Commun. Comput. Phys.
doi: 10.4208/cicp.060109.130110a

Vol. 8, No. 1, pp. 226-248
July 2010

Convergence Analysis for Stochastic Collocation Methods to Scalar Hyperbolic Equations with a Random Wave Speed

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Received 6 January 2009; Accepted (in revised version) 13 January 2010

Communicated by Jan S. Hesthaven

Available online 23 February 2010

Abstract. For a simple model of a scalar wave equation with a random wave speed, Gottlieb and Xiu [Commun. Comput. Phys., 3 (2008), pp. 505-518] employed the generalized polynomial chaos (gPC) method and demonstrated that when uncertainty causes the change of characteristic directions, the resulting deterministic system of equations is a symmetric hyperbolic system with both positive and negative eigenvalues. Consequently, a consistent method of imposing the boundary conditions is proposed and its convergence is established under the assumption that the expansion coefficients decay fast asymptotically. In this work, we investigate stochastic collocation methods for the same type of scalar wave equation with random wave speed. It will be demonstrated that the rate of convergence depends on the regularity of the solutions; and the regularity is determined by the random wave speed and the initial and boundary data. Numerical examples are presented to support the analysis and also to show the sharpness of the assumptions on the relationship between the random wave speed and the initial and boundary data. An accuracy enhancement technique is investigated following the multi-element collocation method proposed by Foo, Wan and Karniadakis [J. Comput. Phys., 227 (2008), pp. 9572-9595].

AMS subject classifications: 52B10, 65D18, 68U05, 68U07

Key words: Hyperbolic equation, stochastic collocation methods, convergence analysis.

1 Introduction

Recently there has been a growing interest in designing efficient methods for the solution of ordinary/partial differential equations with random inputs. The methods include

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Monte Carlo and sampling based methods, perturbation methods, operator based methods and the generalized polynomial chaos (gPC) method, see, e.g., [1,6,13,14]). Among these methods, the gPC method has become one of the most widely used methods. With gPC, stochastic solutions are expressed as orthogonal polynomials of the input random parameters, and different types of orthogonal polynomials can be chosen to achieve better convergence. It is essentially a spectral representation in random space, and exhibit fast convergence when the solution depends smoothly on the random parameters.

Although the polynomial chaos methods (and gPC method) have been extensively applied to analyze PDEs that contain uncertainties, this approach is rarely applied to hyperbolic systems. Gottlieb and Xiu [7] made the first attempt by considering a simple model of a scalar wave equation with random wave speeds. It was shown that when uncertainty causes the change of characteristic directions, the resulting deterministic system of equations is a symmetric hyperbolic system with both positive and negative eigenvalues. A consistent method of imposing the boundary conditions is proposed. A numerical method based on the gPC method is introduced and its convergence theory is established.

In this work, we also consider the same model scalar wave equation with random wave speed using the stochastic collocation methods. Collocation methods have been studied and used in different disciplines for uncertainty quantification (see, e.g., Tatang [11], Xiu and Hesthaven [15], Keese and Matthies [8] Ganapathysubramanian and Zabaras [5]). In collocation methods one seeks to satisfy the governing differential equations at a discrete set of points, called "nodes", in the corresponding random space. Two of the major approaches of high-order stochastic collocation methods are the Lagrange interpolation approach, see Xiu and Hesthaven [15] and later (independently) in [1], and the pseudo-spectral gPC approach from [13].

Following the methods introduced by Tatang [11], we use the roots of the next higher order polynomial as the points at which the approximation is to be found. Let $\Theta=\{y_k\}_{k=1}^N \in \Gamma$ (the parameter space) be such a set of nodes, where N is the number of nodes. A Lagrange interpolation of the solution $w(x,y)$ can be written as

$$I^N w(x,y)=\sum_{k=1}^{N}\tilde{w}_k(x)F_k(y), \tag{1.1}$$

where

$$F_k\in\mathcal{P}_N,\quad F_i(y_k)=\delta_{ik},\quad 1\le i,k\le N, \tag{1.2}$$

are the Lagrange interpolation polynomials, and $\tilde{w}_k(x):=w(x,y_k)$, $1\le k\le N$, is the value of w at the given node $y_k\in\Theta$.

In this work, we will apply the Lagrange interpolation approach to the model scalar wave equation with random wave speed (see, [7]):

$$\partial_t u(x,t;y(\omega))=c(y(\omega))\partial_x u(x,t;y(\omega)),\quad x\in D\equiv(-1,1),\quad t>0, \tag{1.3}$$

$$u(x,0;y(\omega))=u_0(x;y(\omega)),\qquad x\in D. \tag{1.4}$$

We will adopt a probabilistic framework and model y as a random variable with properly defined probability space $(\Omega,\mathcal{A},\mathcal{P})$, whose event space is Ω and is equipped with *σ-algebra* $\mathcal{A}$ and probability measure $\mathcal{P}$. Let $\rho(y):\Gamma\to\mathbf{R}^+$ be the probability density functions of the random variable $y(\omega),\omega\in\Omega$, and its image $\Gamma\equiv y(\Omega)\in\mathbf{R}$ be intervals in $\mathbf{R}$. In what follows, for simplicity, we just omit the symbol ω and assume that y is in the parametric space $\Gamma\equiv[-1,1]$.

A well-posed set of boundary conditions is given by:

$$\begin{aligned} &u(-1,t;y)=u_L(t;y), \quad c(y)<0,\\ &u(1,t;y)=u_R(t;y), \quad\ \ \ c(y)>0. \end{aligned} \tag{1.5}$$

Eqs. (1.3)-(1.5) complete the set up of the problem.

We now solve problem (1.3)-(1.5) by using the Lagrange interpolation approach. We first choose a set of Gauss-collocation-points $\{y_i\}_{i=0}^N$, that is, $\{y_i\}_{i=0}^N$ are the roots of some polynomial Φ_{N+1}. The commonly seen correspondences between the polynomials $\Phi_k(y)$ and the distribution of the random variable y include Hermite-Gaussian (the original PC expansion), Legendre-uniform, Laguerre-Gamma, etc., cf. [2, 10]. We then solve the following system of equations:

$$\partial_t u(x,t;y_j)=c(y_j)\partial_x u(x,t;y_j), \quad j=0,1,\cdots,N. \tag{1.6}$$

Note that with the collocation method the boundary conditions and the initial conditions can be proposed easily, which is not the case in the Galerkin methods [7]. More precisely, we have

$$\begin{aligned} &u(-1,t;y_j)=u_L(t;y_j), \quad \text{if } c(y_j)<0,\\ &u(1,t;y_j)=u_R(t;y_j), \quad\ \ \ \text{if } c(y_j)>0, \end{aligned} \tag{1.7}$$

together with the initial condition

$$u(x,0;y_j)=u_0(x;y_j). \tag{1.8}$$

Note that in the linear case we can obtain the exact solution $u(\bullet,t;y_j)$ for (1.6)-(1.8). The approximation solution for the original problem (1.3)-(1.5) is then given by

$$u^N(x,t;y)=I_N^y u:=\sum_{k=0}^N u(x,t;y_k)F_k(y), \tag{1.9}$$

where $F_k(y)$ are the standard Lagrange interpolation polynomials defined by (1.2).

2 Regularity in various spaces

2.1 Regularity in H^1

Without lose of generality, we use D to indicate the physical space and Γ the parametric space. Since stochastic functions intrinsically have different structure with respect to y

and with respect to x, the analysis of numerical approximations requires tensor spaces. The details for the definition can be founded in [1]. Following [1], if $u \in L^2 \otimes H^k(D)$, then $u(\cdot,y,t) \in H^k(D)$ a.e. on Γ and $u(x,t,\cdot) \in L^2(\Gamma)$ a.e. on D. Moreover, we have (for every fixed $t<T$) the isomorphism

$$L^2 \otimes H^k(D) \simeq L^2(\Gamma;H^k(D)) \simeq H^k(D;L^2(\Gamma))$$

with the definitions

$$\begin{aligned}&L^2(\Gamma;H^k(D))\\ =&\Big\{v:\Gamma\times D\to R \,|\, v \text{ is strongly measurable and} \int_\Gamma \|v(\cdot,y,t)\|^2_{H^k(D)} < +\infty\Big\},\end{aligned}$$

and

$$\begin{aligned}&H^k(D;L^2(\Gamma))\\ =&\Big\{v:\Gamma\times D\to R \,|\, v \text{ is strongly measurable and} \,\forall|\alpha|\le k, \exists\partial_\alpha v\in L^2(\Gamma)\otimes L^2(D),\\ &\int_\Gamma\int_D \partial_\alpha v\varphi(x,y)dxdy = (-1)^{|\alpha|}\int_\Gamma\int_D v(x,y,t)\partial_\alpha\varphi(x,y)dxdy \quad \forall\varphi\in C_0^\infty(\Gamma\times D)\Big\}.\end{aligned}$$

We also denote

$$\Gamma^+ = \big\{y \,|\, y\in\Gamma, \text{ and } c(y)>0\big\}, \quad \Gamma^- = \big\{y \,|\, y\in\Gamma, \text{ and } c(y)<0\big\}.$$

With the above definitions, we now introduce the following lemma.

Lemma 2.1. *Consider the problem* (1.3)-(1.5). *If the following conditions are satisfied:*

$$\int_\Gamma\int_D \rho(y)\Big(\partial_x u_0(x;y)\Big)^2 dxdy < \infty, \tag{2.1a}$$

$$\int_0^T\int_{\Gamma^+} \frac{\rho(y)}{c(y)}\Big(\partial_t u_R(t;y)\Big)^2 dydt < \infty, \tag{2.1b}$$

$$\int_0^T\int_{\Gamma^-} \frac{\rho(y)}{|c(y)|}\Big(\partial_t u_L(t;y)\Big)^2 dydt < \infty, \tag{2.1c}$$

then

$$\int_\Gamma\int_D \rho(y)u_x^2 dxdy < C(T), \quad 0<t\le T, \tag{2.2}$$

where $\rho(y)>0$ is the probability distribution function and $C(T)$ is a positive constant depending on T.

Proof. Using (1.3) and (1.5), we have, for $c(y)<0$,

$$u_x(-1,t;y) = \frac{1}{c(y)}u_t(-1,t;y) = \frac{1}{c(y)}\partial_t u_L(t;y). \tag{2.3}$$

Similarly, for $c(y)>0$,

$$u_x(1,t;y)=\frac{1}{c(y)}\partial_t u_R(t;y). \tag{2.4}$$

It follows from the governing equation (1.3) that

$$\partial_t(u_x^2)=c(y)\partial_x(u_x^2), \quad x\in D, t>0,$$

which leads to

$$\begin{aligned}\partial_t\int_D\rho(y)u_x^2dx&=\rho(y)c(y)\Big[u_x^2(1,t;y)-u_x^2(-1,t;y)\Big]\\&\le\begin{cases}\rho(y)c(y)u_x^2(1,t;y) & \text{if } c(y)>0,\\ -\rho(y)c(y)u_x^2(-1,t;y) & \text{if } c(y)<0.\end{cases}\end{aligned} \tag{2.5}$$

The above result, together with (2.3) and (2.4), yields

$$\begin{aligned}&\frac{d}{dt}\int_\Gamma\int_D\rho(y)u_x^2dxdy\\ \le&\int_{\Gamma^+}\frac{\rho(y)}{c(y)}\Big(\partial_t u_R(t;y)\Big)^2dy+\int_{\Gamma^-}\frac{\rho(y)}{|c(y)|}\Big(\partial_t u_L(t;y)\Big)^2dy.\end{aligned} \tag{2.6}$$

The desired estimate (2.2) is obtained by integrating the above inequality with respect to t and by using the assumption (2.1). $\square$

Theorem 2.1. *Consider the problem* (1.3)-(1.5). *Assume that there exists a constant C such that*

$$|c'(y)|\le C, \quad \textit{almost everywhere in}\,\Gamma, \tag{2.7}$$

i.e., $c'(y)$ is bounded in the distribution sense in Γ. If the assumption (2.1) *holds and furthermore if*

$$\int_\Gamma\int_D\rho(y)(\partial_y u_0(x;y))^2dxdy<\infty, \tag{2.8a}$$

$$\int_0^T\int_{\Gamma^+}\frac{\rho(y)}{c(y)}(\partial_y u_R(t;y))^2dydt<\infty, \tag{2.8b}$$

$$\int_0^T\int_{\Gamma^-}\frac{\rho(y)}{|c(y)|}(\partial_y u_L(t;y))^2dydt<\infty, \tag{2.8c}$$

then

$$\int_\Gamma\int_D\rho(y)u_y^2dxdy<C(T), \quad 0<t\le T, \tag{2.9}$$

where $C(T)$ is a finite number depending on T.

Proof. Differentiating both sides of (1.3) with respect to y gives

$$(u_y)_t = c'(y)u_x + c(y)(u_y)_x$$

which yields

$$(u_y^2)_t = 2c'(y)u_x u_y + c(y)(u_y^2)_x.$$

Integrating the above equation with respect to x leads to

$$\begin{aligned}
&\partial_t \int_D \rho(y) u_y^2 dx \\
=& 2\rho(y)c'(y)\int_D u_x u_y dx + \rho(y)c(y)\Big[u_y^2(1,t;y) - u_y^2(-1,t;y)\Big] \\
\le& 2\rho(y)c'(y)\int_D u_x u_y dx + \begin{cases} \frac{\rho(y)}{c(y)}(\partial_y u_R)^2(t;y) & \text{if } c(y)\ge 0, \\ -\frac{\rho(y)}{c(y)}(\partial_y u_L)^2(t;y) & \text{if } c(y) < 0, \end{cases}
\end{aligned} \tag{2.10}$$

which yields

$$\begin{aligned}
&\frac{d}{dt}\int_\Gamma \int_D \rho(y) u_y^2 dxdy \\
\le& C\int_\Gamma\int_D \rho(y) u_x^2 dxdy + C\int_\Gamma\int_D \rho(y) u_y^2 dxdy \\
&+ \int_{\Gamma^+} \frac{\rho(y)}{c(y)} (\partial_y u_R(t;y))^2 dy + \int_{\Gamma^-} \frac{\rho(y)}{|c(y)|}(\partial_y u_L(t;y))^2 dy,
\end{aligned} \tag{2.11}$$

where the boundedness assumption of $c'(y)$ is used. The desired estimate (2.9) follows from Lemma 2.1, Gronwall inequality and the assumption (2.8). ☐

2.2 Regularity in H^2

Lemma 2.2. *Consider the problem* (1.3)-(1.5). *If the following conditions are satisfied:*

$$\int_\Gamma\int_D \rho(y)\Big(\partial_{xx}u_0(x;y)\Big)^2 dxdy < \infty, \tag{2.12a}$$

$$\int_0^T \int_{\Gamma^+} \frac{\rho(y)}{c^3(y)}\Big(\partial_{tt} u_R(t;y)\Big)^2 dydt < \infty, \tag{2.12b}$$

$$\int_0^T \int_{\Gamma^-} \frac{\rho(y)}{|c^3(y)|}\Big(\partial_{tt} u_L(t;y)\Big)^2 dydt < \infty, \tag{2.12c}$$

then

$$\int_\Gamma\int_D \rho(y) u_{xx}^2 dxdy < C(T), \quad 0 < t \le T, \tag{2.13}$$

where $\rho(y) > 0$ is the probability distribution function, and $C(T)$ is a positive constant dependent on T.

Proof. It follows from (1.3) that

$$u_{tt}=c(y)(u_t)_x=c(y)(c(y)u_x)_x=c^2(y)u_{xx}.$$

This, together with (1.5), gives, for $c(y)<0$,

$$u_{xx}(-1,t;y)=\frac{1}{c^2(y)}u_{tt}(-1,t;y)=\frac{1}{c^2(y)}\partial_{tt}u_L(t;y). \tag{2.14}$$

Similarly, for $c(y)>0$,

$$u_{xx}(1,t;y)=\frac{1}{c^2(y)}\partial_{tt}u_R(t;y). \tag{2.15}$$

It follows from the governing equation (1.3) that

$$\partial_t(u_{xx}^2)=c(y)\partial_x(u_{xx}^2), \quad x\in D, t>0,$$

which leads to

$$\begin{aligned}\partial_t\int_D\rho(y)u_{xx}^2dx&=\rho(y)c(y)\Big[u_{xx}^2(1,t;y)-u_{xx}^2(-1,t;y)\Big]\\&\le\begin{cases}\rho(y)c(y)u_{xx}^2(1,t;y) & \text{if } c(y)>0,\\ -\rho(y)c(y)u_{xx}^2(-1,t;y) & \text{if } c(y)<0.\end{cases}\end{aligned} \tag{2.16}$$

The above result, together with (2.14) and (2.15), yields

$$\begin{aligned}&\frac{d}{dt}\int_\Gamma\int_D\rho(y)u_{xx}^2dxdy\\ \le&\int_{\Gamma^+}\frac{\rho(y)}{c^3(y)}\Big(\partial_{tt}u_R(t;y)\Big)^2dy+\int_{\Gamma^-}\frac{\rho(y)}{|c(y)^3|}\Big(\partial_{tt}u_L(t;y)\Big)^2dy.\end{aligned} \tag{2.17}$$

The desired estimate (2.13) is obtained by integrating the above inequality with respect to t and by using the assumption (2.12). $\square$

Lemma 2.3. *Consider the problem* (1.3)-(1.5)*. If the assumption* (2.7) *holds and also if*

$$\int_\Gamma\int_D\rho(y)\Big(\partial_{xy}u_0(x;y)\Big)^2dxdy<\infty, \tag{2.18a}$$

$$\int_0^T\int_{\Gamma^+}\frac{\rho(y)}{c(y)}\Big(\partial_{ty}u_R(t;y)-\frac{c'(y)}{c(y)}\partial_t u_R(t;y)\Big)^2dydt<\infty, \tag{2.18b}$$

$$\int_0^T\int_{\Gamma^-}\frac{\rho(y)}{|c(y)|}\Big(\partial_{ty}u_L(t;y)-\frac{c'(y)}{c(y)}\partial_t u_L(t;y)\Big)^2dydt<\infty, \tag{2.18c}$$

then

$$\int_\Gamma\int_D\rho(y)u_{xy}^2dxdy<C(T), \quad 0<t\le T, \tag{2.19}$$

where $\rho(y)>0$ is the probability distribution function, and $C(T)$ is a constant dependent on T.

Proof. It follows from (1.3) that $u_x = u_t/c(y)$, which together with (1.5), gives, for $c(y)<0$,

$$\begin{aligned} u_{xy}(-1,t;y) &= \frac{1}{c(y)}\Big(u_{ty}(-1,t;y) - \frac{c'(y)}{c(y)}u_t(-1,t;y)\Big) \\ &= \frac{1}{c(y)}\Big(\partial_{ty}u_L(t;y) - \frac{c'(y)}{c(y)}\partial_t u_L(t;y)\Big). \end{aligned} \tag{2.20}$$

Similarly, for $c(y)>0$,

$$u_{xy}(1,t;y) = \frac{1}{c(y)}\Big(\partial_{ty}u_R(t;y) - \frac{c'(y)}{c(y)}\partial_t u_R(t;y)\Big). \tag{2.21}$$

It follows from the governing equation (1.3) that

$$\partial_t(u_{xy}^2) = 2c'(y)u_{xx}u_{xy} + c(y)\partial_x(u_{xy}^2), \quad x\in D, \ t>0,$$

which leads to

$$\begin{aligned} &\partial_t \int_D \rho(y)u_{xy}^2 dx \\ =& \int_D 2\rho(y)c'(y)u_{xx}u_{xy}dx + \rho(y)c(y)\Big[u_{xy}^2(1,t;y) - u_{xy}^2(-1,t;y)\Big] \\ \le& \int_D 2\rho(y)c'(y)u_{xx}u_{xy}dx + \begin{cases} \rho(y)c(y)u_{xy}^2(1,t;y) & \text{if } c(y)>0, \\ -\rho(y)c(y)u_{xy}^2(-1,t;y) & \text{if } c(y)<0. \end{cases} \end{aligned} \tag{2.22}$$

The above result, together with (2.7), (2.20) and (2.21), yields

$$\begin{aligned} &\frac{d}{dt}\int_\Gamma\int_D \rho(y)u_{xy}^2 dxdy \\ \le& C\int_\Gamma\int_D \rho(y)u_{xx}^2 dxdy + C\int_\Gamma\int_D \rho(y)u_{xy}^2 dxdy \\ &+ \int_{\Gamma^+}\frac{\rho(y)}{c(y)}\Big(\partial_{ty}u_L(t;y) - \frac{c'(y)}{c(y)}\partial_t u_L(t;y)\Big)^2 dy \\ &+ \int_{\Gamma^-}\frac{\rho(y)}{|c(y)|}\Big(\partial_{ty}u_R(t;y) - \frac{c'(y)}{c(y)}\partial_t u_R(t;y)\Big)^2 dy. \end{aligned} \tag{2.23}$$

The desired estimate (2.19) is obtained by integrating the above inequality with respect to t and by using the assumption (2.18). □

Theorem 2.2. *Consider the problem* (1.3)-(1.5)*. Assume that* (2.7) *holds, i.e., $c'(y)$ is bounded in the distribution sense in Γ. Moreover, we assume that $c''(y)$ is bounded in the distribution sense:*

$$|c''(y)| \le C, \quad \textit{almost everywhere in } \Gamma. \tag{2.24}$$

If the assumptions (2.1), (2.12) *and* (2.18) *hold and furthermore if*

$$\int_\Gamma\int_D \rho(y)(\partial_{yy}u_0(x;y))^2 dxdy<\infty, \tag{2.25a}$$

$$\int_0^T\int_{\Gamma^+}\frac{\rho(y)}{c(y)}\Big(\partial_{yy}u_R(t;y)\Big)^2 dydt<\infty, \tag{2.25b}$$

$$\int_0^T\int_{\Gamma^-}\frac{\rho(y)}{|c(y)|}\Big(\partial_{yy}u_L(t;y)\Big)^2 dydt<\infty, \tag{2.25c}$$

then

$$\int_\Gamma\int_D \rho(y)u_{yy}^2 dxdy<C(T),\quad 0<t\le T, \tag{2.26}$$

where $C(T)$ is a finite number depending on T.

Proof. Differentiating both sides of (1.3) with respect to y twice gives

$$(u_{yy})_t=c''(y)u_x+2c'(y)u_{xy}+c(y)(u_{yy})_x,$$

which yields

$$(u_{yy}^2)_t=4c''(y)u_xu_{yy}+4c'(y)u_{xy}u_{yy}+2c(y)(u_{yy}^2)_x. \tag{2.27}$$

Integrating the above equation with respect to x leads to

$$\begin{aligned}\partial_t\int_D\rho(y)u_y^2dx\le{}&4\rho(y)c''(y)\int_D u_xu_{yy}dx+4\rho(y)c'(y)\int_D u_{xy}u_{yy}dx\\&+2\begin{cases}\frac{\rho(y)}{c(y)}(\partial_{yy}u_R)^2(t;y) & \text{if } c(y)\ge 0,\\ -\frac{\rho(y)}{c(y)}(\partial_{yy}u_L)^2(t;y) & \text{if } c(y)<0,\end{cases}\end{aligned} \tag{2.28}$$

which yields

$$\begin{aligned}&\frac{d}{dt}\int_\Gamma\int_D\rho(y)u_{yy}^2dxdy\\ \le{}& C\int_\Gamma\int_D\rho(y)u_x^2dxdy+C\int_\Gamma\int_D\rho(y)u_{yy}^2dxdy+C\int_\Gamma\int_D\rho(y)u_{xy}^2dxdy\\&+\int_{\Gamma^+}\frac{\rho(y)}{c(y)}\Big(\partial_{yy}u_R(t;y)\Big)^2dy+\int_{\Gamma^-}\frac{\rho(y)}{|c(y)|}\Big(\partial_{yy}u_L(t;y)\Big)^2dy,\end{aligned} \tag{2.29}$$

where the boundedness assumption of $c'(y)$ and $c''(y)$ is used. The desired estimate (2.26) follows from Lemmas 2.1-2.3, the Gronwall inequality and the assumption (2.25). □

Remark 2.1. It is clear that if the boundary data and the initial data satisfy some further assumptions, then the solution of the problem (1.3)-(1.5) should have higher regularity. A more detailed set of conditions can be found following the above procedures, which will be omitted in this paper.

2.3 Regularity in BV space

Using a similar trick used in the proof of Lemma 2.1, we can obtain the following result.

Lemma 2.4. *Consider the problem* (1.3)-(1.5). *If the following conditions are satisfied:*

$$\int_\Gamma\int_D \rho(y)|\partial_x u_0(x;y)|dxdy<+\infty, \tag{2.30a}$$

$$\int_0^T\int_{\Gamma^+}\frac{\rho(y)}{c(y)}|\partial_t u_R(t;y)|dydt<+\infty, \tag{2.30b}$$

$$\int_0^T\int_{\Gamma^-}\frac{\rho(y)}{|c(y)|}|\partial_t u_L(t;y)|dydt<+\infty, \tag{2.30c}$$

then we have

$$\int_\Gamma\int_D \rho(y)|\partial_x u(x,t;y)|dxdy<C(T),\quad 0<t\le T, \tag{2.31}$$

where $C(T)$ is a constant dependent on T.

Theorem 2.3. *Consider the problem* (1.3)-(1.5). *Assume that the assumption* (2.7) *holds, i.e., $c'(y)$ is bounded in the distribution sense. If the assumption* (2.30) *holds and furthermore if*

$$\int_\Gamma\int_D \rho(y)|\partial_y u_0(x;y)|dxdy<+\infty, \tag{2.32a}$$

$$\int_0^T\int_{\Gamma^+}\frac{\rho(y)}{|c(y)|}|\partial_y u_R(t;y)|dydt<+\infty, \tag{2.32b}$$

$$\int_0^T\int_{\Gamma^-}\frac{\rho(y)}{|c(y)|}|\partial_y u_L(t;y)|dydt<+\infty, \tag{2.32c}$$

then we have

$$\int_\Gamma\int_D \rho(y)|u_y(x,t;y)|dxdy<C(T),\quad 0<t\le T, \tag{2.33}$$

where $C(T)$ is a finite number depending on T.

Proof. Differentiating both sides of (1.3) with respect to y and multiplying the resulting equation by $\mathrm{sgn}(u_y)$ yield

$$|u_y|_t=c'(y)u_x\mathrm{sgn}(u_y)+c(y)|u_y|_x, \tag{2.34}$$

where $\mathrm{sgn}(u_y)$ gives the sign of u_y. Integrating the above equation with respect to x leads to

$$\begin{aligned}
&\partial_t\int_D\rho(y)|u_y|dx\\
=&\rho(y)c'(y)\int_D u_x\mathrm{sgn}(u_y)dx+\rho(y)c(y)\Big[|u_y(1,t;y)|-|u_y(-1,t;y)|\Big]\\
\le&\rho(y)c'(y)\int_D u_x\mathrm{sgn}(u_y)dx+\begin{cases}\frac{\rho(y)}{|c(y)|}|\partial_y u_R(t;y)| & \text{if } c(y)\ge 0,\\ \frac{\rho(y)}{|c(y)|}|\partial_y u_L(t;y)| & \text{if } c(y)<0,\end{cases}
\end{aligned}\tag{2.35}$$

which yields

$$\frac{d}{dt}\int_\Gamma\int_D \rho(y)|u_y|dxdy \le C\int_\Gamma\int_D \rho(y)|u_x|dxdy \\ +\int_{\Gamma^+}\frac{\rho(y)}{|c(y)|}|\partial_y u_R(t;y)|dy+\int_{\Gamma^-}\frac{\rho(y)}{|c(y)|}|\partial_y u_L(t;y)|dy, \tag{2.36}$$

where the boundedness assumption of $c'(y)$ is used. The desired estimate (2.33) follows from Lemma 2.4, the Gronwall inequality and the assumption (2.32). ☐

3 Convergence of the collocation method

Given a function f, its expectation is defined by

$$E[f]=\int_\Gamma\int_D \rho(y)f(x,y)dydx; \tag{3.1}$$

and its mean square is defined by

$$M[f]=\left(\int_\Gamma\int_D \rho(y)f(x,y)^2dydx\right)^{1/2}. \tag{3.2}$$

Lemma 3.1. ([2], p. 289. Estimates for the interpolation error.) *Assume a given function $w(y)$ satisfies $w^{(m)}\in L^2(-1,1)$ and denote I_Nw its interpolation polynomial associated with the $(N+1)$-point Gauss, or Gauss-Radau, or Gauss-Lobatto points $\{y_j\}_{j=0}^N$, namely,*

$$I_Nw(y)=\sum_{i=0}^N w(y_i)F_i(y). \tag{3.3}$$

Then for $m\le N$ the following estimate holds

$$\|w-I_Nw\|_{L^2(D)}\le CN^{-m}\|w^{(m)}\|_{L^2(-1,1)}. \tag{3.4}$$

Theorem 3.1. *Let u be the solution of (1.3)-(1.5) and u^N be the stochastic collocation solution of (1.9). If the assumptions in Theorem 2.1 are satisfied, then the following estimates on the mean-square and mean errors hold:*

$$e_{ms}(u-u^N):=M[u-u^N]\le C(T)N^{-1},\quad 0<t\le T, \tag{3.5}$$
$$e_{mean}(u-u^N):=E[|u-u^N|]\le C(T)N^{-1},\quad 0<t\le T, \tag{3.6}$$

where $C(T)$ is a constant depending on T but independent of N.

Proof. For any fixed x, it follows from Lemma 3.1 that

$$\int_\Gamma \rho(y)\left(u(x,t;y)-u^N(x,t;y)\right)^2 dy \le CN^{-2}\int_\Gamma \rho(y)u_y^2 dy. \tag{3.7}$$

Integrating the above inequality with respect to x and using Theorem 2.1 yield the desired estimate (3.5). As for (3.6), it follows from a standard inequality $\|w\|_{L^1}\le C\|w\|_{L^2}$. □

If the given data satisfies higher smoothness properties, higher rate of convergence can be obtained. More precisely, we have the following result; whose proof is similar to that of Theorem 3.1 and is therefore omitted.

Theorem 3.2. *Let u be the solution of* (1.3)-(1.5) *and u^N be the stochastic collocation solution of* (1.9). *If the assumptions in Theorem* 2.2 *are satisfied, then the following estimates hold:*

$$e_{ms}(u-u^N)\le C(T)N^{-2}, \quad 0<t\le T, \tag{3.8}$$
$$e_{mean}(u-u^N)\le C(T)N^{-2}, \quad 0<t\le T, \tag{3.9}$$

where $C(T)$ is a constant depending on T but independent of N.

We now discuss the mean error for the case that the exact solution belongs to BV. Let

$$Q_N(f)=\sum_{i=1}^N w_{i,N} f(y_{i,N})$$

be the Gaussian quadrature formula associated with the weight function ρ with N notes ($w_{i,N}$ are the weights), and the reminder is defined by

$$R_N(f)=\int_{-1}^1 \rho(y) f dy - Q_N(f).$$

Lemma 3.2. ([4]) *Let f be function of total bounded variation $V(f)$ over $[-1,1]$. Then for the remainder R_n of the Gaussian formula Q_n, we have*

$$|R_N(f)|\le \frac{\pi}{2N+1}V(f). \tag{3.10}$$

Using Theorem 2.3 and the above lemma, we can obtain the following error estimate.

Theorem 3.3. *Let u be the solution of* (1.3)-(1.5) *and u^N be the stochastic collocation solution of* (1.9). *If the assumptions in Theorem* 2.3 *are satisfied, then for $0<t\le T$ we have the following estimate for the mean error:*

$$e_{mean}:=E[|u-u^N|]\le C(T)N^{-1}, \quad 0<t\le T, \tag{3.11}$$

where $C(T)$ is a constant depending on T but independent of N.

4 Numerical examples

In this section we present some numerical examples to support the theoretical results derived above. In all computations, y is a random variable uniformly distributed in Γ, and the corresponding simple points are the Legendre-Gauss points.

4.1 Example with $u \in H^1, H^2, H^3$ in the random space

Consider the following problem

$$u_t = y u_x \tag{4.1}$$

with the following three initial conditions

$$\begin{aligned} u(x,0;y) &= \sin(x) + 4\mathrm{sgn}(y)y \quad && -1 < x, y < 1; \\ u(x,0;y) &= \sin(x) + 4\mathrm{sgn}(y)y^2 \quad && -1 < x, y < 1; \\ u(x,0;y) &= \sin(x) + 4\mathrm{sgn}(y)y^3 \quad && -1 < x, y < 1, \end{aligned}$$

The corresponding boundary conditions are

$$\begin{cases} u(-1,t;y) = \sin(-1+yt) + 4y & y > 0, \\ u(1,t;y) = \sin(1+yt) - 4y & y < 0; \end{cases}$$
$$\begin{cases} u(-1,t;y) = \sin(-1+yt) + 4y^2 & y > 0, \\ u(1,t;y) = \sin(1+yt) - 4y^2 & y < 0; \end{cases}$$
$$\begin{cases} u(-1,t;y) = \sin(-1+yt) + 4y^3 & y > 0, \\ u(1,t;y) = \sin(1+yt) - 4y^3 & y < 0. \end{cases}$$

It can be checked that the exact solutions for the above three initial-boundary value problems are:

$$u(x,t;y) = \sin(x+yt) + 4\mathrm{sgn}(y)y; \tag{4.2}$$
$$u(x,t;y) = \sin(x+yt) + 4\mathrm{sgn}(y)y^2; \tag{4.3}$$
$$u(x,t;y) = \sin(x+yt) + 4\mathrm{sgn}(y)y^3, \tag{4.4}$$

which can be verified to belong to H^1, H^2, H^3 respectively. In fact, the initial conditions given above only belong to H^1, H^2, H^3 respectively.

Fig. 1 presents the mean-square and mean errors against the number of nodes. It is clear from Fig. 1 that the corresponding convergence rates for the mean-square errors are 1, 2, and 3, respectively, which agrees well with the theoretical predictions. The rate for the mean errors seems better than the theoretical predictions, which implies that the estimate may not be sharp for the mean errors.

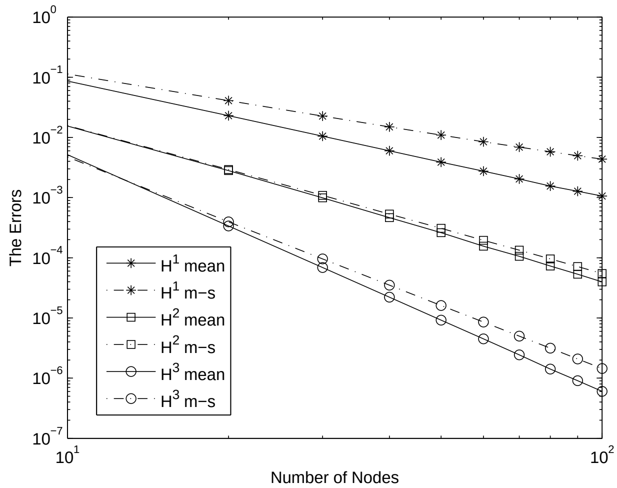

Figure 1: Example of Section 4.1: mean-square errors for solutions with different regularity.

4.2 A smooth problem

We consider the problem with a periodic boundary condition in physical space:

$$\begin{cases} u_t(x,t;y)=yu_x(x,t;y) & 0<x<2\pi,\ t>0, \\ u(x,0;y)=\cos(y) & 0<x<2\pi. \end{cases}$$

The boundary conditions of the type (1.5) are given so that the exact solution is of the form

$$u(x,t;y)=\cos(x-yt).$$

It can be verified that the exact solution belongs to $H_y^{(m)}(-1,1)$ for any given positive integer m. Consequently, it is expected that exponential rate of convergence can be obtained. In Fig. 2, we plot the mean-square error against the order N for several time levels. The exponential rate of convergence of convergence is observed. Also the errors are proportional to the increase of t, and this phenomenon is a severe problem for the polynomial chaos method. Wan and Karniadakis had a paper discussing such an issue [12].

4.3 BV solution in the random space

Consider $u_t=yu_x$ with the initial condition

$$\begin{aligned} &u(x,0;y)=\sin(x)+1 && -1<x<1, \quad y>0, \\ &u(x,0;y)=\sin(x)+1+\alpha && -1<x<1, \quad y<0, \end{aligned} \tag{4.5}$$

and the boundary condition

$$\begin{aligned} &u(1,t;y)=\sin(1+yt)+1 && y>0, \\ &u(-1,t;y)=\sin(-1+yt)+1+\alpha && y<0. \end{aligned} \tag{4.6}$$

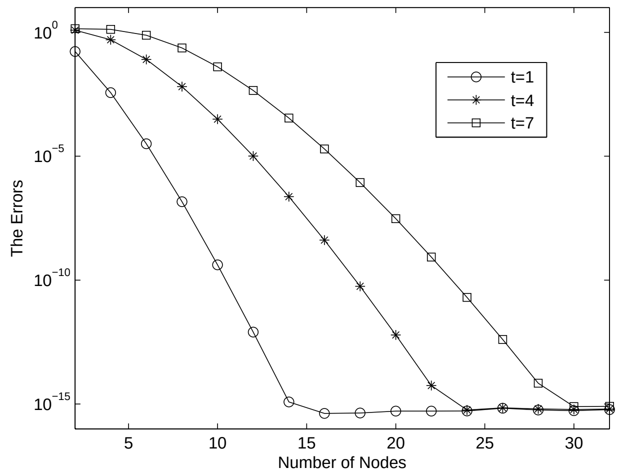

Figure 2: Example of Section 4.2: mean-square errors against the order N for a smooth solution.

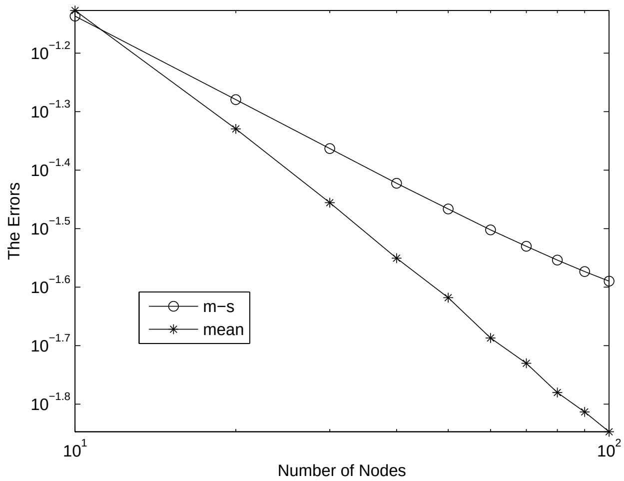

Figure 3: Example of Section 4.3 with $\alpha=0.4$: error of mean (lower curve) and mean-square (upper curve) at $t=2$. For the BV solutions, the rates for the mean error and the mean-square are 1 and 0.5, respectively.

The exact solution is of the form

$$\begin{cases} u(x,t;y)=\sin(x+yt)+1 & y>0, \\ u(x,t;y)=\sin(x+yt)+1+\alpha & y<0. \end{cases}$$

If $\alpha \neq 0$, then the solution is not in H^1 but in BV in the random space. More precisely, the initial condition (4.5) satisfies (2.32a) but not (2.8a). Then we can obtain the error estimates in Theorem 3.3 but not those in Theorem 3.1.

Fig. 3 presents both the mean-square and mean errors. It is observed that the mean

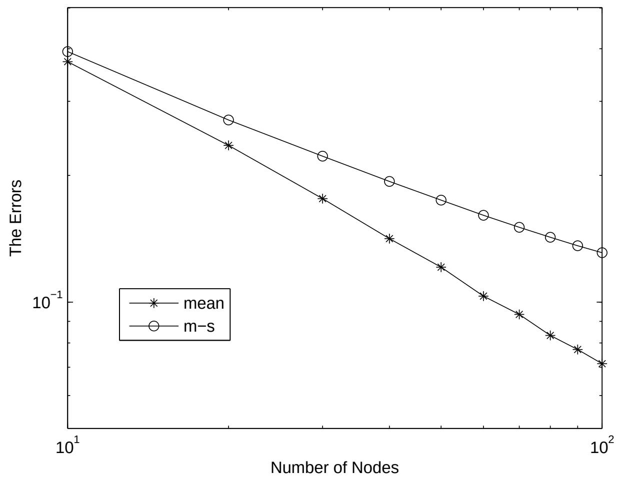

Figure 4: Example of Section 4.4: error of mean (lower curve) and mean-square (upper curve) at $t=2$. For the non-BV solutions, the rates for the mean error and the mean-square are 0.5 and 0.25, respectively.

error is of order one which verifies Theorem 3.3. However, the mean-square error is not of convergence order one; in fact the results indicate a half-order.

4.4 Solution not in BV

Consider

$$\begin{cases} u_t(x,t;y)=yu_x(x,t;y) & -1<x<1, \quad t>0, \\ u(x,0;y)=\sin(\pi x), \end{cases} \tag{4.7}$$

with the boundary condition

$$\begin{cases} u(1,t;y)=t & y>0, \\ u(-1,t;y)=-t & y<0. \end{cases}$$

It can be verified that the above problem does not satisfy (2.1b)-(2.1c); so it is expected that the order-one rate of convergence for the mean error may not be held for this simple problem. In Fig. 4, we plot the mean and mean-square errors. The rates of convergence for both mean and mean-square errors become smaller; which are about 0.5 and 0.25 respectively.

To have a better understanding of the numerical approximations for this non-BV solution, we compare numerical errors for several cases with different solution regularity. We begin with smoother solutions; Figs. 6 and 5 present the exact solution and the errors of the numerical solution for the H^1 and BV solutions. In both cases, the large errors occur at the line $y=0$. In Fig. 7, similar illustrations are made for the non-BV solutions at two different time levels. It is observed that the large numerical errors occur near two lines:

$$x+yt=1, \quad y>0, \qquad x+yt=-1, \quad y<0. \tag{4.8}$$

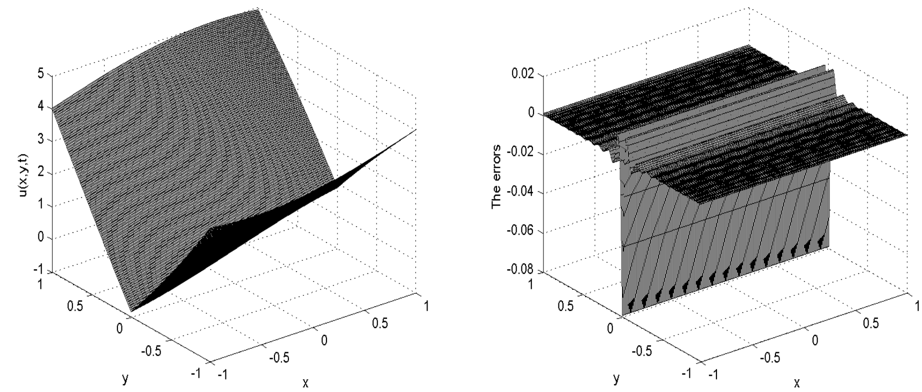

Figure 5: Example of Section 4.2 (the case that the exact solution is in H^1): the exact solution (left) and the numerical errors (right) at $t=1$ with $N=50$.

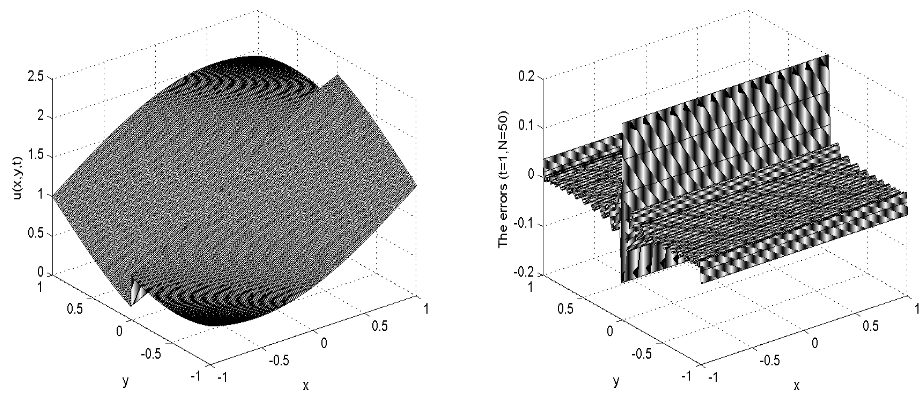

Figure 6: Example of Section 4.3 (the case that the exact solution is in BV): the exact solution (left) and the numerical errors (right) at $t=1$ with $N=50$.

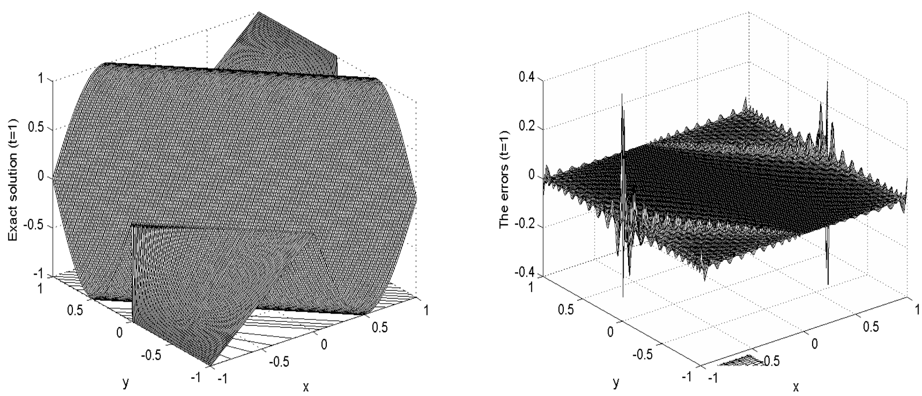

Figure 7: Example of Section 4.4 (the case that the exact solution is not in BV): the exact solution (left) and the numerical errors (right) at $t=1$ with $N=50$.

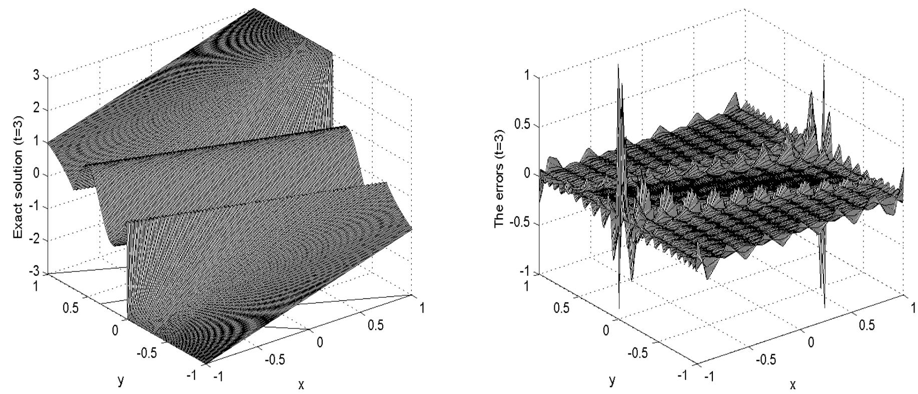

Figure 8: Same as Fig. 7, except at $t=3$.

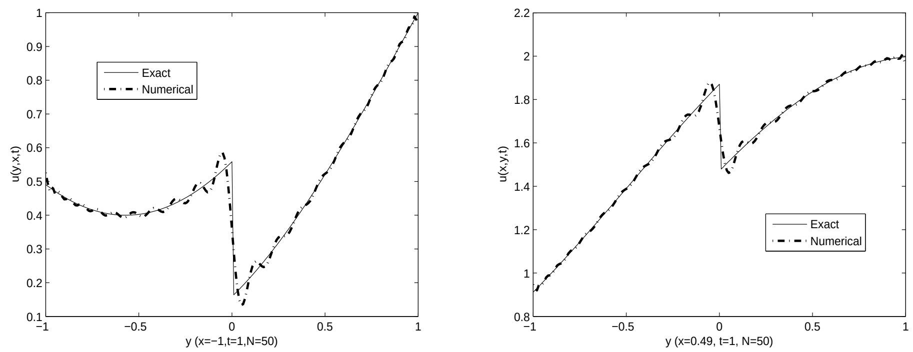

Figure 9: Example of Section 4.3 (the solution is in BV): the exact and numerical solutions at $t=1$, $x=-1$ (left) and at $t=1$, $x=0.5$ (right). $N=50$.

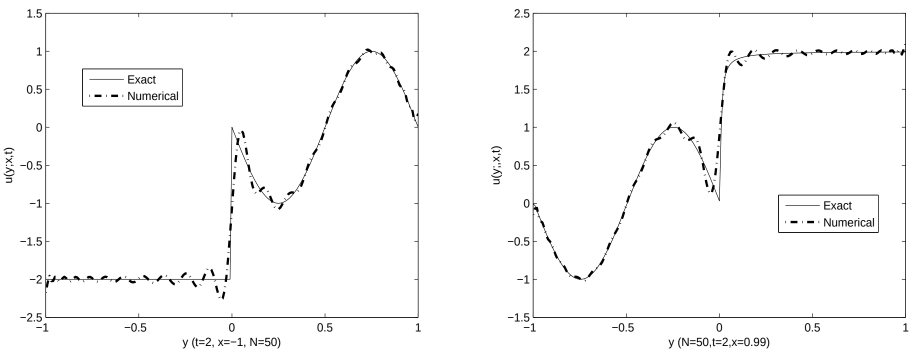

Figure 10: Example of Section 4.4 (the solution is non-BV): the exact and numerical solutions at $t=2$, $x=-1$ (left) and at $t=2$, $x=0.99$ (right). $N=50$.

The errors for some fixed x-values are presented in Figs. 9 and 10. The results again suggest that the accuracy of approximation is highly relevant to the solution regularity.

5 An accuracy enhancement technique

For examples used in the previous section, a simple analysis shows that the non-smoothness occurs because the random speed $c(y)$ changes sign. In this case, the accuracy can be enhanced by a domain decomposition idea, suggested recently by Foo et al. [3]. In this section, we will improve the accuracy for the examples used in the last section where $c(y)$ changes sign at $y=0$. The approach used can be easily extended to the general one-dimensional case when $c(y)$ changes sign more than one time.

Without lose generality, let us consider the case that $c(y)$ changes sign only once in $(-1,1)$. In this case, we can separate the set for $y([-1,1])$ into two subsets $B_1,B_2,$, and in each subset B_i we can use an affine mapping to map the collocation points into the subset. Then we can use the same collocation methodology used in the previous sections to obtain the approximate solutions in each subset separately. More precisely, for each subset, we have the local approximate solution u_i satisfying

$$u_i(x,t;y)=I^N_{B^i}u(x,y,t)=\sum_{j=1}^{r}u(x,q^i_j)L^i_j(y), \tag{5.1}$$

where the points q^i_j are mapped collocation points in B_i. Finally, we can get the approximation by

$$\tilde{u}(x,y,t)=\sum_{i=1}^{2}I_{B^i}u(x,y)\mathbf{I}_{\{y\in B^i\}}, \tag{5.2}$$

where $\mathbf{I}_{\{y\in B^i\}}$ is the conventional characteristic function.

We subsequently consider the computation of statistics and define the conditional probability density function in each element:

$$\eta^i=\frac{\rho(y)}{\int_{B^i}\rho(y)dy}. \tag{5.3}$$

The local mean of a function u is defined by

$$E^i[u(x,t;y)]=E[u(x,t;y)\,|\,y\in B_i]=\int_{B_i}u(x,t;y)\eta^i dy. \tag{5.4}$$

Using the cubature rule over each element, we can easily compute the approximate local mean of u as

$$E^i_a[\tilde{u}(x,t;y)]=\sum_{j=1}^{r}u(x,q^i_j)w_j\approx E^i_a[\tilde{u}(x,t;y)]. \tag{5.5}$$

Finally, the approximate global mean can be assembled from the local means via the Bayes' formula

$$E_a[\tilde{u}(x,t;y)]=\sum_{i=1}^{2}E^i_aP(y\in B_i)\approx E[\tilde{u}(x,t;y)], \tag{5.6}$$

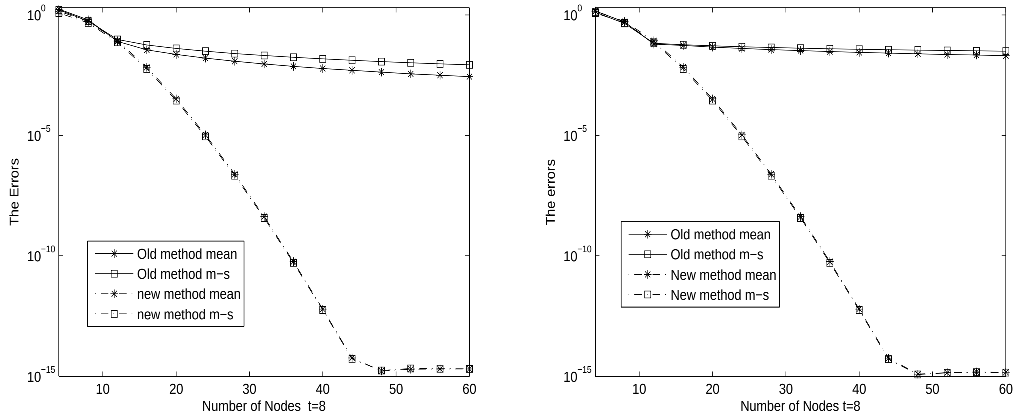

Figure 11: Convergence of the accuracy enhancement method (5.3) with the comparison with the classical approach, for the H^1-solution (left) and the BV-solution (right), at $t=8$.

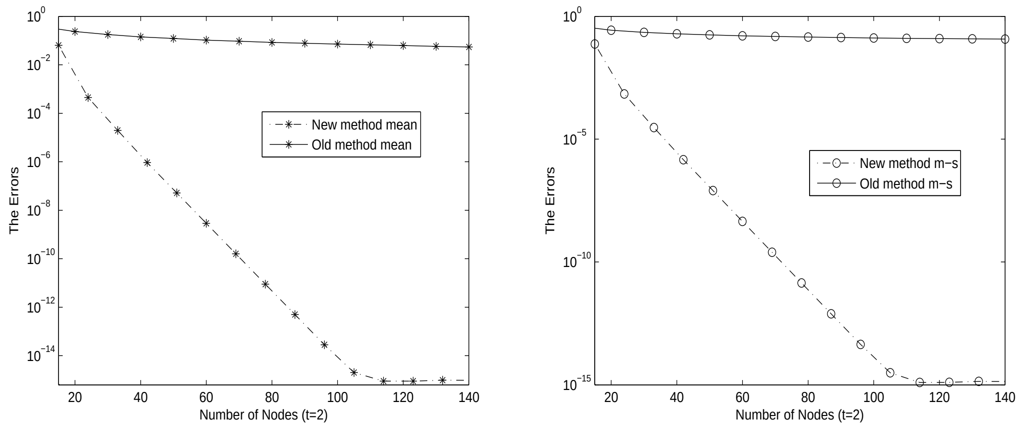

Figure 12: Convergence of the new method for the non-BV solution; left is the mean error and the right is mean-square error; for the non-BV solution in the last section at $t=2$.

and other statistics can be computed by the same procedure. We plot in Fig. 11 the mean and mean-square errors for the H^1-solution and the BV-solution given in the last section by using the new method (5.3), which are compared with those obtained by using the classical collocation approaches described in the last section. It is observed that the method (5.3) recovers the spectral accuracy.

For the non-BV case, we separate the solution domain according to the two lines (4.8) which yields three sub-domains. Then similar approach to (5.3) can be used. For comparison, the cases of Figs. 9 and 10 are re-plotted using the results obtained by the corresponding accuracy enhancement methods; indeed it is observed from Figs. 13-14 that much more accurate numerical approximations are obtained with the new strategy.

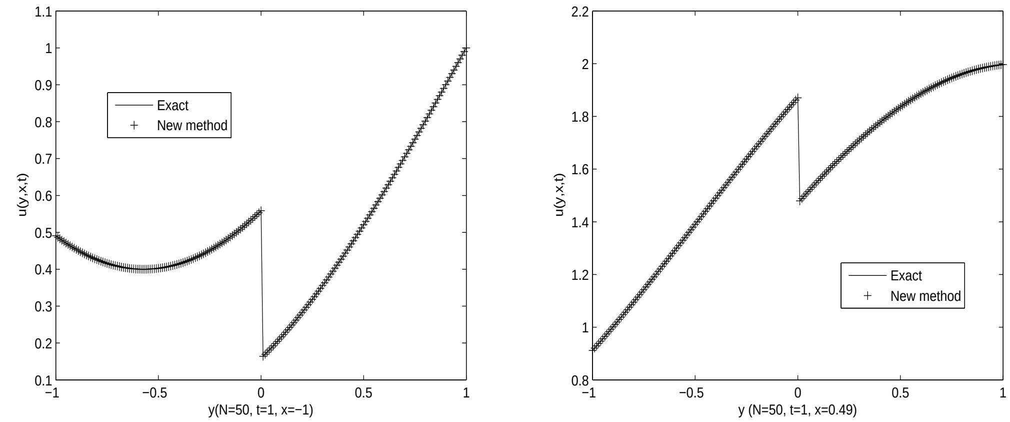

Figure 13: Same as Fig. 9, except with the accuracy enhancement method.

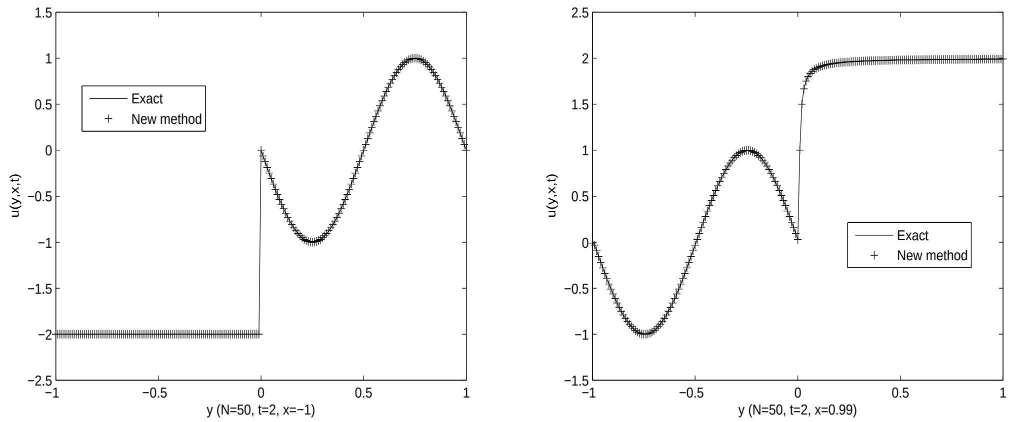

Figure 14: Same as Fig. 10, except with the accuracy enhancement method.

While we should mention here that the methods above might not be easy to apply for more complex problems.

6 Concluding remarks

There are some advantages of using stochastic collocation method, e.g., its implementation is simple and the method seems convenient to handle nonlinear or more complicated problems. Compared to stochastic Galerkin methods, the collocation methods generally result in a larger number of equations than a typical Galerkin method; however, these equations are easier to solve as they are completely decoupled and require only repetitive runs of a deterministic solver. Such a property makes the collocation methods more attractive for problems with complicated governing equations.

Stochastic methods for hyperbolic system with uncertainty are still in the early stage of development. This paper provides a preliminary investigation on the stochastic collocation method for a simple model of a scalar wave equation with random wave speed. It has been demonstrated that the rate of convergence depends not only on the initial data and boundary conditions, but also on the random wave speed.

There are some further issues arising from the analysis. For example, as demonstrated in Section 4.3, it seems that a half-rate of convergence exists for the mean-square error for BV solutions; but this is not covered in the present theoretical framework. Some theoretical analysis for the accuracy enhancement techniques proposed in Section 5 is also needed.

Acknowledgments

The authors are very grateful for Prof. Dongbin Xiu for many helpful discussions and communications. The research was supported by Hong Kong Research Grant Council CERG grants, Hong Kong Baptist University FRG grants, and the Natural Science Foundation of China (G10729101).

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