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The Direct Discontinuous Galerkin (DDG) Method for Diffusion with Interface Corrections

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Abstract. Based on a novel numerical flux involving jumps of even order derivatives of the numerical solution, a direct discontinuous Galerkin (DDG) method for diffusion problems was introduced in [H. Liu and J. Yan, SIAM J. Numer. Anal. 47(1) (2009), 475-698]. In this work, we show that higher order ($k \geq 4$) derivatives in the numerical flux can be avoided if some interface corrections are included in the weak formulation of the DDG method; still the jump of 2*nd* order derivatives is shown to be important for the method to be efficient with a fixed penalty parameter for all p^k elements. The refined DDG method with such numerical fluxes enjoys the optimal $(k+1)$th order of accuracy. The developed method is also extended to solve convection diffusion problems in both one- and two-dimensional settings. A series of numerical tests are presented to demonstrate the high order accuracy of the method.

AMS subject classifications: 35K05, 35K15, 65N12, 65N30

Key words: Diffusion, discontinuous Galerkin methods, stability, numerical flux.

1 Introduction

This paper is the continuation of our project, initiated in [26], of developing a direct discontinuous Galerkin (DDG) method for diffusion problems. Here we focus on the diffusion equation of the form

$$\partial_t U - \nabla \cdot (A(U)\nabla U) = 0, \quad \Omega \times (0,T), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$, the matrix $A(U) = (a_{ij}(U))$ is symmetric and positive definite, and U is an unknown function of (x,t). The method will also be extended to convection-diffusion problems and their invariants.

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The Discontinuous Galerkin (DG) method is a finite element method using a completely discontinuous piecewise polynomial space for the numerical solution and the test functions. One main advantage of the DG method was the flexibility afforded by local approximation spaces combined with the suitable design of numerical fluxes crossing cell interfaces. The application to hyperbolic problems has been quite successful since it was originally introduced by Reed and Hill [28] in 1973 for neutron transport equations. A major development of the DG method for nonlinear hyperbolic conservation laws is carried out by Cockburn, Shu, and collaborators in a series of papers [13, 17, 18, 20]. We refer to [11, 16, 21] for reviews and further references.

However, the application of the DG method to diffusion problems has been a challenging task because of the subtle difficulty in defining appropriate numerical fluxes for diffusion terms, see e.g. [30]. There have been several DG methods suggested in literature to solve the problem, including the method originally proposed by Bassi and Rebay [4] for compressible Navier-Stokes equations, its generalization called the local discontinuous Galerkin (LDG) methods introduced in [19] by Cockburn and Shu and further studied in [6,7,12,15]; as well as the method introduced by Baumann-Oden [5,27]. Also in the 1970s, Galerkin methods for elliptic and parabolic problems using discontinuous finite elements, called the *interior penalty* (IP) methods, were independently introduced and studied; see, e.g., [1,3,34]. We refer to [2] for a unified analysis of DG methods for elliptic problems and background references for the IP methods.

In this article we are interested in the effect of test functions on interface treatments, and accordingly we introduce a refined version of the DDG method proposed in [26]. To illustrate the idea, we consider the scalar one-dimensional diffusion equation

$$u_t = u_{xx},$$

and formulate the DDG method based on the direct weak formulation

$$\int_{I_j} u_t v dx - \widehat{(u_x)} v \Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} + \int_{I_j} u_x v_x dx = 0,$$

where I_j is the j-th computational cell, and v is the test function. In [26] we presented the following numerical flux

$$\widehat{u_x} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x [u_{xx}] + \beta_2 (\Delta x)^3 [u_{xxxx}] + \cdots, \tag{1.2}$$

which involves the average $\overline{u_x}$ and the jumps of even order derivatives of u. This numerical flux satisfies the following desired properties: it (i) is consistent for the smooth solution u; (ii) is conservative in the sense of its being single valued at the interface; (iii) ensures the L^2-stability; and (iv) enforces the high order accuracy of the method

It was shown in [26] that for piecewise p^k polynomial approximations, kth order of accuracy of the DDG method is ensured if the numerical flux is admissible. Numerical experiments in [26] also showed that the use of term $(\Delta x)^{2m-1}[\partial_x^{2m} u]$ $(m = 0, 1, \cdots, [\frac{k}{2}])$ in

(1.2) does lead to the optimal $(k+1)$th order of accuracy. However, how to select an appropriate $\beta_m (m \geq 2)$ to ensure the optimal accuracy remains an unsettled issue.

An observation that motivated this paper is that the derivative of the test function does contribute to the interface flux when higher order elements are used. The goal of this work is to refine the DDG method by using interface corrections so that an optimal accuracy can still be achieved for all k using the numerical flux (1.2) with $\beta_j = 0, j \geq 2$, i.e.,

$$\widehat{u_x} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x [u_{xx}]. \tag{1.3}$$

To this end, we consider the refined weak formulation of the DDG scheme

$$\int_{I_j} u_t v dx - (\widehat{u_x}) v \Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} + \int_{I_j} u_x v_x dx + \frac{1}{2}[u](v_x)^-_{j+\frac{1}{2}} + \frac{1}{2}[u](v_x)^+_{j-\frac{1}{2}} = 0. \tag{1.4}$$

The role of interface correction terms has long been recognized in literature. This dates back to the classical symmetric Interior Penalty method originally introduced by Arnold in [1] for parabolic problems, see also [3, 34] (in the format of so-called primal formulation). More recent works such as those by Van Leer and Nomura in [33], Gassner et al. in [23], and Cheng and Shu in [8] use the weak formulation derived from repeated integration by parts for the diffusion term, and hence also involve v_x in the interface treatments.

The main flavor of our method distinguished from existing ones lies in the flux formulation (1.3), which when combined with the weak formulation (1.4) leads to a class of novel DDG schemes. A special case when $\beta_1 = 0$ reduces to the classical symmetric IP method. In this article we have two objectives:

(i) to analyze the DDG method with interface corrections and present numerical results to show its optimal performance;

(ii) to compare with the classical symmetric IP method.

For (i), our numerical results of the DDG method with interface corrections show that the numerical flux $\widehat{u_x}$ (1.3) with a fixed β_0 enables us to obtain $(k+1)$th order of accuracy for all p^k polynomial approximations, e.g. $\beta_0 = 2$ for p^k up to $k = 9$.

For (ii), we compare with the classical IP method, that is (1.4) with flux

$$\widehat{u_x} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x}. \tag{1.5}$$

It is known that the penalty parameter (β_0 in (1.5)) depends on the order of the polynomial p^k, and needs to be large enough to stabilize the scheme, especially for high order approximations, see, e.g., [22, 29]. Here we reconfirm the need of penalization when $\beta_1 = 0$, and obtain a sharp bound for β_0 in Lemma 2.2 with the assistance of the admissibility condition.

The most attractive feature of the DDG method is the simplicity and optimal accuracy obtained with a simple flux formula, which allows one to further develop DG schemes of high accuracy for more complex problems.

This rest of the paper is organized as follows. In Section 2, we introduce the DDG method with interface corrections for one-dimensional problems. For this model problem, the main idea of devising the method and the scheme formulation are presented. We then prove stability of the DDG scheme for any admissible numerical fluxes. In Section 3, we extend the DDG method to nonlinear convection diffusion equations in both one and two-dimensional problems in which U is a scalar and $A=(a_{ij})_{d\times d}$ is a positive and semi-definite matrix. In Section 4, we present a series of numerical results to validate the refined DDG method. Finally, some concluding remarks are given in Section 5.

2 One-dimensional case

Our new DDG algorithm for diffusion consists of an addition of interface corrections, upon the one proposed in [26], and hence allows a wider choice of numerical fluxes for obtaining the optimal accuracy. Discretization in time with a matching accuracy is obtained by an appropriate Runge-Kutta solver.

2.1 Scheme formulation

We begin with the one-dimensional linear diffusion

$$U_t-U_{xx}=0 \quad \text{in } \Omega\times(0,T), \tag{2.1}$$

subject to initial data

$$U(x,0)=U_0(x) \quad \text{on } \Omega, \tag{2.2}$$

and periodic boundary conditions.

First we partition the domain Ω into computational cells $\Omega=\cup_{j=1}^{N} I_j$, with mesh $\{I_j=[x_{j-1/2},x_{j+1/2}]\}$ of uniform size $\Delta x=x_{j+1/2}-x_{j-1/2}$. We seek an approximation u to U such that for any time $t\in[0,T]$, $u\in\mathbb{V}_{\Delta x}$,

$$\mathbb{V}_{\Delta x}:=\left\{v\in L^2(\Omega):\ v|_{I_j}\in P^k(I_j),\ j=1,\cdots,N\right\},$$

where $P^k(I_j)$ denotes the space of polynomials on I_j with degree at most k. Set

$$\widehat{u_x}v\Big|_{j-\frac{1}{2}}^{j+\frac{1}{2}}:=(\widehat{u_x})_{j+\frac{1}{2}}v^-_{j+\frac{1}{2}}-(\widehat{u_x})_{j-\frac{1}{2}}v^-_{j-\frac{1}{2}}.$$

The DDG method introduced in [26] can be written as

$$\int_{I_j}u_t v dx-\widehat{u_x}v\Big|_{j-\frac{1}{2}}^{j+\frac{1}{2}}+\int_{I_j}u_x v_x dx=0, \tag{2.3}$$

with numerical flux $\widehat{u_x}$ defined by

$$\widehat{u_x}=\beta_0\frac{[u]}{\Delta x}+\overline{u_x}+\sum_{m=1}^{\lfloor k/2\rfloor}\beta_m(\Delta x)^{2m-1}[\partial_x^{2m}u], \tag{2.4}$$

where $\beta_0,\beta_1,\cdots,\beta_{[k]/2}$ are coefficients to be chosen to ensure the stability of the scheme. Here the following notations have been used:

$$u^{\pm}=u(x\pm 0,t),\quad [u]=u^+-u^-,\quad \overline{u}=\frac{u^++u^-}{2}.$$

The admissibility of β_i's and their effects on the numerical accuracy were studied in [26], in which a numerical flux of the following form was tested

$$\widehat{u_x}=\beta_0\frac{[u]}{\Delta x}+\overline{u_x}+\frac{1}{12}\Delta x[u_{xx}]. \tag{2.5}$$

This scheme with $\beta_0=1$ was numerically shown to produce optimal accuracy of $(k+1)th$ order for $k\leq 3$, as well as for $k=odd(>3)$ with a slightly larger β_0.

For $k\geq 4$, instead of relying on high order terms such as $\beta_2[\Delta x]^3[u_{xxxx}]$, in this paper we turn to a refined DDG method with inclusion of interface corrections:

$$\int_{I_j}u_tvdx-\widehat{u_x}v\Big|_{j-\frac{1}{2}}^{j+\frac{1}{2}}+\int_{I_j}u_xv_xdx+\frac{1}{2}[u](v_x)^-_{j+1/2}+\frac{1}{2}[u](v_x)^+_{j-1/2}=0, \tag{2.6}$$

$$\int_{I_j}u(x,0)v(x)dx=\int_{I_j}U_0(x)v(x)dx, \tag{2.7}$$

where the numerical flux $\widehat{u_x}$ is still (2.5) or more general

$$\widehat{u_x}=\beta_0\frac{[u]}{\Delta x}+\overline{u_x}+\beta_1\Delta x[u_{xx}]. \tag{2.8}$$

The motivation of including extra interface terms in (2.6) stems from our observation that the test function $v\in\mathbb{V}_{\Delta x}$ is chosen being none zero only inside each cell I_j (or independent from cell to cell), the slope of the test function will contribute at interfaces whenever $[u]$ is non-zero. It is possible to add interface corrections involving even higher order derivatives of v for larger k.

We recall that $\beta_1=1/12$ was identified through a procedure suggested in [26] by using the Stirling interpolation formula based on four symmetric points

$$x_{j+\frac{1}{2}}\pm\frac{1}{2}\Delta x,\quad x_{j+\frac{1}{2}}\pm\Delta x.$$

When evaluating the derivative of the obtained 3rd order polynomial at the cell interface $x_{j+1/2}$, we obtain the numerical flux formula

$$D_xu=\frac{7}{6}\frac{[u]}{\Delta x}+\overline{u_x}+\frac{\Delta x}{12}[u_{xx}]. \tag{2.9}$$

Therefore in this work we shall use the flux (2.5) for the refined DDG scheme. For non-uniform mesh, Δx needs to be understood as $(\Delta x_j+\Delta x_{j+1})/2$. The 1D scheme is now well defined.

We prove in next section that a large class of β_i' exists for the stability of the DDG method. Note that the scheme (2.6) with (2.8) when $\beta_1=0$ reduces to the classical symmetric DG method, and in such a case sufficiently large β_0 is indeed needed to penalize the interface jumps [1], see Lemma 2.2 below.

2.2 Admissibility and stability

As usual for the DG method the guiding principle for the choice of numerical flux is the stability requirement. Following [26] we adopt the following admissibility criterion:

Definition 2.1 (Admissibility)**.** *We call a numerical flux $\widehat{u}_x$ of the form* (2.8) *admissible if there exists a $\gamma\in(0,1)$ and $0<\alpha\le 1$ such that*

$$\begin{aligned}&\gamma\sum_{j=1}^N\int_{I_j}u_x^2(x,t)dx+\sum_{j=1}^N(\widehat{u_x})_{j+1/2}[u]_{j+1/2}+\sum_{j=1}^N[u]_{j+1/2}(\overline{u_x})_{j+1/2}\\ &\ge\alpha\sum_{j=1}^N\frac{[u]^2_{j+1/2}}{\Delta x}\end{aligned}\tag{2.10}$$

holds for any piecewise polynomials of degree k, i.e. $u\in\mathbb{V}_{\Delta x}$.

This admissibility ensures the stability of the DDG method.

Theorem 2.1 (Energy stability)**.** *Consider the DDG scheme* (2.6)-(2.7)*. If the numerical flux* (2.8) *is admissible as described in* (2.10)*, then we have*

$$\begin{aligned}&\frac{1}{2}\int_0^1u^2(x,T)dx+(1-\gamma)\int_0^T\sum_{j=1}^N\int_{I_j}u_x^2(x,t)dxdt+\alpha\int_0^T\sum_{j=1}^N\frac{[u]^2}{\Delta x}dt\\ &\le\frac{1}{2}\int_0^1U_0^2(x)dx.\end{aligned}\tag{2.11}$$

This can be proved by summation of (2.6) with $v=u$ over $j\in\{1,\cdots,N\}$, and using the admissibility condition (2.10).

In the following two lemmas we show that there is indeed a large set of β_i', making (2.8) admissible fluxes for polynomial approximations of any given degree k.

Lemma 2.1. *Consider the one-dimensional linear diffusion* (2.1)*. The numerical flux* (2.8) *is admissible for any piecewise polynomial of degree $k\ge 0$ provided $(\beta_0,\beta_1)=(1,0)$ when $k=0$, and for any $k>0$*

$$\beta_0\ge\alpha+\frac{1}{\gamma}M(k,\beta_1),\tag{2.12a}$$

where

$$M(k,\beta_1)=\max_{u\in P^k(I_j)}\frac{\Delta x\sum_j\left(\overline{u}_x+\frac{\beta_1}{2}\Delta x[u_{xx}]\right)^2}{\sum_j\int_{I_j}u_x^2dx}. \tag{2.12b}$$

Proof. Note for $k=0$, $(\beta_0,\beta_1)=(1,0)$ is admissible since $\beta_0\geq\alpha$. For admissibility condition to hold when $k\geq 1$, it suffices to select (β_0,β_1) so that the underlying flux (2.8) is admissible locally around each cell, i.e.,

$$\gamma\Delta x\int_{I_j}u_x^2dx+(2\overline{u_x}+\beta_1\Delta x[u_{xx}])[u]\Delta x+(\beta_0-\alpha)[u]^2\geq 0,\quad k\geq 1.$$

This is ensured to hold for all $u|_{I_j}\in P^k(I_j)$ provided

$$(2\overline{u_x}+\beta_1\Delta x[u_{xx}])^2(\Delta x)^2-4(\beta_0-\alpha)\gamma\Delta x\int_{I_j}u_x^2dx\leq 0.$$

Summation of this inequality over all index $j\in\{1,\cdots,N\}$ yields

$$\beta_0\geq\alpha+\frac{1}{\gamma}\frac{\Delta x\sum_j\left(\overline{u}_x+\frac{\beta_1}{2}\Delta x[u_{xx}]\right)^2}{\sum_j\int_{I_j}u_x^2dx}.$$

Maximization of the right hand side over all $u|_{I_j}\in P^k(I_j)$ gives (2.12a). $\square$

Unfortunately, here we cannot theoretically analyze the optimal accuracy obtained from the use of $\beta_1\neq 0$. However, through numerical tests we show that the β_1 term indeed provides a leverage to compensate the β_0 term. Moreover, there exists a large class of β_0 and β_1 that lead to optimal order of accuracy with p^k polynomial approximations. For instance, we find the following results for p^2 quadratic polynomial approximations when applied to the 1D heat equation with initial data $u_0=sinx$: the numerical flux with pairs (β_0,β_1) inside the polygon area in Fig. 1 all gives optimal 3rd order of accuracy. The numerical result also indicates the following facts:

- β_0 cannot be too small, $\beta_0>1$.
- When β_0 increases, β_1 can be lowered to maintain the same order of accuracy.
- A smaller absolute error is achieved with $\beta_1=\frac{1}{12}$, which is the optimal choice when no interface correction is used in [26]. For example, we observe that $(\beta_0,\beta_1)=(2,\frac{1}{12})$ gives smaller errors than $(\beta_0,\beta_1)=(3,0)$. Also $(\beta_0,\beta_1)=(4,\frac{1}{12})$ gives smaller errors compared to $(\beta_0,\beta_1)=(4,0)$.

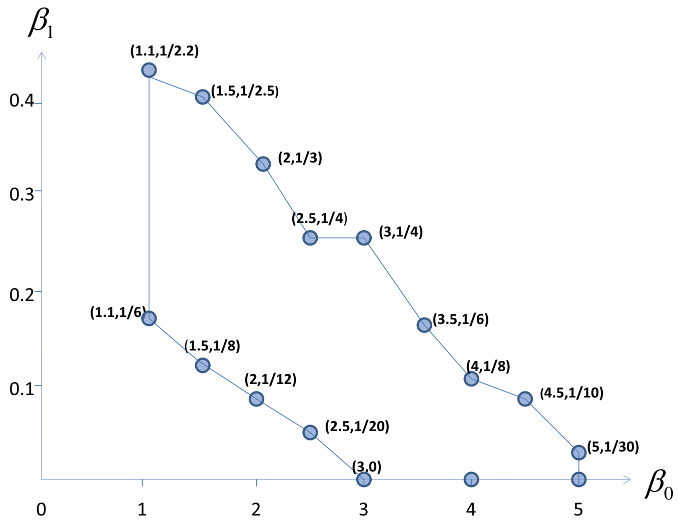

Figure 1: Numerically tested β_0 and β_1 area for p^2 approximation.

Remark 2.1. The recovery procedure introduced in [33] could be used as an alternative way to identify a pair (β_0,β_1). For example, their recovery procedure for p^2 elements gives $(\frac{15}{4},\frac{9}{240})$. This pair is indeed in the polygon area in Fig. 1, but it does not seem to give the optimal accuracy for the DDG method presented in [26].

It is known from the classical penalty method that when $\beta_1=0$, β_0 needs to be large enough to stabilize the scheme. Next we reconfirm this using the notion of admissibility.

Lemma 2.2. *For a given $k\geq 1$, (2.8) with $\beta_1=0$ is admissible if*

$$\beta_0 \geq \alpha+\frac{1}{4\gamma}\lambda_{\max}(H^{-1/2}OH^{-1/2}), \tag{2.13}$$

where H is the Hilbert matrix $H=\left(\frac{1}{m+l-1}\right)$ of size k and O is a $k\times k$ matrix with each entry to be 1.

Proof. From the proof in Lemma 2.1 we have

$$M(k,0)=\max_{u\in P^k(I_j)}\frac{\Delta x\sum_j \overline{u}_x^2}{\sum_j\int_{I_j}u_x^2dx}=\max_{v\in P^{k-1}(I_j)}\frac{\Delta x\sum_j \overline{v}^2}{\sum_j\int_{I_j}v^2dx}.$$

Set $v|_{I_j}=\sum_{m=1}^k a_m^j\xi_j^{m-1}$ with base functions chosen to be $\xi_j=(x-x_{j-1/2})/\Delta x$ for $x\in I_j$. Here a_m^j is the coefficient of the base function in cell I_j. Thus the average of v at the interface $x=x_{j+1/2}$ is

$$\bar{v}\Big|_{x_{j+1/2}}=\frac{1}{2}\left(\sum_{m=1}^k a_m^j\cdot 1^{m-1}+\sum_{m=1}^k a_m^{j+1}\cdot 0^{m-1}\right)=\frac{1}{2}\sum_{m=1}^k a_m^j.$$

On the other hand we have

$$\int_{I_j} v^2 dx = \Delta x \int_0^1 \left(\sum_{m=1}^{k} a_m^j \cdot \xi^{m-1} \right)^2 d\xi = \Delta x \sum_{m,l=1}^{k} a_m^j a_l^j \frac{1}{m+l-1}.$$

These together lead to

$$M(k,0) = \frac{1}{4} \max_{a_m^j, m=1,\cdots,k} \frac{\sum_{j=1}^{N} \sum_{m,l=1}^{k} a_m^j a_l^j}{\sum_{j=1}^{N} \sum_{m,l=1}^{k} a_m^j a_l^j \frac{1}{m+l-1}} = \frac{1}{4} \max_{a^j \in \mathbb{R}^k} \frac{\sum_{j=1}^{N} a^j \cdot O a^j}{\sum_{j=1}^{N} a^j \cdot H a^j},$$

where

$$a^j = (a_1^j, \cdots, a_k^j)^\top, \quad O_{m,l} = 1, \quad H_{m,l} = \frac{1}{m+l-1}.$$

This is clearly bounded from above by

$$\frac{1}{4} \max_{a \in \mathbb{R}^k} \frac{a \cdot O a}{a \cdot H a} = \frac{1}{4} \max_{y \in \mathbb{R}^k} \frac{y \cdot \left(H^{-1/2} O H^{-1/2} \right) y}{\|y\|_2^2},$$

where we have used the fact that the Hilbert matrix H is symmetric and positive definite to transform a via $y = H^{1/2} a$. Due to symmetry of the matrix $H^{-1/2} O H^{-1/2}$, the induced spectral norm is just the largest eigenvalue of this matrix, as claimed. □

From this result we now specify the choice of β_0 for each fixed k, and compare with our numerical results. For instance, we take $\alpha = 1$ and $\gamma = 1/2$, and let β_0 to be an integer as

$$\beta_0 = \left[\frac{1}{2} \lambda_{\max} \left(H^{-1/2} O H^{-1/2} \right) \right] + 2, \tag{2.14}$$

where $[\cdot]$ denotes the integer part. Some calculation for k up to 9 shows that $\beta_0 = [k^2/2] + 2$, which is summarized in Table 1.

Table 1: analytical estimation on $\beta_0(k)$.

k	1	2	3	4	5	6	7	8	9
β_0	2	4	6	10	14	20	26	34	42

The numerical results for $\beta_1 = 0$ is consistent with those given in Table 1. With β_1 non-vanishing we numerically show that $(k+1)$th order of accuracy is obtained for p^k polynomials with a fixed β_0. For instance, optimal accuracy is observed for all k up to 9 when taking $(\beta_0, \beta_1) = (2, \frac{1}{12})$ in our numerical tests.

Remark 2.2. The bound for β_0 stated in Lemma 2.1 is implicit in terms of β_1, but it shows the existence of a large set of admissible numerical fluxes. On the other hand the bound (2.13) obtained in Lemma 2.2 when $\beta_1=0$ is sharp and depends on k.

Remark 2.3. A similar calculation for $M(k,\beta_1)$ yields an estimate about as (2.14) but with O replaced by

$$O_{ml}=\left(1-\frac{\beta_1}{2}(m-1)\right)\left(1-\frac{\beta_1}{2}(l-1)\right).$$

For some $\beta_1>0$ this indeed leads to a smaller β_0 than (2.14).

2.3 Error estimates

Now we turn to the question of the quality of the approximate solution defined by (2.6). Following the notations in [26], we define the energy norm

$$|||v(\cdot,t)|||:=\left(\int_0^1 v^2dx+(1-\gamma)\int_0^t\sum_{j=1}^N\int_{I_j}v_x^2dxd\tau+\alpha\int_0^t\sum_{j=1}^N\frac{[v]^2}{\Delta x}d\tau\right)^{1/2} \tag{2.15}$$

with $\gamma\in(0,1)$ and $\alpha>0$. Admissibility of the numerical flux and the approximation properties of the space $\mathbb{V}_{\Delta x}$ enable us to obtain the following error estimate.

Theorem 2.2 (Error estimate)**.** *Let $e:=u-U$ be the error between the exact solution U and the numerical solution u of the DDG method with interface correction* (2.6)*. If the numerical flux* (2.8) *is admissible* (2.10)*, then the energy norm of the error satisfies*

$$|||e(\cdot,T)|||\le C|||\partial_x^{k+1}U(\cdot,T)|||(\Delta x)^k, \tag{2.16}$$

where $C=C(k,\gamma,\alpha)$ is a constant depending on k,γ,α but is independent of U and Δx.

Proof. We mimic the proof of Theorem 3.2 in [26] but only sketch main steps. Let $\mathbb{P}$ be the L^2 projection defined as $\mathbb{P}(U)(x)$ in $\mathbb{V}_{\Delta x}$ such that,

$$\int_{I_j}(\mathbb{P}(U)(x)-U(x))v(x)dx=0, \quad \forall v\in\mathbb{V}_{\Delta x}.$$

Then the error

$$e=u-\mathbb{P}(U)+\mathbb{P}(U)-U=\mathbb{P}(e)-(U-\mathbb{P}(U)) \tag{2.17}$$

satisfies

$$|||e(\cdot,T)|||\le|||\mathbb{P}(e)(\cdot,T)|||+|||(U-\mathbb{P}(U))(\cdot,T)|||. \tag{2.18}$$

It suffices to estimate $|||\mathbb{P}(e)(\cdot,T)|||$ since

$$|||(U-\mathbb{P}(U))(\cdot,T)|||\le C|||\partial_x^{k+1}U(\cdot,T)|||(\Delta x)^k,$$

from Lemma 3.2 in [26].

The key relation such as (3.13) in [26] remains hold for (2.6):

$$\mathbb{B}(\mathbb{P}(e),\mathbb{P}(e))=\mathbb{B}(U-\mathbb{P}(U),\mathbb{P}(e)), \tag{2.19}$$

where $\mathbb{B}(w,v)$ is the bilinear form defined by

$$\mathbb{B}(w,v)=\int_0^T\int_0^1 w_t(x,t)v(x,t)dx+\int_0^T\sum_{j=1}^N\int_{I_j}w_x(x,t)v_x(x,t)dxdt+\Theta(T,w,v) \tag{2.20}$$

with

$$\Theta(T,w,v)=\int_0^T\sum_{j=1}^N(\widehat{w_x}[v])_{j+1/2}dt+\int_0^T\sum_{j=1}^N(\overline{v_x}[w])_{j+1/2}dt. \tag{2.21}$$

Admissibility of the numerical flux (2.10) ensures that left hand side of (2.19) is bounded from below

$$\mathbb{B}(\mathbb{P}(e),\mathbb{P}(e))\geq|||\mathbb{P}(e)(\cdot,T)|||^2-\frac{1}{2}||\mathbb{P}(e)(\cdot,T)||^2. \tag{2.22}$$

The right hand side of (2.19) reads as

$$\begin{aligned}\mathbb{B}(U-\mathbb{P}(U),\mathbb{P}(e))=&\int_0^T\int_0^1(U-\mathbb{P}(U))_t\mathbb{P}(e)dxdt\\&+\int_0^T\sum_{j=1}^N\int_{I_j}(U-\mathbb{P}(U))_x(\mathbb{P}(e))_xdxdt+\Theta(T,(U-\mathbb{P}(U)),\mathbb{P}(e)),\end{aligned} \tag{2.23}$$

with

$$\begin{aligned}&\Theta(T,(U-\mathbb{P}(U)),\mathbb{P}(e))\\=&\int_0^T\sum_{j=1}^N\Big(\widehat{(U-\mathbb{P}(U))_x}[\mathbb{P}(e)]\Big)_{j+1/2}dt+\int_0^T\sum_{j=1}^N\Big(\overline{\mathbb{P}(e)_x}[U-\mathbb{P}(U)]\Big)_{j+1/2}dt.\end{aligned}$$

The second term in Θ is a new term beyond those given in [26, page 687]. To estimate this we need to bound the trace for $\mathbb{P}(e)$ by the integral on I_j:

$$\sum_{j=1}^N[\overline{\mathbb{P}(e)_x}(x_{j\pm1/2})]^2\leq\frac{C}{\Delta x}\sum_{j=1}^N\int_{I_j}(\mathbb{P}(e)_x)^2dx,$$

which is valid for some constant $C\sim k^2$ for P^k elements. See [10] for the construction of inverse-inequalities on finite element spaces.

Hence the second term in Θ is bounded by

$$\begin{aligned}
&\int_0^T \sum_{j=1}^N \Big(\overline{\mathbb{P}(e)_x} [U - \mathbb{P}(U)] \Big)_{j+1/2} dt \\
\le & \frac{(1-\gamma)\Delta x}{4C} \int_0^T \sum_{j=1}^N (\overline{\mathbb{P}(e)_x})^2_{j+1/2} dt + \frac{C}{(1-\gamma)} \int_0^T \sum_{j=1}^N \frac{[U-\mathbb{P}(U)]^2}{\Delta x} dt \\
\le & \frac{(\gamma-1)}{4} \int_0^T \sum_{j=1}^N \|\mathbb{P}(e)_x\|^2_{I_j} dt + \frac{C}{(1-\gamma)} \int_0^T \sum_{j=1}^N \frac{[U-\mathbb{P}(U)]^2}{\Delta x} dt.
\end{aligned}$$

This when combined with the estimates of other terms given in [26, page 686] leads to the estimate of $|||\mathbb{P}(e)(\cdot,T)||| \le C|||\partial_x^{k+1} U(\cdot,T)|||(\Delta x)^k$ as desired. □

3 Extensions

In this section we extend the refined DDG method to nonlinear convection diffusion problems and multi-dimensional problems.

3.1 One-dimensional convection diffusion equations

We consider the nonlinear convection diffusion equation,

$$U_t + f(U)_x - (a(U)U_x)_x = 0 \quad \text{in } \Omega \times (0,T), \tag{3.1}$$

subject to initial data $U(x,0) = U_0(x)$ and periodic boundary conditions. The diffusion coefficient $a(U) > 0$ is non-negative.

Take $v \in \mathbb{V}_{\Delta x}$ as the test function, the DDG scheme with interface correction is defined as the following,

$$\begin{aligned}
&\int_{I_j} u_t v dx + (\widehat{f(u)} - \widehat{a(u)u_x}) v \Big|_{j-\frac{1}{2}}^{j+\frac{1}{2}} - \int_{I_j} (f(u) - a(u)u_x) v_x dx \\
&\quad + \frac{1}{2}[b(u)](v_x)^-_{j+1/2} + \frac{1}{2}[b(u)](v_x)^+_{j-1/2} = 0,
\end{aligned} \tag{3.2}$$

where $b(u) = \int_0^u a(s)ds$, $\widehat{f(u)} - \widehat{a(u)u_x}$ is the numerical flux to be chosen.

For the convection part we may choose any entropy satisfying numerical flux, for example, the Lax-Friedrichs flux,

$$\widehat{f(u)} = \hat{f}(u^-, u^+) = \frac{1}{2}\big(f(u^-) + f(u^+) - \theta(u^+ - u^-)\big), \tag{3.3}$$

where $\theta = \max_{u \in [u^-, u^+]} |f'(u)|$.

For the diffusion part, the numerical flux is chosen as

$$\widehat{a(u)u_x}=\beta_0\frac{[b(u)]}{\Delta x}+\overline{b(u)_x}+\beta_1\Delta x[b(u)_{xx}]. \tag{3.4}$$

Here β_0 and β_1 are taken the same as those for the linear case.

3.2 Multi-dimensional extensions

We now formulate the refined DDG method for multi-D problems. Since numerical flux for convection can be treated via dimension-wise extension, we present schemes only for nonlinear diffusion problems, for simplicity, in the two-dimensional setting. Let

$$J_1=a_{11}(U)U_x+a_{12}(U)U_y, \quad J_2=a_{21}(U)U_x+a_{22}(U)U_y.$$

Then two-dimensional equation can be written as

$$U_t-\partial_x J_1-\partial_y J_2=0, \quad \text{in } \Omega=[0,1]^2\subset\mathbb{R}^2, \tag{3.5}$$

subject to initial data $U(x,y,0)=U_0(x,y)$ and periodic boundary conditions. The diffusion coefficient matrix (a_{ij}) is assumed to be symmetric and positive definite.

Let a partition of Ω be denoted by rectangular meshes

$$\Omega=\sum_{j,k}^{N,M} I_{j,k}, \quad I_{j,k}=I_j\times I_k=[x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}]\times[y_{k-\frac{1}{2}},y_{k+\frac{1}{2}}]$$

of uniform mesh sizes $\Delta=\max(\Delta x,\Delta y)$. We denote the finite element space by

$$\mathbb{V}_\Delta=\Big\{v:\ v|_{I_{j,k}}\in P^k,\ \forall I_{j,k}\subset\Omega\Big\}, \tag{3.6}$$

where P^k is a polynomial of degree at most k.

Set $b_{ij}(u)=\int_0^u a_{ij}(s)ds$. Then the DDG scheme on each computational cell can be written as

$$\iint_{I_{j,k}} u_t v dxdy-\int_{I_k}\widehat{J_1}v\Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}dy-\int_{I_j}\widehat{J_2}v\Big|_{y_{k-\frac{1}{2}}}^{y_{k+\frac{1}{2}}}dx+\iint_{I_{j,k}}(J_1v_x+J_2v_y)dxdy+B=0, \tag{3.7}$$

$$\iint_{I_{j,k}} u(x,y,0)v(x,y)dxdy=\iint_{I_{j,k}} U_0(x,y)v(x,y)dxdy, \tag{3.8}$$

where the boundary correction is

$$\begin{aligned}B=\frac{1}{2}\bigg\{&\int_{I_k}([b_{11}(u)]v_x^-)_{x_{j+\frac{1}{2}}}+([b_{11}(u)]v_x^+)_{x_{j-\frac{1}{2}}}dy\\&+\int_{I_j}([b_{22}(u)]v_y^-)_{y_{k+\frac{1}{2}}}+([b_{22}(u)]v_y^+)_{y_{k-\frac{1}{2}}}dx\bigg\}.\end{aligned}$$

Here and below

$$[u]_{x_{j+\frac{1}{2}}} := u(x^+_{j+\frac{1}{2}},y,t) - u(x^-_{j+\frac{1}{2}},y,t), \quad [u]_{y_{k+\frac{1}{2}}} = u(x,y^+_{k+\frac{1}{2}},t) - u(x,y^-_{k+\frac{1}{2}},t).$$

The numerical flux $\widehat{J_i}$ is defined by

$$\widehat{J_1}\Big|_{x_{j+\frac{1}{2}}} = \beta_0 \frac{[b_{11}(u)]}{\Delta x} + \overline{b_{11}(u)_x + b_{12}(u)_y} + \beta_1 \Delta x [b_{11}(u)_{xx}], \tag{3.9}$$

$$\widehat{J_2}\Big|_{y_{k+\frac{1}{2}}} = \beta_0 \frac{[b_{22}(u)]}{\Delta y} + \overline{b_{21}(u)_x + b_{22}(u)_y} + \beta_1 \Delta y [b_{22}(u)_{yy}], \tag{3.10}$$

where we take (β_0,β_1) as obtained for one-dimensional case. The 2D algorithm is now well defined.

An appropriate choice of (β_0,β_1) is to ensure the L^2-stability of the method.

Theorem 3.1 (Energy stability)**.** *Assume that for $p \in \mathbb{R}$, $\exists \gamma$ and γ^* such that the eigenvalues of matrix $(a_{ij}(p))$ lie between $[\gamma,\gamma^*]$. Consider the refined DDG scheme with numerical flux* (3.9)-(3.10)*. Then the numerical solution satisfies*

$$\begin{aligned}
&\iint_\Omega u^2(x,y,T)dxdy + \int_0^T \sum_{I_{j,k}} \iint_{I_{j,k}} (J_1 u_x + J_2 u_y) dxdydt \\
&\quad + \gamma\beta_0 \int_0^T \left(\sum_{I_j} \int_{I_j} \frac{\sum_{k=1}^M [u]^2_{y_{k+1/2}}}{\Delta y} dx + \sum_{I_k} \int_{I_k} \frac{\sum_{j=1}^N [u]^2_{x_{j+1/2}}}{\Delta x} dy \right) dt \le \iint_\Omega U_0^2(x,y) dxdy,
\end{aligned} \tag{3.11}$$

provided β_0 is suitably large.

This stability result can be proved by following the similar argument as that explored in [26]. Details are omitted.

Up to now, we have taken the method of lines approach and have left t continuous. For time discretization we use the explicit third order TVD Runge-Kutta method [31, 32] to match the accuracy in space.

Finally we discuss how to define numerical fluxes for unstructured meshes. Let $\{K\}$ be shape- regular meshes. The DDG scheme with interface correction for the 2D heat equation is defined by

$$\int_K u_t v dxdy + \int_K \nabla u \cdot \nabla v dxdy - \int_{\partial K} \widehat{u_\mathbf{n}} v^{int_K} ds + \frac{1}{2} \int_{\partial K} [u] v^{int_K}_\mathbf{n} ds = 0, \tag{3.12}$$

where $\mathbf{n} = (n_x,n_y)$ is the outward normal unit along the cell boundary ∂K, v^{int_K} denotes v evaluated from inside K and $u_\mathbf{n} = \nabla u \cdot \mathbf{n}$ is differentiation in $\mathbf{n}$ direction. The numerical flux remains of the same form:

$$\widehat{u_\mathbf{n}} = \beta_0 \frac{[u]}{\Delta} + \overline{\frac{\partial u}{\partial \mathbf{n}}} + \beta_1 \Delta [u_{\mathbf{nn}}], \tag{3.13}$$

where $\Delta = diam\{K\}$ is the mesh size and

$$[u] = u^{ext_K} - u^{int_K} \quad \text{and} \quad \overline{u_{\mathbf{n}}} = \frac{{u_{\mathbf{n}}}^{ext_K} + {u_{\mathbf{n}}}^{int_K}}{2}.$$

Here u^{ext_K} represents u evaluated from outside of K (inside the neighboring cell).

If the cell boundaries are straight lines, such as the triangular meshes, the above numerical flux reduces to

$$\widehat{u_{\mathbf{n}}} = \widehat{u_x} n_x + \widehat{u_y} n_y$$

with

$$\widehat{u_x} = \beta_0 \frac{[u]}{\Delta} n_x + \overline{u_x} + \beta_1 \Delta \left([u_{xx}] n_x + [u_{xy}] n_y \right),$$
$$\widehat{u_y} = \beta_0 \frac{[u]}{\Delta} n_y + \overline{u_y} + \beta_1 \Delta \left([u_{yx}] n_x + [u_{yy}] n_y \right).$$

In other words:

$$\widehat{u_{\mathbf{n}}} = \beta_0 \frac{[u]}{\Delta} + \overline{u_x n_x + u_y n_y} + \beta_1 \Delta \Big([u_{xx}] n_x^2 + 2[u_{xy}] n_x n_y + [u_{yy}] n_y^2 \Big).$$

4 Numerical examples

In this section we provide a few numerical examples to illustrate the accuracy and capacity of the DDG method with interface corrections. We would like to illustrate the high order accuracy of the method through these numerical examples from one-dimensional to two-dimensional linear and nonlinear problems.

Example 4.1. 1D heat equation.

$$U_t - U_{xx} = 0, \quad x \in (0, 2\pi), \tag{4.1}$$

with initial condition $U(x,0) = \sin(x)$ and periodic boundary conditions.

In this example we will use this model equation to test the performance of the new DDG method. Two different numerical fluxes are investigated, one is to take $\beta_1 = 0$ and the other is to take $\beta_1 \neq 0$ in (2.8).

The numerical flux (2.8) with $\beta_1 = 0$ reduces to the following,

$$\widehat{u_x} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x}. \tag{4.2}$$

In light of the admissibility studied in Section 2 we know that β_0 is a parameter depending on the degree of the approximation polynomial. We need to choose β_0 big enough to stabilize the scheme for high order approximations. We refer to Table 1 for suitable choices of β_0 with different k. The DDG method based on p^k polynomial approximations

Table 2: 1D heat equation with numerical flux (4.2). L^2 and L^∞ errors at $t=1.0$. p^k polynomial approximations with $k=0,1,2,3,4$.

β_0	k		N=10	N=20		N=40		N=80	
			error	error	order	error	order	error	order
1	0	L^2	4.86e-02	2.38e-02	1.03	1.19e-02	1.00	5.90e-03	1.00
		L^∞	1.17e-01	5.80e-02	1.02	2.89e-02	1.00	1.45e-02	1.00
2	1	L^2	5.36E-03	1.41e-03	1.93	3.57e-04	1.98	8.96e-05	1.99
		L^∞	1.03e-02	2.87e-03	1.84	7.46e-04	1.94	1.88e-04	1.98
4	2	L^2	3.21e-04	3.73e-05	3.10	4.56e-06	3.02	5.68e-07	3.00
		L^∞	1.31e-03	1.60e-04	3.03	1.98e-05	3.00	2.48e-06	3.00
6	3	L^2	2.48e-05	1.56e-06	3.98	9.78e-08	3.99	6.11e-09	3.99
		L^∞	5.47e-05	3.60e-06	3.92	2.31e-07	3.96	1.45e-08	3.99
10	4	L^2	4.03e-07	1.03e-08	5.29	3.01e-10	5.09	9.24e-12	5.02
		L^∞	1.83e-06	5.62e-08	5.02	1.75e-09	5.00	5.46e-11	5.00

Table 3: 1D heat equation with numerical flux (4.3). p^k polynomial approximations with $k=2,3,4,5,6,7$.

k		N=10	N=20		N=40		N=80	
		error	error	order	error	order	error	order
2	L^2	3.73e-04	4.65e-05	3.00	5.80e-06	3.00	7.25e-07	3.00
	L^∞	7.21e-04	9.11e-05	2.98	1.14e-05	2.99	1.43e-06	2.99
3	L^2	2.59e-05	1.58e-06	4.03	9.80e-08	4.00	6.12e-09	4.00
	L^∞	6.17e-05	3.72e-06	4.05	2.33e-07	3.99	1.46e-08	3.99
4	L^2	2.13e-06	7.60e-08	4.80	2.47e-09	4.94	7.79e-11	4.98
	L^∞	6.74e-06	2.46e-07	4.77	8.03e-09	4.93	2.54e-10	4.98
		N=4	N=8		N=12		N=16	
5	L^2	1.13e-05	1.88e-07	5.91	1.67e-08	5.96	3.00e-09	5.98
	L^∞	1.97e-05	3.39e-07	5.86	3.03e-08	5.96	5.42e-09	5.98
6	L^2	7.30e-07	4.22e-09	7.43	2.34e-10	7.14	3.07e-11	7.05
	L^∞	1.23e-06	1.09e-08	6.82	7.25e-10	6.67	1.02e-10	6.83
7	L^2	8.43e-08	3.87e-10	7.77	1.57e-11	7.91	1.75e-12	7.63
	L^∞	1.28e-07	5.92e-10	7.76	2.40e-11	7.91	2.65e-12	7.65

with $k=0,1,2,3,4$ are tested and $(k+1)$th order of accuracy is obtained. L^2 and L^∞ errors are listed in Table 2. Note that in this and the remaining examples, L^∞ error is obtained by evaluating on 200 sample points per cell. The second test is to use numerical flux (2.8) with $\beta_1 \neq 0$. As suggested in Section 2 we use the following numerical flux in the DDG scheme (2.6),

$$\widehat{u_x} = 2\frac{[u]}{\Delta x} + \overline{u_x} + \frac{\Delta x}{12}[u_{xx}]. \tag{4.3}$$

p^k polynomial approximations with $k=2,\cdots,9$ are tested and with h refinement optimal $(k+1)$th order of accuracy is obtained. To save the space here we only list the errors and orders for $k=2,3,4,5,6,7$ in Table 3. Note including term $[u_{xx}]$ in the numerical flux does

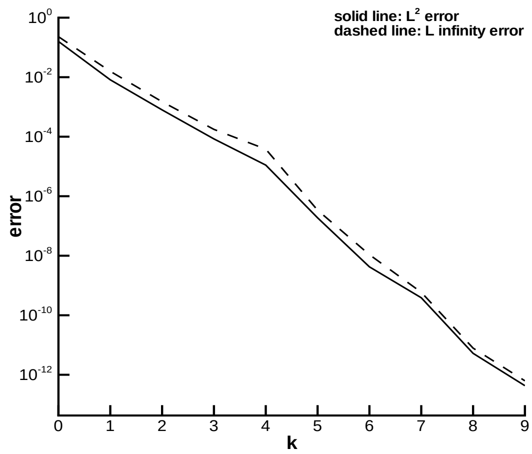

Figure 2: Convergence of the DDG scheme under p-refinement with fixed mesh $N=8$.

relieve the dependence of β_0 on k. We use fixed $\beta_0=2$ for all p^k polynomial approximations up to $k=9$. We also check the p-convergence of the DDG scheme, namely with fixed mesh and increased degree of approximating polynomial we obtain exponential convergence rate, see Fig. 2. Similar to the p-convergence studied in [24], we plot the error against polynomial degree k on the linear-log scale and an almost straight line is observed with increased degree of approximating polynomials. Furthermore, we investigate this scheme on nonuniform mesh and still $(k+1)$th order of accuracy is obtained with refined mesh. Errors and orders are listed in Table 4. For the nonuniform mesh, the partition of the domain $[0,2\pi]$ consists of repeated pattern of $1.1\Delta x$ and $0.9\Delta x$ for odd and even number of index $i=1,\cdots,N$, where $\Delta x=2\pi/N$ with even number N.

Table 4: 1D heat equation with numerical flux (4.3) on non uniform mesh. p^k approximations with $k=2,3,4$.

k		N=10	N=20		N=40		N=80	
		error	error	order	error	order	error	order
2	L^2	4.07e-04	5.04e-05	3.01	6.29e-06	3.00	7.86e-07	3.00
	L^∞	9.37e-04	1.19e-04	2.97	1.50e-05	2.99	1.88e-06	3.00
3	L^2	3.10e-05	1.91e-06	4.02	1.19e-07	4.00	7.43e-09	4.00
	L^∞	8.84e-05	5.46e-06	4.01	3.44e-07	3.99	2.16e-08	3.99
4	L^2	4.07e-06	1.69e-07	4.59	5.76e-09	4.87	1.84e-10	4.97
	L^∞	1.39e-05	5.89e-07	4.56	2.02e-08	4.87	6.46e-10	4.96

The above comparison indicates that the term $[u_{xx}]$ is important for the scheme to remain optimally accurate for high order approximations with a fixed β_0. Hence in the rest examples we use numerical flux (4.3) only.

Example 4.2. 1D fully nonlinear equation.

$$U_t+UU_x-\frac{1}{2}(UU_x)_x=0, \quad x\in(0,1), \tag{4.4}$$

with initial condition $U(x,0)=e^x$ and boundary condition $U(0,t)=1$ and $U(1,t)=e$. The exact solution is given as $U(x,t)=e^x$.

For the nonlinear diffusion term we use the following numerical flux,

$$\widehat{uu_x}=\frac{1}{2}\widehat{(u^2)_x}=\frac{1}{2}\left(2\frac{[u^2]}{\Delta x}+\overline{(u^2)_x}+\frac{\Delta x}{12}[(u^2)_{xx}]\right).$$

We conduct the DDG scheme on this fully nonlinear equation and obtain $(k+1)$th order of accuracy for p^k approximations. Errors and orders are listed in Table 5.

Table 5: 1D fully nonlinear equation (4.4). L^2 and L^∞ errors at $t=0.5$. p^k polynomial approximation with $k=0,1,2,3$.

k		N=10	N=20		N=30		N=40	
		error	error	order	error	order	error	order
0	L^2	1.08e-01	5.54e-02	0.97	3.72e-02	0.98	2.81e-02	0.98
	L^∞	2.11e-01	1.08e-01	0.97	7.24e-02	0.98	5.45e-02	0.99
1	L^2	1.19e-03	2.99e-04	1.99	1.33e-04	2.00	7.49e-05	2.00
	L^∞	3.13e-03	8.40e-04	1.90	3.83e-04	1.94	2.18e-04	1.96
2	L^2	1.02e-05	1.28e-06	2.99	3.80e-07	3.00	1.60e-07	3.00
	L^∞	2.15e-05	2.72e-06	2.99	8.06e-07	3.00	3.40e-07	3.00
3	L^2	5.95e-08	3.72e-09	4.00	7.35e-10	3.98	2.34e-10	3.99
	L^∞	1.39e-07	8.88e-09	3.97	1.77e-09	3.92	5.74e-10	3.95

Example 4.3. 1D nonlinear diffusion equation with nonsmooth solution.

$$U_t-(3U^2U_x)_x=0, \quad \text{in}\,[-12,12], \tag{4.5}$$

with initial condition

$$U(x,0)=\begin{cases} \sqrt{3-\frac{x^2}{12}}, & |x|<6, \\ 0, & |x|\geq 6, \end{cases}$$

and zero boundary conditions. The solution is a non smooth wave propagating with a finite speed. We compute the DDG quadratic approximation for this nonlinear diffusion problem up to $t=2$ and plot the result in Fig. 3. We use the following nonlinear numerical flux

$$\widehat{3u^2u_x}=\widehat{(u^3)_x}=\beta_0\frac{[u^3]}{\Delta x}+\overline{(u^3)_x}+\beta_1\Delta x[(u^3)_{xx}].$$

with $\beta_0=2$ and $\beta_1=1/12$. As expected our DDG method has the capability to sharply capture the corners with discontinuous derivatives. Also compared to the symmetric IP method, our scheme is more accurate in the smooth area and again the symmetric IP method needs $\beta_0=8$ or bigger to stabilize the scheme for this nonlinear problem.

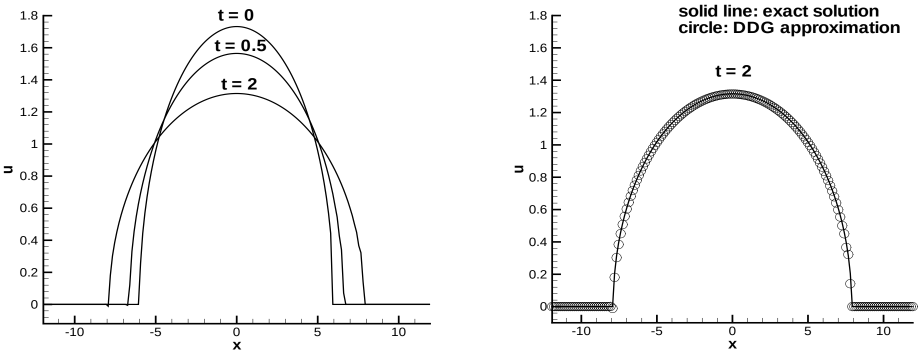

Figure 3: 1D nonlinear diffusion equation (4.5). p^2 approximation with mesh N=180. Left: DDG solution at $t=0,0.5,2$. Right: comparison to exact solution at $t=2$.

Example 4.4. 2D linear convection diffusion equation.

$$U_t+c(U_x+U_y)-\mu(U_{xx}+U_{yy})=0, \quad (x,y)\in(0,2\pi)\times(0,2\pi), \tag{4.6}$$

with initial condition $U(x,y,0)=\sin(x+y)$ and periodic boundary conditions. The exact solution is $U(x,y,t)=e^{-2\mu t}\sin(x+y-2ct)$. In this example we take $c=1$ and $\mu=1$. Accuracy test is performed on a $N\times N$ rectangular mesh. The DDG scheme with interface corrections is implemented at $t=0.5$. Similar to 1D problems, we choose fixed $\beta_0=2$ in the numerical flux for all p^k polynomials. In the x-direction, we take

$$\widehat{u_x}|_{x_{j+1/2}}=\left(2\frac{[u]}{\Delta x}+\overline{u_x}+\frac{\Delta x}{12}[u_{xx}]\right)\Big|_{x_{j+1/2}}.$$

A similar formula in the y-direction is applied. Again $(k+1)$th order of accuracy is obtained with piecewise p^k polynomial approximations. L^2 and L^∞ errors and orders are listed in Table 6.

Example 4.5. 2D anisotropic diffusion equation.

$$U_t+c(U_x+U_y)-\mu(U_{xx}+U_{yy}+U_{xy})=0, \quad (x,y)\in(0,2\pi)\times(0,2\pi) \tag{4.7}$$

with initial condition $U(x,y,0)=\sin(x+y)$ and periodic boundary conditions. The exact solution is $U(x,y,t)=e^{-3\mu t}\sin(x+y-2ct)$. We use this example to test the DDG scheme for diffusion problem with non-isotropic term. For the mixed term u_{xy} we use numerical flux (3.9) on rectangular meshes. In this example we take $c=1$ and $\mu=0.01$. p^k polynomial approximations with $k=0,1,2,3$ are tested and $(k+1)$th order of accuracy is obtained. Errors and orders are listed in Table 7.

Table 6: 2D linear convection diffusion equation (4.6). L^2 and L^∞ errors at $t=0.5$. p^k polynomial approximation with $k=0,1,2,3$.

k		N=10	N=20		N=40		N=80	
		error	error	order	error	order	error	order
0	L^2	8.58e-02	4.74e-02	0.86	2.50e-02	0.92	1.28e-02	0.96
	L^∞	2.23e-01	1.22e-01	0.88	6.26e-02	0.96	3.17e-02	0.98
1	L^2	1.11e-02	2.73e-03	2.02	6.71e-04	2.02	1.66e-04	2.01
	L^∞	5.49e-02	1.39e-02	1.98	3.49e-03	1.99	8.84e-04	1.98
2	L^2	1.13e-03	1.40e-04	3.01	1.74e-05	3.00	2.18e-06	3.00
	L^∞	7.64e-03	9.58e-04	2.99	1.17e-04	3.03	1.46e-05	3.00
3	L^2	1.33e-04	8.16e-06	4.02	5.06e-07	4.01	3.15e-08	4.00
	L^∞	7.03e-04	4.49e-05	3.97	2.87e-06	3.97	1.82e-07	3.98

Table 7: 2D anisotropic case (4.7). L^2 and L^∞ errors at $t=0.3$. p^k polynomial approximations with $k=0,1,2,3$.

k		N=10	N=20		N=40		N=80	
		error	error	order	error	order	error	order
0	L^2	2.03e-01	1.07e-01	0.92	5.56e-02	0.95	2.88e-02	0.95
	L^∞	6.10e-01	3.12e-01	0.97	1.60e-01	0.96	8.09e-02	0.98
1	L^2	3.62e-02	9.44e-03	1.94	2.39e-03	1.98	5.93e-04	2.00
	L^∞	2.26e-01	6.11e-02	1.88	1.54e-02	1.99	3.78e-03	2.02
2	L^2	3.90e-03	4.48e-04	3.12	5.50e-05	3.02	6.69e-06	3.03
	L^∞	3.44e-02	4.29e-03	3.00	5.02e-04	3.09	5.75e-05	3.12
3	L^2	4.44e-04	2.51e-05	4.14	1.55e-06	4.02	9.51e-08	4.02
	L^∞	3.49e-03	2.72e-04	3.68	1.46e-05	4.21	7.86e-07	4.20

Example 4.6. 2D incompressible Navier-Stokes equation in vorticity formulation. In this example we consider two-dimensional incompressible Navier-Stokes equation in vorticity based formulation

$$\omega_t+\nabla\cdot(\mathbf{u}\omega)=\frac{1}{Re}\triangle\omega, \quad (x,y)\in(0,2\pi)\times(0,2\pi). \tag{4.8}$$

Again we use this example to check the high order accuracy of the DDG method. To simplify the computation, we take incompressible velocity field $\mathbf{u}=(u,v)$ as a given function. Here $(u(x,y,t),v(x,y,t))=e^{-\frac{2t}{Re}}(-\cos x\sin y,\sin x\cos y)$ and the exact solution is known as $\omega(x,y,t)=2e^{-\frac{2t}{Re}}\cos x\sin y$, see [9]. Periodic boundary conditions are applied and we take the Reynolds number $Re=100$. We compute the solution at time $t=1$. L^2 and L^∞ errors are listed in Table 8 and we obtain $(k+1)$th order of accuracy with p^k polynomial approximations.

Example 4.7. 2D Buckley-Leverett equation. Finally, we consider the two-dimensional convection diffusion equation [25].

$$u_t+f(u)_x+g(u)_y=\epsilon(u_{xx}+u_{yy}), \quad (x,y)\in(-1.5,1.5)\times(-1.5,1.5). \tag{4.9}$$

Table 8: 2D Navier-Stokes equation in vorticity formulation (4.8). L^2 and L^∞ errors at $t=1.0$. p^k approximations with $k=1,2,3$.

k		N=16	N=32		N=64		N=128	
		error	error	order	error	order	error	order
1	L^2	2.19e-02	5.13e-03	2.09	1.25e-03	2.04	3.06e-04	2.02
	L^∞	1.16e-01	2.59e-02	2.16	6.25e-03	2.05	1.47e-03	2.09
2	L^2	1.34e-03	1.55e-04	3.11	1.85e-05	3.06	2.19e-06	3.07
	L^∞	1.10e-02	1.19e-03	3.20	1.32e-04	3.17	1.49e-05	3.14
3	L^2	3.23e-04	1.07e-05	4.91	4.62e-07	4.52	2.33e-08	4.31
	L^∞	2.09e-03	5.73e-05	5.18	2.30e-06	4.63	1.18e-07	4.28

The nonlinear convection terms are given as

$$f(u)=\frac{u^2}{u^2+(1-u)^2},$$
$$g(u)=f(u)(1-5(1-u)^2),$$

and the initial condition is taken as

$$u(x,y,0)=\begin{cases} 1, & x^2+y^2<0.5, \\ 0, & \text{otherwise}. \end{cases} \tag{4.10}$$

This is the two-dimensional Buckley-Leverett equation with small diffusion. Essentially it is a convection dominated problem with non-convex flux functions. Here we take $\epsilon=0.01$. We compute the DDG solution with p^1 polynomial approximations up to $t=0.5$ with mesh size $N\times N=100\times 100$. In Fig. 4 we show the solution slice at $y=0.75$ and the solution contours in $(0,1)$. Fig. 5 shows the $3D$ outlook of the solution.

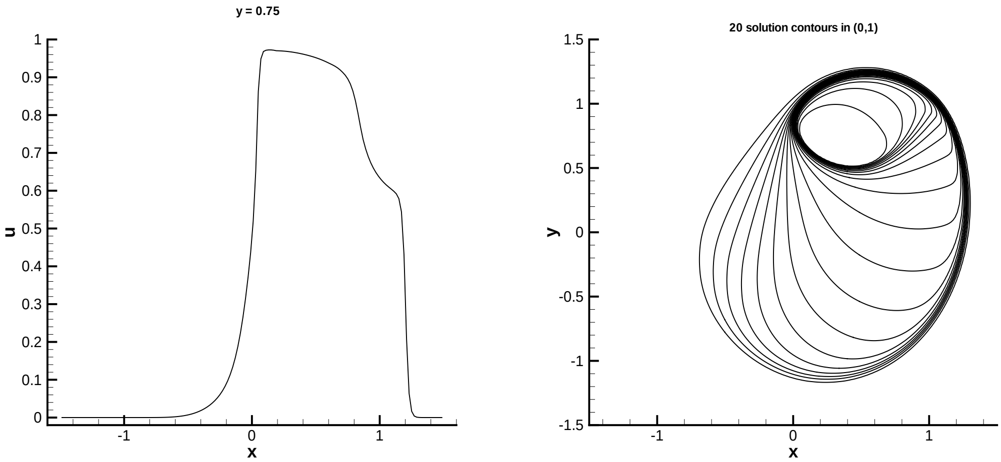

Figure 4: 2D Buckley-Leverett equation (4.9). Left: solution slice at $y=0.75$. Right: solution contours.

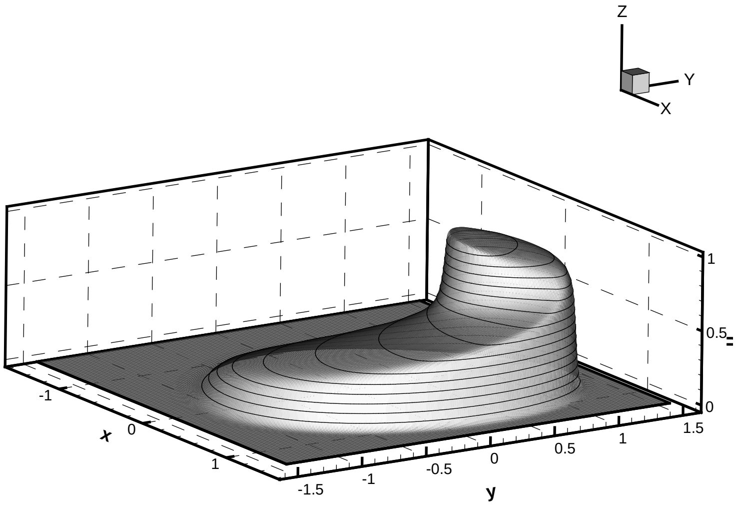

Figure 5: 2D Buckley-Leverett equation (4.9). Solution at $t=0.5$ on 100×100 mesh.

In our simulation we observed that when ϵ is relatively large, say $\epsilon=0.1$, the scheme is stable and accurate. When smaller ϵ is used, we observe some instability phenomena which is related to the steep shock fronts. Here we use slope limiters as introduced in [14] to stabilize the scheme. Again, the DDG method shows its capability to obtain high resolution solutions across sharp transition areas and gives satisfactory results.

5 Concluding remarks

Built upon the DDG method introduced in [26], we have presented a refined direct discontinuous Galerkin (DDG) method for diffusion problems. We include extra interface corrections in the scheme formulation with numerical flux involving only up to second order derivatives of the numerical solution. The refined DDG scheme has the advantage of obtaining optimal accuracy of $(k+1)$th order for all p^k elements. We prove that there exists a large class of coefficients (β_0,β_1) in the numerical flux formulation,

$$\widehat{u_x}=\beta_0[u]/\Delta x+\overline{u_x}+\beta_1\Delta x[u_{xx}],$$

ensuring the stability of the scheme. We also confirm that when $\beta_1=0$, β_0 has to be big enough to guarantee the scheme stability; actually we estimate the precise dependence of β_0 on the polynomial degree k. Extensions of the method to convection diffusion problems in both one- and two-dimensional settings are given. Finally we carry out a series of numerical tests from linear to nonlinear, one-dimensional to two-dimensional problems to demonstrate the high order accuracy of the method. Our numerical results show that $\beta_1\neq0$ does provide a leverage to compensate the β_0 term, thus a fixed β_0 can be used for all p^k polynomials.

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