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The Monotone Robin-Robin Domain Decomposition Methods for the Elliptic Problems with Stefan-Boltzmann Conditions

Wenbin Chen[1,2,*], Jin Cheng[1], Masahiro Yamamoto[3] and Weili Zhang[1]

[1] *School of Mathematical Sciences and Key Laboratory of Computational Physics (MOE), Fudan University, Shanghai, 200433, China.*
[2] *Shanghai Key Laboratory of Intelligent Information Processing, Fudan University, Shanghai, 200433, China.*
[3] *Department of Mathematical Sciences, University of Tokyo, Tokyo, 153-8914, Japan.*

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Abstract. This paper is concerned with the elliptic problems with nonlinear Stefan-Boltzmann boundary condition. By combining with the monotone method, the Robin-Robin domain decomposition methods are proposed to decouple the nonlinear interface and boundary condition. The monotone properties are verified for both the multiplicative and the additive domain decomposition methods. The numerical results confirm the theoretical analysis.

AMS subject classifications: 35J65, 65N55

Key words: Nonlinear Stefan-Boltzmann condition, monotone methods, Robin-Robin domain decomposition method.

1 Model problems

Let u be the solution of Laplace equation with nonlinear Stefan-Boltzmann boundary condition arising from the steel-making industry:

$$-\Delta u = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma_N, \tag{1.2}$$

$$\lambda \frac{\partial u}{\partial \mathbf{n}} = -\sigma(u^4 - u_e^4) \quad \text{on } \Gamma_e, \tag{1.3}$$

$$u = u_s \quad \text{on } \Gamma_s, \tag{1.4}$$

*Corresponding author. *Email addresses:* `wbchen@fudan.edu.cn` (W. Chen), `jcheng@fudan.edu.cn` (J. Cheng), `myama@ms.u-tokyo.ac.jp` (M. Yamamoto), `zhangwl@fudan.edu.cn` (W. Zhang)

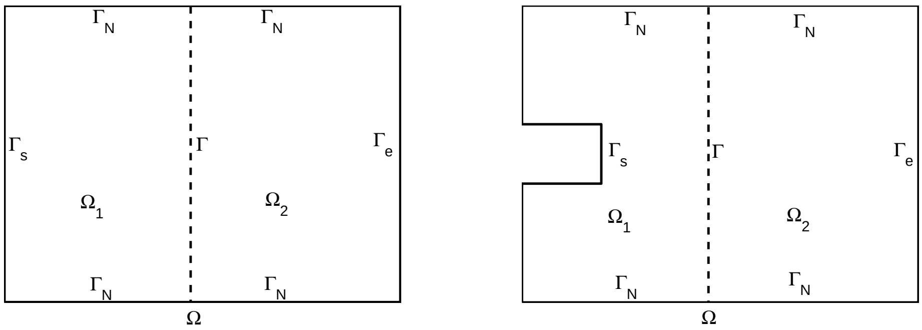


Figure 1: Domain with composite heat-resistant materials and partial corroded domain.

where u represents the temperature of the heat-resistant materials, $u_s>0$ is the temperature of the melting-steel, $u_e>0$ is the temperature of the exterior air , and the temperature of the steel is higher than the temperature of the exterior air. $\Omega=\bigcup\Omega_i$ is the domain made of composite heat-resistant materials, and the heat conduction coefficient of the material λ may be different in the every subdomain Ω_i. The Boltzmann thermal fourth power law (1.3) is imposed on the exterior boundary surrounded by air, and σ is the Boltzmann radiation coefficient. Let $u_i=u|_{\Omega_i}$ and $\lambda_i=\lambda|_{\Omega_i}$. Then

$$\lambda_1\frac{\partial u_1}{\partial \mathbf{n}_1}\Big|_\Gamma+\lambda_2\frac{\partial u_2}{\partial \mathbf{n}_2}\Big|_\Gamma=0 \tag{1.5}$$

at the interface boundary $\Gamma=\overline{\Omega}_1\cap\overline{\Omega}_2$ according to the heat transfer law, here $\mathbf{n}_1$ is the outer unit normal vector from Ω_1 to Ω_2, and $\mathbf{n}_2$ from Ω_2 to Ω_1. Specially, another relationship

$$\lambda_1\frac{\partial u_1}{\partial \mathbf{n}_1}=-\sigma(u_1^4-u_2^4) \tag{1.6}$$

is observed by the experiments which is different from the conventional condition $u_1|_\Gamma=u_2|_\Gamma$. This condition (1.6) is explained as following: the interface Γ is an approximation of very thin layer which is filled with air, and the Boltzmann thermal fourth power law is applied to the heat transfer between the high temperature materials and air, and (1.6) can be obtained by removing the variable of the air temperature.

In the steel-making procedure, the boundary Γ_0 may be corroded after long-time high temperature heat process [6,7,30], the detection of the corrosion is very important, and stable and efficient solvers for the problem (1.1)-(1.6) are the base of any corrosion detected algorithm.

Among the various techniques for the nonlinear partial differential equations, the monotone method is one powerful tool to obtain the existence, uniqueness and other properties of the solutions [5,15,19,26]. Moreover, by using the technique of upper and lower solutions, efficient algorithms can be constructed to solve the nonlinear equations, and the numerical solutions of the iterative sequences will converge monotonically and linearly to the solutions of partial difference equations (see [8, 10, 14–16, 18, 19, 21]). In order to speed up the convergence rate of the iterative method, an accelerated method is proposed in [17] to obtain quadratic convergence in the sense of Euclid norm.

On the other hand, the domain decomposition methods (DDMs) are naturally suitable for our composite material problems (see the monographs [23, 27] and references therein). With the DDM framework, the code can be simplified and modularized even for the nonlinear problems. Specially the Robin-Robin DDM can be used to decouple our nonlinear Robin-type interface condition. For the linear problems, the convergences of the Robin-Robin DDMs have been analyzed in [4, 11, 22, 24]. While for the nonlinear problems, the researches mainly focused on the overlapping Schwarz methods and the nonlinear right hand side (see [1–3, 12, 13, 29]).

In this paper, we follow Pao's monotone method [14–17] to solve the elliptic system in single domain, and give the convergence analysis in the sense of discrete maximum norm. For the composite material problems, we use the Robin-Robin domain decomposition methods to solve the composite material problems, the nonlinear interface and boundary conditions are dealt by the monotone methods and the monotone properties of the iterative sequences in the DDMs can be proven as the case of the single domain.

The paper is organized as follows. One single domain problem is considered in Section 2, and the upper and lower solutions are introduced, the monotone Picard-FD method and Newton-FD method are constructed, and the convergence rate for both methods are proven in sense of discrete maximum norm. In Section 3, the additive Robin-Robin DDM and multiplicative Robin-Robin DDM are proposed to solve the coupled nonlinear interface condition, and the monotone property is verified for both methods. The numerical results are given in Section 4, which confirm our theoretical analysis. Finally the short conclusions are given in Section 5.

2 The monotone methods in one single domain

In this section, we consider the elliptic equations in one single domain Ω:

$$\mathcal{L}u = f(x,u) \quad \text{in } \Omega, \tag{2.1}$$

$$\mathcal{B}u = g(x,u) \quad \text{on } \partial\Omega, \tag{2.2}$$

where $\mathcal{L} = -\Delta$, and $\mathcal{B}$ represents the boundary operator which is the identity operator I at the Dirichlet boundary Γ_s, and the unit normal derivative operator $\frac{\partial}{\partial \mathbf{n}}$ at the Neumann boundary Γ_N and nonlinear Robin boundary (Stefan-Boltzmann) Γ_e. By introducing two auxiliary functions $c(x,u)$ and $b(x,u)$, one iterative sequence $u^{(k)}$ can be generated from the initial guess $u^{(0)}$:

$$\mathcal{L}u^{(k)} + c(x,u^{(k-1)})u^{(k)} = f(x,u^{(k-1)}) + c(x,u^{(k-1)})u^{(k-1)} \quad \text{in } \Omega, \tag{2.3}$$

$$\mathcal{B}u^{(k)} + b(x,u^{(k-1)})u^{(k)} = g(x,u^{(k-1)}) + b(x,u^{(k-1)})u^{(k-1)} \quad \text{on } \partial\Omega. \tag{2.4}$$

Denote the upper sequence $\{\overline{u}^{(k)}\}$ with the upper solution $\tilde{u}$ as initial guess, and the lower sequence $\{\underline{u}^{(k)}\}$ with the lower solution $\hat{u}$ as initial guess respectively. We recall that the upper solution satisfies the inequalities:

$$\mathcal{L}u \geq f(x,u) \quad \text{in } \Omega, \tag{2.5}$$
$$\mathcal{B}u \geq g(x,u) \quad \text{on } \partial\Omega, \tag{2.6}$$

and the lower solution satisfies the reversed inequalities in (2.5)-(2.6). We assume that the upper solution $\tilde{u}$ and lower solution $\hat{u}$ are ordered, i.e, $\tilde{u} \geq \hat{u}$, and denote the sector $J = \langle \hat{u}, \tilde{u} \rangle$ all the function set of v in $C(\overline{\Omega})$ such that $\hat{u} \leq v \leq \tilde{u}$.

There are two special ways to choose the auxiliary functions $c(x,u)$ and $b(x,u)$ which leads to two kind of iterative methods:

- Picard-type method: here $c(x,u)$ and $b(x,u)$ are constants or functions independent of u or k, and

  $$c(x) \geq \max\Big\{0, \sup_{u\in J}\{-f'_u\}\Big\}, \quad b(x) \geq \max\Big\{0, \sup_{u\in J}\{-g'_u\}\Big\},$$

  where f'_u (or g'_u) the partial derivative of f (or g) with respect to u;
- Newton-type method: here $c(x,u) = -f'_u(x,u)$ and $b(x,u) = -g'_u(x,u)$.

For convenience, we suppress the dependence of the functions $f(x,u), g(x,u), c(x,u)$ and $b(x,u)$ on x, and denote them by $f(u), g(u), c(u), b(u)$ or c,b if $c(x,u)$ and $b(x,u)$ are independent of u.

In the Picard-type method, it can be verified that for $\hat{u} \leq u_2 \leq u_1 \leq \tilde{u}$,

$$f(u_1) - f(u_2) \geq -c(u_1 - u_2), \quad g(u_1) - g(u_2) \geq -b(u_1 - u_2). \tag{2.7}$$

Therefore the upper and lower sequences own the monotone property ([15], Lemmas 4.1, 4.2):

$$\underline{u}^{(0)} \leq \underline{u}^{(k)} \leq \underline{u}^{(k+1)} \leq \underline{u} \leq \bar{u} \leq \bar{u}^{(k+1)} \leq \bar{u}^{(k)} \leq \bar{u}^{(0)}, \tag{2.8}$$

where the minimal solution $\underline{u}$ is the limit of the lower sequence $\underline{u}^{(k)}$ and the maximal solution $\bar{u}$ is the limit of the upper sequence $\bar{u}^{(k)}$, respectively.

Obviously, in Newton-type method, the motivation to modify the definition of the auxiliary functions $c(x,u)$ and $b(x,u)$ is to get the quadratic convergence, while it is needed to be checked when the monotone property (2.8) holds. In [17], Pao proposed another way to accelerate the iterative process:

$$c^{(k)} = \max\Big\{0, \max\Big\{-\frac{\partial f}{\partial u}, \underline{u}^{(k)} \leq u \leq \bar{u}^{(k)}\Big\}\Big\}, \tag{2.9a}$$
$$b^{(k)} = \max\Big\{0, \max\Big\{-\frac{\partial g}{\partial u}, \underline{u}^{(k)} \leq u \leq \bar{u}^{(k)}\Big\}\Big\}. \tag{2.9b}$$

Therefore the monotone property (2.8) is considered in the first place. The following lemma demonstrates the monotone property of the iterative sequences.

Lemma 2.1. *Assume that $c(u)$,$b(u)$ are bounded nonnegative function in $\Omega\times J$. If the functions f and g satisfy*

$$f(u)-f(v)\geq -c(u)(u-v), \quad g(u)-g(v)\geq -b(u)(u-v) \tag{2.10}$$

for any $\hat{u}\leq v\leq u\leq \tilde{u}$, then the sequence of (2.3)-(2.4) *has the monotone property:*

$$\hat{\hat{u}}\leq \overline{u}^{(k+1)}\leq \overline{u}^{(k)}\leq \bar{u}^{(0)}, \tag{2.11}$$

where $\hat{\hat{u}}\leq\tilde{u}$ is one lower solution, and $\overline{u}^{(k)}$ is one upper solution of (2.1)-(2.2). *And if the functions f and g satisfy*

$$f(u)-f(v)\geq -c(v)(u-v), \quad g(u)-g(v)\geq -b(v)(u-v) \tag{2.12}$$

for any $\hat{u}\leq v\leq u\leq \tilde{u}$, then the lower sequences of (2.3)-(2.4) *has the monotone property:*

$$\underline{u}^{(0)}\leq \underline{u}^{(k)}\leq \underline{u}^{(k+1)}\leq \overline{\overline{u}}, \tag{2.13}$$

where $\overline{\overline{u}}\geq\hat{u}$ is one upper solution, and $\underline{u}^{(k)}$ is one lower solution of (2.1)-(2.2). *Moreover, if both conditions hold, then $\bar{u}^{(k)}$ and $\underline{u}^{(k)}$ are the ordered upper and lower solutions, respectively.*

Let us briefly discuss the conditions (2.10) and (2.12). Note that $c\geq 0$ and these two conditions hold if f is monotonically nondecreasing with respect to u: $f(u)\geq f(v)$ for $v\leq u$. Otherwise we can use

$$f(u)-f(v)\geq \max\{-c(u),-c(v)\}(u-v), \quad \hat{u}\leq v\leq u\leq \tilde{u}. \tag{2.14}$$

So if c does not depend on u, then the conditions (2.10) and (2.12) are just the condition (2.7). In [28], Wang proposed a different accelerated monotone iteration process. By carefully considering the effect of the smallest eigenvalue of the stiffness matrix, two side monotone iteration sequences like (2.8) can be still constructed.

Now let us be back to our model problem, where $f(x,u)\equiv 0$ and $g(x,u)=-\sigma(u^4-u_e^4)$. Since $u_s\geq u_e$, it is easily checked that $\tilde{u}=\max_x u_s$ is the upper solution and $\hat{u}=\min_x u_e$ is the lower solution. And if the Newton method is considered, it is better to be used in *one-side* way, that is to say, to use the upper sequences to solve this problem, and the monotone property can be guaranteed by Lemma 2.1. Moreover, our choice for $b^{(k)}$ is same as Pao's choice, i.e., (2.9b).

2.1 Numerical schemes and convergence results

In the numerical method, the finite difference method is used to discretize the Laplacian operator. Insider the domain Ω, the five-point finite difference scheme is used to approximate the Laplace operator. Let h be the stepsize and the finite difference solution u_h for the problem (2.1)-(2.2) satisfies the equation:

$$-\Delta_h u_h = f_h(u_h) \quad \text{in} \;\; \Omega_h, \tag{2.15}$$

and $u_{0,j}=(u_s)_{0,j}$ at the nodal points of $\Gamma_{0,h}$, and

$$4u_{i,N}-2u_{i,N-1}-u_{i+1,N}-u_{i-1,N}=h^2 f_{i,N} \quad \text{on } \Gamma_{N,h}, \tag{2.16a}$$
$$4u_{i,0}-2u_{i,1}-u_{i+1,0}-u_{i-1,0}=h^2 f_{i,0} \quad \text{on } \Gamma_{N,h}, \tag{2.16b}$$
$$4u_{N,j}-2u_{N-1,j}-u_{N,j+1}-u_{N,j-1}=h^2 f_{N,j}+2hg_{N,j} \quad \text{on } \Gamma_{e,h}. \tag{2.16c}$$

Denote $\Omega_h,\Gamma_{0,h},\Gamma_{N,h}$ and $\Gamma_{e,h}$ the corresponding sets of the nodal points, and the mirror method is used to get second order approximation at the nodal points of $\Gamma_{N,h}$ and $\Gamma_{e,h}$. We may rewrite the above algebraic equations as

$$\mathcal{L}_h u_h = f_h(u_h) \quad \text{in } \Omega_h, \tag{2.17}$$
$$\mathcal{B}_h u_h = g_h(u_h) \quad \text{on } \partial\Omega_h. \tag{2.18}$$

By combining the finite difference method with the Picard or Newton processes (2.3)-(2.4), the nonlinear system (2.17)-(2.18) can be solved iteratively: If the initial values $u_h^{(0)}=u_{i,j}^{(0)}$ are known, find the finite difference solution $u_h^{(k)}=u_{i,j}^{(k)}$ by the following iteration process

$$\mathcal{L}_h u_h^{(k)}+c_h^{(k-1)}u_h^{(k)}=f_h^{(k-1)}+c_h^{(k-1)}u_h^{(k-1)} \quad \text{in } \Omega_h, \tag{2.19}$$
$$\mathcal{B}_h u_h^{(k)}+b_h^{(k-1)}u_h^{(k)}=g_h^{(k-1)}+b_h^{(k-1)}u_h^{(k-1)} \quad \text{on } \partial\Omega_h. \tag{2.20}$$

Here

$$f_h^{(k-1)}=f(x_{ij},u_{i,j}^{(k-1)}), \quad g_h^{(k-1)}=g(x_{ij},u_{i,j}^{(k-1)}), \quad c_h=c(x_{ij}), \quad b_h=b(x_{ij}),$$

in the Picard-FD sequence or

$$c_h^{(k-1)}=c(x_{ij},u_{ij}^{(k-1)}), \quad b_h^{(k-1)}=b(x_{ij},u_{ij}^{(k-1)})$$

in the Newton-FD sequence.

In both methods, denote $\{\overline{u}_h^{(k)}\}$ (or $\{\underline{u}_h^{(k)}\}$) the sequence $\{u_h^{(k)}\}$ if the initial $u_h^{(0)}$ value is taken by discrete upper solution $\tilde{u}_h$ (or discrete low solution $\hat{u}_h$ respectively). Similar monotone property holds in the discrete cases: If the condition (2.10) holds, then the sequences $\{\overline{u}_h^{(k)}\}$ has the monotone property:

$$\hat{\hat{u}}_h \leq \overline{u}_h^{(k+1)} \leq \overline{u}_h^{(k)} \leq \overline{u}_h, \tag{2.21}$$

where $\hat{\hat{u}}_h \leq \tilde{u}_h$ is a discrete lower solution, and $\overline{u}_h^{(k)}$ is also a upper solution of (2.17)-(2.18). And if the condition (2.12) holds, then the sequences $\{\underline{u}_h^{(k)}\}$ has the monotone property:

$$\hat{u}_h \leq \underline{u}_h^{(k)} \leq \underline{u}_h^{(k+1)} \leq \overline{\overline{u}}_h, \tag{2.22}$$

where $\overline{\overline{u}}_h \geq \hat{u}_h$ is a discrete upper solution, and $\underline{u}^{(k)}$ is also a discrete lower solution of (2.17)-(2.18).

Note that the upper sequence $\overline{u}_h^{(k)}$(or the low sequence) converge monotonically to the discrete maximal solution u_h of (2.17)-(2.18) (or the discrete minimal solution respectively). Pao has obtained the quadratic convergence of the Newton-FD sequence in the sense of Euclid norm(discrete 2-norm), here we show that linear convergence of the Picard-FD sequence and the quadratic convergence of the Newton-FD sequence in the discrete maximum norm. For any grid function v_h on the nodal point set D, the discrete maximum norm $|v_h|_D = \max_{x_{i,j}\in D}|v_{i,j}|_D$. In the following two results, we assume that $f'_u, g'_u \leq 0$ for $u \in J$, and then the solution u_h is unique. This condition can be slightly weakened (see [17]).

Theorem 2.1. *Let $\overline{u}_h^{(k)}$ be the upper sequence of Picard-FD iteration scheme. If $f'_u, g'_u \leq 0$ for $u \in J$ and* (2.10) *is satisfied, then there exists $\rho < 1$ such that*

$$|\overline{u}_h^{(k+1)} - u_h|_{\overline{\Omega}} \leq \rho |\overline{u}_h^{(k)} - u_h|_{\overline{\Omega}}. \tag{2.23}$$

Proof. Let $F(u) = f(u) + c(x)u$, $G(u) = g(u) + c(x)u$, and $e_h^{(k)} = \overline{u}_h^{(k)} - u_h$. Then $e_h^{(k)}$ satisfies the error equations:

$$-\Delta_h e_h^{(k+1)} + c_h e_h^{(k+1)} = F_h(\overline{u}_h^{(k)}) - F_h(u_h), \quad \text{in } \Omega_h, \tag{2.24}$$

$e^{(k+1)}$ vanishes at the nodal points of $\Gamma_{0,h}$,

$$\begin{aligned} &\left(4 + h^2 c_{N,j} + 2hb_{N,j}\right) e_{N,j}^{(k+1)} - 2e_{N-1,j}^{(k+1)} - e_{N,j+1}^{(k+1)} - e_{N,j-1}^{(k+1)} \\ &= h^2 F_h(\overline{u}_{N,j}^{(k)}) + 2hG_h(\overline{u}_{N,j}^{(k)}) - h^2 F_h(u_h) - 2hG_h(u_h) \quad \text{on } \Gamma_{e,h}, \end{aligned} \tag{2.25}$$

and

$$(4 + h^2 c_{i,N}) e_{i,N}^{(k+1)} - 2e_{i,N-1}^{(k+1)} - e_{i+1,N}^{(k+1)} - e_{i-1,N}^{(k+1)} = h^2 \left(F_h(\overline{u}_h^{(k)}) - F_h(u_h)\right)_{i,N}, \tag{2.26}$$

$$(4 + h^2 c_{i,0}) e_{i,0}^{(k+1)} - 2e_{i,1}^{(k+1)} - e_{i+1,0}^{(k+1)} - e_{i-1,0}^{(k+1)} = h^2 \left(F_h(\overline{u}_h^{(k)}) - F_h(u_h)\right)_{i,0} \tag{2.27}$$

at the points of $\Gamma_{N,h}$. If we assume that $|e_h^{(k+1)}|$ attains its maximum value at $x_{i,j}$:

$$|e_{i,j}^{(k+1)}| = |e_h^{(k+1)}|_{\overline{\Omega}}, \tag{2.28}$$

then by the discrete maximum principle [9,25], at this point $x_{i,j}$, $|f'_u| \neq 0$ (if $x_{i,j} \in \Omega_h$) and $|f'_u| + |g'_u| \neq 0$ (if $x_{i,j} \in \partial\Omega_h$). If $x_{i,j} \in \Omega_h$, we can assume that $c_{i,j} > 0$. Otherwise, since $c = \max\{0, \sup_{u\in J}\{-f'_u\}\}$, we have $F_h(\overline{u}_h^{(k)}) - F_h(u_h) = 0$ at the point $x_{i,j}$. Then by the discrete maximum principles, the neighbor points of $x_{i,j}$ also attain the same maximum value, and we can move the point $x_{i,j}$ to another point such that $c_{i,j}>0$. Moreover, if $e_h^{(k+1)}$ attains its maximum at $x_{i,j}\in\Omega$, then $-\Delta_h e_{i,j}^{(k+1)}\geq 0$ and

$$e_{i,j}^{(k+1)}\leq\frac{1}{c_{i,j}}\Big(F_h(\overline{u}_h^{(k)})-F_h(u_h)\Big)=\frac{1}{c_{i,j}}\left(\frac{\partial f}{\partial u}(\xi_{i,j})+c_{i,j}\right)e_{i,j}^{(k)},\tag{2.29}$$

where $u_h\leq\xi_h\leq u^{(k)}$. This estimation also holds if $x_{i,j}\in\Gamma_{N,h}$. Let us consider $x_{i,j}\in\Gamma_{e,h}$. Note that

$$\begin{aligned}&h^2\Big(F_h(\overline{u}_h^{(k)})-F_h(u_h)\Big)+2h\Big(G_h(\overline{u}_h^{(k)})-G_h(u_h)\Big)\\&=\Big(h^2\left(f_u'(\xi_h)+c_h\right)+2h\left(g_u'(\xi_h)+b_h\right)\Big)e_h^{(k)}.\end{aligned}\tag{2.30}$$

If $e_h^{(k+1)}$ attains its maximum at $x_{N,j}\in\Gamma_{e,h}$, then

$$e_{N,j}^{(k+1)}\leq\frac{1}{h^2c_{N,j}+2hb_{N,j}}\Big(h^2\left(f_u'(\xi_{N,j})+c_{N,j}\right)+2h\left(g_u'(\xi_{N,j})+b_{N,j}\right)\Big)e_{N,j}^{(k)}.\tag{2.31}$$

Now define the mesh-function ρ_{ij}:

$$\rho_{i,j}=\begin{cases}\dfrac{1}{c_{i,j}}f_u'(\xi_{i,j})+1, & x_{i,j}\in\Omega_h\cup\Gamma_{a,h},\\ \dfrac{1}{hc_{N,j}+2b_{N,j}}\Big(hf_u'(\xi_{N,j})+2g_u'(\xi_{N,j})\Big)+1, & x_{i,j}\in\Gamma_{e,h}.\end{cases}\tag{2.32}$$

Therefore the above estimates can be rewritten as: if $e_h^{(k+1)}$ attains the maximum at $x_{i,j}$, then

$$0\leq|e_h^{(k+1)}|_{\overline{\Omega}}\leq\rho_{i,j}e_{i,j}^{(k)}.\tag{2.33}$$

From the definitions of b and c, we have $\rho_{i,j}<1$. Now set $\rho=\min\rho_{i,j}$, we have

$$|e_h^{(k+1)}|_{\overline{\Omega}}\leq\rho|e_h^{(k)}|_{\overline{\Omega}},\tag{2.34}$$

which is the desired result. □

If $f(x,u)$ is independent of u, i.e., $f(x,u)=f(x)$, then we can set $c=0$, and $e_h^{(k)}$ only attains its maximum values on the boundary nodal points of $\Gamma_{e,h}$, and

$$\rho_{i,j}=\frac{1}{b_{N,j}}\frac{\partial g}{\partial u}(\xi_{N,j})+1.$$

Now if we consider the Stefan-Boltzmann condition $g(u)=-\sigma(u^4-u_e^4)$, then $g'(u)=-4\sigma u^3$. Note that $u_h\leq\xi_h\leq\overline{u}_h^{(k)}$. Consequently, $g'(\xi_h)\leq-4\sigma u_h^3$. Moreover if we take

$$b=\sup\{-g'(u),\tilde{u}\leq u\leq\overline{u}\},$$

then $b_h=4\sigma\overline{u}^3$, and $\rho\leq 1-|u_h^3/\overline{u}^3|_{\overline{\Gamma}_b}$.

Theorem 2.2. *Assume that f'_u and $g'_u \le 0$ for $u \in J$ and* (2.10) *is satisfied, f''_u and g''_u are continuous and bounded for $u \in J$. Let $\overline{u}_h^k$ be the upper sequence of Picard-FD iteration scheme. Then there exists an C such that*

$$|\overline{u}_h^{(k+1)} - u_h|_{\overline{\Omega}} \le C|\overline{u}_h^{(k)} - u_h|^2_{\overline{\Omega}}. \tag{2.35}$$

Proof. Let $e_h^{(k)} = \overline{u}_h^{(k)} - u_h$. Then $e_h^{(k)}$ also satisfies the error equations (2.24)-(2.27) if we replace c_h by $c_h^{(k)}$ and b_h by $b_h^{(k)}$. Let us expand $f(u_h)$ at $u_h^{(k)}$:

$$f(u_h) = f(u_h^{(k)}) + f'(u_h^{(k)})(u_h - u_h^{(k)}) + \frac{f''(\xi)}{2}(u_h - u_h^{(k)})^2. \tag{2.36}$$

Then

$$F(u_h^{(k)}) - F(u_h) = -\frac{f''(\xi)}{2}(u_h - u_h^{(k)})^2. \tag{2.37}$$

In a similar manner of Lemma 2.1, one mesh grid function can be defined:

$$\alpha_h^{(k)} = \begin{cases} \dfrac{f''_u(\xi)}{2f'_u(u_h^{(k)})}, & x_{i,j} \in \Omega_h \cup \Gamma_{a,h}, \\ \dfrac{1}{2(hf'_u(u_h^{(k)}) + 2g'_u(u_h^{(k)}))}\Big(hf''_u(\xi) + 2g''_u(\xi)\Big), & x_{i,j} \in \Gamma_{e,h}, \end{cases} \tag{2.38}$$

and $\alpha_h^{(k)} = 0$ at the points where the denominators $f'_u(u_h^{(k)})$ or $hf''_u + 2g''_u$ vanishes. Then we have

$$|\overline{u}_h^{(k+1)} - u_h|_{\overline{\Omega}} \le |\alpha_h^{(k)}|_{\overline{\Omega}} |\overline{u}_h^{(k)} - u_h|^2_{\overline{\Omega}}. \tag{2.39}$$

Define

$$\alpha_1 = \frac{\sup\limits_{u\in J}|f''_u|}{2\min\limits_{u\in J}|f'_u|}, \quad \alpha_2 = \frac{\sup\limits_{u\in J}|hf''_u + 2g''_u|}{2\min\limits_{u\in J}\big(|hf'_u(u_h^{(k)}) + 2g'_u|\big)},$$

and $C = \max\{\alpha_1, \alpha_2\}$ if α_1 and α_2 are finite or takes the finite one. Then $|\alpha_h^{(k)}|_{\overline{\Omega}} \le C$, which proves the lemma. □

3 Domain decomposition methods for the composite materials problems

The non-overlapping domain decomposition methods are naturally suitable to split the coupled interface boundary condition (1.6). Two Robin-Robin DDMs can be constructed here: one is the multiplicative version and the another is additive one.

Multiplicative Robin-Robin domain decomposition method: Staring form the initial guesses $u^{(-\frac{1}{2})}$ and $u_2^{(0)}$, for $n\geq 0$, define

$$u^{(n)}=\begin{cases} u_1^{(n-\frac{1}{2})}, & \Omega_1,\\ u_2^{(n)}, & \Omega_2,\end{cases}$$

where $u_1^{(n+\frac{1}{2})}$ is firstly obtained by solving

$$-\Delta u_1^{(n+\frac{1}{2})}=0 \quad \text{in } \Omega_1, \tag{3.1}$$

$$\lambda_1\frac{\partial u_1^{(n+\frac{1}{2})}}{\partial \mathbf{n}_1}=-\sigma\Big((u_1^{(n+\frac{1}{2})})^4-(u_2^{(n)})^4\Big) \quad \text{on } \Gamma, \tag{3.2}$$

with the Dirichlet boundary condition (1.2) on Γ_s and the Neumann boundary condition (1.4) on $\Gamma_N\cap\overline{\Omega}_2$. Then $u_2^{(n+1)}$ is updated by

$$-\Delta u_2^{(n+1)}=0 \quad \text{in } \Omega_2, \tag{3.3}$$

$$\lambda_2\frac{\partial u_2^{(n+1)}}{\partial \mathbf{n}_2}=-\sigma\Big((u_2^{(n+1)})^4-(u_1^{(n+\frac{1}{2})})^4\Big) \quad \text{on } \Gamma, \tag{3.4}$$

and with the Neumann boundary condition (1.4) on $\Gamma_N\cap\overline{\Omega}_2$ and nonlinear Robin boundary condition (1.5) on Γ_a.

In the additive version, one function $b(x)$ defined on Γ and Γ_a will be introduced and be assumed that $b(x)>0$.

Additive Robin-Robin domain decomposition method: Staring form the initial guesses $u_1^{(0)}$ and $u_2^{(0)}$, for $n\geq 0$, define

$$u^{(n)}=\begin{cases} u_1^{(n)}, & \Omega_1,\\ u_2^{(n)}, & \Omega_2,\end{cases}$$

where $u_1^{(n+1)}$ is obtained by solving

$$-\Delta u_1^{(n+1)}=0 \quad \text{in } \Omega_1, \tag{3.5}$$

$$\lambda_1\frac{\partial u_1^{(n+1)}}{\partial \mathbf{n}_1}+b^{(n)}u_1^{(n+1)}=-\sigma\Big((u_1^{(n)})^4-(u_2^{(n)})^4\Big)+b^{(n)}u_1^{(n)} \quad \text{on } \Gamma, \tag{3.6}$$

with the Dirichlet boundary condition (1.2) on Γ_s and the Neumann boundary condition

(1.4) on $\Gamma_N\cap\overline{\Omega}_2$, and $u_2^{(n+1)}$ is updated by

$$-\Delta u_2^{(n+1)}=0 \quad \text{in } \Omega_2, \tag{3.7}$$

$$\lambda_2\frac{\partial u_2^{(n+1)}}{\partial \mathbf{n}_2}+b^{(n)}u_2^{(n+1)}=-\sigma\Big((u_2^{(n)})^4-(u_1^{(n)})^4\Big)+b^{(n)}u_2^{(n)} \quad \text{on } \Gamma, \tag{3.8}$$

$$\lambda_2\frac{\partial u_2^{(n+1)}}{\partial \mathbf{n}_2}+b^{(n)}u_2^{(n+1)}=-\sigma\Big((u_2^{(n)})^4-(u_e)^4\Big)+b^{(n)}u_2^{(n)} \quad \text{on } \Gamma_e, \tag{3.9}$$

with the Neumann boundary condition (1.4) on $\Gamma_N\cap\overline{\Omega}_2$.

It is important to choose the initial value $u^{(0)}$ in both algorithms. In the multiplicative version, $u^{(0)}$ is one of the upper solution, while in the additive version, $u^{(0)}$ is chosen as one big constant. In both algorithms, we can prove that the sequence $u^{(n)}$ decreases monotonically:

$$0\le u^{(n+1)}\le u^{(n)}\le u^{(0)}. \tag{3.10}$$

So $u^{(n)}$ converges monotonically to one function $u(x)$, and $u(x)$ can be proven to be the solution of our model.

In the following results, we will assume that the boundary of Ω is smooth enough such that Hopf's lemma (the *strong maximum principle*, see [5,20]) holds. The smoothness of the boundary can be weakened by some sophisticated mathematical tools.

Lemma 3.1. *Assume that $\sigma(x)>0$ and $u\in C^2(\Omega)\cap C^1(\overline{\Omega})$ satisfies*

$$-\Delta u=f(x) \quad \textit{in } \Omega, \tag{3.11}$$

$$\frac{\partial u}{\partial \mathbf{n}}=-\sigma u^4+g(x), \quad \textit{on } \partial\Omega. \tag{3.12}$$

If $\overline{u}$ is one upper solution and $\underline{u}$ is one lower solution, then

$$\underline{u}\le u\le \overline{u}. \tag{3.13}$$

Proof. Let $w=u-\overline{u}$. Then w satisfies:

$$-\Delta w\le 0 \quad \text{in } \Omega, \tag{3.14}$$

$$\frac{\partial w}{\partial \mathbf{n}}\le -\sigma(u^4-\overline{u}^4) \quad \text{on } \partial\Omega. \tag{3.15}$$

Now if $w>0$ at some points in $\overline{\Omega}$, and arrives its maximum values at x_0, then by the weak maximum principle, x_0 must be on the boundary $\partial\Omega$. Moreover, at this point, $\frac{\partial w}{\partial \mathbf{n}}(x_0)>0$ by the strong maximum principle. From the assumption $w(x_0)>0$ and the inequality (3.15), we have

$$\frac{\partial w}{\partial \mathbf{n}}(x_0)\le -\sigma(u^4(x_0)-\overline{u}^4(x_0))<0.$$

Consequently, $w\le 0$ by contradiction, that is to say, $u\le\overline{u}$. The other inequality can be obtained similarly. □

Remark 3.1. By using this lemma, three useful results can be obtained:

- (Positivity) If $f \geq 0$ and $g \geq 0$, then $u \geq 0$.
- (Boundedness) If $f=0$ and $g(x)=\sigma v^4$ with non-negative function v, then

$$\min_{x\in\partial\Omega} v \leq u \leq \max_{x\in\partial\Omega} v. \tag{3.16}$$

- If $u,v \geq 0$ and $u,v \in C^2(\Omega)\cap C^1(\overline{\Omega})$ satisfy

$$-\Delta(u-v)\geq 0 \qquad \text{in } \Omega, \tag{3.17}$$

$$\frac{\partial(u-v)}{\partial \mathbf{n}} \geq -\sigma(u^4-v^4) \quad \text{on } \partial\Omega, \tag{3.18}$$

  then $u \geq v$ on $\bar{\Omega}$.

We note that the lemma still holds for boundary conditions mixed with linear Dirichlet and Neumann boundary conditions.

Theorem 3.1. *If $u^{(0)}$ is one upper solution of* (1.1)-(1.6), *and $u_1^{(-\frac{1}{2})}=u_2^{(0)}$ on Γ, then the monotone property* (3.10) *holds for the multiplicative DDM algorithm.*

Proof. Let $w^{(1)}=u^{(0)}-u^{(1)}$. Then $w_1^{(1)}=u_1^{(-\frac{1}{2})}-u_1^{(\frac{1}{2})}$ in Ω_1 satisfies

$$-\Delta w_1^{(1)} \geq 0 \quad \text{in } \Omega_1, \tag{3.19}$$

$$\lambda_1 \frac{\partial w_1^{(1)}}{\partial \mathbf{n}_1} \geq -\sigma\big((u_1^{(-\frac{1}{2})})^4-(u_1^{(\frac{1}{2})})^4\big) \quad \text{on } \Gamma, \tag{3.20}$$

with the constraints

$$w_1^{(1)} \geq 0 \quad \text{on } \Gamma_s \qquad \text{and} \qquad \frac{\partial w_1^{(1)}}{\partial \mathbf{n}_1} \geq 0 \quad \text{on } \Gamma_N.$$

Then we have $w_1^{(1)} \geq 0$ by Lemma 3.1, i.e.,

$$u_1^{(-\frac{1}{2})} \geq u_1^{(\frac{1}{2})}. \tag{3.21}$$

Furthermore, $w_2^{(1)}=u_2^{(0)}-u_2^{(1)}$ in Ω_2 satisfies

$$-\Delta w_2^{(1)} \geq 0 \quad \text{in } \Omega_2, \tag{3.22a}$$

$$\lambda_2 \frac{\partial w_2^{(1)}}{\partial \mathbf{n}_2} \geq -\sigma\Big((u_2^{(0)})^4-(u_2^{(1)})^4\Big)+\sigma\Big((u_1^{(-\frac{1}{2})})^4-(u_1^{(\frac{1}{2})})^4\Big) \quad \text{on } \Gamma, \tag{3.22b}$$

with the constraints

$$w_1^{(1)}\geq 0 \quad \text{on } \Gamma_s, \qquad \frac{\partial w_1^{(1)}}{\partial \mathbf{n}}\geq 0 \quad \text{on } \Gamma_N$$

and

$$\lambda_2 \frac{\partial w_2^{(1)}}{\partial \mathbf{n}_2}\geq -\sigma\Big((u_2^{(0)})^4-u_2^{(1)})^4\Big) \quad \text{on } \Gamma_e.$$

It follows from (3.21) that

$$\lambda_2 \frac{\partial w_2^{(1)}}{\partial \mathbf{n}_2}\geq -\sigma\Big((u_2^{(0)})^4-u_2^{(1)})^4\Big) \quad \text{on } \Gamma.$$

Consequently, $w_2^{(1)}\geq 0$ by Lemma 3.1 again.

Now we assume the monotone property holds for $n\geq 1$. Let $w^{(n+1)}=u^{(n)}-u^{(n+1)}$. Then $-\Delta w_1^{(n+1)}=0$ in Ω_1, and

$$\begin{aligned}\lambda_1 \frac{\partial w_1^{(n+1)}}{\partial \mathbf{n}_1}&\geq -\sigma\Big((u_1^{(n-\frac{1}{2})})^4-(u_1^{(n+\frac{1}{2})})^4\Big)+\sigma\Big((u_2^{(n-1)})^4-(u_2^{(n)})^4\Big)\\ &\geq -\sigma\Big((u_1^{(n-\frac{1}{2})})^4-(u_1^{(n+\frac{1}{2})})^4\Big) \qquad \text{on } \Gamma.\end{aligned} \tag{3.23}$$

Then we have

$$u_1^{(n-\frac{1}{2})}\geq u_1^{(n+\frac{1}{2})}.$$

Similarly, $-\Delta w_2^{(n+1)}=0$ in Ω_2 and

$$\begin{aligned}\lambda_2 \frac{\partial w_2^{(n+1)}}{\partial \mathbf{n}_2}&\geq -\sigma\Big((u_2^{(n)})^4-(u_2^{(n+1)})^4\Big)+\sigma\Big((u_1^{(n-\frac{1}{2})})^4-(u_1^{(n+\frac{1}{2})})^4\Big)\\ &\geq -\sigma\Big((u_2^{(n)})^4-(u_2^{(n+1)})^4\Big) \qquad \text{on } \Gamma.\end{aligned} \tag{3.24}$$

Similar inequality holds on the nonlinear boundary Γ_e, and we have

$$u_2^{(n)}\geq u_2^{(n+1)}.$$

Then the monotone property holds for $n+1$, and the lemma holds by induction. $\square$

Theorem 3.2. *If $u^{(0)}$ is one upper solution of* (1.1)-(1.6), *$u_1^{(-0)}=u_2^{(0)}$ on Γ, and $b^{(n)}\geq 4\sigma(u^{(n)})^3$, then the monotone property* (3.10) *holds for the additive DDM algorithm.*

Proof. Let $w^{(1)}=u^{(0)}-u^{(1)}$. According to the definition of $u^{(0)}$ and $u^{(1)}$, it is easy to verify that $-\Delta w^{(1)}=0$ in $\Omega_1\cup\Omega_2$, and

$$\lambda_i \frac{\partial w_i^{(1)}}{\partial \mathbf{n}_i}+b^{(0)}w_i^{(1)}\geq 0 \quad \text{on } \Gamma\cup\Gamma_e. \tag{3.25}$$

Then $w^{(1)} \geq 0$, i.e.,

$$u_i^{(1)} \leq u_i^{(0)} \quad \text{for } \ i=1,2.$$

Now we assume the monotone property holds for $n \geq 1$, and let $w^{(n+1)} = u^{(n)} - u^{(n+1)}$. Then $-\Delta w_1^{(n+1)} = 0$ in Ω_1, and

$$\begin{aligned}\lambda_1 \frac{\partial w^{(n+1)_1}}{\partial \mathbf{n}_1} + b^{(n)} w_1^{(n+1)} =& -\sigma\Big((u_1^{(n-1)})^4 - (u_1^{(n)})^4 \Big) + b^{(n-1)} \Big(u_1^{(n-1)} - u_1^{(n)} \Big) \\ &+ \sigma\Big((u_2^{(n-1)})^4 - (u_2^{(n)})^4 \Big) \quad \text{in } \ \Omega. \end{aligned} \tag{3.26}$$

By the assumption of the monotone property for n, we have

$$u_i^{(n-1)} \geq u_i^{(n)} \quad \text{on } \ \Gamma.$$

With the condition $b^{(n)} \geq 4\sigma(u^{(n)})^3$, we have

$$\lambda_1 \frac{\partial w_1^{(n+1)}}{\partial \mathbf{n}_1} + b^{(n)} w_1^{(n+1)} \geq 0 \quad \text{on } \ \Gamma. \tag{3.27}$$

So we have

$$w_1^{(n+1)} \geq 0 \quad \text{in } \ \Omega_1$$

by Lemma 3.1. In a similar manner, we can prove that

$$w_2^{(n+1)} \geq 0 \quad \text{in } \ \Omega_2.$$

The lemma is proven by induction. □

We add some comments on our Robin-Robin domain decomposition methods: Though $u_1^{-1/2}$ in the multiplicative algorithm only appears in the proof, it is hidden in the practical algorithm. Note that for the multiplicative version, one Picard-FD or Newton-FD algorithm is still needed to solve the nonlinear system in single subdomain Ω_1 (or Ω_2). So the initial guess is needed for the Picard-FD algorithm or Newton-FD algorithm, which is just $u_1^{-1/2}$ in Ω_1 and $u^{(0)}$ in Ω_2. Moreover, the nonnegative function $b(x)$ or $b^{(n)}$ are also needed to guarantee the monotone property. In the multiplicative algorithm, at each step in one single subdomain, k-steps Picard-FD or Newton-FD iteration can be implemented. Specially, for $k=1$, the algorithm becomes one serial version of the additive DDM algorithm.

4 Numerical experiments

4.1 Numerical experiments for one domain

Here we set $\Omega = (0,1) \times (0,1)$, $\lambda = \sigma = 1$ and $g(x,u) = -(u^4 - u_e^4)$ in the Stefan-Boltzmann condition. Two exact solutions are used to check the numerical behaviors in the monotone iterations for the elliptic problems with Stefan-Boltzmann conditions:

1. one is $u=0.1e^{\pi(1-x)}\cos(\pi y)$ and $f(x)=0$, $u_s=0.1\pi e^{\pi}\cos(\pi y)+2$, $u_e^4=(0.1\cos(\pi y)+2)^4-0.1\pi\cos(\pi y)$;
2. another one is $u=\frac{5}{2}-\frac{1}{2}x^2$, and $f=1$, $u_s=5/2$, $u_e=\sqrt[4]{15}$.

We note that the finite difference equations are exactly satisfied for the second solution $u=5/2-x^2/2$. The first exact solution will be used if we do not point out explicitly. The nonlinear term only appears in the boundary condition, therefore $c=0$ in the Picard-FD scheme and Newton-FD scheme.

Convergence behavior of the two schemes. By choosing an upper solution as the initial function for the iteration process, the sequences $\overline{u}_h^{(k)}$ can be obtained by using the Picard-FD method or Newton-FD method. Fig. 2 shows the convergence behaviors of the Picard-FD scheme ($b=20$) and Newton-FD scheme, for both solutions 1 and 2. The relative errors for both schemes decrease monotonically with respect to iteration steps and the errors stagnate at some error level after a certain iterative steps. Moreover, the relative error of the Newton method reaches more quickly to the error level than the Picard method.

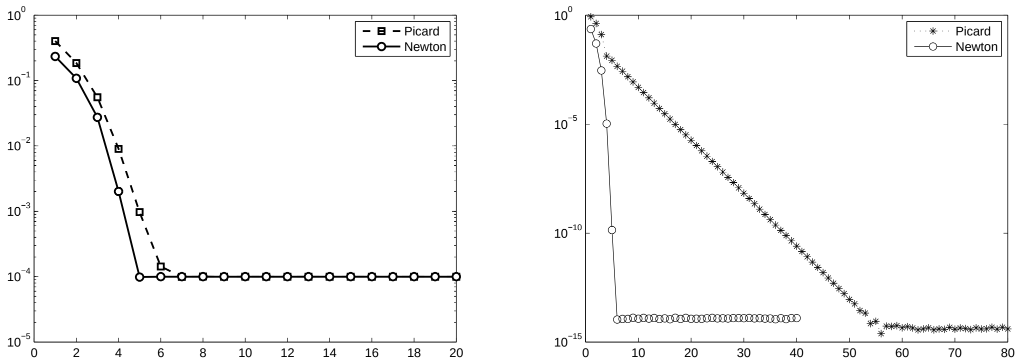
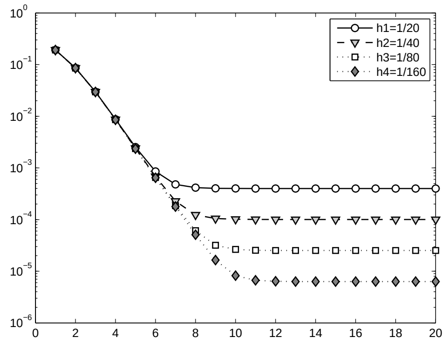


Figure 2: Convergence behavior of Picard-FD method and Newton-FD method: Left (solution 1), right figure (solution 2).

In order to explain the stagnation phenomenon of the relative errors, we used the second exact solution $u=5/2-x^2/2$. The relationships of the relative errors with the iterative steps are plotted in right figure of Fig. 2, which clearly demonstrate the linear convergence of the Picard-FD method and quadratical convergence of the Newton-FD method. In this case, the finite difference solution is just the exact solution if we do not consider the truncation error.

The influence of the meshsize. From the above two experiments, we know that the relative error will be in "stagnation state" with the increment of the iterative steps k for one non-quadratic polynomial solutions, for example, for the first exact solution. The left figure of Fig. 3 implies that the relative errors of the Picard-FD scheme at the finial



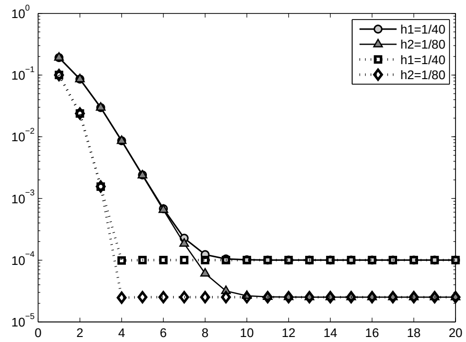


Figure 3: Left (the relative errors of Picard-FD scheme with the meshsize), right (comparison with Newton-FD methods).

"stagnation state" can be reduced if the smaller mesh-sizes h are used. Same results can be observed for the Newton-FD method (see the right figure of Fig. 3).

Now the difference mesh-size h is used to compute the relative errors in the final "stagnation state", the results are listed in the following table:

Mesh-size	$h=1/10$	$h=1/20$	$h=1/40$	$h=1/80$	$h=1/160$
Relative error	1.566e-3	3.986e-4	1.003e-4	2.514e-5	6.291e-6

From the table, we conclude that the relative error in the stagnation state converges like $\mathcal{O}(h^2)$, which is consistent with the error estimate of the finite difference method.

The numerical experiments for the Laplace equation with the nonlinear Stefan-Boltzmann conditions confirms our theoretical analysis: the upper-solution sequence $\overline{u}_h^n$ of the Picard-FD method converges linearly to the finite difference solution u_h,

$$|\overline{u}_h^{(k+1)}-u_h|_{\overline{\Omega}}\leq\rho|\overline{u}_h^{(k)}-u_h|_{\overline{\Omega}},\quad \rho<1; \tag{4.1}$$

and the upper-solution sequence $\overline{u}_h^n$ of Newton-FD method quadratically converge to the finite difference solution u_h,

$$|\overline{u}_h^{(k+1)}-u_h|_{\overline{\Omega}}\leq C|\overline{u}_h^{(k)}-u_h|^2_{\overline{\Omega}}. \tag{4.2}$$

Both methods have second order accuracy after sufficiently large iteration steps:

$$|\overline{u}_h^{(k)}-u|_{\overline{\Omega}}=\mathcal{O}(h^2). \tag{4.3}$$

Generally, the exact solution is not known, and the following stop criterion often used: For one given tolerance ε, the iterative process will be stopped if

$$|\overline{u}_h^{(k)}-\overline{u}_h^{(k-1)}|_{\Omega}/|\overline{u}_h^{(k-1)}|_{\Omega}\geq\varepsilon. \tag{4.4}$$

In practice, $\varepsilon=\mathcal{O}(h^2)$ may be enough to guarantee the convergence.

The advantage of the Picard-FD method is that the stiffness matrix is the same at every iterative step, which has to be changed in each step of the Newton-FD method. On the other hand, the Newton-FD method converges faster. Moreover, the technique of the upper-lower solutions guarantees the global convergence of the Newton-FD method.

4.2 The Robin-Robin DDMs

In the composite material problems, we set $\Omega_1=(0,1)\times(0,1)$ and $\Omega_2=(1,2)\times(0,1)$. The exact solution are

$$\begin{aligned} u_1 &= 0.2(1-x)+(\cos(\pi y)+2.2)^{1/4} \quad \text{in } \Omega_1,\\ u_2 &= 0.2(1-x)+(\cos(\pi y)+2)^{1/4} \quad \text{in } \Omega_2. \end{aligned}$$

Note that u_1 and u_2 are not harmonic functions, and $-\Delta u_i=f_i$ with

$$\begin{aligned} f_1(x,y) &= \frac{3}{16}\pi^2(\cos(\pi y)+2.2)^{-7/4}\sin(\pi y)^2+\frac{1}{4}\pi^2(\cos(\pi y)+2.2)^{-3/4}\cos(\pi y),\\ f_2(x,y) &= \frac{3}{16}\pi^2(\cos(\pi y)+2)^{-7/4}\sin(\pi y)^2+\frac{1}{4}\pi^2(\cos(\pi y)+2)^{-3/4}\cos(\pi y),\\ u_e &= \Big(\big(-\frac{1}{5}+(\cos(\pi y)+2)^{\frac{1}{4}}\big)^4-\frac{1}{5}\Big)^{\frac{1}{4}}. \end{aligned}$$

It can be verified that $u^{(0)}=-\frac{3}{2}x^2+3x+1.54$ is one upper solution, and $u^{(0)}|_\Gamma=2.04$ is taken as the initial guess. $b^{(n)}=4\sigma(u^{(n)})^3$ is chosen in the additive DDM and in the Newton-FD method for the multiplicative DDM.

Monotone property. Here we demonstrate the monotone property (3.10). First Fig. 4 shows the difference of $u^{(1)}-u^{(2)}$ in the multiplicative Robin-Robin DDM. The error is found positive, which means $u^{(2)}\le u^{(1)}$, and the maximum values attains on the interface

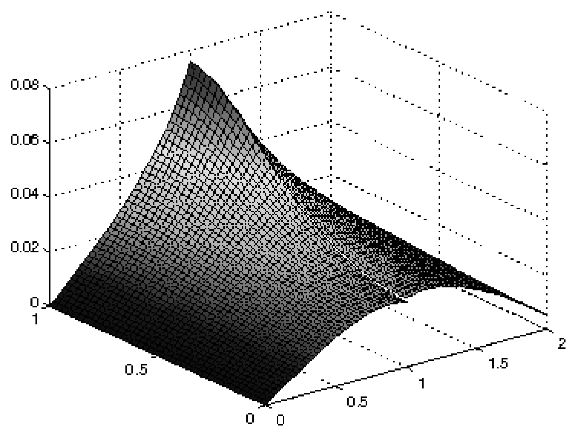


Figure 4: The difference of the $u^{(1)}-u^{(2)}$.

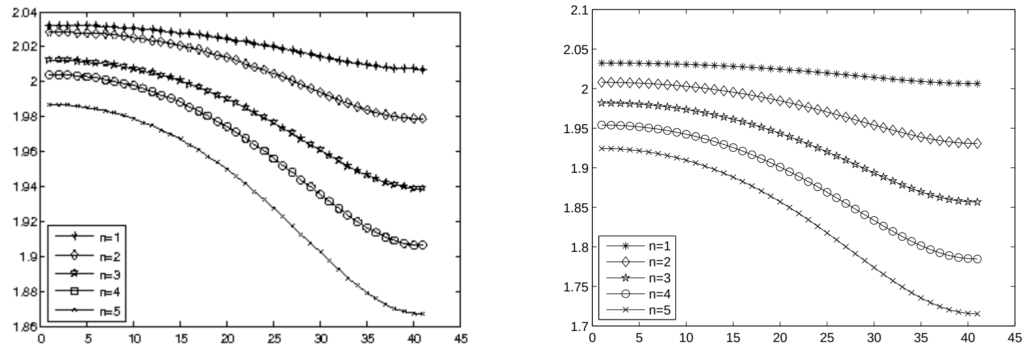


Figure 5: The values $u_1^{(n)}$ on the interface boundary for $n=1,\cdots,5$ (Left: additive DDM, right: multiplicative DDM.

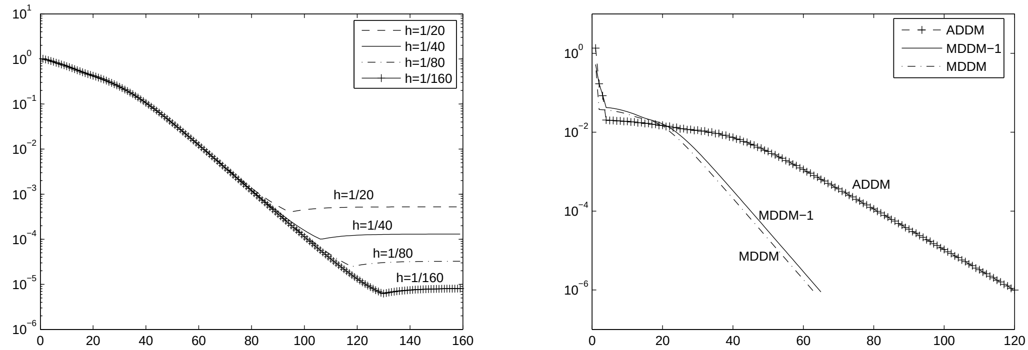


Figure 6: Left: Converge to the exact solution by different mesh-sizes, right: Iterative behaviors of three methods.

Γ. Let us check the monotone property on the interface Γ. In Fig. 5, the values of $u^{(n)}$ at the interface Γ decrease monotonically same as the exact solutions. Note that the monotone property in the whole domain can be guaranteed by Lemma 3.1.

Convergence results. Here we study the error: $e_h^{(n)} = u_h^{(n)} - u$. For different mesh-size $h=\frac{1}{20},\frac{1}{40},\frac{1}{80}$ and $\frac{1}{160}$, the max-norms of the errors $|e_h^{(n)}|_\Omega$ are plotted in Fig. 6. In the left figure of Fig. 6, the multiplicative domain decomposition method (MDDM) is used, and the error tolerance is 10^{-8}. It is observed that the errors stagnate in some error level after enough iterations, which is explained in the experiments for the single domain. Similar behaviors can be observed for the additive domain decomposition method (ADDM) and the MDDM with one inner-iteration (MDDM-1). Moreover, the three methods have almost the same error level if the same mesh-size h is used.

For fixed $h=\frac{1}{40}$, we compare the iterative behaviors in the right figure of Fig. 6, where the relative iterative errors $|u_h^{(n+1)} - u_h^{(n)}|_\Omega / |u_h^{(n+1)}|_\Omega$ are plotted for ADDM, MDDM and

MDDM-1. For each method, the relative iterative errors drop fast at beginning, then slow and finally the asymptotical linear convergence can be observed. The MDDM and the MDDM-1 are faster than the ADDM, and the MDDM-1 is slightly slower than the MDDM. The Newton-FD methods will run $2\sim4$ times for the MDDM and only one time for the MDDM-1. The advantage of the ADDM is the potential of parallelization.

5 Conclusion

In this paper, the monotone method and the Robin-Robin domain decomposition methods are combined to solve the elliptic problems with nonlinear Stefan-Boltzmann conditions. In single domain, both the Picard-FD method and Newton-FD method are efficient to obtain the sequences which converge monotonically to the exact solution. For the composite materials problems, the Robin-Robin DDMs can be implemented easily, and the monotone properties still hold in each iterative step. Our methods can be generalized to the parabolic case, which will be reported elsewhere. Moreover, the convergence behaviors of the DDMs will be further studied in future works.

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