,\beta}(x) = x^\alpha e^{-\beta x}$, $\alpha > -1, \beta > 0$. Especially, $\omega_\beta(x) = \omega_{0,\beta}(x) = e^{-\beta x}$. The scaled Laguerre polynomial of degree l is defined by

$$\mathcal{L}_l^{(\beta)}(x) = \frac{1}{l!} e^{\beta x} \partial_x^l (x^l e^{-\beta x}).$$

To design proper pseudospectral method for exterior problems, we shall use the orthogonal system of Laguerre functions, defined by

$$\tilde{\mathcal{L}}_l^{(\beta)}(x) = e^{-\frac{1}{2}\beta x} \mathcal{L}_l^{(\beta)}(x), \quad l = 0,1,2,\cdots. \tag{2.1}$$

The set of $\tilde{\mathcal{L}}_l^{(\beta)}(x)$ is a complete $L^2(\Lambda)-$ orthogonal system, see [15].

Let $\Lambda_1 = \{ x \mid 1 < x < \infty \}$ and $\omega^1_{\alpha,\beta}(x) = \omega_{\alpha,\beta}(x-1)$. In particular, $\omega^1_\beta(x) = \omega_\beta(x-1)$. For any positive integer N, $\mathcal{P}_N(\Lambda_1)$ stands for the set of all polynomials of degree at most

N. Let $\xi^{(\beta)}_{R,N,\Lambda_1,j}$, $0\le j\le N$, be the zeros of polynomial $(x-1)\partial_x\mathcal{L}^{(\beta)}_{N+1}(x-1)$, which are arranged in ascending order. Denote by $\omega^{(\beta)}_{R,N,\Lambda_1,j}$, $0\le j\le N$, the corresponding Christoffel numbers such that

$$\int_{\Lambda_1}\phi(x)\omega^1_\beta(x)dx=\sum_{j=0}^{N}\phi\left(\xi^{(\beta)}_{R,N,\Lambda_1,j}\right)\omega^{(\beta)}_{R,N,\Lambda_1,j},\quad \forall\phi\in\mathcal{P}_{2N}(\Lambda_1). \tag{2.2}$$

Furthermore, let

$$\tilde{\xi}^{(\beta)}_{R,N,\Lambda_1,j}=\xi^{(\beta)}_{R,N,\Lambda_1,j},\quad \tilde{\omega}^{(\beta)}_{R,N,\Lambda_1,j}=e^{\beta\left(\xi^{(\beta)}_{R,N,\Lambda_1,j}-1\right)}\omega^{(\beta)}_{R,N,\Lambda_1,j}.$$

We also introduce the following discrete inner product and norm,

$$(u,v)_{N,\beta,\Lambda_1}=\sum_{j=0}^{N}u\left(\tilde{\xi}^{(\beta)}_{R,N,\Lambda_1,j}\right)v\left(\tilde{\xi}^{(\beta)}_{R,N,\Lambda_1,j}\right)\tilde{\omega}^{(\beta)}_{R,N,\Lambda_1,j},\quad ||u||_{N,\beta,\Lambda_1}=(u,u)^{\frac{1}{2}}_{N,\beta,\Lambda_1}.$$

Next, we set

$$\mathcal{Q}_{N,\beta}(\Lambda_1)=\left\{\,e^{-\frac{1}{2}\beta(x-1)}\psi\mid\psi\in\mathcal{P}_N(\Lambda_1)\right\}.$$

By virtue of (2.2), it is easy to show that

$$(\phi,\psi)_{\Lambda_1}=(\phi,\psi)_{N,\beta,\Lambda_1},\quad\forall\phi\in\mathcal{Q}_{m,\beta}(\Lambda_1),\quad\psi\in\mathcal{Q}_{2N-m,\beta}(\Lambda_1),\quad 0\le m\le 2N, \tag{2.3}$$

$$||\phi||_{N,\beta,\Lambda_1}=||\phi||_{\Lambda_1},\quad\forall\phi\in\mathcal{Q}_{N,\beta}(\Lambda_1). \tag{2.4}$$

Let $\bar{\Lambda}_1=\Lambda_1\cup\{x=1\}$. For any $u\in C(\bar{\Lambda}_1)$, the Laguerre-Gauss-Radau interpolation $\tilde{\mathcal{I}}_{R,N,\beta,\Lambda_1}u\in\mathcal{Q}_{N,\beta}(\Lambda_1)$ is determined uniquely by

$$\tilde{\mathcal{I}}_{R,N,\beta,\Lambda_1}u(\tilde{\xi}^{(\beta)}_{R,N,\Lambda_1,j})=u(\tilde{\xi}^{(\beta)}_{R,N,\Lambda_1,j}),\quad 0\le j\le N. \tag{2.5}$$

In the forthcoming discussions, we denote by c a generic positive constant which does not depend on M,N,β and any function. We know from [12] that if $u\in L^2(\Lambda_1)\cap C(\bar{\Lambda}_1)$, $\partial^r_x(e^{\frac{1}{2}\beta(x-1)}u)\in L^2_{\omega^1_{r,\beta}}(\Lambda_1)\cap L^2_{\omega^1_{r-1,\beta}}(\Lambda_1)$ with integer $r\ge 1$, then

$$\begin{aligned}||\tilde{\mathcal{I}}_{R,N,\beta,\Lambda_1}u-u||_{\Lambda_1}\le{}& c(\beta N)^{\frac{1-r}{2}}(\beta^{-1}||\partial^r_x(e^{\frac{1}{2}\beta(x-1)}u)||_{\omega^1_{r-1,\beta},\Lambda_1}\\ &+(1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}}||\partial^r_x(e^{\frac{1}{2}\beta(x-1)}u)||_{\omega^1_{r,\beta},\Lambda_1}).\end{aligned} \tag{2.6}$$

Moreover, we use (2.3) and (2.6) to derive that for any $\phi\in\mathcal{Q}_{N,\beta}(\Lambda_1)$ and integer $r\ge 1$,

$$\begin{aligned}&|(u,\phi)_{\Lambda_1}-(u,\phi)_{N,\beta,\Lambda_1}|=|(\tilde{\mathcal{I}}_{R,N,\beta,\Lambda_1}u-u,\phi)_{\Lambda_1}|\\ \le{}& c(\beta N)^{\frac{1-r}{2}}(\beta^{-1}||\partial^r_x(e^{\frac{1}{2}\beta(x-1)}u)||_{\omega^1_{r-1,\beta},\Lambda_1}\\ &+(1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}}||\partial^r_x(e^{\frac{1}{2}\beta(x-1)}u)||_{\omega^1_{r,\beta},\Lambda_1})||\phi||_{\Lambda_1}.\end{aligned} \tag{2.7}$$

Remark 2.1. If $u\in H^r_{(x-1)^r}(\Lambda_1)$, then the norm $||\partial^r_x(e^{\frac{1}{2}\beta(x-1)}u)||_{\omega^1_{r,\beta},\Lambda_1}$ is finite.

2.2 Legendre-Gauss-Lobatto interpolation

We now consider the Legendre-Gauss-Lobatto interpolation on the interval $I_1=\{\, y \mid |y|<1\}$. For integer $r\geq 0$, we define the space $H^r(I_1)$ and its norm $||u||_{r,I_1}$ in the usual way. The inner product and norm of $L^2(I_1)$ are denoted by $(u,v)_{I_1}$ and $||u||_{I_1}$, respectively.

We denote by $L_l(y)$ the Legendre polynomial of degree l. The set of all Legendre polynomials is a complete $L^2(I_1)$-orthogonal system.

For integer $M\geq 0$, $\mathcal{P}_M(I_1)$ stands for the set of all polynomials of degree at most M. The orthogonal projection P_{M,I_1}: $L^2(I_1)\rightarrow\mathcal{P}_M(I_1)$ is defined by

$$(P_{M,I_1}u-u,\phi)_{I_1}=0, \quad \forall\phi \in\mathcal{P}_M(I_1).$$

If $\left(1-y^2\right)^{\frac{r}{2}}\partial_y^r u\in L^2(I_1)$ with integer $r\geq 0$, then (cf. [8,11])

$$||P_{M,I_1}u-u||_{I_1}\leq cM^{-r}||\left(1-y^2\right)^{\frac{r}{2}}\partial_y^r u||_{I_1}. \tag{2.8}$$

Next, let $\zeta_{L,M,I_1,k}$ be the roots of polynomial $(1-y^2)\partial_y L_M(y), 0\leq k\leq M$, which are arranged in ascending order. The corresponding Christoffel numbers are denoted by $\rho_{L,M,I_1,k}, 0\leq k\leq M$. We also introduce the discrete inner product and norm as

$$(u,v)_{M,I_1}=\sum_{k=0}^{M}u(\zeta_{L,M,I_1,k})v(\zeta_{L,M,I_1,k})\rho_{L,M,I_1,k}, \quad ||u||_{M,I_1}=(u,u)_{M,I_1}^{\frac{1}{2}}.$$

We have that (cf. [3,6])

$$(\phi,\psi)_{M,I_1}=(\phi,\psi)_{I_1}, \quad \forall\,\phi\psi\in\mathcal{P}_{2M-1}(I_1), \tag{2.9}$$

$$||\phi||_{I_1}\leq ||\phi||_{M,I_1}\leq\sqrt{2+\frac{1}{M}}||\phi||_{I_1}, \quad \forall\,\phi\in\mathcal{P}_M(I_1). \tag{2.10}$$

Let $\bar{I}_1=I_1\cup\{y=\pm 1\}$. For any $u\in C(\bar{I}_1)$, the Legendre-Gauss-Lobatto interpolation $\mathcal{I}_{L,M,I_1}u\in\mathcal{P}_M(I_1)$ is determined uniquely by

$$\mathcal{I}_{L,M,I_1}u(\zeta_{L,M,I_1,k})=u(\zeta_{L,M,I_1,k}), \quad 0\leq k\leq M.$$

We have from (2.10) of [16] that if $u\in C(\bar{I}_1)$, $(1-y^2)^{\frac{r-1}{2}}\partial_y^r u\in L^2(I_1)$ and integer $r\geq 1$, then

$$||\mathcal{I}_{L,M,I_1}u-u||_{I_1}\leq cM^{-r}||(1-y^2)^{\frac{r-1}{2}}\partial_y^r u||_{I_1}. \tag{2.11}$$

Finally, we use (2.8)-(2.11) to deduce that for any $\phi\in\mathcal{P}_M(I_1)$ and integer $r\geq 1$,

$$\begin{aligned}|(u,\phi)_{I_1}-(u,\phi)_{M,I_1}| &\leq c(||P_{M-1,I_1}u-u||_{I_1}+||P_{M-1,I_1}u-\mathcal{I}_{L,M,I_1}u||_{M,I_1})||\phi||_{I_1}\\ &\leq c(||P_{M-1,I_1}u-u||_{I_1}+||u-\mathcal{I}_{L,M,I_1}u||_{I_1})||\phi||_{I_1}\\ &\leq cM^{-r}||(1-y^2)^{\frac{r-1}{2}}\partial_y^r u||_{I_1}||\phi||_{I_1}.\end{aligned} \tag{2.12}$$

2.3 Mixed Laguerre-Legendre interpolation on Ω_1

We now consider the mixed Laguerre-Legendre interpolation on the subdomain $\Omega_1 = I_1\times\Lambda_1$. Let $\chi(x,y)$ be a certain weight function. We define the weighted space $L^2_\chi(\Omega_1)$ as usual, with the inner product $(u,v)_{\chi,\Omega_1}$ and the norm $||u||_{\chi,\Omega_1}$. We omit the subscript $\chi(x,y)$ in the notations whenever $\chi(x,y)\equiv 1$.

Let $V_{M,N,\beta}(\Omega_1)=\mathcal{P}_M(I_1)\otimes\mathcal{Q}_{N,\beta}(\Lambda_1)$. The meanings of $\tilde{\xi}^{(\beta)}_{R,N,\Lambda_1,j}, \tilde{\omega}^{(\beta)}_{R,N,\Lambda_1,j}, \zeta_{L,M,I_1,k}$ and $\rho_{L,M,I_1,k}$ are the same as before. The discrete inner product and norm are given by

$$(u,v)_{M,N,\beta,\Omega_1}=\sum_{k=0}^{M}\sum_{j=0}^{N} u\left(\zeta_{L,M,I_1,k},\tilde{\xi}^{(\beta)}_{R,N,\Lambda_1,j}\right) v\left(\zeta_{L,M,I_1,k},\tilde{\xi}^{(\beta)}_{R,N,\Lambda_1,j}\right)\rho_{L,M,I_1,k}\tilde{\omega}^{(\beta)}_{R,N,\Lambda_1,j},$$
$$||u||_{M,N,\beta,\Omega_1}=(u,u)^{\frac{1}{2}}_{M,N,\beta,\Omega_1}.$$

With the aid of (2.3), (2.4), (2.9) and (2.10), we have that

$$\begin{aligned}&(\phi,\psi)_{M,N,\beta,\Omega_1}=(\phi,\psi)_{\Omega_1}, \quad \forall\,\phi,\psi\in V_{M-1,N,\beta}(\Omega_1),\\ &||\phi||_{\Omega_1}\le||\phi||_{M,N,\beta,\Omega_1}\le\sqrt{2+\frac{1}{M}}\,||\phi||_{\Omega_1}, \quad \forall\,\phi\in V_{M,N,\beta}(\Omega_1).\end{aligned}\tag{2.13}$$

Let

$$\Omega_{1,M,N,\beta}=\left\{\left(\zeta_{L,M,I_1,k},\tilde{\xi}^{(\beta)}_{R,N,\Lambda_1,j}\right), 0\le k\le M, 0\le j\le N\right\}.$$

The mixed Laguerre-Legendre interpolation $I_{M,N,\beta,\Omega_1}u\in V_{M,N,\beta}(\Omega_1)$ is determined by

$$I_{M,N,\beta,\Omega_1}u(x,y)=u(x,y), \quad (x,y)\in\Omega_{1,M,N,\beta}.$$

For describing approximation errors, we introduce the notation with integers $r,q\ge 1$,

$$\begin{aligned}\mathbb{C}^{q,r}_{M,\beta,\Omega_1}(u)=&\int_{I_1}\left(\beta^{-2}||\partial^r_x\left(g_\beta(x)u\right)||^2_{\omega^1_{r-1,\beta},\Lambda_1}+(1+\beta^{-1})||\partial^r_x\left(g_\beta(x)u\right)||^2_{\omega^1_{r,\beta},\Lambda_1}\right)dy\\ &+M^{-2}\int_{I_1}\left(\beta^{-2}||\partial^r_x\left(g_\beta(x)\partial_y u\right)||^2_{\omega^1_{r-1,\beta},\Lambda_1}+(1+\beta^{-1})||\partial^r_x\left(g_\beta(x)\partial_y u\right)||^2_{\omega^1_{r,\beta},\Lambda_1}\right)dy\\ &+\int_{\Lambda_1}||\left(1-y^2\right)^{\frac{q-1}{2}}\partial^q_y u||^2_{I_1}dx.\end{aligned}$$

where

$$g_\beta(x)=e^{\beta(x-1)/2}.\tag{2.14}$$

Let ϑ be the identity operator. In fact, $I_{M,N,\beta,\Omega_1}u=\mathcal{I}_{L,M,I_1}(\tilde{\mathcal{I}}_{R,N,\beta,\Lambda_1}u)$, and so

$$I_{M,N,\beta,\Omega_1}u-u=D_1(u)+D_2(u)+D_3(u),$$

where

$$D_1(u)=\tilde{\mathcal{I}}_{R,N,\beta,\Lambda_1}u-u, \quad D_2(u)=\mathcal{I}_{L,M,I_1}u-u, \quad D_3(u)=(\mathcal{I}_{L,M,I_1}-\vartheta)(\tilde{\mathcal{I}}_{R,N,\beta,\Lambda_1,}-\vartheta)u.$$

Thanks to (2.6), we deduce that

$$||D_1(u)||^2_{\Omega_1} \le c(\beta N)^{1-r} \int_{I_1} \Big(\beta^{-2} ||\partial_x^r \left(g_\beta(x)u\right)||^2_{\omega^1_{r-1,\beta},\Lambda_1} \\ +(1+\beta^{-1})\ln N ||\partial_x^r \left(g_\beta(x)u\right)||^2_{\omega^1_{r,\beta},\Lambda_1} \Big) dy,$$

where $g_\beta(x)$ is defined by (2.14). Using (2.11) gives

$$||D_2(u)||^2_{\Omega_1} \le cM^{-2q} \int_{\Lambda_1} ||(1-y^2)^{\frac{q-1}{2}} \partial_y^q u||^2_{I_1} dx.$$

Moreover, we use (2.11) with $r=1$ to obtain

$$||D_3(u)||^2_{\Omega_1} \le cM^{-2} ||D_1(\partial_y u)||^2_{\Omega_1}.$$

Finally, a combination of previous statements leads to that

$$||I_{M,N,\beta,\Omega_1} u - u||^2_{\Omega_1} \le c\left(M^{-2q} + (\beta N)^{1-r} \ln N\right) \mathbb{C}^{q,r}_{M,\beta,\Omega_1}(u), \tag{2.15}$$

provided that $\mathbb{C}^{q,r}_{M,\beta,\Omega_1}(u)$ is finite. Moreover, by using (2.13) and (2.15), we derive that for any $\phi \in V_{M,N,\beta}(\Omega_1)$ and integers $q,r \ge 1$,

$$\begin{aligned} &|(u,\phi)_{\Omega_1} - (u,\phi)_{M,N,\beta,\Omega_1}| \\ \le& |(u,\phi)_{\Omega_1} - (I_{M-1,N,\beta,\Omega_1} u,\phi)_{\Omega_1}| + |(I_{M,N,\beta,\Omega_1} u - I_{M-1,N,\beta,\Omega_1} u,\phi)_{M,N,\beta,\Omega_1}| \\ \le& c\Big(||I_{M-1,N,\beta,\Omega_1} u - u||_{\Omega_1} + ||I_{M,N,\beta,\Omega_1} u - I_{M-1,N,\beta,\Omega_1} u||_{\Omega_1} \Big) ||\phi||_{\Omega_1} \\ \le& c\left(M^{-q} + (\beta N)^{\frac{1-r}{2}} (\ln N)^{\frac{1}{2}}\right) \left(\mathbb{C}^{q,r}_{M,\beta,\Omega_1}(u)\right)^{\frac{1}{2}} ||\phi||_{\Omega_1}. \end{aligned} \tag{2.16}$$

2.4 Two-dimensional Laguerre interpolation on Ω_2

We first consider the Laguerre interpolation on $\Lambda_2 = \{\, y \mid 1 < y < \infty\}$. Let

$$\omega^2_{\alpha,\beta}(y) = \omega_{\alpha,\beta}(y-1), \quad \omega^2_\beta(y) = \omega_\beta(y-1).$$

For integer $N > 0$, $\mathcal{P}_N(\Lambda_2)$ denotes the set of all polynomials of degree at most N.

Let $\zeta^{(\beta)}_{R,N,\Lambda_2,j}$ be the zeros of polynomial $(y-1)\partial_y \mathcal{L}^{(\beta)}_{N+1}(y-1)$, and $\omega^{(\beta)}_{R,N,\Lambda_2,j}$ be the corresponding Christoffel numbers such that

$$\int_{\Lambda_2} \phi(y) \omega^2_\beta(y) dy = \sum_{j=0}^{N} \phi\left(\zeta^{(\beta)}_{R,N,\Lambda_2,j}\right) \omega^{(\beta)}_{R,N,\Lambda_2,j}, \quad \forall \phi \in \mathcal{P}_{2N}(\Lambda_2).$$

We set $\tilde{\xi}^{(\beta)}_{R,N,\Lambda_2,j}=\xi^{(\beta)}_{R,N,\Lambda_2,j}$ and $\tilde{\omega}^{(\beta)}_{R,N,\Lambda_2,j}=e^{\beta(\xi^{(\beta)}_{R,N,\Lambda_2,j}-1)}\omega^{(\beta)}_{R,N,\Lambda_2,j}$. Furthermore,

$$(u,v)_{N,\beta,\Lambda_2}=\sum_{j=0}^{N}u\left(\tilde{\xi}^{(\beta)}_{R,N,\Lambda_2,j}\right)v\left(\tilde{\xi}^{(\beta)}_{R,N,\Lambda_2,j}\right)\tilde{\omega}^{(\beta)}_{R,N,\Lambda_2,j},\quad ||u||_{N,\beta,\Lambda_2}=(u,u)^{\frac{1}{2}}_{N,\beta,\Lambda_2}.$$

Next, let $\mathcal{Q}_{N,\beta}(\Lambda_2)=\left\{\, e^{-\frac{1}{2}\beta(y-1)}\psi \mid \psi\in\mathcal{P}_N(\Lambda_2)\right\}$. We have that

$$(\phi,\psi)_{\Lambda_2}=(\phi,\psi)_{N,\beta,\Lambda_2},\quad \forall\phi\in\mathcal{Q}_{m,\beta}(\Lambda_2),\ \psi\in\mathcal{Q}_{2N-m,\beta}(\Lambda_2),\quad 0\le m\le 2N, \tag{2.17}$$
$$||\phi||_{N,\beta,\Lambda_2}=||\phi||_{\Lambda_2},\quad \forall\phi\in\mathcal{Q}_{N,\beta}(\Lambda_2). \tag{2.18}$$

Let $\bar{\Lambda}_2=\Lambda_2\cup\{y=1\}$. For $u\in C(\bar{\Lambda}_2)$, the Laguerre-Gauss-Radau interpolation $\tilde{\mathcal{I}}_{R,N,\beta,\Lambda_2}u\in\mathcal{Q}_{N,\beta}(\Lambda_2)$ is determined by

$$\tilde{\mathcal{I}}_{R,N,\beta,\Lambda_2}u\left(\tilde{\xi}^{(\beta)}_{R,N,\Lambda_2,j}\right)=u\left(\tilde{\xi}^{(\beta)}_{R,N,\Lambda_2,j}\right),\quad 0\le j\le N.$$

If $u\in L^2(\Lambda_2)\cap C(\bar{\Lambda}_2)$, $\partial_y^r\left(e^{\frac{1}{2}\beta(y-1)}u\right)\in L^2_{\omega^2_{r,\beta}}(\Lambda_2)\cap L^2_{\omega^2_{r-1,\beta}}(\Lambda_2)$ and integer $r\ge 1$, then

$$\begin{aligned}||\tilde{\mathcal{I}}_{R,N,\beta,\Lambda_2}u-u||_{\Lambda_2}\ \le\ & c(\beta N)^{\frac{1-r}{2}}\Big(\beta^{-1}||\partial_y^r\left(g_\beta(y)u\right)||_{\omega^2_{r-1,\beta},\Lambda_2}\\ &+(1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}}||\partial_y^r\left(g_\beta(y)u\right)||_{\omega^2_{r,\beta},\Lambda_2}\Big).\end{aligned} \tag{2.19}$$

where $g_\beta(y)$ is defined by (2.14). Moreover, for any $\phi\in\mathcal{Q}_{N,\beta}(\Lambda_2)$ and integer $r\ge 1$,

$$\begin{aligned}|(u,\phi)_{\Lambda_2}-(u,\phi)_{N,\beta,\Lambda_2}|\ \le\ & c(\beta N)^{\frac{1-r}{2}}\Big(\beta^{-1}||\partial_y^r\left(g_\beta(y)u\right)||_{\omega^2_{r-1,\beta},\Lambda_2}\\ &+(1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}}||\partial_y^r\left(g_\beta(y)u\right)||_{\omega^2_{r,\beta},\Lambda_2}\Big)||\phi||_{\Lambda_2}.\end{aligned} \tag{2.20}$$

We now turn to the two-dimensional Laguerre interpolation on the subdomain $\Omega_2=\Lambda_1\times\Lambda_2$. We define the weighted space $L^2_\chi(\Omega_2)$ as usual, with the inner product $(u,v)_{\chi,\Omega_2}$ and the norm $||u||_{\chi,\Omega_2}$. We omit the subscript χ in the notations whenever $\chi(x,y)\equiv 1$.

Let $V_{N,N,\beta}(\Omega_2)=\mathcal{Q}_{N,\beta}(\Lambda_1)\otimes\mathcal{Q}_{N,\beta}(\Lambda_2)$. The corresponding discrete inner product and norm are given by

$$\begin{aligned}&(u,v)_{N,N,\beta,\Omega_2}=\sum_{j=0}^{N}\sum_{k=0}^{N}u\left(\tilde{\xi}^{(\beta)}_{R,N,\Lambda_1,j},\tilde{\xi}^{(\beta)}_{R,N,\Lambda_2,k}\right)v\left(\tilde{\xi}^{(\beta)}_{R,N,\Lambda_1,j},\tilde{\xi}^{(\beta)}_{R,N,\Lambda_2,k}\right)\tilde{\omega}^{(\beta)}_{R,N,\Lambda_1,j}\tilde{\omega}^{(\beta)}_{R,N,\Lambda_2,k},\\ &||u||_{N,N,\beta,\Omega_2}=(u,u)^{\frac{1}{2}}_{N,N,\beta,\Omega_2}.\end{aligned}$$

By virtue of (2.3), (2.4), (2.17) and (2.18), we have that

$$(\phi,\psi)_{N,N,\beta,\Omega_2}=(\phi,\psi)_{\Omega_2},\quad \forall\ \phi,\psi\in V_{N,N,\beta}(\Omega_2), \tag{2.21}$$
$$||\phi||_{N,N,\beta,\Omega_2}=||\phi||_{\Omega_2},\quad \forall\ \phi\in V_{N,N,\beta}(\Omega_2). \tag{2.22}$$

Let

$$\Omega_{2,N,N,\beta}=\left\{\left(\tilde{\xi}^{(\beta)}_{R,N,\Lambda_1,j},\tilde{\xi}^{(\beta)}_{R,N,\Lambda_2,k}\right),0\le j\le N,0\le k\le N\right\}.$$

The two-dimensional Laguerre interpolation $I_{N,N,\beta,\Omega_2}u\in V_{N,N,\beta}(\Omega_2)$ is determined by

$$I_{N,N,\beta,\Omega_2}u(x,y)=u(x,y),\quad (x,y)\in\Omega_{2,N,N,\beta}. \tag{2.23}$$

For deriving the approximation errors, we introduce the notation with integer $r\ge 1$,

$$\begin{aligned}
&\mathbb{C}^r_{\beta,\Omega_2}(u)=\int_{\Lambda_1}\left(\beta^{-2}||\partial_y^r\left(g_\beta(y)u\right)||^2_{\omega^2_{r-1,\beta},\Lambda_2}+(1+\beta^{-1})||\partial_y^r\left(g_\beta(y)u\right)||^2_{\omega^2_{r,\beta},\Lambda_2}\right)dx\\
&+\int_{\Lambda_2}y\left((1+\beta^{-2})||\partial_x^r\left(g_\beta(x)u\right)||^2_{\omega^1_{r-1,\beta},\Lambda_1}+(\beta^2+\beta^{-1})||\partial_x^r\left(g_\beta(x)u\right)||^2_{\omega^1_{r,\beta},\Lambda_1}\right)dy\\
&+\int_{\Lambda_2}y\left((\beta^{-2}+\beta^{-4})||\partial_x^r\left(g_\beta(x)\partial_y u\right)||^2_{\omega^1_{r-1,\beta},\Lambda_1}+(1+\beta^{-3})||\partial_x^r\left(g_\beta(x)\partial_y u\right)||^2_{\omega^1_{r,\beta},\Lambda_1}\right)dy.
\end{aligned}$$

Let ϑ be the same as before. Then,

$$I_{N,N,\beta,\Omega_2}u-u=D_1(u)+D_2(u)+D_3(u),$$

with

$$\begin{aligned}
&D_1(u)=\tilde{\mathcal{I}}_{R,N,\beta,\Lambda_1}u-u,\quad D_2(u)=\tilde{\mathcal{I}}_{R,N,\beta,\Lambda_2}u-u,\\
&D_3(u)=(\tilde{\mathcal{I}}_{R,N,\beta,\Lambda_2,}-\vartheta)(\tilde{\mathcal{I}}_{R,N,\beta,\Lambda_1,}-\vartheta)u.
\end{aligned}$$

We use (2.6) and (2.19) to deduce that

$$\begin{aligned}
||D_1(u)||^2_{\Omega_2}\le c(\beta N)^{1-r}\int_{\Lambda_2}&\Big(\beta^{-2}||\partial_x^r\left(g_\beta(x)u\right)||^2_{\omega^1_{r-1,\beta},\Lambda_1}\\
&+(1+\beta^{-1})\ln N||\partial_x^r\left(g_\beta(x)u\right)||^2_{\omega^1_{r,\beta},\Lambda_1}\Big)dy,\\
||D_2(u)||^2_{\Omega_2}\le c(\beta N)^{1-r}\int_{\Lambda_1}&\Big(\beta^{-2}||\partial_y^r\left(g_\beta(y)u\right)||^2_{\omega^2_{r-1,\beta},\Lambda_2}\\
&+(1+\beta^{-1})\ln N||\partial_y^r\left(g_\beta(y)u\right)||^2_{\omega^2_{r,\beta},\Lambda_2}\Big)dx.
\end{aligned}$$

Next, using (2.19) with $r=1$ gives that

$$\begin{aligned}
||D_3(u)||^2_{\Omega_2}\;\le\;& c\int_{\Lambda_1}\left(||D_1(u)||^2_{\Lambda_2}+\beta^{-2}||D_1(\partial_y u)||^2_{\Lambda_2}\right)dx\\
&+c(1+\beta^{-1})\ln N\int_{\Omega_2}(y-1)\left(\beta^2 D_1^2(u)+D_1^2(\partial_y u)\right)dxdy\\
\le\;& c||D_1(u)||^2_{\Omega_2}+c\beta^{-2}||D_1(\partial_y u)||^2_{\Omega_2}\\
&+c\beta^2(1+\beta^{-1})\ln N\int_{\Lambda_2}(y-1)||D_1(u)||^2_{\Lambda_1}dy\\
&+c(1+\beta^{-1})\ln N\int_{\Lambda_2}(y-1)||D_1(\partial_y u)||^2_{\Lambda_1}dy.
\end{aligned}$$

We can also use (2.6) to estimate $||D_1(u)||^2_{\Lambda_1}$ and $||D_1(\partial_y u)||^2_{\Lambda_1}$ similarly. Finally, a combination of previous statements leads to that

$$||I_{N,N,\beta,\Omega_2}u-u||^2_{\Omega_2}\le c(\beta N)^{1-r}(\ln N)^2\mathbb{C}^r_{\beta,\Omega_2}(u), \tag{2.24}$$

provided that $\mathbb{C}^r_{\beta,\Omega_2}(u)$ is finite. Furthermore, we can use (2.17) and (2.24) to derive that for any $\phi\in V_{N,N,\beta}(\Omega_2)$ and integer $r\ge 1$,

$$|(u,\phi)_{\Omega_2}-(u,\phi)_{N,N,\Omega_2}|\le c(\beta N)^{\frac{1-r}{2}}\ln N(\mathbb{C}^r_{\beta,\Omega_2}(u))^{\frac{1}{2}}||\phi||_{\Omega_2}. \tag{2.25}$$

2.5 Interpolations on other subdomains

We first consider the Laguerre interpolations on the infinite intervals $\Lambda_3=\{\ x\ |\ x<-1\}$ and $\Lambda_4=\{\ y\ |\ y<-1\}$. Let $\omega^3_{\alpha,\beta}(x)=\omega_{\alpha,\beta}(-x-1)$ and $\omega^4_{\alpha,\beta}(y)=\omega_{\alpha,\beta}(-y-1)$. In particular, $\omega^3_\beta(x)=\omega_\beta(-x-1)$ and $\omega^4_\beta(y)=\omega_\beta(-y-1)$. For any positive integer N, $\mathcal{P}_N(\Lambda_k)$ stands for the set of all polynomials of degree at most N. For simplicity, we denote x by z_3, and y by z_4.

Let $\xi^{(\beta)}_{R,N,\Lambda_k,j}$ be the zeros of polynomial $(-z_k-1)\partial_{z_k}\mathcal{L}^{(\beta)}_{N+1}(-z_k-1)$, and $\omega^{(\beta)}_{R,N,\Lambda_k,j}$ be the corresponding Christoffel numbers. We set

$$\tilde{\xi}^{(\beta)}_{R,N,\Lambda_k,j}=\xi^{(\beta)}_{R,N,\Lambda_k,j}\quad\text{and}\quad\tilde{\omega}^{(\beta)}_{R,N,\Lambda_k,j}=e^{-\beta(\xi^{(\beta)}_{R,N,\Lambda_k,j}+1)}\omega^{(\beta)}_{R,N,\Lambda_k,j}.$$

We introduce the discrete inner product $(u,v)_{N,\beta,\Lambda_k}$ and the discrete norm $||u||_{N,\beta,\Lambda_k}$ in the usual way. Besides,

$$\mathcal{Q}_{N,\beta}(\Lambda_k)=\left\{e^{\frac{1}{2}(z_k+1)}\psi\ |\ \psi\in\mathcal{P}_N(\Lambda_k)\right\}.$$

Now, let $\bar{\Lambda}_3=\Lambda_3\cup\{x=-1\}$ and $\bar{\Lambda}_4=\Lambda_4\cup\{y=-1\}$. For any $u\in C(\bar{\Lambda}_k)$, the Laguerre-Gauss-Radau interpolations $\tilde{\mathcal{I}}_{R,N,\beta,\Lambda_k}u\in\mathcal{Q}_{N,\beta}(\Lambda_k)$, are determined by

$$\tilde{\mathcal{I}}_{R,N,\beta,\Lambda_k}u\left(\tilde{\xi}^{(\beta)}_{R,N,\Lambda_k,j}\right)=u\left(\tilde{\xi}^{(\beta)}_{R,N,\Lambda_k,j}\right),\quad 0\le j\le N,\quad k=3,4.$$

We can estimate $||\tilde{\mathcal{I}}_{R,N,\beta,\Lambda_k}u-u||_{\Lambda_k}$ and $|(u,\phi)_{\Lambda_k}-(u,\phi)_{N,\beta,\Lambda_k}|$ as in Subsection 2.1.

We next turn to Legendre-Gauss-Lobatto interpolation on the $I_2=\{\ x\ |\ |x|<1\}$. For integer $M\ge 0$, $\mathcal{P}_M(I_2)$ stands for the set of all polynomials of degree at most M. Let $\zeta_{L,M,I_2,k}$ be the roots of polynomial $(1-x^2)\partial_x L_M(x), 0\le k\le M$, which are arranged in ascending order. The corresponding Christoffel numbers are denoted by $\rho_{L,M,I_2,k}, 0\le k\le M$. We also introduce the discrete inner product $(u,v)_{M,I_2}$ and the discrete norm $||u||_{M,I_2}=(u,u)^{\frac{1}{2}}_{M,I_2}$ as usual.

For any $u\in C(\bar{I}_2)$, the Legendre-Gauss-Lobatto interpolation $\mathcal{I}_{L,M,I_2}u\in\mathcal{P}_M(I_2)$ is determined by

$$\mathcal{I}_{L,M,I_2}u(\zeta_{L,M,I_2,k})=u(\zeta_{L,M,I_2,k}),\quad 0\le k\le M.$$

We can estimate $||\mathcal{I}_{L,M,I_2}u-u||_{I_2}$ and $|(u,\phi)_{I_2}-(u,\phi)_{M,I_2}|$ as in Subsection 2.2.

We now turn to the interpolation on the subdomains Ω_j, $3\le j\le 8$. Indeed, $\Omega_3=I_2\times\Lambda_2$, $\Omega_4=\Lambda_3\times\Lambda_2$, $\Omega_5=\Lambda_3\times I_1$, $\Omega_6=\Lambda_3\times\Lambda_4$, $\Omega_7=I_2\times\Lambda_4$ and $\Omega_8=\Lambda_1\times\Lambda_4$. Therefore, we take the finite dimensional spaces as follows,

$$V_{M,N,\beta}(\Omega_3)=\mathcal{P}_M(I_2)\otimes\mathcal{Q}_{N,\beta}(\Lambda_2),\qquad V_{N,N,\beta}(\Omega_4)=\mathcal{Q}_{N,\beta}(\Lambda_3)\otimes\mathcal{Q}_{N,\beta}(\Lambda_2),$$
$$V_{M,N,\beta}(\Omega_5)=\mathcal{Q}_{N,\beta}(\Lambda_3)\otimes\mathcal{P}_M(I_1),\qquad V_{N,N,\beta}(\Omega_6)=\mathcal{Q}_{N,\beta}(\Lambda_3)\otimes\mathcal{Q}_{N,\beta}(\Lambda_4),$$
$$V_{M,N,\beta}(\Omega_7)=\mathcal{P}_M(I_2)\otimes\mathcal{Q}_{N,\beta}(\Lambda_4),\qquad V_{N,N,\beta}(\Omega_8)=\mathcal{Q}_{N,\beta}(\Lambda_1)\otimes\mathcal{Q}_{N,\beta}(\Lambda_4).$$

We can define the discrete inner product $(\phi,\psi)_{M,N,\beta,\Omega_j}$ and the discrete norm $||\phi||_{M,N,\beta,\Omega_j}$ in the usual way. Like (2.13) and (2.21), we have that

$$(\phi,\psi)_{M,N,\beta,\Omega_j}=(\phi,\psi)_{\Omega_j},\quad \forall\,\phi,\psi\in V_{M-1,N,\beta}(\Omega_j),\quad j=3,5,7, \tag{2.26}$$
$$||\phi||_{\Omega_j}\le||\phi||_{M,N,\beta,\Omega_j}\le\sqrt{2+M^{-1}}\,||\phi||_{\Omega_j},\quad \forall\,\phi\in V_{M,N,\beta}(\Omega_j),\quad j=3,5,7, \tag{2.27}$$
$$(\phi,\psi)_{N,N,\beta,\Omega_j}=(\phi,\psi)_{\Omega_j},\quad \forall\,\phi,\psi\in V_{N,N,\beta}(\Omega_j),\quad j=4,6,8. \tag{2.28}$$

Next, we denote the sets of interpolation nodes on Ω_j, by $\Omega_{j,M,N,\beta}, j=3,5,7$ and $\Omega_{j,N,N,\beta}, j=2,4,6$. For instance,

$$\Omega_{3,M,N,\beta}=\left\{\left(\zeta_{L,M,I_2,k},\tilde{\xi}^{(\beta)}_{R,N,\Lambda_2,j}\right),0\le k\le M,0\le j\le N\right\},$$
$$\Omega_{4,N,N,\beta}=\left\{\left(\tilde{\xi}^{(\beta)}_{R,N,\Lambda_3,j},\tilde{\xi}^{(\beta)}_{R,N,\Lambda_2,k}\right),0\le k\le N,0\le j\le N\right\},\ \text{etc.}$$

The mixed Laguerre-Legendre interpolations $I_{M,N,\beta,\Omega_j}u\in V_{M,N,\beta}(\Omega_j), j=3,5,7$, and the two-dimensional Laguerre interpolations $I_{N,N,\beta,\Omega_j}u\in V_{N,N,\beta}(\Omega_j), j=4,6,8$, are determined by

$$I_{M,N,\beta,\Omega_j}u(x,y)=u(x,y),\qquad (x,y)\in\Omega_{j,M,N,\beta},\qquad j=3,5,7,$$
$$I_{N,N,\beta,\Omega_j}u(x,y)=u(x,y),\qquad (x,y)\in\Omega_{j,N,N,\beta},\qquad j=4,6,8.$$

We introduce the following notations with integers $q,r\ge 1$,

$$\begin{aligned}\mathbb{C}^{q,r}_{M,\beta,\Omega_3}(u)=&\int_{I_2}\left(\beta^{-2}q^{(1,1)}_{2,0}(x)+(1+\beta^{-1})q^{(1,2)}_{2,0}(x)\right)dx\\&+M^{-2}\int_{I_2}\left(\beta^{-2}q^{(1,1)}_{2,1}(x)+(1+\beta^{-1})q^{(1,2)}_{2,1}(x)\right)dx+\int_{\Lambda_2}||(1-x^2)^{\frac{q-1}{2}}\partial_x^q u||^2_{I_2}dy,\\ \mathbb{C}^{q,r}_{M,\beta,\Omega_5}(u)=&\int_{I_1}\left(\beta^{-2}q^{(2,1)}_{3,0}(y)+(1+\beta^{-1})q^{(2,2)}_{3,0}(y)\right)dy\\&+M^{-2}\int_{I_1}\left(\beta^{-2}q^{(2,1)}_{3,1}(y)+(1+\beta^{-1})q^{(2,2)}_{3,1}(y)\right)dy+\int_{\Lambda_3}||(1-y^2)^{\frac{q-1}{2}}\partial_y^q u||^2_{I_1}dx,\\ \mathbb{C}^{q,r}_{M,\beta,\Omega_7}(u)=&\int_{I_2}\left(\beta^{-2}q^{(3,1)}_{4,0}(x)+(1+\beta^{-1})q^{(3,2)}_{4,0}(x)\right)dx\\&+M^{-2}\int_{I_2}\left(\beta^{-2}q^{(3,1)}_{4,1}(x)+(1+\beta^{-1})q^{(3,2)}_{4,1}(x)\right)dx+\int_{\Lambda_4}||(1-x^2)^{\frac{q-1}{2}}\partial_x^q u||^2_{I_2}dy,\end{aligned}$$

$$
\begin{aligned}
\mathbb{C}^r_{\beta,\Omega_4}(u)=&\int_{\Lambda_2}\Big(\beta^{-2}q^{(2,1)}_{3,0}(y)+(1+\beta^{-1})q^{(2,2)}_{3,0}(y)\Big)dy\\
&-\int_{\Lambda_3}x\Big((1+\beta^2)q^{(1,1)}_{2,0}(x)+(\beta^2+\beta^{-1})q^{(1,2)}_{2,0}(x)\Big)dx\\
&-\int_{\Lambda_3}x\Big((\beta^{-2}+\beta^{-4})q^{(1,1)}_{2,1}(x)+(1+\beta^{-3})q^{(1,2)}_{2,1}(x)\Big)dx,\\
\mathbb{C}^r_{\beta,\Omega_6}(u)=&\int_{\Lambda_3}\Big(\beta^{-2}q^{(3,1)}_{4,0}(x)+(1+\beta^{-1})q^{(3,2)}_{4,0}(x)\Big)dx\\
&-\int_{\Lambda_4}y\Big((1+\beta^2)q^{(2,1)}_{3,0}(y)+(\beta^2+\beta^{-1})q^{(2,2)}_{3,0}(y)\Big)dy\\
&-\int_{\Lambda_4}y\Big((\beta^{-2}+\beta^{-4})q^{(2,1)}_{3,1}(y)+(1+\beta^{-3})q^{(2,2)}_{3,1}(y)\Big)dy,\\
\mathbb{C}^r_{\beta,\Omega_8}(u)=&\int_{\Lambda_4}\Big(\beta^{-2}q^{(4,1)}_{1,0}(y)+(1+\beta^{-1})q^{(4,2)}_{1,0}(y)\Big)dy\\
&+\int_{\Lambda_1}x\Big((1+\beta^2)q^{(3,1)}_{4,0}(x)+(\beta^2+\beta^{-1})q^{(3,2)}_{4,0}(x)\Big)dx\\
&+\int_{\Lambda_1}x\Big((\beta^{-2}+\beta^{-4})q^{(3,1)}_{4,1}(x)+(1+\beta^{-3})q^{(3,2)}_{4,1}(x)\Big)dx,
\end{aligned}
$$

where

$$
\begin{aligned}
&q^{(1,1)}_{j,k}(x)=||\partial^r_y\Big(e^{\frac12\beta(y-1)}\partial^k_x u\Big)||^2_{\omega^j_{r-1,\beta},\Lambda_j}, \quad q^{(1,2)}_{j,k}(x)=||\partial^r_y\Big(e^{\frac12\beta(y-1)}\partial^k_x u\Big)||^2_{\omega^j_{r,\beta},\Lambda_j},\\
&q^{(2,1)}_{j,k}(y)=||\partial^r_x\Big(e^{-\frac12\beta(x+1)}\partial^k_y u\Big)||^2_{\omega^j_{r-1,\beta},\Lambda_j}, \quad q^{(2,2)}_{j,k}(y)=||\partial^r_x\Big(e^{-\frac12\beta(x+1)}\partial^k_y u\Big)||^2_{\omega^j_{r,\beta},\Lambda_j},\\
&q^{(3,1)}_{j,k}(x)=||\partial^r_y\Big(e^{-\frac12\beta(y+1)}\partial^k_x u\Big)||^2_{\omega^j_{r-1,\beta},\Lambda_j}, \quad q^{(3,2)}_{j,k}(x)=||\partial^r_y\Big(e^{-\frac12\beta(y+1)}\partial^k_x u\Big)||^2_{\omega^j_{r,\beta},\Lambda_j},\\
&q^{(4,1)}_{j,k}(y)=||\partial^r_x\Big(e^{\frac12\beta(x-1)}\partial^k_y u\Big)||^2_{\omega^j_{r-1,\beta},\Lambda_j}, \quad q^{(4,2)}_{j,k}(y)=||\partial^r_x\Big(e^{\frac12\beta(x-1)}\partial^k_y u\Big)||^2_{\omega^j_{r,\beta},\Lambda_j}.
\end{aligned}
$$

Following the same line as in the derivations of (2.15) and (2.24), we verify that

$$||I_{M,N,\beta,\Omega_j}u-u||^2_{\Omega_j}\le c(M^{-2q}+(\beta N)^{1-r}\ln N)\mathbb{C}^{q,r}_{M,\beta,\Omega_j}(u), \quad j=3,5,7, \tag{2.29}$$

$$||I_{N,N,\beta,\Omega_j}u-u||^2_{\Omega_j}\le c(\beta N)^{1-r}(\ln N)^2\mathbb{C}^r_{\beta,\Omega_j}(u), \quad j=4,6,8. \tag{2.30}$$

Moreover, if $\phi\in V_{M,N,\beta}(\Omega_j), j=3,5,7$, $\phi\in V_{N,N,\beta}(\Omega_j), j=4,6,8$, and integers $q,r\ge 1$, then

$$
\begin{aligned}
&|(u,\phi)_{\Omega_j}-(u,\phi)_{M,N,\beta,\Omega_j}|\\
\le& c\Big(M^{-q}+(\beta N)^{\frac{1-r}{2}}(\ln N)^{\frac12}\Big)\Big(\mathbb{C}^{q,r}_{M,\beta,\Omega_j}(u)\Big)^{\frac12}||\phi||_{\Omega_j}, \quad j=3,5,7, \qquad (2.31)\\
&|(u,\phi)_{\Omega_j}-(u,\phi)_{N,N,\beta,\Omega_j}|\\
\le& c(\beta N)^{\frac{1-r}{2}}\ln N\Big(\mathbb{C}^r_{\beta,\Omega_j}(u)\Big)^{\frac12}||\phi||_{\Omega_j}, \quad j=4,6,8. \qquad (2.32)
\end{aligned}
$$

2.6 Composite interpolation on exterior domain Ω.

We are now in position to study the composite interpolation on the whole domain Ω, which will be used in the next section. The weighted space $L^2_\chi(\Omega)$ is defined in the usual way, with the inner product $(u,w)_{\chi,\Omega}$ and the norm $||u||_{\chi,\Omega}$. We omit the subscript χ in the notations whenever $\chi(x,y)\equiv 1$. Further, let

$$V_{M,N,\beta}(\Omega)=H^1_0(\Omega)\cap S_M,$$

where

$$S_p=\{\phi\colon\ \phi\mid \phi|_{\Omega_j}\in V_{p,N,\beta}(\Omega_j), j=1,3,5,7, \text{ and } \phi|_{\Omega_j}\in V_{N,N,\beta}(\Omega_j), j=2,4,6,8\}.$$

We introduce the discrete inner product and norm as

$$(u,v)_{M,N,\beta,\Omega}=\sum_{j=1,3,5,7}(u,v)_{M,N,\beta,\Omega_j}+\sum_{j=2,4,6,8}(u,v)_{N,N,\beta,\Omega_j},\quad ||u||_{M,N,\beta,\Omega}=(u,u)^{\frac{1}{2}}_{M,N,\beta,\Omega}.$$

By virtue of (2.13), (2.21) and (2.26)-(2.28), we have that

$$(\phi,\psi)_{M,N,\beta,\Omega}=(\phi,\psi)_\Omega,\quad \forall\ \phi,\psi\in S_{M-1}, \tag{2.33}$$

$$||\phi||_\Omega\le||\phi||_{M,N,\beta,\Omega}\le\sqrt{2+M^{-1}}\ ||\phi||_\Omega,\quad \forall\ \phi\in S_M. \tag{2.34}$$

The composite interpolation $I_{M,N,\beta,\Omega}u(x,y)\in S_M$ is defined as

$$I_{M,N,\beta,\Omega}u(x,v)=\begin{cases} I_{M,N,\beta,\Omega_j}u(x,v), & \text{on } \Omega_j,\ j=1,3,5,7,\\ I_{N,N,\beta,\Omega_j}u(x,v), & \text{on } \Omega_j,\ j=2,4,6,8.\end{cases}$$

For description of approximation errors, we introduce the following notation,

$$\mathbb{C}^{q,r}_{M,\beta,\Omega}(u)=\sum_{j=1,3,5,7}\mathbb{C}^{q,r}_{M,\beta,\Omega_j}(u)+\sum_{j=2,4,6,8}\mathbb{C}^{r}_{\beta,\Omega_j}(u).$$

By using Eqs. (2.15), (2.24), (2.29) and (2.30), we conclude that

Theorem 2.1. *If $u\in C(\Omega)$ and $\mathbb{C}^{q,r}_{M,\beta,\Omega}(u)$ is finite for integers $r>1$, $q\ge 0$, then*

$$||I_{M,N,\beta,\Omega}u-u||^2_\Omega\le c\left(M^{-2q}+(\beta N)^{1-r}(\ln N)^2\right)\mathbb{C}^{q,r}_{M,\beta,\Omega}(u). \tag{2.35}$$

Moreover, with the aid of Eqs. (2.16), (2.25), (2.31) and (2.32), we deduce that

$$|(u,\phi)_\Omega-(u,\phi)_{M,N,\beta,\Omega}|\le c\left(M^{-q}+(\beta N)^{\frac{1-r}{2}}\ln N\right)\left(\mathbb{C}^{q,r}_{M,\beta,\Omega}(u)\right)^{\frac{1}{2}}||\phi||_\Omega. \tag{2.36}$$

3 Composite pseudospectral method for exterior problems

In this section, we propose the composite pseudospectral method for exterior problems.

3.1 A steady problem

We first consider the following problem

$$\begin{cases} -\Delta W(x,y)+\mu W(x,y)=F(x,y), \quad \mu>0, & (x,y)\in\Omega, \\ W(x,y)=g(x,y), & (x,y) \text{ on } \Gamma, \\ W(x,y)\to 0, & |x| \text{ or } |y|\to\infty, \end{cases} \tag{3.1}$$

where $g(1,y)=g_1(y)$, $g(x,1)=g_2(x)$, $g(-1,y)=g_3(y)$ and $g(x,-1)=g_4(x)$. Suppose that g_j satisfy the consistent condition, namely, $g_1(-1)=g_4(1)$, $g_1(1)=g_2(1)$, $g_2(-1)=g_3(1)$ and $g_3(-1)=g_4(-1)$. Then $g(x,y)$ is continuous on boundary Γ. Let (cf. [13])

$$\begin{aligned} \widetilde{W}(x,y) = & \frac{1}{4}e^{(1-x^2)(1-y^2)}\Big(2(1-y)g_4(x)+2(1+y)g_2(x)+2(1-x)g_3(y)+2(1+x)g_1(y) \\ & -(1-x)(1-y)g_3(-1)-(1-x)(1+y)g_2(-1) \\ & -(1+x)(1-y)g_4(1)-(1+x)(1+y)g_1(1)\Big). \end{aligned}$$

It can be checked that $\widetilde{W}(x,y)=W(x,y)$ on Γ, and $\widetilde{W}(x,y)\to 0$ if $|x|$ or $|y|\to\infty$. Further, let

$$W(x,y)=U(x,y)+\widetilde{W}(x,y), \quad f(x,y)=F(x,y)+\Delta\widetilde{W}(x,y)-\mu\widetilde{W}(x,y).$$

Then (3.1) can be rewritten as

$$\begin{cases} -\Delta U(x,y)+\mu U(x,y)=f(x,y), \quad \mu>0, & (x,y)\in\Omega, \\ U(x,y)=0, & (x,y) \text{ on } \Gamma, \\ U(x,y)\to 0, & |x| \text{ or } |y|\to\infty. \end{cases} \tag{3.2}$$

Let $\mu\geq 0$, we define the bilinear form as

$$a_\mu(u,v)=(\partial_x u,\partial_x v)_\Omega+(\partial_y u,\partial_y v)_\Omega+\mu(u,v)_\Omega, \quad \forall u,v\in H_0^1(\Omega).$$

A weak formulation of (3.2) is to seek solution $U\in H_0^1(\Omega)$ such that

$$a_\mu(U,v)=(f,v)_\Omega, \quad \forall v\in H_0^1(\Omega). \tag{3.3}$$

The composite pseudospectral scheme for (3.3) is to find $u_{M,N}\in V_{M,N,\beta}(\Omega)$ such that

$$a_{M,N,\beta,\mu}(u_{M,N},\phi)=(f,\phi)_{M,N,\beta,\Omega}, \quad \forall\phi\in V_{M,N,\beta}(\Omega), \tag{3.4}$$

where

$$a_{M,N,\beta,\mu}(u,\phi)=(\partial_x u,\partial_x w)_{M,N,\beta,\Omega}+(\partial_y u,\partial_y w)_{M,N,\beta,\Omega}+\mu(u,w)_{M,N,\beta,\Omega}.$$

In order to estimate the error of numerical solution, we need some preparations. The projection ${}_*P^1_{M,N,\beta,\mu,\Omega}: H_0^1(\Omega)\to V_{M,N,\beta}(\Omega)$ is defined by

$$a_\mu({}_*P^1_{M,N,\beta,\mu,\Omega}u-u,\phi)_\Omega=0, \quad \forall\phi\in V_{M,N,\beta}(\Omega).$$

We shall use the follow notations with integers $q,r\geq 1$,

$$
\begin{aligned}
\mathbb{B}^{q,r}_{\beta,\Omega_1}(u)=&(\beta+\beta^{-2})\int_{I_1}\Big(q^{(4,1)}_{1,0}(y)+q^{(4,1)}_{1,1}(y)+q^{(5,1)}_{1}(y)\Big)dy+\int_{\Lambda_1}q^{(6,1)}(x)dx,\\
\mathbb{B}^{q,r}_{\beta,\Omega_3}(u)=&(\beta+\beta^{-2})\int_{I_2}\Big(q^{(1,1)}_{2,0}(x)+q^{(1,1)}_{2,1}(x)+q^{(5,2)}_{2}(x)\Big)dx+\int_{\Lambda_2}q^{(6,2)}(y)dy,\\
\mathbb{B}^{q,r}_{\beta,\Omega_5}(u)=&(\beta+\beta^{-2})\int_{I_1}\Big(q^{(2,1)}_{3,0}(y)+q^{(2,1)}_{3,1}(y)+q^{(7,1)}(y)\Big)dy+\int_{\Lambda_3}q^{(6,1)}(x)dx,\\
\mathbb{B}^{q,r}_{\beta,\Omega_7}(u)=&(\beta+\beta^{-2})\int_{I_2}\Big(q^{(3,1)}_{4,0}(x)+q^{(3,1)}_{4,1}(x)+q^{(7,2)}(x)\Big)dx+\int_{\Lambda_4}q^{(6,2)}(y)dy,\\
\mathbb{B}^{r}_{\beta,\Omega_2}(u)=&(\beta+\beta^{-2})\Big(\int_{\Lambda_2}\Big(q^{(4,1)}_{1,0}(y)+q^{(4,1)}_{1,1}(y)\Big)dy+\int_{\Lambda_1}\Big(q^{(1,1)}_{2,0}(x)+q^{(1,1)}_{2,1}(x)\Big)dx\Big)\\
&+(\beta+\beta^{-4})\int_{\Lambda_1}e^{-\beta(x-1)}||\partial^r_y(e^{\frac12\beta(y-1)}\partial_x\Big(e^{\frac12\beta(x-1)}u\Big)||^2_{\omega^2_{r-1,\beta},\Lambda_2}dx,\\
\mathbb{B}^{r}_{\beta,\Omega_4}(u)=&(\beta+\beta^{-2})\Big(\int_{\Lambda_3}\Big(q^{(1,1)}_{2,0}(x)+q^{(1,1)}_{2,1}(x)\Big)dx+\int_{\Lambda_2}\Big(q^{(2,1)}_{3,0}(y)+q^{(2,1)}_{3,1}(y)\Big)dy\Big)\\
&+(\beta+\beta^{-4})\int_{\Lambda_2}e^{-\beta(y-1)}||\partial^r_x(e^{-\frac12\beta(x+1)}\partial_x\Big(e^{\frac12\beta(y-1)}u\Big)||^2_{\omega^3_{r-1,\beta},\Lambda_3}dy,\\
\mathbb{B}^{r}_{\beta,\Omega_6}(u)=&(\beta+\beta^{-2})\Big(\int_{\Lambda_4}\Big(q^{(2,1)}_{3,0}(y)+q^{(2,1)}_{3,1}(y)\Big)dy+\int_{\Lambda_3}\Big(q^{(3,1)}_{4,0}(x)+q^{(3,1)}_{4,1}(x)\Big)dx\Big)\\
&+(\beta+\beta^{-4})\int_{\Lambda_3}e^{\beta(x+1)}||\partial^r_y(e^{-\frac12\beta(y+1)}\partial_x\Big(e^{-\frac12\beta(x+1)}u\Big)||^2_{\omega^4_{r-1,\beta},\Lambda_4}dx,\\
\mathbb{B}^{r}_{\beta,\Omega_8}(u)=&(\beta+\beta^{-2})\Big(\int_{\Lambda_1}\Big(q^{(3,1)}_{4,0}(x)+q^{(3,1)}_{4,1}(x)\Big)dx+\int_{\Lambda_4}\Big(q^{(4,1)}_{1,0}(y)+q^{(4,1)}_{1,1}(y)\Big)dy\Big)\\
&+(\beta+\beta^{-4})\int_{\Lambda_4}e^{\beta(y+1)}||\partial^r_x(e^{\frac12\beta(x-1)}\partial_x\Big(e^{-\frac12\beta(y+1)}u\Big)||^2_{\omega^1_{r-1,\beta},\Lambda_1}dy,
\end{aligned}
$$

where

$$
\begin{aligned}
&q^{(5,1)}_j(y)=(1-y^2)^{q-1}||\partial_x\Big(e^{\frac12\beta(x-1)}\partial^q_y u\Big)||^2_{\omega^j_\beta,\Lambda_j},\\
&q^{(5,2)}_j(x)=(1-x^2)^{q-1}||\partial_y\Big(e^{\frac12\beta(y-1)}\partial^q_y u\Big)||^2_{\omega^j_\beta,\Lambda_j},\\
&q^{(6,1)}(x)=||(1-y^2)^{\frac{q-1}{2}}\partial_x\partial^q_y u||^2_{I_1}+||(1-y^2)^{\frac{q-1}{2}}\partial^q_y u||^2_{I_1},\\
&q^{(6,2)}(y)=||(1-x^2)^{\frac{q-1}{2}}\partial_y\partial^q_x u||^2_{I_2}+||(1-x^2)^{\frac{q-1}{2}}\partial^q_x u||^2_{I_2},\\
&q^{(7,1)}(y)=(1-y^2)^{q-1}||\partial_x\Big(e^{-\frac12\beta(x+1)}\partial^q_y u\Big)||^2_{\omega^3_\beta,\Lambda_3},\\
&q^{(7,2)}(x)=(1-x^2)^{q-1}||\partial_y\Big(e^{-\frac12\beta(y+1)}\partial^q_x u\Big)||^2_{\omega^4_\beta,\Lambda_4};
\end{aligned}
$$

and

$$
\mathbb{B}^{q,r}_{\beta,\Omega}(u)=\sum_{j=1,3,5,7}\mathbb{B}^{q,r}_{\beta,\Omega_j}(u)+\sum_{j=2,4,6,8}\mathbb{B}^{r}_{\beta,\Omega_j}(u).
$$

According to [13], we have that

$$\begin{aligned}&||\partial_x\left({}_*P^1_{M,N,\beta,\mu,\Omega}u-u\right)||^2_\Omega+||\partial_y\left({}_*P^1_{M,N,\beta,\mu,\Omega}u-u\right)||^2_\Omega+\mu||{}_*P^1_{M,N,\beta,\mu,\Omega}u-u||^2_\Omega\\ \le\,&c(1+\mu)\left(M^{2-2q}+(\beta N)^{1-r}\right)\mathbb{B}^{q,r}_{\beta,\Omega}(u).\end{aligned}\tag{3.5}$$

Now, let $U_{M,N}={}_*P^1_{M,N,\beta,\mu,\Omega}U$. We derive from (3.3) that

$$a_{M,N,\beta,\mu}(U_{M,N},\phi)+\sum_{j=1}^{2}G_j(\phi)=(f,\phi)_{M,N,\beta,\Omega},\quad\forall\phi\in V_{M,N,\beta}(\Omega),\tag{3.6}$$

where

$$G_1(\phi)=a_\mu(U,\phi)-a_{M,N,\beta,\mu}(U_{M,N},\phi),\quad G_2(\phi)=(f,\phi)_{M,N,\beta,\Omega}-(f,\phi)_\Omega.$$

Let $\widetilde{U}_{M,N}=u_{M,N}-U_{M,N}$. Subtracting (3.6) from (3.4), we obtain that

$$a_{M,N,\beta,\mu}(\widetilde{U}_{M,N},\phi)=\sum_{j=1}^{2}G_j(\phi),\quad\forall\phi\in V_{M,N,\beta}(\Omega).\tag{3.7}$$

Taking $\phi=\widetilde{U}_{M,N}$ in (3.7), we deduce that

$$||\partial_x\widetilde{U}_{M,N}||^2_{M,N,\beta,\Omega}+||\partial_y\widetilde{U}_{M,N}||^2_{M,N,\beta,\Omega}+\mu||\widetilde{U}_{M,N}||^2_{M,N,\beta,\Omega}=\sum_{j=1}^{2}G_j(\widetilde{U}_{M,N}).\tag{3.8}$$

Therefore, it suffices to estimate the terms $|G_j(\widetilde{U}_{M,N})|$.

We first use the Cauchy inequality, (2.33), (2.34) and (3.5) to verity that for integers $q,r\ge1$,

$$\begin{aligned}&\left|G_1(\widetilde{U}_{M,N})\right|=\left|a_\mu(U,\widetilde{U}_{M,N})-a_{M,N,\beta,\mu}(U_{M,N},\widetilde{U}_{M,N})\right|\\ \le&\left|\left(\partial_xU-\partial_{x*}P^1_{M-1,N,\beta,\mu,\Omega}U,\partial_x\widetilde{U}_{M,N}\right)_\Omega+\left(\partial_{x*}P^1_{M-1,N,\beta,\mu,\Omega}U-\partial_xU_{M,N},\partial_x\widetilde{U}_{M,N}\right)_{M,N,\beta,\Omega}\right|\\ &+\left|\left(\partial_yU-\partial_{y*}P^1_{M-1,N,\beta,\mu,\Omega}U,\partial_y\widetilde{U}_{M,N}\right)_\Omega+\left(\partial_{y*}P^1_{M-1,N,\beta,\mu,\Omega}U-\partial_yU_{M,N},\partial_y\widetilde{U}_{M,N}\right)_{M,N,\beta,\Omega}\right|\\ &+\mu\left|\left(U-{}_*P^1_{M-1,N,\beta,\mu,\Omega}U,\widetilde{U}_{M,N}\right)_\Omega+\left({}_*P^1_{M-1,N,\beta,\mu,\Omega}U-U_{M,N},\widetilde{U}_{M,N}\right)_{M,N,\beta,\Omega}\right|\\ \le&\,c(1+\mu)\left(M^{2-2q}+(\beta N)^{1-r}\right)\mathbb{B}^{q,r}_{\beta,\Omega}(U)+\frac{1}{2}||\partial_x\widetilde{U}_{M,N})||^2_\Omega\\ &+\frac{1}{2}||\partial_y\widetilde{U}_{M,N})||^2_\Omega+\frac{1}{4}\mu||\widetilde{U}_{M,N}||^2_\Omega.\end{aligned}\tag{3.9}$$

On the other hand, using (2.36) gives that

$$\begin{aligned}&\left|(f,\widetilde{U}_{M,N})_{M,N,\beta,\Omega}-(f,\widetilde{U}_{M,N})_{\Omega}\right|\\ \le&\frac{c}{\mu}\left(M^{2-2q}+(\beta N)^{1-r}(\ln N)^2\right)\mathbb{C}^{q-1,r}_{M,\beta,\Omega}(f)+\frac{1}{4}\mu||\widetilde{U}_{M,N}||^2_{\Omega}.\end{aligned}\tag{3.10}$$

Inserting (3.9)-(3.10) into (3.8), we use (2.34) to obtain

$$\begin{aligned}&||\partial_x\widetilde{U}_{M,N}||^2_{\Omega}+||\partial_y\widetilde{U}_{M,N}||^2_{\Omega}+\mu||\widetilde{U}_{M,N}||^2_{\Omega}\\ \le&c(\mu+\frac{1}{\mu})\left(M^{2-2q}+(\beta N)^{1-r}(\ln N)^2\right)\left(\mathbb{B}^{q,r}_{\beta,\Omega}(U)+\mathbb{C}^{q-1,r}_{M,\beta,\Omega}(f)\right).\end{aligned}$$

Finally, a combination of the above estimate and (3.5) leads to the following result.

Theorem 3.1. *If $U\in H^1_0(\Omega), f\in C(\Omega)$, and $\mathbb{B}^{q,r}_{\beta,\Omega}(U)$ and $\mathbb{C}^{q-1,r}_{M,\beta,\Omega}(f)$ are finite for integers $r>1, q\ge 1$, then*

$$\begin{aligned}&||\partial_x(U-u_{M,N})||^2_{\Omega}+||\partial_y(U-u_{M,N})||^2_{\Omega}+\mu||U-u_{M,N}||^2_{\Omega}\\ \le&c(\mu+\frac{1}{\mu})\left(M^{2-2q}+(\beta N)^{1-r}(\ln N)^2\right)\left(\mathbb{B}^{q,r}_{\beta,\Omega}(U)+\mathbb{C}^{q-1,r}_{M,\beta,\Omega}(f)\right).\end{aligned}\tag{3.11}$$

3.2 A unsteady problem

Let d be a constant. We consider the unsteady problem

$$\begin{cases}\partial_t W(x,y,t)=\Delta W(x,y,t)+dW(x,y,t)+F(x,y,t), & (x,y)\in\Omega,\ 0<t\le T,\\ W(x,y,t)=g(x,y,t), & (x,y)\text{ on }\Gamma,\ 0<t\le T,\\ W(x,y,t)\to 0, & |x|\text{ or }|y|\to\infty,\ 0<t\le T,\\ W(x,y,0)=W_0, & (x,v)\text{ on }\Omega\cup\Gamma,\end{cases}\tag{3.12}$$

where $g(1,y,t)=g_1(y,t)$, $g(x,1,t)=g_2(x,t)$, $g(-1,y,t)=g_3(y,t)$ and $g(x,-1,t)=g_4(x,t)$. We also assume that $g_1(-1,t)=g_4(1,t)$, $g_1(1,t)=g_2(1,t)$, $g_2(-1,t)=g_3(1,t)$ and $g_3(-1,t)=g_4(-1,t)$. Thus, $g(x,y,t)$ is continuous on the boundary Γ. Set

$$\begin{aligned}\widetilde{W}(x,y,t)=&\frac{1}{4}e^{(1-x^2)(1-y^2)}\Big(2(1-y)g_4(x,t)+2(1+y)g_2(x,t)+2(1-x)g_3(y,t)\\&+2(1+x)g_1(y,t)-(1-x)(1-y)g_3(-1,t)-(1-x)(1+y)g_2(-1,t)\\&-(1+x)(1-y)g_4(1,t)-(1+x)(1+y)g_1(1,t)\Big).\end{aligned}$$

We make the variable transformation

$$W(x,y,t)=U(x,y,t)+\widetilde{W}(x,y,t),\quad f(x,y,t)=F(x,y,t)+\Delta\widetilde{W}(x,y,t)-d\widetilde{W}(x,y,t).$$

Then, (3.12) is reformulated to

$$\begin{cases} \partial_t U(x,y,t) = \Delta U(x,y,t) + dU(x,y,t) + f(x,y,t), & (x,y) \in \Omega,\ 0 < t \leq T, \\ U(x,y,t) = 0, & (x,y) \text{ on } \Gamma,\ 0 < t \leq T, \\ U(x,y,t) \to 0, & |x| \text{ or } |y| \to \infty,\ 0 < t \leq T, \\ U(x,y,0) = U_0 = W_0(x,y) - \widetilde{W}(x,y,0), & (x,y) \text{ on } \Omega \cup \Gamma. \end{cases} \tag{3.13}$$

A weak formulation of (3.13) is to seek solution $U \in L^\infty\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H_0^1(\Omega)\right)$ such that

$$\begin{cases} (\partial_t U(t),u)_\Omega + (\partial_x U(t),\partial_x u)_\Omega + (\partial_y U(t),\partial_y u)_\Omega \\ \qquad = d(U(t),u)_\Omega + (f(t),u)_\Omega, \qquad \forall\, u \in H_0^1(\Omega),\ 0 < t \leq T, \\ U(0) = U_0. \end{cases} \tag{3.14}$$

The composite pseudospectral scheme for (3.14) is to find $u_{M,N} \in V_{M,N,\beta}(\Omega)$ such that

$$\begin{cases} (\partial_t u_{M,N}(t),\phi)_{M,N,\beta,\Omega} + (\partial_x u_{M,N}(t),\partial_x \phi)_{M,N,\beta,\Omega} + (\partial_y u_{M,N}(t),\partial_y \phi)_{M,N,\beta,\Omega} \\ \qquad = d(u_{M,N}(t),\phi)_{M,N,\beta,\Omega} + (f(t),\phi)_{M,N,\beta,\Omega}, \qquad \forall\, \phi \in V_{M,N,\beta}(\Omega),\ 0 < t \leq T, \\ u_{M,N}(0) = I_{M,N,\beta,\Omega} U_0. \end{cases} \tag{3.15}$$

We now deal with the convergence of scheme (3.15). We first consider the case with $d \geq 0$. Let $U_{M,N} = {}_*P^1_{M,N,\beta,\eta,\Omega} U$ and $\eta > 0$. We derive from (3.14) that

$$\begin{cases} (\partial_t U_{M,N}(t),\phi)_{M,N,\beta,\Omega} + (\partial_x U_{M,N}(t),\partial_x \phi)_{M,N,\beta,\Omega} + (\partial_y U_{M,N}(t),\partial_y \phi)_{M,N,\beta,\Omega} \\ \quad + \eta (U_{M,N}(t),\phi)_{M,N,\beta,\Omega} + \sum\limits_{j=1}^{5} G_j(t,\phi) \\ \quad = (d+\eta)(U_{M,N}(t),\phi)_{M,N,\beta,\Omega} + (f(t),\phi)_{M,N,\beta,\Omega},\ \forall\, \phi \in V_{M,N,\beta}(\Omega),\ 0 < t \leq T, \\ U_{M,N}(0) = {}_*P^1_{M,N,\beta,\eta,\Omega} U_0, \end{cases} \tag{3.16}$$

where

$$\begin{aligned} G_1(t,\phi) &= (\partial_t U(t),\phi)_\Omega - (\partial_t U_{M,N}(t),\phi)_{M,N,\beta,\Omega}, \\ G_2(t,\phi) &= (\partial_x U(t),\partial_x \phi)_\Omega - (\partial_x U_{M,N}(t),\partial_x \phi)_{M,N,\beta,\Omega}, \\ G_3(t,\phi) &= (\partial_y U(t),\partial_y \phi)_\Omega - (\partial_y U_{M,N}(t),\partial_y \phi)_{M,N,\beta,\Omega}, \\ G_4(t,\phi) &= d((U_{M,N}(t),\phi)_{M,N,\beta,\Omega} - (U(t),\phi)_\Omega), \\ G_5(t,\phi) &= (f(t),\phi)_{M,N,\beta,\Omega} - (f(t),\phi)_\Omega. \end{aligned}$$

Setting $\widetilde{U}_{M,N} = u_{M,N} - U_{M,N}$ and subtracting (3.16) from (3.15), we obtain that

$$\begin{cases} (\partial_t \widetilde{U}_{M,N}(t),\phi)_{M,N,\beta,\Omega} + (\partial_x \widetilde{U}_{M,N}(t),\partial_x \phi)_{M,N,\beta,\Omega} \\ \quad + (\partial_y \widetilde{U}_{M,N}(t),\partial_y \phi)_{M,N,\beta,\Omega} + \eta (\widetilde{U}_{M,N}(t),\phi)_{M,N,\beta,\Omega} \\ \quad = (d+\eta)(\widetilde{U}_{M,N}(t),\phi)_{M,N,\beta,\Omega} + \sum\limits_{j=1}^{5} G_j(t,\phi),\ \forall\, \phi \in V_{M,N,\beta}(\Omega),\ 0 < t \leq T, \\ \widetilde{U}_{M,N}(0) = I_{M,N,\beta,\Omega} U_0 - {}_*P^1_{M,N,\beta,\eta,\Omega} U_0. \end{cases} \tag{3.17}$$

Take $\phi = 2\widetilde{U}_{M,N}(t)$ in (3.17). Then, we obtain that

$$\begin{aligned}&\partial_t ||\widetilde{U}_{M,N}(t)||^2_{M,N,\beta,\Omega} + 2||\partial_x \widetilde{U}_{M,N}(t)||^2_{M,N,\beta,\Omega} + 2||\partial_y \widetilde{U}_{M,N}(t)||^2_{M,N,\beta,\Omega}\\ &= 2d||\widetilde{U}_{M,N}(t)||^2_{M,N,\beta,\Omega} + 2\sum_{j=1}^{5} G_j(t,\widetilde{U}_{M,N}(t)).\end{aligned} \tag{3.18}$$

We next estimate $|G_j(t,\widetilde{U}_{M,N})|$. Firstly, we use (2.33), (2.34) and (3.5) to deduce that for integers $q,r \geq 1$,

$$\begin{aligned}2\Big|G_1\Big(t,\widetilde{U}_{M,N}(t)\Big)\Big| &= 2\Big|\Big(\partial_t U(t) - {}_*P^1_{M-1,N,\beta,\Omega}\partial_t U(t), \widetilde{U}_{M,N}(t)\Big)_\Omega\\ &\quad + ({}_*P^1_{M-1,N,\beta,\Omega}\partial_t U(t) - \partial_t U_{M,N}(t), \widetilde{U}_{M,N}(t))_{M,N,\beta,\Omega}\Big|\\ &\leq c\frac{1}{\eta}(1+\frac{1}{\eta})\Big(M^{2-2q} + (\beta N)^{1-r}\Big)\mathbb{B}^{q,r}_{\beta,\Omega}(\partial_t U(t)) + \frac{1}{2}\eta||\widetilde{U}_{M,N}(t)||^2_\Omega.\end{aligned} \tag{3.19}$$

Similarly,

$$\begin{aligned}&2\Big|G_2\Big(t,\widetilde{U}_{M,N}(t)\Big) + G_3\Big(t,\widetilde{U}_{M,N}(t)\Big) + G_4\Big(t,\widetilde{U}_{M,N}(t)\Big)\Big|\\ &\leq c(1+\eta)\left(1+d^2\eta^{-2}\right)\Big(M^{2-2q} + (\beta N)^{1-r}\Big)\mathbb{B}^{q,r}_{\beta,\Omega}(U(t))\\ &\quad + ||\partial_x \widetilde{U}_{M,N}(t))||^2_\Omega + ||\partial_y \widetilde{U}_{M,N}(t))||^2_\Omega + \eta||\widetilde{U}_{M,N}(t))||^2_\Omega.\end{aligned} \tag{3.20}$$

Next, using (2.36) yields

$$\begin{aligned}&2|G_5\Big(t,\widetilde{U}_{M,N}(t)\Big)|\\ &\leq \frac{c}{\eta}\Big(M^{2-2q} + (\beta N)^{1-r}(\ln N)^2\Big)\mathbb{C}^{q-1,r}_{M,\beta,\Omega}(f(t)) + \frac{1}{2}\eta||\widetilde{U}_{M,N}(t)||^2_\Omega.\end{aligned} \tag{3.21}$$

Furthermore, with the aid of (2.35) and (3.5), we derive that

$$\begin{aligned}&||\widetilde{U}_{M,N}(0)||^2_\Omega \leq 2||I_{M,N,\beta,\Omega}U_0 - U_0||_\Omega + 2||U_0 - {}_*P^1_{M,N,\beta,\eta,\Omega}U_0||\\ &\leq c(1+\frac{1}{\eta})\Big(M^{2-2q} + (\beta N)^{1-r}(\ln N)^2\Big)\Big(\mathbb{B}^{q,r}_{\beta,\Omega}(U_0) + \mathbb{C}^{q-1,r}_{M,\beta,\Omega}(U_0)\Big).\end{aligned} \tag{3.22}$$

For simplicity of statements, we introduce the following notations,

$$\begin{aligned}&E_{M,N,\beta}(u(t)) = ||u(t)||^2_{M,N,\beta,\Omega} + \int_0^t \Big(||\partial_x u(\xi)||^2_{M,N,\beta,\Omega} + ||\partial_y u(\xi)||^2_{M,N,\beta,\Omega}\Big)d\xi,\\ &E(u(t)) = ||u(t)||^2_\Omega + \int_0^t \left[||\partial_x u(\xi)||^2_\Omega + ||\partial_y u(\xi)||^2_\Omega\right]d\xi.\end{aligned}$$

By substituting (3.19)-(3.21) into (3.18), we obtain

$$\begin{aligned}&\partial_t E_{M,N,\beta}\Big(\widetilde{U}_{M,N}(t)\Big)\\ &\leq 2(d+\eta)E\Big(\widetilde{U}_{M,N}(t)\Big) + c\left((1+d^2)\eta^{-2} + \eta\right)\Big(M^{2-2q} + (\beta N)^{1-r}(\ln N)^2\Big)\\ &\quad \cdot\Big[\mathbb{B}^{q,r}_{\beta,\Omega}(\partial_t U(t)) + \mathbb{B}^{q,r}_{\beta,\Omega}(U(t)) + \mathbb{C}^{q-1,r}_{\beta,\Omega}(f(t))\Big].\end{aligned} \tag{3.23}$$

Due to (2.34), we have that $E(\widetilde{U}_{M,N}(t)) \le cE_{M,N,\beta}(\widetilde{U}_{M,N}(t))$. Thereby, (3.23) reads

$$\begin{aligned}&\partial_t\left(E_{M,N,\beta}\left(\widetilde{U}_{M,N}(t)\right)e^{-2(d+\eta))t}\right)\\&\le c\left((1+d^2)\eta^{-2}+\eta\right)e^{-2(d+\eta))t}\left(M^{2-2q}+(\beta N)^{1-r}(\ln N)^2\right)\\&\quad\cdot\left[\mathbb{B}^{q,r}_{\beta,\Omega}(\partial_t U(t))+\mathbb{B}^{q,r}_{\beta,\Omega}(U(t))+\mathbb{C}^{q-1,r}_{\beta,\Omega}(f(t))\right].\end{aligned} \tag{3.24}$$

Integrating the above inequality with respect to t, and using (2.34) again, we obtain

$$\begin{aligned}E\left(\widetilde{U}_{M,N}(t)\right)&\le E_{M,N,\beta}\left(\widetilde{U}_{M,N}(t)\right)\le c\left(M^{2-2q}+(\beta N)^{1-r}(\ln N)^2\right)\\&\quad\cdot\left(\mathcal{G}_\Omega(U,q,r,\beta,d,\eta,t)+\mathcal{M}_\Omega(U_0,q,r,\beta,d,\eta,t)+\mathcal{F}_\Omega(f,q,r,\beta,d,\eta,t)\right),\end{aligned} \tag{3.25}$$

where

$$\begin{aligned}\mathcal{G}_\Omega(U,q,r,\beta,d,\eta,t)&=\left((1+d^2)\eta^{-2}+\eta\right)e^{2(d+\eta)t}\\&\qquad\int_0^t e^{-2(d+\eta)\xi}\left(\mathbb{B}^{q,r}_{\beta,\Omega}(\partial_t U(\xi))+\mathbb{B}^{q,r}_{\beta,\Omega}(U(\xi))\right)d\xi,\\\mathcal{M}_\Omega(U_0,q,r,\beta,d,\eta,t)&=(1+\frac{1}{\eta})e^{2(d+\eta)t}\left(\mathbb{B}^{q,r}_{\beta,\Omega}(U_0)+\mathbb{C}^{q-1,r}_{\beta,\Omega}(U_0)\right),\\\mathcal{F}_\Omega(f,q,r,\beta,d,\eta,t)&=\left((1+d^2)\eta^{-2}+\eta\right)e^{2(d+\eta)t}\int_0^t e^{-2(d+\eta)\xi}\mathbb{C}^{q-1,r}_{\beta,\Omega}\left(f(\xi)\right)d\xi.\end{aligned}$$

Further, let

$$\begin{aligned}&\mathcal{R}_\Omega(U,q,r,\beta,d,\eta,t)\\&=\mathcal{G}_\Omega(U,q,r,\beta,d,\eta,t)+(1+\eta)\int_0^t\mathbb{B}^{q,r}_{\beta,\eta,\Omega}\left(U(\xi)\right)d\xi+(1+\frac{1}{\eta})\mathbb{B}^{q,r}_{\beta,\eta,\Omega}(U(t)).\end{aligned}$$

Then, a combination of (3.5) and (3.25) leads to the following conclusion.

Theorem 3.2. *If $U\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1_0(\Omega))$, $f\in L^2(0,T;C(\Omega))$, and $\mathcal{R}_\Omega(U,q,r,\beta,d,\eta,t)$, $\mathcal{M}_\Omega(U_0,q,r,\beta,d,\eta,t)$ and $\mathcal{F}_\Omega(f,q,r,\beta,d,\eta,t)$ are finite for integers $r>1,q\ge 1$, then*

$$\begin{aligned}E(U(t)-u_{M,N}(t))&\le c\left(M^{2-2q}+(\beta N)^{1-r}(\ln N)^2\right)\\&\quad\cdot\left(\mathcal{R}_\Omega(U,q,r,\beta,d,\eta,t)+\mathcal{M}_\Omega(U_0,q,r,\beta,d,\eta,t)+\mathcal{F}_\Omega(f,q,r,\beta,d,\eta,t)\right).\end{aligned} \tag{3.26}$$

We next consider the case with $d<0$. In this case, we put $\mu=-d$ and $U_{M,N}={}_*P^1_{M,N,\beta,\mu,\Omega}U$. Then, following the same line as in the derivation of (3.17), we obtain that

$$\begin{cases}\left(\partial_t\widetilde{U}_{M,N}(t),\phi\right)_{M,N,\beta,\Omega}+\left(\partial_x\widetilde{U}_{M,N}(t),\partial_x\phi\right)_{M,N,\beta,\Omega}\\\qquad+\left(\partial_y\widetilde{U}_{M,N}(t),\partial_y\phi\right)_{M,N,\beta,\Omega}+\mu\left(\widetilde{U}_{M,N}(t),\phi\right)_{M,N,\beta,\Omega}\\\qquad=G_1(t,\phi)+G_5(t,\phi)+G_6(t,\phi),\qquad\forall\,\phi\in V_{M,N,\beta}(\Omega),\ 0<t\le T,\\\widetilde{U}_{M,N}(0)=I_{M,N,\beta,\Omega}U_0-{}_*P^1_{M,N,\beta,1,\Omega}U_0,\end{cases} \tag{3.27}$$

where $G_1(t,\phi)$ and $G_5(t,\phi)$ are the same as those in (3.16), and

$$G_6(t,\phi)=a_\mu(U(t),\phi)-a_{M,N,\beta,\mu}(U_{M,N}(t),\phi).$$

Let $\phi=\widetilde{U}_{M,N}(t)$ in (3.27). Like (3.19), (3.20), (3.21) and (3.22), we can verify that

$$\begin{aligned}
&2\Big|G_1\left(t,\widetilde{U}_{M,N}(t)\right)\Big|\le c\left(\mu^{-1}+\mu^{-2}\right)\left(M^{2-2q}+(\beta N)^{1-r}\right)\mathbb{B}^{q,r}_{\beta,\Omega}(\partial_t U(t)+\frac{1}{3}\mu||\widetilde{U}_{M,N}(t)||^2_\Omega,\\
&2\Big|G_5\left(t,\widetilde{U}_{M,N}(t)\right)\Big|\le\frac{c}{\mu}\left(M^{2-2q}+(\beta N)^{1-r}(\ln N)^2\right)\mathbb{C}^{q-1,r}_{M,\beta,\Omega}(f(t))+\frac{1}{3}\mu||\widetilde{U}_{M,N}(t)||^2_\Omega,\\
&2\Big|G_6(t,\widetilde{U}_{M,N}(t))\Big|\le c\left(1+\mu+\mu^{-2}\right)\left(M^{2-2q}+(\beta N)^{1-r}\right)\mathbb{B}^{q,r}_{\beta,\Omega}(U(t))+||\partial_x\widetilde{U}_{M,N}(t))||^2_\Omega\\
&\qquad\qquad +||\partial_y\widetilde{U}_{M,N}(t))||^2_\Omega+\frac{1}{3}\mu||\widetilde{U}_{M,N}(t))||^2_\Omega.
\end{aligned}$$

Besides,

$$||\widetilde{U}_{M,N}(0)||^2_\Omega\le c\left(1+\frac{1}{\mu}\right)\left(M^{2-2q}+(\beta N)^{-r}\right)\left(\mathbb{B}^{q,r}_{\beta,\Omega}(U_0)+\mathbb{C}^{q-1,r}_{M,\beta,\Omega}(U_0)\right).$$

Let

$$E_\mu(u(t))=||u(t)||^2_\Omega+\int_0^t\left(||\partial_x u(\xi)||^2_\Omega+||\partial_y u(\xi)||^2_\Omega+\mu||u(\xi)||^2_\Omega\right)d\xi.$$

Then, by an argument similar to the derivation of (3.26), we reach the error estimate.

Theorem 3.3. *If $U\in L^\infty\left(0,T;L^2(\Omega)\right)\cap L^2\left(0,T;H^1_0(\Omega)\right),f\in L^2(0,T;C(\Omega))$, and the right side of the following inequality is finite for integers $r>1,q\ge 1$, then*

$$\begin{aligned}
&E_\mu\Big(U(t)-u_{M,N}(t)\Big)\\
\le\, & c\left(M^{2-2q}+(\beta N)^{1-r}\ln^2 N\right)\Big[\left(\mu+\mu^{-2}\right)e^t\int_0^t e^{-\xi}\left(\mathbb{B}^{q,r}_{\beta,\Omega}(\partial_t U(\xi))+\mathbb{B}^{q,r}_{\beta,\Omega}(U(\xi))\right)d\xi\\
&+(1+\mu)\int_0^t\mathbb{B}^{q,r}_{\beta,\eta,\Omega}(U(\xi))d\xi+(1+\frac{1}{\mu})\mathbb{B}^{q,r}_{\beta,\eta,\Omega}(U(t))\Big]\\
&+(1+\frac{1}{\mu})\left(\mathbb{B}^{q,r}_{\beta,\eta,\Omega}(U_0)+\mathbb{C}^{q-1,r}_{M,\beta,\Omega}(U_0)\right)+(\mu+\mu^{-2})e^t\int_0^t e^{-\xi}\mathbb{C}^{q-1,r}_{\beta,\Omega}(f(\xi)))d\xi. \qquad (3.28)
\end{aligned}$$

4 Numerical results

We first consider scheme (3.4). In order to derive an efficient algorithm which leads to a symmetrical and spare discrete system, we need two classes of basis functions. The first class of basis functions correspond to the subdomains. For instance, the basis function $\phi^{(j,\beta)}_{m,l}(x,y)$, corresponding to the subdomain Ω_j, vanishes on its boundary and other subdomains. The second class of basis functions correspond to the common boundaries of adjacent subdomains. For example, the basis function $\psi^{(j,\beta)}_l(x,y)$, corresponding to the common boundary $\Gamma_{j,j+1}$, does not vanish on this boundary. But it vanishes on the boundaries $\Gamma_{j-1,j}$ and $\Gamma_{j+1,j+2}$, and vanishes on the subdomains $\Omega_k, k\neq j,j+1$. In addition, all basis functions are polynomials on the related subdomains. Therefore, they are in $V_{M,N,\beta}(\Omega)$. The explicit expressions of $\phi_{m,l}^{(j,\beta)}(x,y)$ and $\psi_l^{(j,\beta)}(x,y)$ were given in [13]. For examples,

$$\phi_{m,l}^{(1,\beta)}(x,y)=\begin{cases}\left(\tilde{\mathcal{L}}_l^{(\beta)}(x-1)-\tilde{\mathcal{L}}_{l+1}^{(\beta)}(x-1)\right)(L_m(y)-L_{m+2}(y)), & \text{in } \Omega_1,\\ 0, & \text{otherwise},\end{cases}$$

$$\phi_{m,l}^{(2,\beta)}(x,y)=\begin{cases}\left(\tilde{\mathcal{L}}_l^{(\beta)}(x-1)-\tilde{\mathcal{L}}_{l+1}^{(\beta)}(x-1)\right)\left(\tilde{\mathcal{L}}_m^{(\beta)}(y-1)-\tilde{\mathcal{L}}_{m+1}^{(\beta)}(y-1)\right), & \text{in } \Omega_2,\\ 0, & \text{otherwise},\end{cases}$$

$$\psi_l^{(1,\beta)}(x,y)=\begin{cases}\frac{1}{2}(1+y)e^{\frac{1}{2}\beta(1-y)}\left(\tilde{\mathcal{L}}_l^{(\beta)}(x-1)-\tilde{\mathcal{L}}_{l+1}^{(\beta)}(x-1)\right), & \text{in } \Omega_1\cup\Omega_2,\\ 0, & \text{otherwise}.\end{cases}$$

In actual computation, we expand the numerical solution $u_{M,N}(x,y)$ as

$$\begin{aligned}u_{M,N}(x,y)=&\sum_{j=1,5}\sum_{l=0}^{N-2}\sum_{m=0}^{M-4}a_{m,l}^{(j)}\phi_{m,l}^{(j,\beta)}(x,y)+\sum_{j=3,7}\sum_{m=0}^{M-4}\sum_{l=0}^{N-2}a_{l,m}^{(j)}\phi_{l,m}^{(j,\beta)}(x,y)\\&+\sum_{j=2,8}\sum_{l=0}^{N-2}\sum_{m=0}^{N-2}a_{m,l}^{(j)}\phi_{m,l}^{(j,\beta)}(x,y)+\sum_{j=4,6}\sum_{m=0}^{N-2}\sum_{l=0}^{N-2}a_{l,m}^{(j)}\phi_{l,m}^{(j,\beta)}(x,y)\\&+\sum_{j=1,4,5,8}\sum_{l=0}^{N-2}b_l^{(j)}\psi_l^{(j,\beta)}(x,y)+\sum_{j=2,3,6,7}\sum_{m=0}^{N-2}b_m^{(j)}\psi_m^{(j,\beta)}(x,y).\end{aligned}\tag{4.1}$$

Inserting (4.1) into (3.4), we obtain a symmetrical and spare discrete system with the unknown coefficients $a_{m,l}^{(j)}$ and $b_l^{(j)}$. For shortening the paper, we leave the detail to the readers. The errors of numerical solutions will be measured by the discrete norm $E_{M,N}^*=||U-u_{M,N}||_{M,N,\beta,\Omega}$.

We now use scheme (3.4) to solve (3.3) with the test function:

$$\begin{aligned}U(x,y)=&\Big(x^2|y^2-1|(y^2-1)^\gamma+(y^2-1)^{\gamma+1}+y^2|x^2-1|(x^2-1)^\gamma\\&+(x^2-1)^{\gamma+1}\Big)e^{-3\sqrt{x^2+1}-3\sqrt{y^2+1}-3\gamma+7}.\end{aligned}$$

It can be checked that if $r-1<\gamma<r$, then $U\in H^r(\Omega)$. But U is not in $H^{r+1}(\Omega)$. Therefore, if $\gamma>0$, then $U\in H^r(\Omega)$ with $r>1$. In this case, Theorem 3.1 ensures the convergence of scheme (3.4). In opposite, if $\gamma=0$, then $U\in H^1(\Omega)$ at most, and scheme (3.4) might not be convergent.

In Fig. 1, we plot the numerical errors $\log_{10}E_{M,N}^*$ with $\gamma=1,2,3$, with various modes $M=N$ and different parameter β. They demonstrate that the errors decay very fast as M becomes large. This fact agrees with the analysis well. It seems that scheme (3.4) with

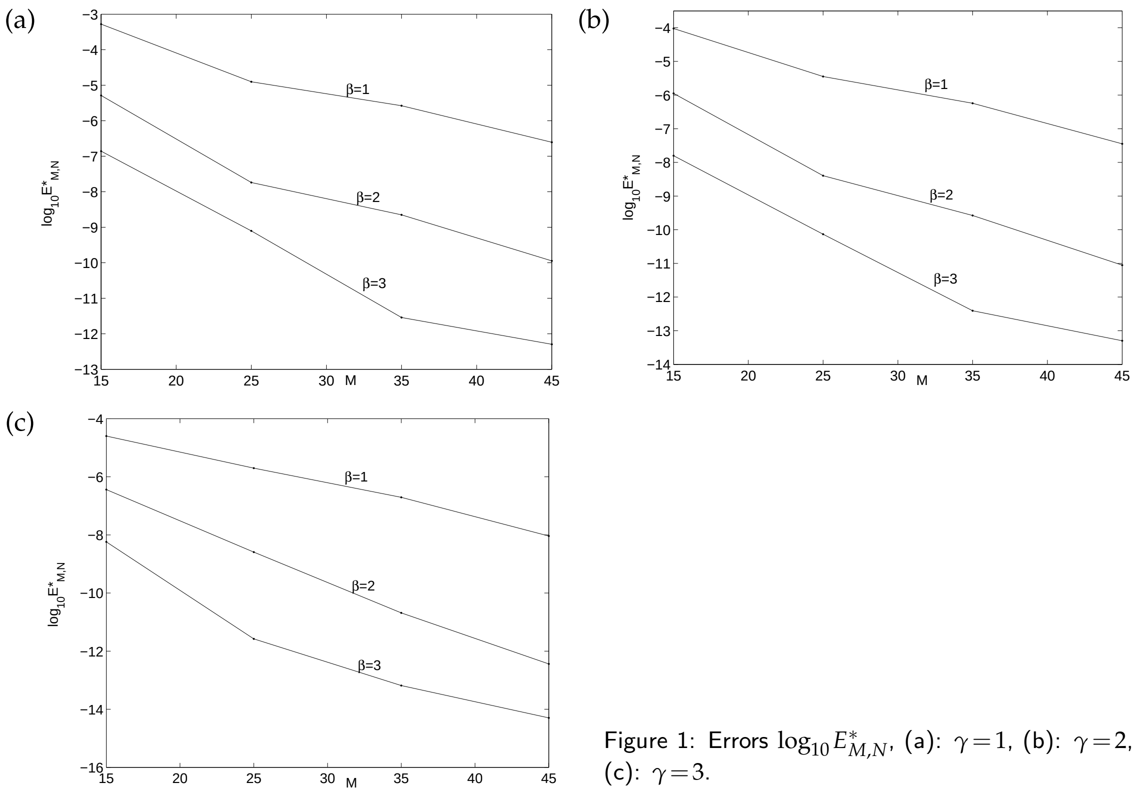

Figure 1: Errors $\log_{10} E^*_{M,N}$, (a): $\gamma=1$, (b): $\gamma=2$, (c): $\gamma=3$.

reasonably larger β provides more accurate numerical results. However, how to choose the best parameter β is still an open problem. Roughly speaking, if the exact solution decays faster as $|x|$ or $|y|$ increases, then it is reasonable to take bigger β. Furthermore, Fig. 1 indicates that for the same modes $M=N$ and the same parameter β, the numerical results with bigger γ are more accurate than those with smaller γ. In other words, the more regular the exact solutions, the smaller the numerical errors. The previous statements show that scheme (3.4) possesses the spectral accuracy.

We now turn to scheme (3.15). We use the Crank-Nicolson discretization in time t, with the mesh step τ. The corresponding fully discrete scheme is as follows,

$$\begin{cases} \dfrac{1}{\tau}\Big(u_{M,N,\tau}(t+\tau)-u_{M,N,\tau}(t),\phi\Big)_{M,N,\beta,\Omega}+\dfrac{1}{2}\Big(\partial_x(u_{M,N,\tau}(t+\tau) \\ \quad +u_{M,N,\tau}(t)),\partial_x\phi\Big)_{M,N,\beta,\Omega}+\Big(\partial_y(u_{M,N,\tau}(t+\tau)+u_{M,N,\tau}(t)),\partial_y\phi\Big)_{M,N,\beta,\Omega} \\ \quad =\dfrac{1}{2}d\Big(u_{M,N,\tau}(t+\tau)+u_{M,N,\tau}(t),\phi\Big)_{M,N,\beta,\Omega}+\dfrac{1}{2}\Big(f(t+\tau)+f(t),\phi\Big)_{M,N,\beta,\Omega}, \\ \qquad\qquad\qquad \forall\,\phi\in V_{M,N,\beta}(\Omega),\ \ t=\tau,2\tau,\cdots,T-\tau, \\ u_{M,N,\tau}(0)=I_{M,N,\beta,\Omega}U_0. \end{cases} \tag{4.2}$$

In actual computation, we expand the numerical solution $u_{M,N,\tau}(x,y,t)$ as in (4.1). But

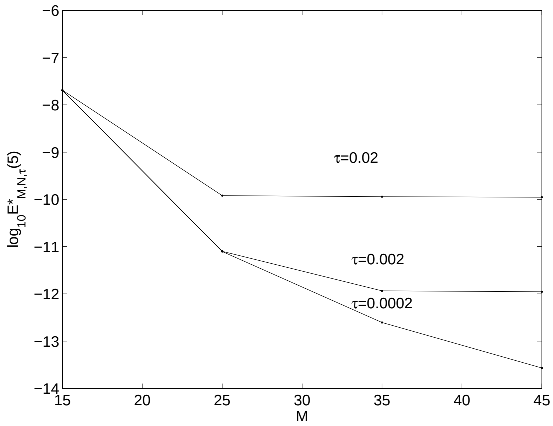

Figure 2: Errors $\log_{10} E^*_{M,N,\tau}(5)$.

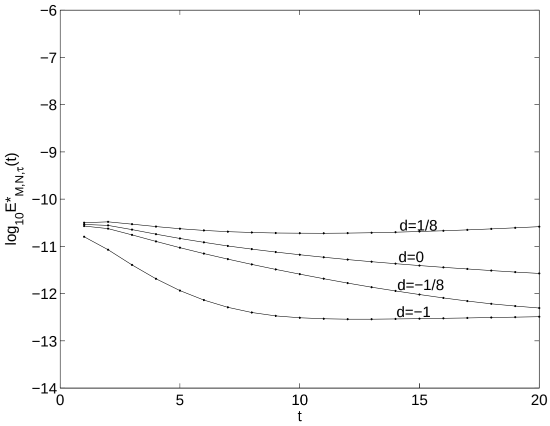

Figure 3: Errors $\log_{10} E^*_{M,N.\tau}(t)$.

all coefficients depend on t. Substituting such expansion into (4.2), we derive an efficient algorithm. At each time step, we need to solve a system with the unknown coefficients $a^{(j)}_{m,l}(t)$ and $b^{(j)}_{l}(t)$. The numerical error at time t is measured by the quantity $E^*_{M,N,\tau}(t) = ||U(t) - u_{M,N,\tau}(t)||_{M,N,\beta,\Omega}$.

We now use scheme (4.2) to solve (3.14) with the test function

$$U(x,y,t) = \Big(x^2|y^2-1|(y^2-1)^\gamma + (y^2-1)^{\gamma+1} + y^2|x^2-1|(x^2-1)^\gamma + (x^2-1)^{\gamma+1} \Big) \cdot (x+y+\sqrt{t+1}) e^{-3\sqrt{x^2+1}-3\sqrt{y^2+1}-3\gamma+7}.$$

In Fig. 2, we plot the numerical errors $\log_{10} E^*_{M,N,\tau}(t)$ at $t=5$, with $\gamma=\beta=3$, $d=-1$ and various values of $M=N$ and τ. It is observed that the errors decay very fast when $M=N$ increase and τ decreases, as predicted by the error estimate (3.28). This fact agrees very well with the analysis. In Fig. 3, we plot the numerical errors $\log_{10} E^*_{M,N,\tau}(t)$ with $\gamma=\beta=3, M=N=35, \tau=0.002$ and $d=-1,-\frac{1}{8},0,\frac{1}{8}$. They indicate the stability of scheme (4.2). But for large positive d, the numerical errors might increase fast as t increases. In fact, in this case, the continuous version (3.14) is also less stable.

5 Concluding remarks

In this paper, we investigated the composite Laguerre-Legendre pseudospectral method for exterior problems with a square obstacle. To do this, we divided the unbounded domain into eight unbounded subdomains. Then we used the mixed Laguerre-Legendre interpolations and two-dimensional Laguerre interpolations for the underlying problems on different subdomains. We also used two classes of specific basis functions so that the numerical solutions match properly on the common boundaries of adjacent subdomains. Moreover, they lead to symmetrical and sparse discrete systems which can be resolved efficiently. We provided the composite Laguerre-Legendre pseudospectral schemes for two model problems. The numerical results demonstrated the spectral accuracy in space of these schemes and confirm the theoretical analysis well. The main idea and techniques used in this work are also applicable to other exterior problems.

In this paper, we developed the mixed Laguerre-Legendre interpolations and two-dimensional Laguerre interpolations on unbounded domains. Some sharp results were established. These results play important role in various pseudospectral methods for unbounded domains, especially for exterior problems. We also developed a technique for pseudospectral methods coupled with domain decompositions, which are useful for spectral element method, p-version of finite element method and other high order methods with complex geometry.

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