

A Two-Scale Asymptotic Analysis of a Time-Harmonic Scattering Problem with a Multi Layered Thin Periodic Domain

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Abstract. The scope of this paper is to show how a two-scale asymptotic analysis, based on a superposition principle, allows us to derive high order approximate boundary conditions for a scattering problem of a time-harmonic wave by a thin and tangentially periodic multi-layered domain. The periods are assumed of the same order of the thickness. New terms like memory effect and variance-covariance ones are observed contrarily to the laminar case. As a result, optimal error estimates are obtained.

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1 Introduction

In industrial word, a wide variety of materials are coated or connected by a thin structure. For example, electronic devices, patch antennas, radar absorbing paints, self-focusing lens are some illustrations of this situation. Many authors have been devoted to solve the problem of the coating effect by developing robust methods for approximating the solution inside the thin layer, see [3, 8, 9, 16, 17, 25–27] and the references therein. Their main approach consists of constructing an equivalent boundary (or transmission) condition which is able to memorize the effect of the thin shell in an approximate way. Our motivation in this paper is to show how this memory effect can be captured in the case of the scattering of a time-harmonic wave by an obstacle coated by a multi layered thin periodic domain, the periods are small of same order of the thickness. More precisely, besides the non commutativity of a two-step procedure, i.e., homogenization for a fixed

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thickness δ followed by an asymptotic analysis for small δ , or vice-versa, it is shown that neither the one neither the other is able to give an answer in our case. So, inspired from the two-scale convergence technique [7, 20, 21] which takes full advantage of the periodicity information, a suitable superposition of test functions oscillating at same order of δ is used to derive correctly some variance-covariance terms. The idea is similar to the one used in [24] for rough surfaces when small details are not visible within a standard homogenization technique. Nevertheless, such an approach leads to a loss of a half power of δ in the rate of convergence when compared with the case of an homogeneous thin layer. This loss is due essentially to a compensation rule that keeps traces of some lower order terms in the proof of the convergence theorem (cf., e.g., [3]). Finally, it is shown how to optimize it by the use of the simple but clever trick (cf., e.g., [28]) making it possible to obtain optimal estimates from non-optimal ones and the existence of the ansatz up to next order only.

In Section 2, a brief description of the model is presented for a 2D situation. In Section 3, a two-scale asymptotic analysis (cf., e.g., [3, 10, 13, 21, 24]) with respect to the thickness and the period permits to justify the terms in the periodic ansatz proposed and a first convergence theorem is obtained for the Neumann case. In Section 4, apparently new to our knowledge, some approximate boundary conditions are derived until the second order which makes the difference significative with respect to the homogeneous or even the laminar cases (cf., e.g., [9, 29]). Mainly, a convergence result is proved and it is shown how to optimize it providing more regularity on the data.

2 The model setting

In all what follows, standards tools from the functional analysis of PDE(s) and differential geometry background are used without comments (cf., e.g., [11, 12, 15]). Let $\Omega_{\delta, \infty}$ be an exterior domain in \mathbb{R}^2 with boundary Γ_δ (compact C^∞ manifold) such that $\Omega_{\delta, \infty} = \Omega_\delta^+ \cup \Gamma \cup \Omega_\infty$. Γ is an interface parallel to Γ_δ and δ is a non-negative small parameter. $\Omega_\delta^+ = \{x \in \Omega_{\delta, \infty} : d(x, \Gamma) \leq \delta\}$ represents the thin layer of thickness δ and Ω_∞ is the exterior domain to the coated scatterer. Let $f \in L^2(\mathbb{R}^2)$ compactly supported in Ω_∞ . From now on, v^+ (respectively v^-) will denote the restriction of a distribution v defined on $\Omega_{\delta, \infty}$ to the subset Ω_δ^+ (respectively Ω_∞). The problem is to find a complex valued distribution u_δ solution of the scattering problem:

$$\Delta u_\delta^- + k^2 u_\delta^- = -f, \quad (2.1)$$

$$\operatorname{div}(\alpha_\delta \nabla u_\delta^+) + k^2 \beta_\delta u_\delta^+ = 0, \quad (2.2)$$

$$u_\delta^+|_\Gamma = u_\delta^-|_\Gamma, \quad (2.3)$$

$$\alpha_\delta \partial_{\mathbf{n}} u_\delta^+|_\Gamma = \partial_{\mathbf{n}} u_\delta^-|_\Gamma, \quad (2.4)$$

$$\lim_{|x| \rightarrow \infty} |x| \left(\nabla u_\delta^- \cdot \frac{x}{|x|} - i k u_\delta^- \right) = 0, \quad (2.5)$$

with either a Neumann boundary condition

$$N: \partial_{\mathbf{n}} u_{\delta}^+ |_{\Gamma_{\delta}} = 0 \quad (2.6)$$

or a Dirichlet condition

$$D: u_{\delta}^+ |_{\Gamma_{\delta}} = 0 \quad (2.7)$$

on the scatterer's boundary. \mathbf{n} will designate different unitary outwardly normal vectors to the corresponding boundaries (oriented all towards the scatterer $\Omega_0 = \mathbb{R}^2 - \bar{\Omega}_{\delta, \infty}$). α_{δ} and β_{δ} are two regular functions related to the contrast and refractive index properties of the periodic coating. They are expected to be periodic in the tangential direction with a small period $\epsilon = d\delta$, of the same order of the thickness parameter. Without loss of generality, one can take $d = 1$. In addition, one assumes the following uniform estimations:

$$c_1 \leq \|\alpha_{\delta}\|_{L^{\infty}(\Omega_{\delta}^+)} \leq c_2, \quad (2.8)$$

$$0 \leq \|\beta_{\delta}\|_{L^{\infty}(\Omega_{\delta}^+)} \leq c_2, \quad (2.9)$$

where $0 < c_1 < c_2$ are two constants independent of δ (sufficiently small). $k > 0$ denotes the wave number. The relations (2.3) and (2.4) are transmission conditions at the interface between the thin layer Ω_{δ}^+ and the exterior domain Ω_{∞} . Finally, (2.5) is the far field Sommerfeld outgoing radiation condition which will be denoted in all what follows by $S.R.C(\cdot)$. The system (2.2)-(2.7) describes the scattering of a time-harmonic wave problem for a TM (or TE) polarization in electromagnetic or a soft (or hard) obstacle in acoustics, according to the boundary conditions considered in (2.6) or (2.7). Standard techniques using Rellich lemma and the Fredholm alternative show the existence and uniqueness of a strong solution in the space $H_{loc}^2(\bar{\Omega}_{\delta, \infty})$ (see [9, 23, 30]).

3 Two-scale asymptotic analysis

Let us focus our analysis on the Neumann boundary condition, the Dirichlet's one is being more straightforward to handle. Using tangential and normal coordinates $(s, t) \in \Gamma \times (0, 1)$ in the tubular manifold Ω_{δ}^+ , and the Dirichlet-Neumann operator $S_{k, \rho}$ associated to the exterior Helmholtz equation for large radius ρ , one obtains the variational formulation of the system (2.2)-(2.6) in a fixed and bounded domain as follows:

$$u_{\delta} \in X_N; \quad \forall v \in X_N, \quad (3.1a)$$

$$\delta a^+(\delta, u_{\delta}^+, v^+) + \delta b^+(\delta, u_{\delta}^+, v^+) + a_k^-(u_{\delta}^-, v^-) = \int_{\Omega} f v^- dx, \quad (3.1b)$$

where the bilinear forms

$$a^+(\delta, u, v) = \int_{\Gamma \times (0,1)} \alpha_\delta \left[\left(1 + \frac{t\delta}{R(s)}\right)^{-1} \partial_s u \partial_s v + \frac{1}{\delta^2} \left(1 + \frac{t\delta}{R(s)}\right) \partial_t u \partial_t v \right] ds dt, \tag{3.2}$$

$$b^+(\delta, u, v) = -k^2 \int_{\Gamma \times (0,1)} \beta_\delta \left(1 + \frac{t\delta}{R(s)}\right) uv ds dt, \tag{3.3}$$

$$a_k^-(u, v) = \int_{\Omega} \nabla u \nabla v dx - k^2 \int_{\Omega} uv dx + \langle S_{k,\rho} u, v \rangle_{-\frac{1}{2}, \frac{1}{2}, \Sigma} \tag{3.4}$$

are defined on the Hilbert space

$$X_N = \left\{ (v^+, v^-) \in H^1(\Gamma \times (0,1)) \times H^1(\Omega) : v^+(\cdot, 0) = v^-(\Gamma) \right\}.$$

Σ is the truncation circle of radius ρ and Ω the bounded annular domain delimited by Σ and Γ . $R(s)$ is the curvature radius of Γ at s . Finally, $\alpha_\delta(s, t) = \alpha(s, t, s/\delta)$ and $\beta_\delta(s, t) = \beta(s, t, s/\delta)$ are some δ -periodic coefficients, i.e. α and β are two functions 1-periodic with respect to the third variable $y = s/\delta$. Following the authors in [2,9,10], one writes formally the expansion:

$$u_\delta^-(x) = u_0^-(x) + \delta u_1^-(x) + \dots + \delta^j u_j^-(x) + \dots, \tag{3.5}$$

$$u_\delta^+(s, t) = u_0^+\left(s, t, \frac{s}{\delta}\right) + \delta u_1^+\left(s, t, \frac{s}{\delta}\right) + \dots + \delta^j u_j^+\left(s, t, \frac{s}{\delta}\right) + \dots, \tag{3.6}$$

where the terms u_j^+ are expected to be in the space $H^1(\Gamma \times (0,1), H_\#^1(0,1))$, such that the following mean transmission condition holds:

$$\int_0^1 u_j^+(\cdot, 0, y) dy = u_j^-(\cdot). \tag{3.7}$$

Let $H_\#^1(0,1)_{/\mathbb{R}}$ be the space of functions ϕ in $H^1(0,1)$ 1-periodic with respect to $y = s/\delta$ such that $\int_0^1 \phi dy = 0$. From now on, one will denote by $\langle \cdot \rangle_\eta$ the arithmetic mean in the variable $\eta \in (0,1)$ of a function, i.e. $\langle \cdot \rangle_\eta = \int_0^1 \cdot d\eta$. The main idea in the construction of the periodic ansatz (3.6) consists in the superposition principle through the splitting

$$u_j^+\left(s, t, \frac{s}{\delta}\right) = \hat{u}_j^+(s, t) + \check{u}_j^+\left(s, \frac{s}{\delta}\right) \tag{3.8}$$

in such a way that the cross derivative $\partial_t \partial_y u_j^+ = 0$. Thus, test functions on $\Gamma \times (0,1)$ must behave like

$$v_\delta^+(s, t) = v^+(s, t) + \delta v_1^+\left(s, \frac{s}{\delta}\right), \tag{3.9a}$$

$$v^+ \in X; v_1^+ \in H^1\left(\Gamma, H_\#^1(0,1)_{/\mathbb{R}}\right). \tag{3.9b}$$

The inverse of $(1+t\delta/R)$ can be expanded by the exact relation

$$\left(1 + \frac{t\delta}{R}\right)^{-1} = 1 - \delta \left(\frac{t}{R}\right) + \dots + \delta^j \left(-\frac{t}{R}\right)^j + \delta^{j+1} \frac{\left(-\frac{t}{R}\right)^{j+1}}{1 + \frac{t\delta}{R}}, \quad (3.10)$$

and plugging formally (3.5), (3.6) and (3.9) in Eq. (3.1), one obtains by taking the mean on the variable y in $(0,1)$ and matching increasing powers of δ a hierarchy of variational equations. In so doing, it will be shown that u_j^- is the unique solution of the following scattering problem at order j

$$\begin{cases} u_j^- \in H_{loc}^1(\bar{\Omega}_\infty), \\ \Delta u_j^- + k^2 u_j^- = -f_j: D'(\Omega_\infty), \\ \partial_n u_j^- = N_j(u_0^-, u_1^-, \dots, u_{j-1}^-): D'(\Gamma), \\ \text{S.R.C}(u_j^-), \end{cases} \quad (3.11)$$

where $f_0 = f$ and $f_j = 0$ for $j \geq 1$. The operators N_j involves tangential derivatives and will be determined (see [9] for the non periodic case) with the help of the following two lemmas.

Lemma 3.1. *Let, $p, g \in L^2(\Gamma \times (0,1))$ such that $\partial_s g \in L^2(\Gamma \times (0,1))$. Then the solution l of the following variational equation:*

$$\begin{cases} L\phi^+ = \int_{\Gamma \times (0,1)} l \partial_t \phi^+ ds dt + \int_{\Gamma \times (0,1)} (g \partial_s \phi^+ + p \phi^+) ds dt = 0, \\ \forall \phi^+ \in H_{0,\Gamma}^1(\Gamma \times (0,1)) \stackrel{\text{def}}{=} \{ \phi^+ \in H^1(\Gamma \times (0,1)) : \phi^+(\cdot, 0) = 0, \Gamma \} \end{cases}$$

is given explicitly by

$$l(s,t) = - \int_t^1 (p(s,\xi) - \partial_s g(s,\xi)) d\xi. \quad (3.12)$$

In addition, if $\phi^+(\cdot, 0) \neq 0$, then

$$L\phi^+ = - \int_\Gamma l(s,0) \phi^+(s,0) ds. \quad (3.13)$$

Lemma 3.2. *Let $p_1, g_1 \in L^2(\Gamma, L_\#^2(0,1)_{/\mathbb{R}})$ such that $\partial_s g_1 \in L^2(\Gamma, L_\#^2(0,1)_{/\mathbb{R}})$. Then the solution l_1 of the following variational equation:*

$$L\phi_1^+ = \int_{\Gamma \times (0,1)} l_1 \partial_y \phi_1^+ ds dy + \int_{\Gamma \times (0,1)} (g_1 \partial_s \phi_1^+ + p_1 \phi_1^+) ds dy = 0: \forall \phi_1^+ \in H^1(\Gamma, H_\#^1(0,1)_{/\mathbb{R}})$$

is given explicitly by

$$l_1(s,y) = l_1(s,0) + \int_0^y (p_1(s,\xi) - \partial_s g_1(s,\xi)) d\xi - y \langle p_1(s,\cdot) - \partial_s g_1(s,\cdot) \rangle_y \quad (3.14)$$

and $l_1 \in L^2(\Gamma, H_\#^1(0,1))$.

3.1 The hierarchy of variational equations

3.1.1 Determination of u_0

Matching terms in δ^{-1} , one gets for any $v^+ \in H^1(\Gamma \times (0,1))$

$$\int_{\Gamma \times (0,1)} \langle \alpha \rangle_y \partial_t u_0^+ \partial_t v^+ ds dt = 0. \tag{3.15}$$

As for the homogeneous case [9], Lemma 3.1 shows with the choice $l = \langle \alpha \rangle_y \partial_t u_0^+$, $p = g = 0$ that

$$\partial_t u_0^+ = 0. \tag{3.16}$$

Next, matching the terms in δ^0 gives for any $(v^+, v^-) \in X_N, v_1^+ \in H^1(\Gamma, H_{\#}^1(0,1)_{/\mathbb{R}})$:

$$\begin{aligned} & \int_{\Gamma \times (0,1)} \int_0^1 \alpha [\partial_t u_1^+ \partial_t v^+ + \partial_y u_0^+ (\partial_s v^+ + \partial_y v_1^+)] dy ds dt + a_k^-(u_0^-, v^-) \\ & = \int_{\Omega} f v^- dx. \end{aligned} \tag{3.17}$$

Choosing test functions $v^+ = v^- = 0$, one obtains:

$$\int_{\Gamma} \int_0^1 \langle \alpha \rangle_t \partial_y u_0^+ \partial_y v_1^+ dy ds = 0. \tag{3.18}$$

Now, Lemma 3.2 gives with the choice $l_1 = \langle \alpha \rangle_t \partial_y u_0^+$, $p_1 = g_1 = 0$:

$$\partial_y (\langle \alpha \rangle_t \partial_y u_0^+) = 0 \tag{3.19}$$

and using the y -periodicity of l_1 one obtains the second expected result:

$$\partial_y u_0^+ = 0. \tag{3.20}$$

Consequently, (3.20) and (3.18) lead to:

$$u_0^+(s, \cdot, \cdot) = u_0^-(s) : \forall s \in \Gamma. \tag{3.21}$$

As a result, u_0^- is the unique solution to the scattering problem (3.11) for $j = 0$ with the Neumann boundary condition on Γ :

$$\partial_n u_0^- = 0. \tag{3.22}$$

Remark 3.1. As expected, at order 0 the effect of the thin layer is completely neglected. From now on, one will use frequently the elliptic regularity (cf., e.g. [6]) for the Helmholtz equation $\Delta v + k^2 v = g$, i.e. for any $s \geq 0$, if $g \in H^s(\Omega)$ then $v \in H^{s+\frac{3}{2}}(\Gamma)$.

As a consequence, if $f \in L^2(\Omega)$ then $u_0^- \in H^{\frac{3}{2}}(\Gamma)$ and by virtue of (3.21) one obtains $u_0^+ \in H^1(\Gamma \times (0,1), H_{\#}^1(0,1))$.

3.1.2 Determination of u_1

Applying Lemma 3.1 to (3.17) one obtains under (3.20) the identity $\langle \alpha \rangle_y \partial_t u_1^+ = 0$, i.e.,

$$\partial_t u_1^+ = 0. \quad (3.23)$$

As a result, matching terms in δ^1 gives for any $(v^+, v^-) \in X_N$ and $v_1^+ \in H^1(\Gamma, H_{\#}^1(0,1)_{/\mathbb{R}})$

$$\begin{aligned} & \int_{\Gamma \times (0,1)} \int_0^1 \alpha [\partial_t u_2^+ \partial_t v^+ + (\partial_s u_0^- + \partial_y u_1^+) (\partial_s v^+ + \partial_y v_1^+)] dy ds dt \\ & - k^2 \int_{\Gamma \times (0,1)} \int_0^1 \beta u_0^- v^+ dy ds dt + a_k^- (u_1^-, v^-) = 0. \end{aligned} \quad (3.24)$$

Thus, if $v^- = v^+ = 0$ then

$$\int_{\Gamma} \int_0^1 \langle \alpha \rangle_t (\partial_s u_0^- + \partial_y u_1^+) \partial_y v_1^+ dy ds = 0. \quad (3.25)$$

Similarly, Lemma 3.2 shows with the choice $l_1 = \langle \alpha \rangle_t (\partial_s u_0^- + \partial_y u_1^+)$, $p_1 = g_1 = 0$ that $l_1(s, \cdot) = l_1(s, 0)$. Next, using the y periodicity of u_1^+ and taking the mean in y one obtains directly the first harmonic-moment in y of the arithmetic one in t of the contrast α , designated by $\alpha_0 = \langle 1 / \langle \alpha \rangle_t \rangle_y^{-1}$, such that

$$\langle \alpha \rangle_t (\partial_s u_0^- + \partial_y u_1^+) = \alpha_0(s) \partial_s u_0^-. \quad (3.26)$$

Consequently, if $v_1^+ = 0$ in Eq. (3.24), then

$$\begin{aligned} & \int_{\Gamma \times (0,1)} \langle \alpha \rangle_y \partial_t u_2^+ \partial_t v^+ ds dt + \int_{\Gamma \times (0,1)} \left\langle \frac{\alpha}{\langle \alpha \rangle_t} \right\rangle_y \alpha_0 \partial_s u_0^- \partial_s v^+ ds dt \\ & - k^2 \int_{\Gamma \times (0,1)} \langle \beta \rangle_y u_0^- v^+ ds dt + a_k^- (u_1^-, v^-) = 0. \end{aligned} \quad (3.27)$$

Thus, if $v^+ \in H_{0,\Gamma}^1(\Gamma \times (0,1))$ and $v^- = 0$, then

$$\begin{aligned} & \int_{\Gamma \times (0,1)} \langle \alpha \rangle_y \partial_t u_2^+ \partial_t v^+ ds dt + \int_{\Gamma \times (0,1)} \left\langle \frac{\alpha}{\langle \alpha \rangle_t} \right\rangle_y \alpha_0 \partial_s u_0^- \partial_s v^+ ds dt \\ & - k^2 \int_{\Gamma \times (0,1)} \langle \beta \rangle_y u_0^- v^+ ds dt = 0. \end{aligned} \quad (3.28)$$

Hence, Lemma 3.1 with the choice $l = \langle \alpha \rangle_y \partial_t u_2^+$, $p = -k^2 \langle \beta \rangle_y u_0^-$ and $g = \left\langle \frac{\alpha}{\langle \alpha \rangle_t} \right\rangle_y \alpha_0 \partial_s u_0^-$ gives:

$$l(s, t) = \int_t^1 \left(\partial_s \left\langle \frac{\alpha}{\langle \alpha \rangle_t} \right\rangle_y \alpha_0 \partial_s u_0^- + k^2 \langle \beta \rangle_y u_0^- \right) d\zeta, \quad (3.29)$$

$$l(s, 0) = \partial_s \alpha_0 \partial_s u_0^- + k^2 \beta_0 u_0^-, \quad (3.30)$$

where α_0 is recuperated, thanks to the identity $\langle \alpha / \langle \alpha \rangle_t \rangle_t = 1$. The coefficient $\beta_0 = \langle \beta \rangle_{y,t}$ is the first arithmetic moment of β with respect to the two scales y and t . As a result, if $v^+(\cdot, 0) \neq 0$, then Eq. (3.24) shows that u_1^- is the unique solution to the scattering problem (3.11) for $j=1$ with the Neumann boundary condition on Γ :

$$\partial_n u_1^- = \partial_s \alpha_0 \partial_s u_0^- + k^2 \beta_0 u_0^- \tag{3.31}$$

From Remark 3.1, if $f \in H^{\frac{1}{2}}(\Omega)$ then $u_0^- \in H^2(\Gamma)$, i.e., $\partial_n u_1^- \in L^2(\Gamma)$ and consequently $u_1^- \in H^1(\Omega)$. The regularity of u_1^+ is completed by Eq. (3.26) which gives

$$\partial_y u_1^+ = \left(\frac{\alpha_0}{\langle \alpha \rangle_t} - 1 \right) \partial_s u_0^- \tag{3.32}$$

and with the help of (3.23) $u_1^+ \in H^1(\Gamma \times (0,1), H^1_{\#}(0,1))$.

Remark 3.2. As in homogenization technique [10], one can evaluate u_1^+ in terms of $\partial_s u_0^-$ (according to (3.7) and (3.8)) through the decoupling relation $u_1^+ = u_1^- + w_1(s,y) \partial_s u_0^-$ where $w_1 \in H^1(\Gamma, H^1_{\#}(0,1)_{/\mathbb{R}})$ is an auxiliary variable, unique solution of the following so called basic cell equation at first order in the asymptotic analysis:

$$\partial_y [\langle \alpha \rangle_t (1 + \partial_y w_1)] = 0. \tag{3.33}$$

Nevertheless, this cell equation is not necessary in our analysis since (3.32) gives the desired decoupling.

3.1.3 Determination of u_2

Matching terms in δ^2 and using previous properties of u_0 and u_1 (essentially the relation (3.26)) gives for any $(v^+, v^-) \in X_N$ and $v_1^+ \in H^1(\Gamma, H^1_{\#}(0,1)_{/\mathbb{R}})$

$$\begin{aligned} & \int_{\Gamma \times (0,1)} \langle \alpha \rangle_y \left(\frac{t}{R(s)} \partial_t u_2^+ + \partial_t u_3^+ \right) \partial_t v^+ ds dt \\ & + \int_{\Gamma \times (0,1)} \int_0^1 \alpha \left(-\frac{t}{R(s)} \frac{\alpha_0}{\langle \alpha \rangle_t} \partial_s u_0^- + (\partial_s u_1^+ + \partial_y u_2^+) \right) \partial_s v^+ dy ds dt \\ & - k^2 \int_{\Gamma \times (0,1)} \left(\langle \beta u_1^+ \rangle_y + \frac{t}{R(s)} \langle \beta \rangle_y u_0^- \right) v^+ ds dt + a_k^- (u_2^-, v^-) \\ & + \int_{\Gamma} \int_0^1 \left(-\frac{\alpha_0}{R(s)} \frac{\langle t \alpha \rangle_t}{\langle \alpha \rangle_t} \partial_s u_0^- + \langle \alpha \rangle_t (\partial_s u_1^+ + \partial_y u_2^+) \right) \partial_y v_1^+ dy ds \\ & - k^2 \int_{\Gamma} \int_0^1 \langle \beta \rangle_t u_0^- v_1^+ dy ds = 0. \end{aligned} \tag{3.34}$$

Similarly, if $v^- = v^+ = 0$, then

$$\begin{aligned} & \int_{\Gamma} \int_0^1 \left(-\frac{\alpha_0}{R(s)} \frac{\langle t \alpha \rangle_t}{\langle \alpha \rangle_t} \partial_s u_0^- + \langle \alpha \rangle_t (\partial_s u_1^+ + \partial_y u_2^+) \right) \partial_y v_1^+ dy ds \\ & - k^2 \int_{\Gamma} \int_0^1 \langle \beta \rangle_t u_0^- v_1^+ dy ds = 0. \end{aligned} \tag{3.35}$$

Now, Lemma 3.2 shows with the choice

$$l_1 = -\frac{\alpha_0}{R(s)} \frac{\langle t\alpha \rangle_t}{\langle \alpha \rangle_t} \partial_s u_0^- + \langle \alpha \rangle_t (\partial_s u_1^+ + \partial_y u_2^+), \quad g_1 = 0, \quad p_1 = -k^2 \langle \beta \rangle_t u_0^-$$

that $l_1 \in L^2(\Gamma, H_{\#}^1(0,1))$ and

$$\begin{aligned} l_1(s,0) &= l_1(s,y) - \left[\int_0^y (p_1(s,\xi) - \partial_s g_1(s,\xi)) d\xi - y \langle p_1(s,\cdot) - \partial_s g_1(s,\cdot) \rangle_y \right] \\ &= \langle \alpha \rangle_t (\partial_s u_1^+ + \partial_y u_2^+) - \frac{\alpha_0}{R(s)} \frac{\langle t\alpha \rangle_t}{\langle \alpha \rangle_t} \partial_s u_0^- + k^2 \left(\int_0^y \langle \beta \rangle_t dy' - y\beta_0 \right) u_0^-. \end{aligned} \quad (3.36)$$

Moreover, $\langle \partial_s u_1^+ \rangle_y = \partial_s u_1^-$ by virtue of (3.7) and (3.23). Then taking the mean in y of $l_1(\cdot,0) / \langle \alpha \rangle_t$ gives easily (with the help of the y periodicity of u_2^+)

$$\frac{l_1(\cdot,0)}{\alpha_0} = \partial_s u_1^- - \frac{\alpha_0}{R(s)} \left\langle \frac{\langle t\alpha \rangle_t}{\langle \alpha \rangle_t^2} \right\rangle_y \partial_s u_0^- + k^2 \left\langle \frac{\int_0^y \langle \beta \rangle_t dy' - y\beta_0}{\langle \alpha \rangle_t} \right\rangle_y u_0^- \quad (3.37)$$

and with the help of (3.36) and (3.37) one obtains directly

$$(\partial_s u_1^+ + \partial_y u_2^+) = \frac{\alpha_0}{\langle \alpha \rangle_t} \left(\partial_s u_1^- + \frac{\tilde{\alpha}}{R(s)} \partial_s u_0^- - k^2 \tilde{\beta} u_0^- \right), \quad (3.38)$$

where the coefficients $\tilde{\alpha}$ and $\tilde{\beta}$ are given by

$$\tilde{\alpha} = \frac{\langle t\alpha \rangle_t}{\langle \alpha \rangle_t} - \alpha_0 \left\langle \frac{\langle t\alpha \rangle_t}{\langle \alpha \rangle_t^2} \right\rangle_y, \quad (3.39)$$

$$\tilde{\beta} = \frac{\int_0^y \langle \beta(s,\cdot,y') \rangle_t dy' - y\beta_0}{\alpha_0} - \left\langle \frac{\int_0^y \langle \beta(s,\cdot,y') \rangle_t dy' - y\beta_0}{\langle \alpha \rangle_t} \right\rangle_y. \quad (3.40)$$

In other hand, $\langle \beta u_1^+ \rangle_y$ is determined as follow: Integrating (3.32) one obtains $u_1^+ = u_1^- + w_1(s,y) \partial_s u_0^-$, where

$$w_1 = \alpha_0 \int_0^y \frac{dy'}{\langle \alpha \rangle_t} - y - \left\langle \alpha_0 \int_0^y \frac{dy'}{\langle \alpha \rangle_t} - y \right\rangle_y.$$

Consequently,

$$\langle \beta u_1^+ \rangle_y = \langle \beta \rangle_y u_1^- + \langle \beta w_1 \rangle_y \partial_s u_0^-. \quad (3.41)$$

Now, if $v_1^+ = 0$ in Eq. (3.34), then for any $v = (v^+, v^-) \in X_N$

$$\begin{aligned} &\int_{\Gamma \times (0,1)} \langle \alpha \rangle_y \left(\frac{t}{R(s)} \partial_t u_2^+ + \partial_t u_3^+ \right) \partial_t v^+ ds dt \\ &+ \int_{\Gamma \times (0,1)} \int_0^1 \alpha \frac{\alpha_0}{\langle \alpha \rangle_t} \left(\partial_s u_1^- + \frac{\alpha' - t}{R(s)} \partial_s u_0^- - k^2 \beta' u_0^- \right) \partial_s v^+ dy ds dt \\ &- k^2 \int_{\Gamma \times (0,1)} \left(\langle \beta \rangle_y u_1^- + \langle \beta w_1 \rangle_y \partial_s u_0^- + \frac{t}{R(s)} \langle \beta \rangle_y u_0^- \right) v^+ ds dt + a_k^- (u_2^-, v^-) = 0. \end{aligned} \quad (3.42)$$

This last equation is solved for $v^- = 0$ by Lemma 3.1 with the choice

$$\begin{aligned} l &= \langle \alpha \rangle_y \left(\frac{t}{R(s)} \partial_t u_2^+ + \partial_t u_3^+ \right), \\ p &= -k^2 \left[\langle \beta \rangle_y \left(u_1^- + \frac{t}{R} u_0^- \right) + \langle \beta w_1 \rangle_y \partial_s u_0^- \right], \\ g &= \alpha_0 \left\langle \frac{\alpha}{\langle \alpha \rangle_t} \right\rangle_y \partial_s u_1^- + \frac{\alpha_0}{R} \left\langle \frac{(\alpha' - t)\alpha}{\langle \alpha \rangle_t} \right\rangle_y \partial_s u_0^- - k^2 \alpha_0 \left\langle \frac{\alpha \beta'}{\langle \alpha \rangle_t} \right\rangle_y u_0^- \end{aligned}$$

and gives $l(s,0) = - \int_0^1 (p(s,\xi) - \partial_s g(s,\xi)) d\xi$, i.e.,

$$\begin{aligned} l(s,0) &= \partial_s \alpha_0 \partial_s u_1^- + k^2 \beta_0 u_1^- + \partial_s \left(-\frac{\alpha_{1\#}}{R} \right) \partial_s u_0^- + k^2 \left(\frac{\beta_1}{R} \right) u_0^- \\ &\quad + k^2 (\beta_{\#} \partial_s u_0^- - \partial_s (C_{\alpha\beta} u_0^-)), \end{aligned} \tag{3.43}$$

where the coefficients $\alpha_{1\#}$, β_1 , $\beta_{\#}$ and $C_{\alpha\beta}$ are given by

$$\begin{aligned} \alpha_{1\#}(s) &= \alpha_0 \left\langle \frac{\alpha(t - \tilde{\alpha})}{\langle \alpha \rangle_t} \right\rangle_{y,t}, \\ \beta_1(s) &= \langle \beta \rangle_{y,t}, \quad \beta_{\#}(s) = \langle \beta w_1 \rangle_{y,t}, \\ C_{\alpha\beta}(s) &= \alpha_0 \left\langle \frac{\alpha \tilde{\beta}}{\langle \alpha \rangle_t} \right\rangle_{y,t}, \end{aligned}$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are defined in (3.39) and (3.40). As a result, if $v^- \neq 0$, then Eq. (3.42) shows that u_2^- is the unique solution to the scattering problem (3.11) for $j=2$ with the Neumann boundary condition on Γ :

$$\begin{aligned} \partial_n u_2^- &= \partial_s \alpha_0 \partial_s u_1^- + k^2 \beta_0 u_1^- + \partial_s \left(-\frac{\alpha_{1\#}}{R} \right) \partial_s u_0^- \\ &\quad + k^2 \left[\left(\frac{\beta_1}{R} - \partial_s C_{\alpha\beta} \right) + (\beta_{\#} - C_{\alpha\beta}) \partial_s \cdot \right] u_0^-. \end{aligned} \tag{3.44}$$

Consequently, if $f \in H^{\frac{3}{2}}(\Omega)$ then $u_0^- \in H^3(\Gamma)$, i.e., $u_1^- \in H^2(\Gamma)$, which leads to $\partial_n u_2^- \in L^2(\Gamma)$. Thus, $u_2^- \in H^1(\Omega)$ and by (3.38) and the regularity of the coefficients α and β , one obtains u_2^+ in $H^1(\Gamma \times (0,1), H_{\#}^1(0,1))$.

Remark 3.3. It is easy to check that in the case of homogeneous layers [9], both $C_{\alpha\beta}$ and $\beta_{\#}$ are vanishing while $\alpha_{1\#}$ and β_1 are reduced respectively to the first order moments in t of α and β as obtained in [29]. They seem to outline a major difference, if one compares (3.44) with the laminar case [16, 29], i.e., $\epsilon = o(\delta)$. More precisely, one can see them as memory terms (see magnetization effect in the case of electromagnetism). For example, if one considers a thin multi-layered domain tangentially periodic, then $\beta_{\#}$ captures the

memory effect inside each layer while $\alpha_{1\#}$ deals with the variance in the periodicity between the different layers. The coefficient $C_{\alpha\beta}$ represents a covariance term which exists only when both α and β are periodic.

Remark 3.4. Following Remark 3.2, one can introduce a second order auxiliary variable (according to the superposition principle and the mean transmission condition) $w_2 \in H^2(\Gamma, H_{\#}^1(0,1)_{/\mathbb{R}})$ such that $u_2^+ = w_2 \partial_s u_1^+ + (1-t)u_2^-$ and w_2 will be the solution of a second order non-homogeneous differential equation so called basic cell equation at second order in the asymptotic analysis. In fact, these equations are solved implicitly by Lemma 3.2.

3.2 Convergence analysis for the Neumann case

Due to the unboundedness of the tangential derivatives, the convergence result for the truncated ansatz

$$\phi_{\delta}^j = u_0 + \delta u_1 + \dots + \delta^j u_j \quad (3.45)$$

will be stated in the larger Hilbert space

$$Y_N = \left\{ (v^+, v^-) \in H^1(0,1, L^2(\Gamma)) \times H^1(\Omega) : v^+(\cdot, 0) = v^- \Gamma \right\}$$

and is based essentially on the stability argument of Bendali-Lemrabet [9].

Theorem 3.1. For any $j=0,1,2$, there exists a constant c independent of δ and the source term f , such that:

$$\|u_{\delta} - \phi_{\delta}^0\|_Y \leq c\delta^{\frac{1}{2}} \|f\|_{0,\Omega}, \quad (3.46)$$

$$\|u_{\delta} - \phi_{\delta}^j\|_Y \leq c\delta^j \|f\|_{j-\frac{1}{2},\Omega}, \quad j=1,2. \quad (3.47)$$

Proof. The letter c will denote a generic constant for different estimations. Clearly, the estimate (3.46) is simpler to establish. Indeed, it will be sufficient to estimate the linear form defined on X_N by

$$L_{\delta}^{(0)}v = \delta a^+ (\delta, u_{\delta}^+ - u_0^+, v^+) + \delta b^+ (\delta, u_{\delta}^+ - u_0^+, v^+) + a_k^- (u_{\delta}^- - u_0^-, v^-).$$

Equations verified by u_{δ} and u_0 lead directly to

$$L_{\delta}^{(0)}v = -\delta \int_{\Omega^+} \alpha_{\delta} \left(1 + \frac{t\delta}{R(s)}\right)^{-1} \partial_s u_0^- \partial_s v^+ ds dt + \delta k^2 \int_{\Omega^+} \beta_{\delta} \left(1 + \frac{t\delta}{R(s)}\right) u_0^- v^+ ds dt. \quad (3.48)$$

Note that if $f \in L^2(\Omega)$ then $u_0 \in X_N$. Hence, under the uniform estimations (2.8) and (2.9), there exists a constant c independent of δ (small enough) and $f \in L^2(\Omega)$ such that:

$$\left| L_{\delta}^{(0)}v \right| \leq c \|f\|_{0,\Omega} \left(\delta \|\partial_s v^+\|_{0,\Gamma \times (0,1)} + \delta \|v^+\|_{0,\Gamma \times (0,1)} \right). \quad (3.49)$$

Afterward, by the following standard argument

$$\|v^+\|_{0,\Gamma \times (0,1)} \leq c \left(\|\partial_t v^+\|_{0,\Gamma \times (0,1)} + \|v^-\|_{1,\Omega} \right), \tag{3.50}$$

one obtains the desired estimation for $j=0$, i.e.,

$$\left| L_\delta^{(0)} v \right| \leq c \|f\|_{0,\Omega} \delta^{\frac{1}{2}} \left(\delta^{\frac{1}{2}} \|\partial_s v^+\|_{0,\Gamma \times (0,1)} + \delta^{-\frac{1}{2}} \|\partial_t v^+\|_{0,\Gamma \times (0,1)} + \|v^-\|_{1,\Omega} \right). \tag{3.51}$$

Finally, the proof of (3.46) is achieved by the stability theorem in [9].

Similarly, the inequality (3.47) for $j=1$ holds if one estimates the following linear form defined on X_N by

$$\begin{aligned} L_\delta^{(1)} v &= \delta a^+ (\delta, u_\delta^+ - u_0^+ - \delta u_1^+, v^+) + \delta b^+ (\delta, u_\delta^+ - u_0^+ - \delta u_1^+, v^+) \\ &\quad + a_k^- (u_\delta^- - u_0^- - \delta u_1^-, v^-) \\ &= L_\delta^{(0)} v - \delta^2 a^+ (\delta, u_1^+, v^+) - \delta^2 b^+ (\delta, u_1^+, v^+) - \delta a_k^- (u_1^-, v^-). \end{aligned}$$

Thus, based on the derivation rule $\partial_s \rightarrow \partial_s + \frac{1}{\delta} \partial_y$ and the equations satisfied by u_δ, u_0 and u_1 (essentially the decoupling relation (3.32)), one obtains

$$\begin{aligned} L_\delta^{(1)} v &= \delta \left(\begin{aligned} & - \int_{\Omega^+} \alpha_\delta \left(1 - \frac{t\delta}{R(s)} + \dots \right) (\partial_s u_0^- + \partial_y u_1^+ (s, t, \frac{s}{\delta})) \partial_s v^+ ds dt \\ & + k^2 \int_{\Omega^+} \beta_\delta \left(1 + \frac{t\delta}{R(s)} \right) u_0^- v^+ ds dt - a_k^- (u_1^-, v^-) \end{aligned} \right) \\ &\quad + \delta^2 \left(\begin{aligned} & - \int_{\Omega^+} \alpha_\delta \left(1 - \frac{t\delta}{R(s)} + \dots \right) \partial_s u_1^+ (s, t, \frac{s}{\delta}) \partial_s v^+ ds dt \\ & + k^2 \int_{\Omega^+} \beta_\delta \left(1 + \frac{t\delta}{R(s)} \right) u_1^+ (s, t, \frac{s}{\delta}) v^+ ds dt \end{aligned} \right). \end{aligned} \tag{3.52}$$

In the homogeneous case [9], the terms weighted by δ in $L_\delta^{(1)} v$ are reduced to zero because $\partial_s u_0^-$ is not coupled with $\partial_y u_1^+$. Nevertheless, although one can control them by the weak convergence property of oscillating functions to their mean values (cf., e.g., [2]), they must decrease the rate of convergence of the solution with a loss of a half power in δ as follow: Let us denote these bad terms of order δ remaining in $L_\delta^{(1)} v$ by

$$B_\delta^{(1)} v = \delta \left(- \int_{\Omega^+} \alpha_\delta \left(\partial_s u_0^- + \partial_y u_1^+ \left(s, t, \frac{s}{\delta} \right) \right) \partial_s v^+ ds dt + k^2 \int_{\Omega^+} \beta_\delta u_0^- v^+ ds dt - a_k^- (u_1^-, v^-) \right). \tag{3.53}$$

Then, the y -derivative of u_1^+ can be handled by writing:

$$\alpha_\delta \left(\partial_s u_0^- + \partial_y u_1^+ \left(s, t, \frac{s}{\delta} \right) \right) = \frac{\alpha_\delta}{\langle \alpha_\delta \rangle_t} \left[\langle \alpha_\delta \rangle_t \left(\partial_s u_0^- + \partial_y u_1^+ \left(s, t, \frac{s}{\delta} \right) \right) \right]$$

which with the help of relation (3.32) becomes

$$\alpha_\delta \left(\partial_s u_0^- + \partial_y u_1^+ \left(s, t, \frac{s}{\delta} \right) \right) = \frac{\alpha_\delta}{\langle \alpha_\delta \rangle_t} \alpha_0 \partial_s u_0^-.$$

Hence

$$B_\delta^{(1)}v = \delta \left(- \int_{\Omega^+} \frac{\alpha_\delta}{\langle \alpha_\delta \rangle_t} \alpha_0 \partial_s u_0^- \partial_s v^+ ds dt + k^2 \int_{\Omega^+} \beta_\delta u_0^- v^+ ds dt - a_k^- (u_1^-, v^-) \right) \quad (3.54)$$

and Eq. (3.27) leads to the compensation rule

$$B_\delta^{(1)}v = \delta \int_{\Omega^+} \langle \alpha \rangle_y \partial_t u_2^+ \partial_t v^+ ds dt + \delta \int_{\Omega^+} \left[\left\langle \frac{\alpha}{\langle \alpha \rangle_t} \right\rangle_y - \frac{\alpha_\delta}{\langle \alpha_\delta \rangle_t} \right] \alpha_0 \partial_s u_0^- \partial_s v^+ ds dt - k^2 \delta \int_{\Omega^+} \left[\langle \beta \rangle_y - \beta \left(s, t, \frac{s}{\delta} \right) \right] u_0^- v^+ ds dt. \quad (3.55)$$

Now, thanks to the weak convergence of $\beta_\delta = \beta(s, t, s/\delta)$ to its mean $\langle \beta \rangle_y$ and of $\alpha_\delta / \langle \alpha_\delta \rangle_t$ to $\langle \alpha / \langle \alpha \rangle_t \rangle_y$ in $L^2(\Omega^+)$ and the regularity assumption $f \in H^{\frac{1}{2}}(\Omega)$ which leads to u_0^- in $H^1(\Gamma)$ and $\langle \alpha \rangle_y \partial_t u_2^+$ in $L^2(\Omega^+)$ (see relation (3.29)), one obtains

$$\int_{\Omega^+} \left[\left\langle \frac{\alpha}{\langle \alpha \rangle_t} \right\rangle_y - \frac{\alpha_\delta}{\langle \alpha_\delta \rangle_t} \right] \alpha_0 \partial_s u_0^- \partial_s v^+ ds dt \rightarrow 0, \quad \text{as } \delta \rightarrow 0, \quad (3.56)$$

$$\int_{\Omega^+} \left[\langle \beta \rangle_y - \beta_\delta \right] u_0^- v^+ ds dt \rightarrow 0, \quad \text{as } \delta \rightarrow 0. \quad (3.57)$$

Consequently, there exists $c > 0$ such that for any $v^+ \in H^1(\Omega^+)$, for any $\varepsilon > 0$, the following estimation holds for any δ (sufficiently small):

$$\left| B_\delta^{(1)}v \right| \leq c \|f\|_{\frac{1}{2}, \Omega} \delta \|\partial_t v^+\|_{0, \Omega^+} + \varepsilon \delta. \quad (3.58)$$

Then, taking $\varepsilon = c \|f\|_{\frac{1}{2}, \Omega} \|v^+\|_{0, \Omega^+}$ strictly positive, one obtains:

$$\left| B_\delta^{(1)}v \right| \leq c \delta \|f\|_{\frac{1}{2}, \Omega} \left(\|\partial_t v^+\|_{0, \Omega^+} + \|v^+\|_{0, \Omega^+} \right). \quad (3.59)$$

The remaining step is straightforward and leads with the help of (3.50) to the estimation:

$$\left| L_\delta^{(1)}v \right| \leq c \|f\|_{\frac{1}{2}, \Omega} \left(\delta^2 \|\partial_s v^+\|_{0, \Omega^+} + \delta \|\partial_t v^+\|_{0, \Omega^+} + \delta \|v^-\|_{1, \Omega} \right). \quad (3.60)$$

At this stage, one can see why the rate of convergence is only in δ . Indeed, (3.60) can not give more than the first order estimation

$$\left| L_\delta^{(1)}v \right| \leq c \|f\|_{\frac{1}{2}, \Omega} \delta \left(\delta^{\frac{1}{2}} \|\partial_s v^+\|_{0, \Omega^+} + \delta^{-\frac{1}{2}} \|\partial_t v^+\|_{0, \Omega^+} + \|v^-\|_{1, \Omega} \right) \quad (3.61)$$

which, in fact, achieves the proof of (3.47) for $j=1$ by the stability theorem in [9]. Finally, let us sketch the proof of (3.47) for $j=2$. As formerly, one must estimate the following

linear form defined on X_N by

$$\begin{aligned} L_\delta^{(2)}v &= \delta a^+ (\delta, u_\delta^+ - u_0^+ - \delta u_1^+ - \delta^2 u_2^+, v^+) + \delta b^+ (\delta, u_\delta^+ - u_0^+ - \delta u_1^+ - \delta^2 u_2^+, v^+) \\ &\quad + a_k^- (u_\delta^- - u_0^- - \delta u_1^- - \delta^2 u_2^-, v^-) \\ &= L_\delta^{(1)}v - \delta^3 a^+ (\delta, u_2^+, v^+) - \delta^3 b^+ (\delta, u_2^+, v^+) - a_k^- (u_2^-, v^-). \end{aligned}$$

In addition to the results obtained in the previous case, one adds the properties and equations related to the term u_2 , uses the derivation rule $\partial_s \rightarrow \partial_s + \frac{1}{\delta} \partial_y$ for the terms in $(s, t, s/\delta)$ and obtains

$$\begin{aligned} L_\delta^{(2)}v &= L_\delta^{(1)}v - \delta^3 \int_{\Omega^+} \alpha_\delta \left(1 - \frac{t\delta}{R(s)} + \dots \right) \left(\partial_s u_2^+ + \frac{1}{\delta} \partial_y u_2^+ \right) \partial_s v^+ ds dt \\ &\quad - \delta \int_{\Omega^+} \alpha_\delta \left(1 + \frac{t\delta}{R(s)} \right) \partial_t u_2^+ \partial_t v^+ ds dt \\ &\quad + \delta^3 k^2 \int_{\Omega^+} \beta_\delta \left(1 + \frac{t\delta}{R(s)} \right) u_2^+ v^+ ds dt - \delta^2 a_k^- (u_2^-, v^-). \end{aligned}$$

Now, the terms in δ^k for $k < 3$ (essentially those containing tangential derivatives) become the bad ones in the case of periodic layers. Fortunately, they are compensated as in (3.55) in the hierarchy of equations by the weak convergence property but once more at the cost of a half power of δ in the rate of convergence. At least, it is not worse to take advantage of this weak convergence together with the properties of u_1 and u_2 (essentially (3.42) and (3.43)) and the regularity assumption $f \in H^{\frac{3}{2}}(\Omega)$ in order to check that $\|v^+\|_{0,\Omega^+}$ still be weighted by only the lower power δ^2 and not more. Doing so and with the help of (3.50) one obtains the estimation

$$\left| L_\delta^{(2)}v \right| \leq c \|f\|_{\frac{3}{2},\Omega} \left(\delta^3 \|\partial_s v^+\|_{0,\Omega^+} + \delta^2 \|\partial_t v^+\|_{0,\Omega^+} + \delta^2 \|v^-\|_{1,\Omega} \right) \tag{3.62}$$

which in turn gives the expected one:

$$\left| L_\delta^{(2)}v \right| \leq c \|f\|_{\frac{3}{2},\Omega} \delta^2 \left(\delta^{\frac{1}{2}} \|\partial_s v^+\|_{0,\Omega^+} + \delta^{-\frac{1}{2}} \|\partial_t v^+\|_{0,\Omega^+} + \|v^-\|_{1,\Omega} \right). \tag{3.63}$$

Consequently, the proof is terminated by the stability theorem in [9]. □

4 Neumann approximate boundary conditions

In the truncated ansatz at order j given by (3.45) one replaces each term u_l^- by $\delta^{j-l} u_j^{EN}$ and obtains for the Neumann condition case the following scattering problem for the unknown u_j^{EN} (referred to Engquist-Nédélec [17])

$$\begin{cases} u_j^{EN} \in H^1(\Omega), \\ \Delta u_j^{EN} + k^2 u_j^{EN} = -f : D'(\Omega), \\ \partial_n u_j^{EN} + Z_\delta^{(j)} u_j^{EN} = 0 : D'(\Gamma), \\ \partial_n u_j^{EN} + S_k u_j^{EN} = 0 : D'(\Sigma), \end{cases} \tag{4.1}$$

where

$$Z_\delta^{(j)} = Z^{(0)} + \delta Z^{(1)} + \dots + \delta^j Z^{(j)} \quad (4.2)$$

is the approximate Dirichlet-Neumann (or impedance boundary) operator at order j related to the thin periodic layer such that

$$Z^{(0)} = 0, \quad (4.3)$$

$$Z^{(1)} = -(\partial_s \alpha_0 \partial_s + k^2 \beta_0), \quad (4.4)$$

$$Z^{(2)} = \partial_s \left(\frac{\alpha_{1\#}}{R} \right) \partial_s - k^2 \left[\left(\frac{\beta_1}{R} - \partial_s C_{\alpha\beta} \right) + (\beta_\# - C_{\alpha\beta}) \partial_s \right] \quad (4.5)$$

and α_0 , β_0 , $\alpha_{1\#}$, β_1 and $\beta_\#$ are the effective-homogenized coefficients of the layer. The rationale of this model is that the truncated ansatz ϕ_δ^{j-} satisfies the same problem (4.1) excepted for the boundary condition on Γ which is not homogeneous and reads as follows

$$\partial_n \phi_\delta^{j-} + Z_\delta^{(j)} \phi_\delta^{j-} = \delta^{j+1} \rho_\delta^j : D'(\Gamma), \quad (4.6)$$

where the right hand side is given by:

$$\rho_\delta^0 = 0, \quad \rho_\delta^1 = -Z^{(1)} u_1^-, \quad (4.7)$$

$$\rho_\delta^2 = -Z^{(1)} u_2^- - Z^{(2)} (u_1^- + \delta u_2^-).$$

The variational formulation of (4.1) reads as follows (cf., e.g., [19])

$$\begin{cases} u_j^{EN} \in V^{(m)}; \forall v \in V^{(m)}, \\ a_k^- (u_j^{EN}, v) + \sum_{l=0}^j \delta^l (a_\Gamma^l (u_j^{EN}, v) + b_\Gamma^l (u_j^{EN}, v)) = \int_\Omega f v dx, \end{cases} \quad (4.8)$$

where $V^{(m)}$ is the Hilbert space defined by

$$V^{(m)} = \left\{ v \in H^1(\Omega); v|_\Gamma \in H^m(\Gamma) \right\} : m=0,1 \quad (4.9)$$

such that $m=0,1,1$ according to $j=0,1,2$ and $a_\Gamma^{(j)}$, $b_\Gamma^{(j)}$ are some continuous bilinear forms defined on $V^{(m)}$ as follows

$$a_\Gamma^{(0)} = b_\Gamma^{(0)} = 0, \quad (4.10)$$

$$a_\Gamma^{(1)}(u, v) = \int_\Gamma \alpha_0 \partial_s u \partial_s v ds, \quad (4.11)$$

$$b_\Gamma^{(1)}(u, v) = -k^2 \int_\Gamma \beta_0 u v ds, \quad (4.12)$$

$$a_\Gamma^{(2)}(u, v) = - \int_\Gamma \frac{\alpha_{1\#}}{R} \partial_s u \partial_s v ds - k^2 \int_\Gamma (\beta_\# - C_{\alpha\beta}) (\partial_s u) v ds, \quad (4.13)$$

$$b_\Gamma^{(2)}(u, v) = -k^2 \int_\Gamma \left(\frac{\beta_1}{R} - \partial_s C_{\alpha\beta} \right) u v ds. \quad (4.14)$$

Remark 4.1. Standard techniques using Rellich Lemma and Fredholm alternative (cf., e.g. [23,30]) lead (for δ small enough) to the existence and uniqueness of a solution to problem (4.8). Indeed, the only new term with respect to [9] (in the case of an homogeneous layer) is the non symmetric one contained in the bilinear form $a_\Gamma^{(2)}$. Fortunately, a dominating power of δ weights this bilinear form and consequently does not affect the dominant coercive part in the Fredholm alternative for δ sufficiently small. As a result, the stability argument in [9] for the problem (4.8) remains true. Consequently, the following theorem holds.

Theorem 4.1. *There exists a constant c independent of δ and the source term f such that the solution u_δ of the variational problem (3.1) and the solution u_j^{EN} of (4.8) satisfies*

$$\|u_\delta^- - u_0^{EN}\|_{1,\Omega} \leq c\delta^{\frac{1}{2}} \|f\|_{0,\Omega}, \tag{4.15}$$

$$\|u_\delta^- - u_j^{EN}\|_{1,\Omega} \leq c\delta^j \|f\|_{j-\frac{1}{2},\Omega} : j=1,2. \tag{4.16}$$

Proof. Since $\phi_\delta^{0-} = u_0^{EN} = u_0^-$, i.e., $\omega_\delta^0 = 0$, (4.15) is a direct consequence of (3.46) obtained in Theorem 3.1. Next, with the help of (3.47) it will be sufficient to estimate the difference $\omega_\delta^j = \phi_\delta^{j-} - u_j^{EN}$ for $j=1$ or 2 . Thus, (4.8) and (4.6) give:

$$a_k^- (\omega_\delta^j, v) + \sum_{l=0}^j \delta^l (a_\Gamma^{(l)} (\omega_\delta^j, v) + b_\Gamma^{(l)} (\omega_\delta^j, v)) = \mathfrak{R}_\delta^{(j)} v, \tag{4.17}$$

where $\mathfrak{R}_\delta^{(j)} v = \delta^{j+1} \int_\Gamma \rho_\delta^j v dx$ and from the definition of ρ_δ^j in (4.7)

$$\begin{aligned} \mathfrak{R}_\delta^{(1)} v &= -\delta^2 \int_\Gamma Z^{(1)} u_1^- v dx, \\ \mathfrak{R}_\delta^{(2)} v &= -\delta^3 \int_\Gamma [Z^{(1)} u_2^- + Z^{(2)} (u_1^- + \delta u_2^-)] v dx. \end{aligned}$$

Note that if $f \in H^{j-\frac{1}{2}}(\Omega)$ then $u_j^- \in H^1(\Gamma)$. Thus, integrating by part on Γ one obtains

$$\begin{aligned} \mathfrak{R}_\delta^{(1)} v &= \delta^2 [a_\Gamma^{(1)} (u_1^-, v) + b_\Gamma^{(1)} (u_1^-, v)], \\ \mathfrak{R}_\delta^{(2)} v &= \delta^3 [a_\Gamma^{(1)} (u_2^-, v) + b_\Gamma^{(1)} (u_2^-, v) + a_\Gamma^2 (u_1^- + \delta u_2^-, v) + b_\Gamma^2 (u_1^- + \delta u_2^-, v)]. \end{aligned}$$

Then, since the bilinear forms $a_\Gamma^{(j)}$ and $b_\Gamma^{(j)}$ are continuous on $V^{(m)}$, there exists a constant c independent of δ (small enough) such that:

$$|\mathfrak{R}_\delta^{(j)} v| \leq c\delta^{j+1} \|f\|_{j-\frac{1}{2},\Omega} \|v\|_{1,\Gamma}.$$

Now, following Remark 4.1 one concludes with the help of the stability theorem in [9] that ω_δ^j satisfies the estimation:

$$\left\| \omega_\delta^j \right\|_{1,\Omega} \leq c\delta^{j+\frac{1}{2}} \|f\|_{j-\frac{1}{2},\Omega}. \quad (4.18)$$

Consequently, by the convergence result Theorem 3.1 and the following decomposition:

$$u_\delta^- - u_j^{EN} = u_\delta^- - \phi_\delta^{j-} + \omega_\delta^j, \quad (4.19)$$

the proof is achieved. \square

Remark 4.2. Surprisingly, the error estimate (4.18) between the approximate solution u_j^{EN} and the truncated ansatz ϕ_δ^{j-} is optimal then the one obtained in Theorem 3.1 where a loss of half power in δ was observed in the rate of convergence. This is actually an advantage for optimizing the error estimate stated in the previous Theorem 4.1 providing only the existence of the asymptotic expansion at order $j+1$ (cf., e.g., [28]). For example, at order $j=1$ one writes

$$\begin{aligned} \|u_\delta^- - (u_0^- + \delta u_1^-)\|_{1,\Omega} &= \|u_\delta^- - (u_0^- + \delta u_1^- + \delta^2 u_2^-) + \delta^2 u_2^-\|_{1,\Omega} \\ &\leq \|u_\delta^- - (u_0^- + \delta u_1^- + \delta^2 u_2^-)\|_{1,\Omega} + \delta^2 \|u_2^-\|_{1,\Omega} \end{aligned}$$

and consequently, the existence of u_2^- in $H^1(\Omega)$ (providing, of course, more regularity on f , i.e. $f \in H^{\frac{3}{2}}(\Omega)$) leads with the help of (3.47) (for $j=2$) and the independence of $\|u_2^-\|_{1,\Omega}$ on δ to the following optimal estimation at order $j=1$:

$$\|u_\delta^- - (u_0^- + \delta u_1^-)\|_{1,\Omega} \leq c\delta^2 \|f\|_{\frac{3}{2},\Omega}.$$

As a result, (4.16) and (4.19) leads to the optimal error estimate

$$\left\| u_\delta^- - u_1^{EN} \right\|_{1,\Omega} \leq c\delta^{\frac{3}{2}} \|f\|_{\frac{3}{2},\Omega},$$

where the half power of δ lost in the rate of convergence in Theorem 3.1 estimation (3.47) is recuperated.

Remark 4.3. The construction of approximate boundary conditions in the Dirichlet's case is more straightforward because of the shifting in the determination of the terms u_j (in fact, matching terms in δ^{j-1} determines completely u_j while the next power δ^j was necessary to achieve this term in the Neumann case). Thus, the new terms involving memory and variance-covariance effect are not observed until order $j=3$.

5 Conclusion

This work is a 2D illustration of the efficiency of the two-scale asymptotic analysis to deal with a double singular perturbation problem. As a result, this affects the rate of convergence by a loss of a half power of δ in the case of thin periodic layers. Furthermore, new terms like memory effects and variance-covariance ones are taken into account within such an approach. The case of an infinite number of thin periodic layers is self-contained in our analysis. However, the treatment of a thin periodic layer with high contrast (for example $\alpha = o(\delta^{-1})$) requires more investigations, as it is known, even in the homogeneous case. Nevertheless, the techniques discussed here are useful to derive effective-homogenized boundary conditions in the case of composite materials, grating, chirality, etc... (cf., [4,5]). Finally, the 3D systems (like full Maxwell's equations) seems to be nontrivial and consequently needs to be handled rigorously using such a multi-scale analysis.

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