

Analytical Solution for Waves Propagation in Heterogeneous Acoustic/Porous Media. Part I: The 2D Case

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Abstract. Thanks to the Cagniard-de Hoop's method we derive the solution to the problem of wave propagation in an infinite bilayered acoustic/poroelastic media, where the poroelastic layer is modelled by the biphasic Biot's model. This first part is dedicated to solution to the two-dimensional problem. We illustrate the properties of the solution, which will be used to validate a numerical code.

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1 Introduction

The computation of analytical solutions for wave propagation problems is of high importance for the validation of numerical computational codes or for a better understanding of the reflexion/transmission properties of the media. Cagniard-de Hoop method [1, 2] is a useful tool to obtain such solutions and permits to compute each type of waves (P wave, S wave, head wave, ...) independently. Although it was originally dedicated to the solution to elastodynamic wave propagation, it can be applied to any transient wave propagation problem in stratified medium. However, as far as we know, few works have been dedicated to the application of this method to poroelastic medium. In [3] the analytical solution to poroelastic wave propagation in an homogeneous 2D medium is

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provided and in [4] the authors compute the analytical expression of the reflected wave at the interface between an acoustic and a poroelastic layer in two dimension but they do not explicit the expression of the transmitted waves. The coupling between acoustic and poroelastic media is of high interest for the simulation of wave propagation for seismics problem in sea bottom or for ultrasound wave propagation in biological tissues, when the human skin can be regarded as a fluid and the bones as a porous medium.

In order to validate computational codes of wave propagation in poroelastic media, we have implemented the codes Gar6more 2D [5] and Gar6more 3D [6] which provide the complete solution (reflected and transmitted waves) of the propagation of wave in stratified 2D or 3D media composed of acoustic/acoustic, acoustic/elastic, acoustic/poroelastic or poroelastic/poroelastic layers. The codes are freely downloadable at

<http://www.spice-rtn.org/library/software/Gar6more2D>

and

<http://www.spice-rtn.org/library/software/Gar6more3D>.

We will focus here on the 2D acoustic/poroelastic case; the three-dimensional and the poroelastic/poroelastic cases will be the object of forthcoming papers. The outline of the paper is as follows: we first present the model problem we want to solve and derive the Green problem from it (Section 1). Then we present the analytical solution to the wave propagation problem in a stratified 2D medium composed of an acoustic and a poroelastic layer (Section 2) and we detail the computation of the solution (Section 3). Finally we show how the analytical solution can be used to validate a numerical code (Section 4).

2 The model problem

We consider an infinite two-dimensional medium ($\Omega = \mathbf{R}^2$) composed of an homogeneous acoustic layer $\Omega^+ = \mathbf{R} \times]0, +\infty[$ and an homogeneous poroelastic layer $\Omega^- = \mathbf{R} \times]-\infty, 0[$ separated by an horizontal interface Γ (see Fig. 1). We first describe the equations in the two layers (Sections 2.1 and 2.2) and the transmission conditions on the interface Γ (Section 2.3). Then we present the Green problem from which we compute the analytical solution (Section 2.4).

2.1 The equation of acoustics

In the acoustic layer we consider the second-order formulation of the wave equation with a point source in space, a regular source function f in time and zero initial conditions:

$$\begin{cases} \ddot{P}^+ - V^{+2} \Delta P^+ = \delta_x \delta_{y-h} f(t), & \text{in } \Omega^+ \times]0, T], \\ \ddot{\mathbf{U}}^+ = -\frac{1}{\rho^+} \nabla P^+, & \text{in } \Omega^+ \times]0, T], \\ P^+(x, y, 0) = 0, \dot{P}^+(x, y, 0) = 0, & \text{in } \Omega^+, \\ \mathbf{U}^+(x, y, 0) = 0, \dot{\mathbf{U}}^+(x, y, 0) = 0, & \text{in } \Omega^+, \end{cases} \quad (2.1)$$

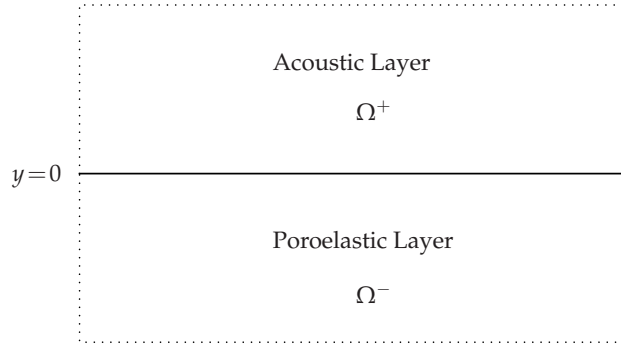


Figure 1: Configuration of the study.

where P^+ is the pressure, \mathbf{U}^+ is the displacement field, V^+ is the celerity of the wave, ρ^+ is the density of the fluid.

2.2 Biot's model

In the second layer we consider the second order formulation of the poroelastic equations [7-9]

$$\left\{ \begin{array}{ll} \rho^- \ddot{\mathbf{U}}_s^- + \rho_f^- \ddot{\mathbf{W}}^- - \nabla \cdot \Sigma^- = 0, & \text{in } \Omega^- \times]0, T], \\ \rho_f^- \ddot{\mathbf{U}}_s^- + \rho_w^- \ddot{\mathbf{W}}^- + \frac{1}{\mathcal{K}^-} \dot{\mathbf{W}}^- + \nabla P^- = 0, & \text{in } \Omega^- \times]0, T], \\ \Sigma^- = \lambda^- \nabla \cdot \mathbf{U}_s^- \mathbf{I}_2 + 2\mu^- \varepsilon(\mathbf{U}_s^-) - \beta^- P^- \mathbf{I}_2, & \text{in } \Omega^- \times]0, T], \\ \frac{1}{m^-} P^- + \beta^- \nabla \cdot \mathbf{U}_s^- + \nabla \cdot \mathbf{W}^- = 0, & \text{in } \Omega^- \times]0, T], \\ \mathbf{U}_s^-(x, 0) = 0, \quad \mathbf{W}^-(x, 0) = 0, & \text{in } \Omega^-, \\ \dot{\mathbf{U}}_s^-(x, 0) = 0, \quad \dot{\mathbf{W}}^-(x, 0) = 0, & \text{in } \Omega^-, \end{array} \right. \quad (2.2)$$

with

$$(\nabla \cdot \Sigma^-)_i = \sum_{j=1}^2 \frac{\partial \Sigma_{ij}^-}{\partial x_j} \quad \forall i=1,2.$$

As usual \mathbf{I}_2 is the identity matrix of $\mathcal{M}_2(\mathbb{R})$ and $\varepsilon(\mathbf{U}_s^-)$ is the solid strain tensor defined by:

$$\varepsilon_{ij}(\mathbf{U}) = \frac{1}{2} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right).$$

In (2.2), the unknowns are: \mathbf{U}_s^- , the displacement field of solid particles; $\mathbf{W}^- = \phi^-(\mathbf{U}_f^- - \mathbf{U}_s^-)$, the relative displacement, \mathbf{U}_f^- being the displacement field of fluid particle and ϕ^- the porosity; P^- , the fluid pressure; Σ^- , the solid stress tensor.

The parameters describing the physical properties of the medium are given by: $\rho^- = \phi^- \rho_f^- + (1 - \phi^-) \rho_s^-$ is the overall density of the saturated medium, with ρ_s^- the density of the solid and ρ_f^- the density of the fluid; $\rho_w^- = a^- \rho_f^- / \phi^-$, where a^- is the tortuosity of the solid matrix; $\mathcal{K}^- = \kappa^- / \eta^-$, where κ^- is the permeability of the solid matrix and η^- is the viscosity of the fluid; m^- and β^- are positive physical coefficients:

$$\beta^- = 1 - K_b^- / K_s^-, \quad m^- = \left[\phi^- / K_f^- + (\beta^- - \phi^-) / K_s^- \right]^{-1},$$

where K_s^- is the bulk modulus of the solid, K_f^- is the bulk modulus of the fluid and K_b^- is the frame bulk modulus; μ^- is the frame shear modulus, and $\lambda^- = K_b^- - 2\mu^- / 3$ is the Lamé constant.

2.3 Transmission conditions

Let \mathbf{n} be the unitary normal vector of Γ outwardly directed to Ω^- . The transmission conditions on the interface between the acoustic and porous medium are [10]:

$$\mathbf{W}^- \cdot \mathbf{n} = (\mathbf{U}^+ - \mathbf{U}_s^-) \cdot \mathbf{n}, \quad (2.3a)$$

$$P^- = P^+, \quad (2.3b)$$

$$\Sigma^- \mathbf{n} = -P^+ \mathbf{n}. \quad (2.3c)$$

2.4 The Green problem

We will not compute directly the solution to (2.1)-(2.3) but the solution to the following Green problem:

$$\ddot{p}^+ - V^{+2} \Delta p^+ = \delta_x \delta_{y-h} \delta_t, \quad \text{in } \Omega^+ \times]0, T], \quad (2.4a)$$

$$\ddot{\mathbf{u}}^+ = -\frac{1}{\rho^+} \nabla p^+, \quad \text{in } \Omega^+ \times]0, T], \quad (2.4b)$$

$$\rho^- \ddot{\mathbf{u}}_s^- + \rho_f^- \ddot{\mathbf{w}}^- - \nabla \cdot \boldsymbol{\sigma}^- = 0, \quad \text{in } \Omega^- \times]0, T], \quad (2.5a)$$

$$\rho_f^- \ddot{\mathbf{u}}_s^- + \rho_w^- \ddot{\mathbf{w}}^- + \frac{1}{\mathcal{K}^-} \dot{\mathbf{w}}^- + \nabla p^- = 0, \quad \text{in } \Omega^- \times]0, T], \quad (2.5b)$$

$$\boldsymbol{\sigma}^- = \lambda^- \nabla \cdot \mathbf{u}_s^- \mathbf{I}_2 + 2\mu^- \boldsymbol{\varepsilon}(\mathbf{u}_s^-) - \beta^- p^- \mathbf{I}_2, \quad \text{in } \Omega^- \times]0, T], \quad (2.5c)$$

$$\frac{1}{m^-} p^- + \beta^- \nabla \cdot \mathbf{u}_s^- + \nabla \cdot \mathbf{w}^- = 0, \quad \text{in } \Omega^- \times]0, T], \quad (2.5d)$$

and

$$\mathbf{w}^- \cdot \mathbf{n} = (\mathbf{u}^+ - \mathbf{u}_s^-) \cdot \mathbf{n}, \quad p^- = p^+, \quad \boldsymbol{\sigma}^- \mathbf{n} = -p^+ \mathbf{n}, \quad \text{on } \Gamma \times]0, T]. \quad (2.6)$$

The solution to (2.1)-(2.3) is then computed from the solution to the Green Problem thanks to a convolution by the source function. For instance we have:

$$P^+(x, y, t) = p^+(x, y, \cdot) * f(\cdot) = \int_0^t p^+(x, y, \tau) f(t - \tau) d\tau$$

(we have similar relations for the other unknowns). We also suppose that the poroelastic medium is non dissipative, i.e the viscosity $\eta^- = 0$. Using Eqs. (2.5c) and (2.5d) we can eliminate σ^- and p^- in (2.5) and we obtain the equivalent system:

$$\rho^- \ddot{\mathbf{u}}_s^- + \rho_f^- \ddot{\mathbf{w}}^- - \alpha^- \nabla(\nabla \cdot \mathbf{u}_s^-) + \mu^- \nabla \times (\nabla \times \mathbf{u}_s^-) - m^- \beta^- \nabla(\nabla \cdot \mathbf{w}^-) = 0, \quad (2.7a)$$

$$\rho_f^- \ddot{\mathbf{u}}_s^- + \rho_w^- \ddot{\mathbf{w}}^- - m^- \beta^- \nabla(\nabla \cdot \mathbf{u}_s^-) - m^- \nabla(\nabla \cdot \mathbf{w}^-) = 0, \quad (2.7b)$$

with $\alpha^- = \lambda^- + 2\mu^- + m^- \beta^{-2}$. Using Eq. (2.4b) the transmission conditions (2.6) are rewritten as:

$$\begin{cases} \ddot{u}_{sy}^- + \ddot{w}_y^- = -\frac{1}{\rho^+} \partial_y p^+, \\ -m^- \beta^- \nabla \cdot \mathbf{u}_s^- - m^- \nabla \cdot \mathbf{w}^- = p^+, \\ \partial_y u_{sx}^- + \partial_x u_{sy}^- = 0, \\ (\lambda^- + m^- \beta^{-2}) \nabla \cdot \mathbf{u}_s^- + 2\mu^- \partial_y u_{sy}^- + m^- \beta^- \nabla \cdot \mathbf{w}^- = -p^+. \end{cases} \quad (2.8)$$

We split the displacement fields \mathbf{u}_s^- and \mathbf{w}^- into irrotational and isovolumic fields (P-wave and S-wave):

$$\mathbf{u}_s^- = \nabla \Theta_u^- + \nabla \times \Psi_u^-; \quad \mathbf{w}^- = \nabla \Theta_w^- + \nabla \times \Psi_w^-. \quad (2.9)$$

Using this last change of variables, we can then rewrite the system (2.7) in the following form:

$$A \ddot{\Theta}^- - B \Delta \Theta^- = 0, \quad y < 0, \quad (2.10a)$$

$$\ddot{\Psi}_u^- - V_s^{-2} \Delta \Psi_u^- = 0, \quad y < 0, \quad (2.10b)$$

$$\ddot{\Psi}_w^- = -\frac{\rho_f^-}{\rho_w^-} \ddot{\Psi}_u^-, \quad y < 0, \quad (2.10c)$$

where $\Theta^- = (\Theta_u^-, \Theta_w^-)^t$, A and B are 2×2 symmetric matrices:

$$A = \begin{pmatrix} \rho^- & \rho_f^- \\ \rho_f^- & \rho_w^- \end{pmatrix}; \quad B = \begin{pmatrix} \lambda^- + 2\mu^- + m^- (\beta^-)^2 & m^- \beta^- \\ m^- \beta^- & m^- \end{pmatrix},$$

and $V_s^- = (\mu \rho_w^- / (\rho^- \rho_w^- - \rho_f^{-2}))^{1/2}$ is the S-wave velocity.

We multiply the first equation of system (2.10) by the inverse of A . The matrix $A^{-1}B$ is diagonalizable: $A^{-1}B = \mathcal{P}D\mathcal{P}^{-1}$, where \mathcal{P} is the change-of-coordinates matrix, $D = \text{diag}(V_{pf}^{-2}, V_{ps}^{-2})$ is the diagonal matrix similar to $A^{-1}B$, V_{pf}^- and V_{ps}^- are respectively the fast P-wave velocity and the slow P-wave velocity ($V_{ps} < V_{pf}$).

Using the change of variables $\Phi^- = (\Phi_{Pf}^-, \Phi_{Ps}^-)^t = \mathcal{P}^{-1}\Theta^-$, we obtain the uncoupled system on fast P-waves, slow P-waves and S-waves:

$$\ddot{\Phi}^- - D\Delta\Phi^- = 0, \quad y < 0, \quad (2.11a)$$

$$\ddot{\Psi}_u^- - V_S^{-2}\Delta\Psi_u^- = 0, \quad y < 0, \quad (2.11b)$$

$$\Psi_w^- = -\frac{\rho_f^-}{\rho_w^-}\Psi_u^-, \quad y < 0. \quad (2.11c)$$

Finally, we obtain the Green problem equivalent to (2.4)-(2.6):

$$\ddot{p}^+ - V^{+2}\Delta p^+ = \delta_x\delta_{y-h}\delta_t, \quad y > 0, \quad (2.12a)$$

$$\ddot{\Phi}_i^- - V_i^{-2}\Delta\Phi_i^- = 0, \quad i \in \{Pf, Ps, S\}, \quad y < 0, \quad (2.12b)$$

$$\mathcal{B}(p^+, \Phi_{Pf}^-, \Phi_{Ps}^-, \Phi_S^-) = 0, \quad y = 0, \quad (2.12c)$$

where we have set $\Phi_S^- = \Psi_u^-$ in order to have similar notations for the Pf , Ps and S waves. The operator \mathcal{B} represents the transmission conditions on Γ :

$$\mathcal{B} \begin{pmatrix} p^+ \\ \Phi_{Pf}^- \\ \Phi_{Ps}^- \\ \Phi_S^- \end{pmatrix} = \begin{bmatrix} \frac{1}{\rho^+}\partial_y & (\mathcal{P}_{11} + \mathcal{P}_{21})\partial_{y_{tt}}^3 & (\mathcal{P}_{12} + \mathcal{P}_{22})\partial_{y_{tt}}^3 & \left(\frac{\rho_f^-}{\rho_w^-} - 1\right)\partial_{x_{tt}}^3 \\ 1 & \frac{m^-(\beta^-\mathcal{P}_{11} + \mathcal{P}_{21})}{V_{Pf}^2}\partial_{tt}^2 & \frac{m^-(\beta^-\mathcal{P}_{12} + \mathcal{P}_{22})}{V_{Ps}^2}\partial_{tt}^2 & 0 \\ 0 & 2\mathcal{P}_{11}\partial_{xy}^2 & 2\mathcal{P}_{12}\partial_{xy}^2 & \partial_{yy}^2 - \partial_{xx}^2 \\ 1 & \mathcal{B}_{42} & \mathcal{B}_{43} & -2\mu^-\partial_{xy}^2 \end{bmatrix} \begin{bmatrix} p^+ \\ \Phi_{Pf}^- \\ \Phi_{Ps}^- \\ \Phi_S^- \end{bmatrix},$$

where $\mathcal{P}_{i,j}$, $i, j = 1, 2$ are the components of the change-of-coordinates matrix \mathcal{P} , \mathcal{B}_{41} and \mathcal{B}_{43} are given by:

$$\mathcal{B}_{42} = \frac{(\lambda^- + m^-\beta^{-2})\mathcal{P}_{11} + m^-\beta^-\mathcal{P}_{21}}{V_{Pf}^{-2}}\partial_{tt}^2 + 2\mu^-\mathcal{P}_{11}\partial_{yy}^2,$$

$$\mathcal{B}_{43} = \frac{(\lambda^- + m^-\beta^{-2})\mathcal{P}_{12} + m^-\beta^-\mathcal{P}_{22}}{V_{Ps}^{-2}}\partial_{tt}^2 + 2\mu^-\mathcal{P}_{12}\partial_{yy}^2.$$

To obtain this operator we have used the transmission conditions (2.8), the change of variables (2.9) and the uncoupled system (2.11).

Moreover, we can determine the solid displacement u_s^- by using the change of variables (2.9) and the fluid displacement u^+ by using (2.4).

3 Expression of the analytical solution

To state our results, we need the following notations and definitions:

1. Definition of the complex square root. For $q \in \mathbb{C} \setminus \mathbb{R}^-$, we use the following definition of the square root $g(q) = q^{1/2}$:

$$g(q)^2 = q \quad \text{and} \quad \Re[g(q)] > 0.$$

The branch cut of $g(q)$ in the complex plane will thus be the half-line defined by $\{q \in \mathbb{R}^-\}$ (see Fig. 2). In the following, we use the abuse of notation $g(q) = i\sqrt{-q}$ for $q \in \mathbb{R}^-$.

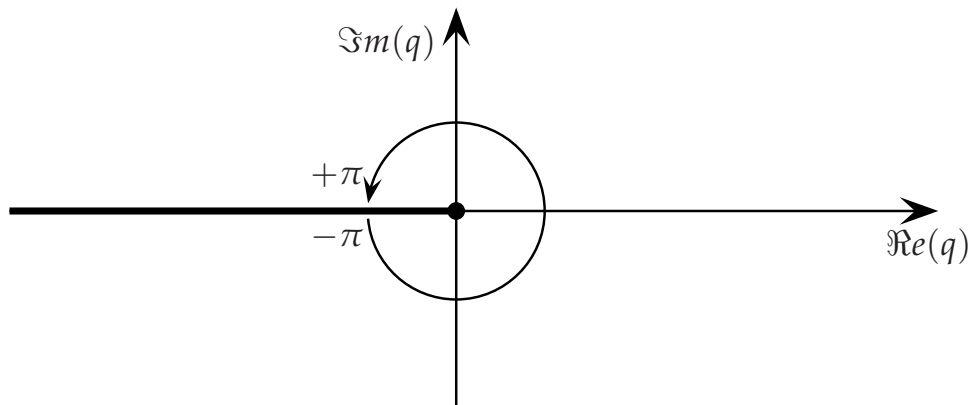


Figure 2: Definition of the function $x \mapsto (x)^{1/2}$.

2. Definition of the functions κ^+ and κ_i^- . For $i \in \{Pf, Ps, S\}$ and $q \in \mathbb{C}$, we define the functions

$$\kappa^+ := \kappa^+(q) = \left(\frac{1}{V+2} + q^2 \right)^{1/2} \quad \text{and} \quad \kappa_i^- := \kappa_i^-(q) = \left(\frac{1}{V_i-2} + q^2 \right)^{1/2}.$$

3. Definition of the reflection and transmission coefficients. For a given $q \in \mathbb{C}$, we denote by $\mathcal{R}(q)$, $\mathcal{T}_{Pf}(q)$, $\mathcal{T}_{Ps}(q)$ and $\mathcal{T}_S(q)$ the solution to the linear system

$$\mathcal{A}(q) \begin{bmatrix} \mathcal{R}(q) \\ \mathcal{T}_{Pf}(q) \\ \mathcal{T}_{Ps}(q) \\ \mathcal{T}_S(q) \end{bmatrix} = -\frac{1}{2\kappa^+ V+2} \begin{bmatrix} \frac{\kappa^+(q)}{\rho^+} \\ 1 \\ 0 \\ 1 \end{bmatrix}, \tag{3.1}$$

where the matrix $\mathcal{A}(q)$ is defined for $q \in \mathbb{C}$ by:

$$\mathcal{A}(q) = \begin{bmatrix} -\frac{\kappa^+(q)}{\rho^+} & \kappa_{P_f}^-(q)(\mathcal{P}_{11} + \mathcal{P}_{21}) & \kappa_{P_s}^-(q)(\mathcal{P}_{12} + \mathcal{P}_{22}) & iq \left(1 - \frac{\rho_f^-}{\rho_w^-}\right) \\ 1 & \frac{m^-}{V_{P_f}^{-2}}(\beta^- \mathcal{P}_{11} + \mathcal{P}_{21}) & \frac{m^-}{V_{P_s}^{-2}}(\beta^- \mathcal{P}_{12} + \mathcal{P}_{22}) & 0 \\ 0 & -2iq\kappa_{P_f}^-(q)\mathcal{P}_{11} & -2iq\kappa_{P_s}^-(q)\mathcal{P}_{12} & (\kappa_S^{-2}(q) + q^2) \\ 1 & \mathcal{A}_{4,2}(q) & \mathcal{A}_{4,3}(q) & 2iq\mu^- \kappa_S^-(q) \end{bmatrix},$$

with

$$\mathcal{A}_{4,2}(q) = \frac{(\lambda^- + m^- \beta^{-2})\mathcal{P}_{11} + m^- \beta^- \mathcal{P}_{21}}{V_{P_f}^{-2}} + 2\mu^- \kappa_{P_f}^-(q)^2 \mathcal{P}_{11},$$

$$\mathcal{A}_{4,3}(q) = \frac{(\lambda^- + m^- \beta^{-2})\mathcal{P}_{12} + m^- \beta^- \mathcal{P}_{22}}{V_{P_s}^{-2}} + 2\mu^- \kappa_{P_s}^-(q)^2 \mathcal{P}_{12}.$$

We also denote by V_{\max} the greatest velocity in the two media:

$$V_{\max} = \max(V^+, V_{P_f}, V_{P_s}, V_S).$$

We can now present the expression of the solution to the Green Problem:

Theorem 3.1. The pressure and the displacement in the top medium are given by

$$\begin{aligned} p^+(x, y, t) &= p_{\text{inc}}^+(x, y, t) + p_{\text{ref}}^+(x, y, t) \text{ and } \mathbf{u}^+(x, y, t) \\ &= \int_0^t v_{\text{inc}}^+(x, y, \tau) d\tau + \int_0^t v_{\text{ref}}^+(x, y, \tau) d\tau, \end{aligned}$$

and the displacement in the bottom medium is given by

$$\mathbf{u}_s^- = \int_0^t v^-(x, y, \tau) d\tau \text{ with } v^- = v_{P_f}^- + v_{P_s}^- + v_S^-,$$

where

• p_{inc}^+ and v_{inc}^+ are respectively the pressure and the velocity of the incident wave and satisfy:

$$\begin{cases} p_{\text{inc}}^+(x, y, t) = \frac{1}{2\pi V^{+2} \sqrt{t^2 - t_0^2}}, & v_{\text{inc},x}^+(x, y, t) = \frac{tx}{2\pi V^{+2} \rho^+ r \sqrt{t^2 - t_0^2}}, \\ v_{\text{inc},y}^+(x, y, t) = \frac{t(y-h)}{2\pi V^{+2} \rho^+ r \sqrt{t^2 - t_0^2}}, & \text{if } t > t_0, \\ p_{\text{inc}}(x, y, t) = 0 \text{ and } v_{\text{inc}}(x, y, t) = 0, & \text{else.} \end{cases}$$

We set here $r = (x^2 + (y-h)^2)^{1/2}$ and $t_0 = r/V^+$ denotes the arrival time of the incident wave.

• p_{ref}^+ and v_{ref}^+ are respectively the pressure and the velocity of the reflected wave and satisfy:

$$\begin{cases} p_{\text{ref}}^+(x,y,t) = -\frac{\Im m[\kappa^+(v(t))\mathcal{R}(v(t))]}{\pi\sqrt{t_0^2-t^2}}, & v_{\text{ref},x}^+(x,y,t) = -\frac{\Im m[iv(t)\kappa^+(v(t))\mathcal{R}(v(t))]}{\pi\rho^+\sqrt{t_0^2-t^2}}, \\ v_{\text{ref},y}^+(x,y,t) = -\frac{\Im m[\kappa^{+2}(v(t))\mathcal{R}(v(t))]}{\pi\rho^+\sqrt{t_0^2-t^2}}, & \text{if } t_h < t \leq t_0 \text{ and } \frac{x}{r} > \frac{V^+}{V_{\text{max}}}, \\ p_{\text{ref}}^+(x,y,t) = \frac{\Re e[\kappa^+(\gamma(t))\mathcal{R}(\gamma(t))]}{\pi\sqrt{t_0^2-t^2}}, & v_{\text{ref},x}^+(x,y,t) = \frac{\Re e[i\gamma(t)\kappa^+(\gamma(t))\mathcal{R}(\gamma(t))]}{\pi\rho^+\sqrt{t_0^2-t^2}}, \\ v_{\text{ref},y}^+(x,y,t) = \frac{\Re e[\kappa^{+2}(\gamma(t))\mathcal{R}(\gamma(t))]}{\pi\rho^+\sqrt{t^2-t_0^2}}, & \text{if } t > t_0, \\ p_{\text{ref}}(x,y,t) = 0 \text{ and } v_{\text{ref}}(x,y,t) = 0, & \text{else.} \end{cases}$$

We set here $r = (x^2 + (y+h)^2)^{1/2}$, $t_0 = r/V^+$ denotes the arrival time of the reflected volume wave,

$$t_h = (y+h)\sqrt{\frac{1}{V^{+2}} - \frac{1}{V_{\text{max}}^2}} + \frac{|x|}{V_{\text{max}}},$$

denotes the arrival time of the reflected head wave and the complex functions $v := v(t)$ and $\gamma := \gamma(t)$ are defined by

$$\begin{cases} v(t) = -i\left(\frac{y+h}{r}\sqrt{\frac{1}{V^{+2}} - \frac{t^2}{r^2}} + \frac{xt}{r^2}\right), & \text{for } t_h < t \leq t_0 \text{ and } x < 0, \\ v(t) = i\left(\frac{y+h}{r}\sqrt{\frac{1}{V^{+2}} - \frac{t^2}{r^2}} - \frac{xt}{r^2}\right), & \text{for } t_h < t \leq t_0 \text{ and } x \geq 0, \end{cases}$$

and

$$\gamma(t) = -i\frac{x}{r^2}t + \frac{y+h}{r}\sqrt{\frac{t^2}{r^2} - \frac{1}{V^{+2}}} \text{ for } t > t_0.$$

- v_{Pf}^- is the velocity of the transmitted Pf wave and satisfies:

$$\left\{ \begin{array}{l} v_{Pf,x}^-(x,y,t) = -\frac{\mathcal{P}_{11}}{\pi} \Re \left[i v(t) \mathcal{T}_{Pf}(v(t)) \frac{dv}{dt}(t) \right], \\ v_{Pf,y}^-(x,y,t) = \frac{\mathcal{P}_{11}}{\pi} \Re \left[\kappa_{Pf}^-(v(t)) \mathcal{T}_{Pf}(v(t)) \frac{dv}{dt}(t) \right], \end{array} \right. \quad \begin{array}{l} \text{if } t_h < t \leq t_0 \\ \text{and } |\Im m(v(t_0))| > \frac{1}{V_{\max}}, \end{array}$$

$$\left\{ \begin{array}{l} v_{Pf,x}^-(x,y,t) = -\frac{\mathcal{P}_{11}}{\pi} \Re \left[i \gamma(t) \mathcal{T}_{Pf}(\gamma(t)) \frac{d\gamma}{dt}(t) \right], \\ v_{Pf,y}^-(x,y,t) = \frac{\mathcal{P}_{11}}{\pi} \Re \left[\kappa_{Pf}^-(\gamma(t)) \mathcal{T}_{Pf}(\gamma(t)) \frac{d\gamma}{dt}(t) \right], \end{array} \right. \quad \text{if } t > t_0,$$

$$v_{Pf}^-(x,y,t) = 0, \quad \text{else.}$$

Here t_0 denotes the arrival time of the Pf wave (its calculation is detailed in appendix) and t_h is the arrival time of the Pf head wave:

$$t_h = -y \sqrt{\frac{1}{V_{Pf}^{-2}} - \frac{1}{V_{\max}^2}} + h \sqrt{\frac{1}{V^{+2}} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}}.$$

For $t_h < t \leq t_0$, the function $v := v(t)$ is defined as the only root of

$$q \in \mathbb{C} \mapsto \mathcal{F}(q,t) = -y \left(\frac{1}{V_{Pf}^{-2}} + q^2 \right)^{1/2} + h \left(\frac{1}{V^{+2}} + q^2 \right)^{1/2} + iqx - t,$$

such that $\Im m \left(x \frac{dv(t)}{dt} \right) \leq 0$. For $t > t_0$, the function $\gamma := \gamma(t)$ is defined as the only root of $q \in \mathbb{C} \mapsto \mathcal{F}(q,t)$ whose real part is positive.

- v_{Ps}^- is the velocity of the transmitted Ps wave and satisfies:

$$\left\{ \begin{array}{l} v_{Ps,x}^-(x,y,t) = -\frac{\mathcal{P}_{12}}{\pi} \Re \left[i v(t) \mathcal{T}_{Ps}(v(t)) \frac{dv}{dt}(t) \right], \\ v_{Ps,y}^-(x,y,t) = \frac{\mathcal{P}_{12}}{\pi} \Re \left[\kappa_{Ps}^-(v(t)) \mathcal{T}_{Ps}(v(t)) \frac{dv}{dt}(t) \right], \end{array} \right. \quad \begin{array}{l} t_h < t \leq t_0 \\ \text{and } |\Im m(v(t_0))| > \frac{1}{V_{\max}}, \end{array}$$

$$\left\{ \begin{array}{l} v_{Ps,x}^-(x,y,t) = -\frac{\mathcal{P}_{12}}{\pi} \Re \left[i \gamma(t) \mathcal{T}_{Ps}(\gamma(t)) \frac{d\gamma}{dt}(t) \right], \\ v_{Ps,y}^-(x,y,t) = \frac{\mathcal{P}_{12}}{\pi} \Re \left[\kappa_{Ps}^-(\gamma(t)) \mathcal{T}_{Ps}(\gamma(t)) \frac{d\gamma}{dt}(t) \right], \end{array} \right. \quad \text{if } t > t_0,$$

$$v_{Ps}^-(x,y,t) = 0, \quad \text{else.}$$

Here t_0 denotes the arrival time of the Ps wave and t_h is the arrival time of the Ps head wave:

$$t_h = -y \sqrt{\frac{1}{V_{Ps}^{-2}} - \frac{1}{V_{\max}^2}} + h \sqrt{\frac{1}{V^{+2}} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}}.$$

For $t_h < t \leq t_0$, the function $v := v(t)$ is defined as the only root of

$$q \in \mathbb{C} \mapsto \mathcal{F}(q, t) = -y \left(\frac{1}{V_{Ps}^{-2}} + q^2 \right)^{1/2} + h \left(\frac{1}{V^{+2}} + q^2 \right)^{1/2} + iqx - t,$$

such that $\Im m \left(x \frac{dv(t)}{dt} \right) \leq 0$. For $t > t_0$, the function $\gamma := \gamma(t)$ is defined as the only root of $q \in \mathbb{C} \mapsto \mathcal{F}(q, t)$ whose real part is positive.

- v_{S^-} is the velocity of the transmitted S wave and satisfies:

$$\begin{cases} v_{S,x}^-(x, y, t) = \frac{1}{\pi} \Re e \left[\kappa_S^-(v(t)) \mathcal{T}_S(v(t)) \frac{dv}{dt}(t) \right], & \text{if } t_h < t \leq t_0 \\ v_{S,y}^-(x, y, t) = \frac{1}{\pi} \Re e \left[i v(t) \mathcal{T}_S(v(t)) \frac{dv}{dt}(t) \right], & \text{and } |\Im m(v(t_0))| > \frac{1}{V_{\max}}, \end{cases}$$

$$\begin{cases} v_{S,x}^-(x, y, t) = \frac{1}{\pi} \Re e \left[\kappa_S^-(\gamma(t)) \mathcal{T}_S(\gamma(t)) \frac{d\gamma}{dt}(t) \right], \\ v_{S,y}^-(x, y, t) = \frac{1}{\pi} \Re e \left[i \gamma(t) \mathcal{T}_S(\gamma(t)) \frac{d\gamma}{dt}(t) \right], \end{cases} \quad \text{if } t > t_0,$$

$$v_{S,x}^-(x, y, t) = 0, \quad \text{else.}$$

Here t_0 denotes the arrival time of the S wave and t_h is the arrival time of the S head wave:

$$t_h = -y \sqrt{\frac{1}{V_S^{-2}} - \frac{1}{V_{\max}^2}} + h \sqrt{\frac{1}{V^{+2}} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}}.$$

For $t_h < t \leq t_0$, the function $v := v(t)$ is defined as the only root of

$$q \in \mathbb{C} \mapsto \mathcal{F}(q, t) = -y \left(\frac{1}{V_S^{-2}} + q^2 \right)^{1/2} + h \left(\frac{1}{V^{+2}} + q^2 \right)^{1/2} + iqx - t,$$

such that $\Im m \left(x \frac{dv(t)}{dt} \right) \leq 0$. For $t > t_0$, the function $\gamma := \gamma(t)$ is defined as the only root of $q \in \mathbb{C} \mapsto \mathcal{F}(q, t)$ whose real part is positive.

Remark 3.1. For the practical computations of the displacement, we won't have to explicitly compute the primitive of the velocities v , which would be rather tedious, since

$$\left(\int_0^t v(\tau) d\tau \right) * f = v * \left(\int_0^t f(\tau) d\tau \right).$$

Therefore, we will only have to compute the primitive of the source function f .

4 Proof of the theorem

To prove the theorem, we use the Cagniard-de Hoop method (see [1, 2, 11–13]), which consists of two steps:

1. We apply a Laplace transform in time,

$$\tilde{u}(x, y, s) = \int_0^{+\infty} u(x, y, t) e^{-st} dt,$$

and a Fourier transform in the x variable,

$$\hat{u}(k_x, y, s) = \int_{-\infty}^{+\infty} \tilde{u}(x, y, s) e^{ik_x x} dx$$

to (2.12) in order to obtain an ordinary differential system whose solution $\hat{\mathcal{G}}(k_x, y, s)$ can be explicitly computed (Section 4.1);

2. we apply an inverse Fourier transform in the x variable to \mathcal{G} :

$$\tilde{\mathcal{G}}(x, y, s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{\mathcal{G}}(k_x, y, s) e^{-ik_x x} dk_x.$$

and, using tools of complex analysis, we turn this inverse Fourier transform into the Laplace transform of some function $\mathcal{H}(x, y, t)$:

$$\tilde{\mathcal{G}}(x, y, s) = \frac{1}{2\pi} \int_0^{+\infty} \mathcal{H}(x, y, t) e^{-st} dt.$$

Then, using the injectivity of the Laplace transform, we identify $\mathcal{G}(x, y, t)$ to $\mathcal{H}(x, y, t)$ (see Section 4.2).

4.1 The solution in the Laplace-Fourier plane

Let us first apply a Laplace transform in time and a Fourier transform in the x variable to (2.12) to obtain

$$\begin{cases} \left(\frac{s^2}{V^{+2}} + k_x^2\right) \hat{p}^+ - \frac{\partial^2 \hat{p}^+}{\partial y^2} = \frac{\delta(y-h)}{V^{+2}}, & y > 0, \\ \left(\frac{s^2}{V_i^{-2}} + k_x^2\right) \hat{\Phi}_i^- - \frac{\partial^2 \hat{\Phi}_i^-}{\partial y^2} = 0, \quad i \in \{Pf, Ps, S\} & y < 0, \\ \hat{\mathcal{B}}(\hat{p}^+, \hat{\Phi}_{Pf}^-, \hat{\Phi}_{Ps}^-, \hat{\Phi}_S^-) = 0 & y = 0, \end{cases} \quad (4.1)$$

where $\hat{\mathcal{B}}$ is the Laplace-Fourier transform of the operator \mathcal{B} .

From the two first equations of (4.1), we deduce that the solution $(\hat{p}^+, (\hat{\Phi}_i^-)_{i \in \{Pf, Ps, S\}})$ is such that

$$\begin{cases} \hat{p}^+ = \hat{p}_{inc}^+ + \hat{p}_{ref}^+ & \hat{p}_{inc}^+(k_x, y, s) = \frac{e^{-s|y-h|\left(\frac{1}{V^{+2}} + \frac{k_x^2}{s^2}\right)^{1/2}}}{sV^{+2}\left(\frac{1}{V^{+2}} + \frac{k_x^2}{s^2}\right)^{1/2}}, \\ \hat{p}_{ref}^+(k_x, y, s) = R(k_x, s) e^{-sy\left(\frac{1}{V^{+2}} + \frac{k_x^2}{s^2}\right)^{1/2}}, \\ \hat{\Phi}_i^-(k_x, y, s) = T_i(k_x, s) e^{sy\left(\frac{1}{V_i^{-2}} + \frac{k_x^2}{s^2}\right)^{1/2}}, \quad i \in \{Pf, Ps, S\}. \end{cases} \quad (4.2)$$

where the coefficients R and T_i are computed by using the last equation of (4.1):

$$\hat{\mathcal{B}}(\hat{p}_{ref}^+, \hat{\Phi}_{Pf}^-, \hat{\Phi}_{Ps}^-, \hat{\Phi}_S^-) = -\hat{\mathcal{B}}(\hat{p}_{inc}^+, 0, 0, 0).$$

Using the expressions of \hat{p}_{inc}^+ , \hat{p}_{ref}^+ and $\hat{\Phi}_i^-$, we obtain that $R(k_x, s)$, $T_{Pf}(k_x, s)$, $T_{Ps}(k_x, s)$, $T_S(k_x, s)$ are solution to

$$\mathcal{A}\left(\frac{k_x}{s}\right) \begin{bmatrix} sR(k_x, s) \\ s^3 T_{Pf}(k_x, s) \\ s^3 T_{Ps}(k_x, s) \\ s^3 T_S(k_x, s) \end{bmatrix} = -\frac{e^{-sh\kappa^+\left(\frac{k_x}{s}\right)}}{2V^{+2}\kappa^+\left(\frac{k_x}{s}\right)} \begin{bmatrix} \kappa^+\left(\frac{k_x}{s}\right) \\ \rho^+ \\ 1 \\ 0 \\ 1 \end{bmatrix}. \quad (4.3)$$

From the definition of the functions $\mathcal{R}(q)$, $\mathcal{T}_{Pf}(q)$, $\mathcal{T}_{Ps}(q)$ and $\mathcal{T}_S(q)$ we deduce that

$$\begin{bmatrix} \mathcal{R}\left(\frac{k_x}{s}\right) \\ \mathcal{T}_{Pf}\left(\frac{k_x}{s}\right) \\ \mathcal{T}_{Ps}\left(\frac{k_x}{s}\right) \\ \mathcal{T}_S\left(\frac{k_x}{s}\right) \end{bmatrix} = \begin{bmatrix} sR(k_x, s) \\ s^3T_{Pf}(k_x, s) \\ s^3T_{Ps}(k_x, s) \\ s^3T_S(k_x, s) \end{bmatrix} e^{sh\kappa^+\left(\frac{k_x}{s}\right)}. \tag{4.4}$$

Finally, we rewrite \hat{p}_{inc}^+ , \hat{p}_{ref}^+ and $\hat{\Phi}_i^-$ under the form

$$\begin{cases} \hat{p}^+ = \hat{p}_{inc}^+ + \hat{p}_{ref}^+ & \hat{p}_{inc}^+(k_x, y, s) = \frac{1}{sV+2\kappa^+\left(\frac{k_x}{s}\right)} e^{-s|y-h|\kappa^+\left(\frac{k_x}{s}\right)}, \\ \hat{p}_{ref}^+(k_x, y, s) = \frac{1}{s} \mathcal{R}\left(\frac{k_x}{s}\right) e^{-s(y+h)\kappa^+\left(\frac{k_x}{s}\right)}, \\ \hat{\Phi}_i^-(k_x, y, s) = \frac{1}{s^3} \mathcal{T}_i\left(\frac{k_x}{s}\right) e^{-s(y\kappa_i^-\left(\frac{k_x}{s}\right) - h\kappa^+\left(\frac{k_x}{s}\right))}, \quad i \in \{Pf, Ps, S\}. \end{cases} \tag{4.5}$$

From (2.4) and (2.9), we deduce the expression of the displacement field:

$$\begin{cases} \hat{u}^+ = \hat{u}_{inc}^+ + \hat{u}_{ref}^+ & \hat{u}_{inc,x}^+ = i \frac{k_x}{\rho^+s^2} \hat{p}_{inc}^+ & \hat{u}_{inc,y}^+ = \text{sign}(h-y) \frac{\kappa^+\left(\frac{k_x}{s}\right)}{\rho^+s} \hat{p}_{inc}^+ \\ \hat{u}_{ref,x}^+ = i \frac{k_x}{\rho^+s^2} \hat{p}_{ref}^+ & \hat{u}_{ref,y}^+ = \frac{\kappa^+\left(\frac{k_x}{s}\right)}{\rho^+s} \hat{p}_{ref}^+ \\ \hat{u}_{sx}^- = -ik_x \mathcal{P}_{11} \hat{\Phi}_{Pf}^- - ik_x \mathcal{P}_{12} \hat{\Phi}_{Ps}^- + s\kappa_S^- \left(\frac{k_x}{s}\right) \Phi_S^-, \\ \hat{u}_{sy}^- = s\kappa_{Pf}^- \left(\frac{k_x}{s}\right) \mathcal{P}_{11} \hat{\Phi}_{Pf}^- + s\kappa_{Ps}^- \left(\frac{k_x}{s}\right) \mathcal{P}_{12} \hat{\Phi}_{Ps}^- + ik_x \Phi_S^-. \end{cases} \tag{4.6}$$

In the following we only detail the computation of $\hat{u}_{sx,Ps}^- = -ik_x \mathcal{P}_{12} \hat{\Phi}_{Ps}^-$, since the computation of the other terms is very similar.

4.2 The Laplace transform of the solution

Let us now apply an inverse Fourier transform in the x variable to $\hat{u}_{sx,Ps}^-$ and set $k_x = qs$ to obtain

$$\tilde{u}_{sx,Ps}^- = - \int_{-\infty}^{+\infty} \frac{iq\mathcal{P}_{12}}{2s\pi} \mathcal{T}_{Ps}(q) e^{-s(-y\kappa_{Ps}^-(q) + h\kappa^+(q) + iqx)} dq = - \frac{\mathcal{P}_{12}}{2s\pi} \int_{-\infty}^{+\infty} \Xi(q) dq, \tag{4.7}$$

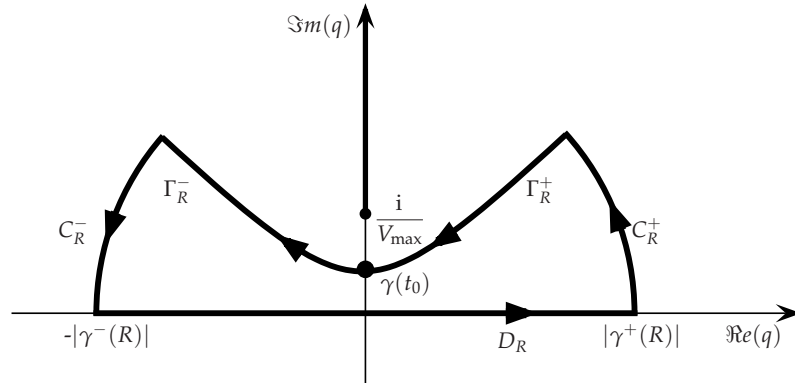


Figure 3: Integration path if $|\gamma(t_0)| < 1/V_{\max}$.

with

$$\Xi(q) = iq \mathcal{T}_{Ps}(q) e^{-s(-y\kappa_{Ps}^-(q) + h\kappa^+(q)) + iqx}.$$

The key point of the Cagniard-de Hoop method is to turn this Fourier integral into a Laplace integral by finding a path Γ in the complex plane such that

$$-y\kappa_{Ps}^-(q) + h\kappa^+(q) + iqx = t \in \mathbb{R}^+, \quad \forall q \in \Gamma.$$

This amounts to compute the roots q of the function

$$\mathcal{F}(q, t) = -y \left(\frac{1}{V_{Ps}^{-2}} + q^2 \right)^{1/2} + h \left(\frac{1}{V^{+2}} + q^2 \right)^{1/2} + iqx - t,$$

for $t \in \mathbb{R}^+$. We recall in appendix A some properties of the function \mathcal{F} ; from Properties A.1, A.3 and A.2 we can define the function $\gamma(t) \in \mathbb{C}$ for $t > t_0$ as the only root of $\mathcal{F}(\cdot, t)$ whose real part is positive, where t_0 is the arrival time of the Ps volume wave (we recall in appendix B the practical computation of t_0). Moreover, for a given $R \in \mathbb{R}^+$, we define

$$\Gamma_R^+ = \{p = \gamma(t) \mid t_0 < t < R\}, \quad \Gamma_R^- = \{p = -\overline{\gamma(t)} \mid t_0 < t < R\} \text{ and } \Gamma^\pm = \lim_{R \rightarrow \infty} \Gamma_R^\pm.$$

We represent the paths Γ_R^\pm in Fig. 3 in the case $x < 0$ (for $x \geq 0$ the path would be similar but in the half plane $\Im m(q) \leq 0$). The shape of Γ^\pm is given by the following property:

Property 4.1. When R tends to infinity, Γ_R^\pm admits an asymptote of equation

$$p = \frac{\pm(y-h) - ix}{(y-h)^2 + x^2}.$$

We now have to decide whether this path intersects the branch cuts of the functions κ^+ and κ_i^- . From the definition of the complex square root, we deduce that the branch cuts of κ^+ are the two half lines

$$\left\{ q \in \mathbb{C} \mid \Re(q) = 0 \text{ and } |\Im(q)| \geq \frac{1}{V^+} \right\}$$

and the branch cuts of κ_i^- are the two half lines

$$\left\{ q \in \mathbb{C} \mid \Re(q) = 0 \text{ and } |\Im(q)| \geq \frac{1}{V_i^-} \right\}.$$

and, since only $\gamma(t_0)$ is purely imaginary, the path cross the branch cuts if and only if

$$|\gamma(t_0)| \geq \frac{1}{V_{\max}}.$$

We have then have to consider two possibilities:

- if $|\gamma(t_0)| \leq 1/V_{\max}$: then Γ_R does not intersect the branch cuts of the functions κ_i^- or κ^+ . We define the segment $D_R = \{p \in \mathbb{R}, |p| \leq |\gamma(R)|\}$ and we close the path by the two arcs of circle C_R^- and C_R^+ of radius $\gamma(R)$ linking Γ_R^\pm and D_R (see Fig. 3). By Cauchy's theorem we have

$$\int_{D_R} \Xi(q) dq + \int_{C_R^+} \Xi(q) dq + \int_{\Gamma_R^-} \Xi(q) dq - \int_{\Gamma_R^+} \Xi(q) dq + \int_{C_R^-} \Xi(q) dq = 0,$$

and, using Jordan's lemma, we obtain:

$$\lim_{R \rightarrow \infty} \int_{C_R^\pm} \Xi(q) dq = 0.$$

so that

$$\int_{-\infty}^{+\infty} \Xi(q) dq = \int_{\Gamma^+} \Xi(q) dq - \int_{\Gamma^-} \Xi(q) dq.$$

Next, using the change of variable $q = \gamma(t)$ on Γ^+ and $q = -\overline{\gamma(t)}$ on Γ^- , we end up with

$$\begin{aligned} & \int_{-\infty}^{+\infty} \Xi(q) dq \\ &= \int_{t_0}^{+\infty} i\gamma(t) \mathcal{T}_{P_s}(\gamma(t)) \frac{d\gamma(t)}{dt} e^{-st} dt - \int_{t_0}^{+\infty} i\overline{\gamma(t)} \mathcal{T}_{P_s}(-\overline{\gamma(t)}) \frac{d\overline{\gamma(t)}}{dt} e^{-st} dt. \end{aligned}$$

From the definition of \mathcal{T}_{P_s} , we check, after some calculation, that

$$\mathcal{T}_{P_s}(-\overline{\gamma(t)}) = -\overline{\mathcal{T}_{P_s}(\gamma(t))}$$

and

$$i\overline{\gamma(t)} \mathcal{T}_{P_s}(-\overline{\gamma(t)}) \frac{d\overline{\gamma(t)}}{dt} = -i\gamma(t) \mathcal{T}_{P_s}(\gamma(t)) \frac{d\gamma(t)}{dt},$$

so that

$$\begin{aligned} \tilde{u}_{s_x, P_s}^-(x, y, s) &= - \int_{t_0}^{+\infty} \frac{P_{12}}{s\pi} \Re e \left(i\gamma(t) \mathcal{T}_{P_s}(\gamma(t)) \frac{d\gamma(t)}{dt} \right) e^{-st} dt \\ &= \int_{t_0}^{+\infty} \left[\int_0^t v_{s,x}^-(x, y, \tau) d\tau \right] e^{-st} dt. \end{aligned}$$

We conclude by using the injectivity of the Laplace transform that

$$u_{s_x, P_s}^-(x, y, t) = \int_0^t v_{s,x}^-(x, y, \tau) d\tau.$$

Remark 4.1. The computation of $\frac{d\gamma(t)}{dt}$ is easily achieved by using the implicit function theorem:

$$\frac{d\gamma(t)}{dt} = \frac{1}{-y\gamma(t) \left(\frac{1}{V_{Ps}^2} + \gamma^2(t) \right)^{-1/2} + h\gamma(t) \left(\frac{1}{V^2} + \gamma^2(t) \right)^{-1/2} + ix}.$$

Since $\mathcal{F}(\cdot, t)$ admits a double root at $t = t_0$, the function $\frac{d\gamma(t)}{dt}$ is singular at this point, however this singularity behaves as [14]

$$\frac{\alpha}{\sqrt{t^2 - t_0^2}}$$

and can therefore be integrated.

• if $|\gamma(t_0)| > 1/V_{\max}$; in this case, the path does intersect the branch cut and we have to consider an additional path Y to bypass the half-lines defined by

$$\left\{ q \in \mathbb{C} \mid \Re e(q) = 0 \text{ and } |\Im m(q)| \geq \frac{1}{V_{\max}} \right\}.$$

This path must obviously satisfy the condition

$$-y\kappa_{P_s}^-(q) + h\kappa^+(q) + iq = t \in \mathbb{R}^+, \forall q \in Y.$$

If $x < 0$, $\Im m(\gamma(t_0)) \geq 0$ (Property A.2), so that $\gamma(t_0)$ lies on the branch cut

$$\left\{ q \in \mathbb{C} \mid \Re e(q) = 0 \text{ and } \Im m(q) > \frac{1}{V_{\max}} \right\}.$$

Therefore, using Properties A.4 and A.5, we define $v(t)$, for $t_h \leq t \leq t_0$, as the only root of $\mathcal{F}(\cdot, t)$ such that

$$\Im m \left(\frac{dv(t)}{dt} \right) = \Im m \left(-yv(t) \left(\frac{1}{V_{Ps}^2} + v^2(t) \right)^{-1/2} + hv(t) \left(\frac{1}{V^2} + v^2(t) \right)^{-1/2} + ix \right) > 0,$$

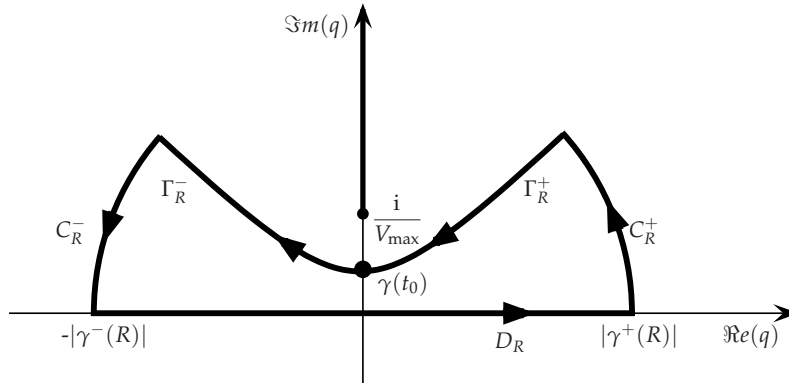


Figure 4: Integration path if $|\gamma(t_0)| > 1/V_{\max}$.

where

$$t_h = -y \sqrt{\frac{1}{V_{Ps}^{-2}} - \frac{1}{V_{\max}^2}} + h \sqrt{\frac{1}{V^{+2}} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}}.$$

If $x > 0$, we define $v(t)$, for $t_h \leq t \leq t_0$, as the only root of $\mathcal{F}(\cdot, t)$ such that

$$\Im m \left(\frac{dv(t)}{dt} \right) = \Im m \left(-yv(t) \left(\frac{1}{V_{Ps}^{-2}} + v^2(t) \right)^{-1/2} + hv(t) \left(\frac{1}{V^{+2}} + v^2(t) \right)^{-1/2} + ix \right) < 0.$$

We are now able to define the paths Y_R^\pm and Y^\pm by:

$$Y_R^\pm = \left\{ v(t) \pm \frac{1}{R} \mid t_h < t < t_0 \right\} \text{ and } Y^\pm = \lim_{R \rightarrow +\infty} Y_R^\pm.$$

We represent the paths Γ_R^\pm and Y_R^\pm in Fig. 4 in the case $x < 0$ (for $x \geq 0$ the path would be similar but in the half plane $\Im m(q) \leq 0$). We define the segment

$$D_R = \left\{ p \in \mathbb{R} \mid |p| \leq |\gamma(R)| \right\}$$

and we close the path by the two arcs of circle C_R^- and C_R^+ of radius $\gamma(R)$ linking Γ_R^\pm and D_R and the half circle c_R linking Y_R^+ to Y_R^- (see Fig. 4). By Cauchy's theorem:

$$\int_{D_R} \Xi(q) dq + \int_{C_R^+} \Xi(q) dq - \int_{\Gamma_R^+} \Xi(q) dq + \int_{\Gamma_R^-} \Xi(q) dq - \int_{Y_R^+} \Xi(q) dq + \int_{Y_R^-} \Xi(q) dq + \int_{C_R^-} \Xi(q) dq + \int_{c_R} \Xi(q) dq = 0.$$

Using once again Jordan's Lemma, we prove that the integrals over C_R^- , C_R^+ and c_R vanish when R tends to infinity and that

$$\int_{-\infty}^{+\infty} \Xi(q) dq = \int_{\Gamma^+} \Xi(q) dq - \int_{\Gamma^-} \Xi(q) dq + \int_{Y^+} \Xi(q) dq - \int_{Y^-} \Xi(q) dq.$$

The calculation of the integral over Γ^\pm is done as in the first case and we only focus on the calculation over Y^\pm . Let us now use the change of variable $q = v_{P_s}^-(t) - 1/R$ on Y_R^- and $q = v_{P_s}^-(t) + 1/R$ on Y_R^+ to obtain:

$$\int_{Y_R^-} \Xi(q) dq = \int_{t_h}^{t_0} i \left(v(t) - \frac{1}{R} \right) \mathcal{T}_{P_s} \left(v(t) - \frac{1}{R} \right) \frac{dv(t)}{dt} e^{-st} dt,$$

$$\int_{Y_R^+} \Xi(q) dq = \int_{t_h}^{t_0} i \left(v(t) + \frac{1}{R} \right) \mathcal{T}_{P_s} \left(v(t) + \frac{1}{R} \right) \frac{dv(t)}{dt} e^{-st} dt.$$

Because of the branch cut, it is clear that

$$\lim_{R \rightarrow +\infty} \mathcal{T}_{P_s}(v(t) + 1/R) \neq \lim_{R \rightarrow +\infty} \mathcal{T}_{P_s}(v(t) - 1/R).$$

However, as $v(t)$ is imaginary,

$$v(t) - 1/R = -\overline{v(t) + 1/R} \text{ and } \mathcal{T}_{P_s}(v(t) - 1/R) = \overline{\mathcal{T}_{P_s}(v(t) + 1/R)},$$

so that

$$\int_{Y^+} \Xi(q) dq - \int_{Y^-} \Xi(q) dq = - \int_{t_h}^{t_0} 2v(t) \Im(\mathcal{T}_{P_s}(v(t))) \frac{dv(t)}{dt} e^{-st} dt. \tag{4.8}$$

Remark 4.2. Following the definition of the square root of a negative number, we made the abuse of notation

$$\lim_{R \rightarrow +\infty} \mathcal{T}_{P_s}(v(t) + 1/R) = \mathcal{T}_{P_s}(v(t)).$$

Since $v_{P_s}^-(t)$ and $\frac{dv_{P_s}^-(t)}{dt}$ are purely imaginary, (4.8) can be rewritten under the form

$$\int_{Y^+} \Xi(q) dq - \int_{Y^-} \Xi(q) dq = \int_{t_h}^{t_0} 2\Re \left(i v(t) \mathcal{T}_{P_s}(v(t)) \frac{dv(t)}{dt} \right) e^{-st} dt.$$

Finally we have

$$\begin{aligned} \tilde{u}_{sx, P_s}^-(x, y, s) &= - \int_{t_h}^{t_0} \frac{\mathcal{P}_{12}}{s\pi} \Re \left(i v(t) \mathcal{T}_{P_s}(v(t)) \frac{dv(t)}{dt} \right) e^{-st} dt \\ &\quad - \int_{t_0}^{+\infty} \frac{\mathcal{P}_{12}}{s\pi} \Re \left(i \gamma(t) \mathcal{T}_{P_s}(\gamma(t)) \frac{d\gamma(t)}{dt} \right) e^{-st} dt \\ &= \int_{t_h}^{+\infty} \left[\int_0^t v_{P_s, x}^-(x, y, \tau) d\tau \right] e^{-st} dt \end{aligned}$$

and we conclude by using the injectivity of the Laplace transform that

$$u_{sx, P_s}^-(x, y, t) = \int_0^t v_{P_s, x}^-(x, y, \tau) d\tau.$$

5 Numerical illustration

To illustrate the use our results, we have compared our analytical solution to a numerical one obtained by Morency and Tromp [15]. We consider an acoustic layer with a density $\rho^+ = 1020 \text{ kg/m}^3$ and a celerity $V^+ = 1500 \text{ m/s}$ on top of a poroelastic layer whose characteristic coefficients are: the solid density $\rho_s^- = 2500 \text{ kg/m}^3$; the fluid density $\rho_f^- = 1020 \text{ kg/m}^3$; the porosity $\phi^- = 0.4$; the tortuosity $a^- = 2$; the solid bulk modulus $K_s^- = 16.0554 \text{ GPa}$; the fluid bulk modulus $K_f^- = 2.295 \text{ GPa}$; the frame bulk modulus $K_b^- = 10 \text{ GPa}$; and the frame shear modulus $\mu^- = 9.63342 \text{ GPa}$. As a result, the celerity of the waves in the poroelastic medium are: for the fast P wave, $V_{pf}^- = 3677 \text{ m/s}$; for the slow P wave, $V_{ps}^- = 1060 \text{ m/s}$; and for the ψ wave, $V_s^- = 2378 \text{ m/s}$.

The source is located in the acoustic layer, at 500m from the interface. It is a point source in space and a fifth derivative of a Gaussian of dominant frequency $f_0 = 15 \text{ Hz}$:

$$f(t) = 4.10^{10} \frac{\pi^2}{f_0^2} \left[9 \left(t - \frac{1}{f_0} \right) + 4 \frac{\pi^2}{f_0^2} \left(t - \frac{1}{f_0} \right)^3 - 4 \frac{\pi^4}{f_0^4} \left(t - \frac{1}{f_0} \right)^5 \right] e^{-\frac{\pi^2}{f_0^2} \left(t - \frac{1}{f_0} \right)^2}.$$

We compute the solution at two receivers, the first one is in the acoustic layer, at 533m from the interface; the second one is in the poroelastic layer, at 533m from the interface; both are located on a vertical line at 400m from the source (see Fig. 5). We represent the y component of the displacement from $t = 0$ to $t = 1$ s. on Fig. 6. The left picture represents the solution at receiver 1 while the right picture represents the solution at receiver 2. On both pictures the blue solid curve is the analytical solution and the red dashed curve is the numerical solution.

Both pictures show a good agreement between the two solutions, which validates the numerical code.

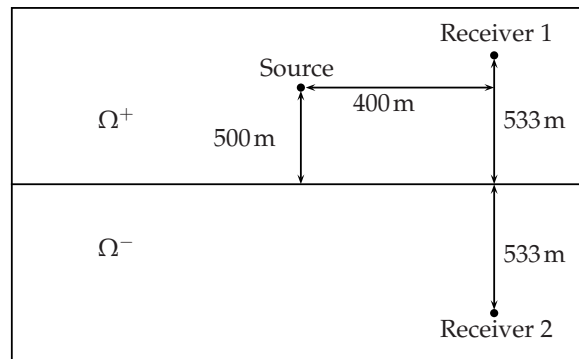


Figure 5: Configuration of the experiment.

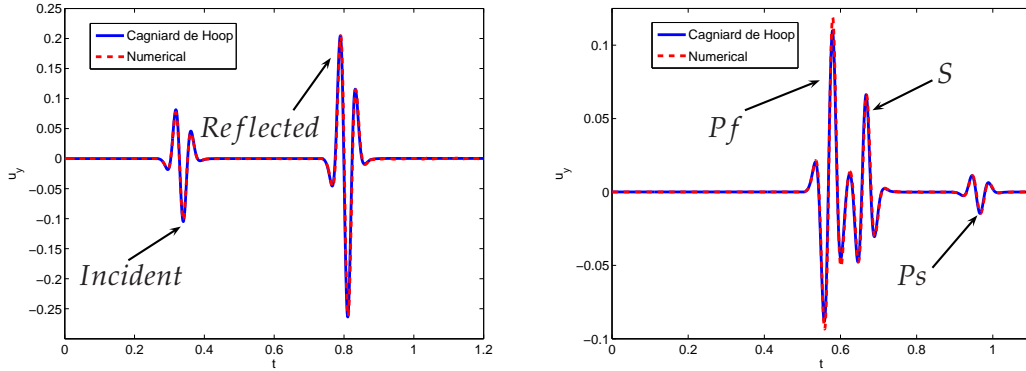


Figure 6: The y component of the displacement at receiver 1 (left) and 2 (right). The blue solid curve is the analytical solution computed by the Cagniard-de Hoop method, the red dashed curve is the numerical solution.

6 Conclusion

We provided the complete solution (reflected and transmitted wave) of the propagation of wave in a stratified 2D medium composed of an acoustic and a poroelastic layer and we used it to validate a numerical code. In a forthcoming paper we will use this solution as a basis to derive the solution in a three dimensional medium. We will also extend the method to the propagation of waves in heterogeneous poroelastic medium in two and three dimensions.

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A Properties of the function $\mathcal{F}(q,t)$

We recall in this section some properties of the function (see [1, 11, 14]):

$$\mathcal{F} := \mathcal{F}(q,t) = -y \left(\frac{1}{V_{Ps}^2} + q^2 \right)^{1/2} + h \left(\frac{1}{V^2} + q^2 \right)^{1/2} + iqx - t,$$

Property A.1. For each $t \in \mathbb{R}^+$, $\mathcal{F}(\cdot, t)$ admits at most two roots.

Property A.2. There is one and only one $t_0 \in \mathbb{R}^+$ such that $\mathcal{F}(\cdot, t_0)$ admits a double root q_0 . This root is purely imaginary and t_0 corresponds to the physical arrival time of the wave. Moreover $\Im m(q_0) \leq 0$ if $x \geq 0$ and $\Im m(q_0) > 0$ if $x < 0$.

Property A.3. For $t > t_0$, $\mathcal{F}(\cdot, t)$ admits exactly two roots, which have the same imaginary part, positive if $x < 0$ and negative else, and an opposite non-zero real part.

Property A.4. If $|\gamma(t_0)| \leq 1/V_{\max}$, there exists a time $t_h \leq t_0$, such that for $t_h \leq t \leq t_0$, the function $\mathcal{F}(\cdot, t)$ admits exactly two imaginary roots $q_1(t)$ and $q_2(t)$. The time t_h is such that

$$\gamma(t_h) = \begin{cases} \frac{i}{V_{\max}}, & \text{if } x < 0, \\ -\frac{i}{V_{\max}}, & \text{if } x \geq 0, \end{cases}$$

and corresponds to the physical arrival time of the head wave. It can be computed by using the relation $\mathcal{F}(i/V_{\max}, t_h) = 0$ (if $x < 0$) or $\mathcal{F}(-i/V_{\max}, t_h) = 0$ (if $x \geq 0$):

$$t_h = -y \sqrt{\frac{1}{V_{Ps}^{-2}} - \frac{1}{V_{\max}^2}} + h \sqrt{\frac{1}{V^+{}^2} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}}.$$

Property A.5. The roots $q_1(t)$ and $q_2(t)$ satisfy

$$\Im m(q_1(t)) \in [1/V_{\max}, \Im m(\gamma(t_0))] \text{ and } \Im m(\partial_t q_1(t)) > 0,$$

$$\Im m(q_2(t)) \in [\Im m(\gamma(t_0)), -1/V_{\max}] \text{ and } \Im m(\partial_t q_2(t)) < 0.$$

B Definition of the arrival time of the transmitted waves

We detail in this section the computation of the arrival time of the transmitted Ps wave at point (x, y) . We first have to determine the fastest path from the source to the point (x, y) : we search a point ξ_0 on the interface between the two media which minimizes the function

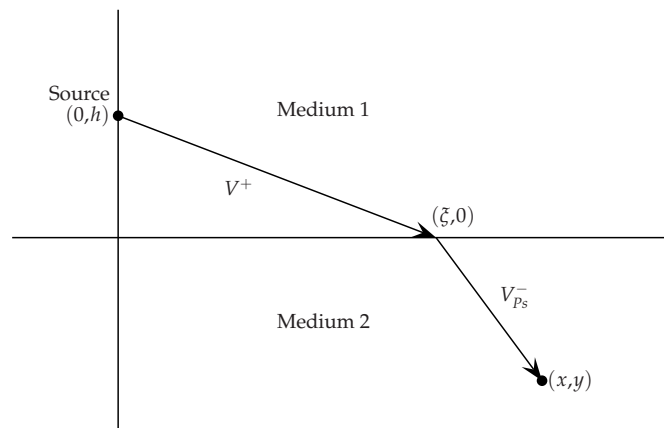
$$t(\xi) = \frac{\sqrt{\xi^2 + h^2}}{V^+} + \frac{\sqrt{(x - \xi)^2 + y^2}}{V_{Ps}^-}$$

(see Fig. 7). This leads us to find ξ_0 such that

$$t'(\xi_0) = \frac{\xi_0}{V^+ \sqrt{\xi_0^2 + h^2}} + \frac{\xi_0 - x}{V_{Ps}^- \sqrt{(x - \xi_0)^2 + y^2}} = 0. \quad (\text{B.1})$$

From a numerical point of view, the solution to this equation is done by computing the roots of the following fourth degree polynomial

$$\left(\frac{1}{V^+{}^2} - \frac{1}{V_{Ps}^{-2}} \right) X^4 + 2x \left(\frac{1}{V_{Ps}^{-2}} - \frac{1}{V^+{}^2} \right) X^3 + \left(\frac{x^2 + y^2}{V^+{}^2} - \frac{x^2 + h^2}{V_{Ps}^{-2}} \right) X^2 + \frac{xh^2}{V_{Ps}^{-2}} X + \frac{x^2 h^2}{V_{Ps}^{-2}},$$

Figure 7: Path of the transmitted i wave.

ξ_0 is thus the only real root of this polynomial located between 0 and x which is also solution to (B.1). Once ξ_0 is computed, we can define

$$t_0 = \frac{\sqrt{\xi_0^2 + h^2}}{V^+} + \frac{\sqrt{(x - \xi_0)^2 + y^2}}{V_{Ps}^-}.$$

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