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SHORT NOTE

On the Connection Between the Spectral Difference Method and the Discontinuous Galerkin Method

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Abstract. In this short note we present a derivation of the Spectral Difference Scheme from a Discontinuous Galerkin (DG) discretization of a nonlinear conservation law. This allows interpretation of the Spectral Difference Scheme as a particular discretization under the quadrature-free nodal DG paradigm. Moreover, it enables identification of the key differences between the Spectral Difference Scheme and standard nodal DG schemes.

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1 Introduction

High-order numerical schemes that are based on locally discontinuous polynomial approximations on standard unstructured meshes are particularly attractive for nonlinear convection-dominated problems in complex geometry. The Discontinuous Galerkin (DG) method [1] is a well-known example. If accuracy requirements are moderate, however, higher order schemes are advantageous only if a particular discretization method supports efficient implementation. Local numerical volume and surface integration required by the DG approach can be considered a drawback in this regard. For nonlinear conservation laws, where integration necessitates explicit evaluation of analytical and numerical flux functions at quadrature points, numerical quadratures are quite irksome, since a generic optimal node placement for a given accuracy is not known for general mesh

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elements, in particular simplex elements. One often uses integration rules based on (singular) tensor products, which oversample the solution in order to achieve the desired order of accuracy [2].

In this context the Spectral Difference Scheme [3–5] has been proposed as a collocation-based method, using local interpolation of the strong form of the equation, with the aim to achieve superior efficiency by avoiding volume and surface quadratures altogether, while maintaining conservation. The Spectral Difference approach extends tensor-product-based collocation approaches that had previously been formulated for quadrilateral meshes [6] to more general unstructured-grid elements.

Another approach is given by "quadrature-free" DG schemes [7] that avoid numerical integration by a suitable projection of the nonlinear analytical and numerical flux functions in the interior and on the surface of mesh elements, respectively, onto finite-dimensional spaces. As a consequence all integrations involve only analytically known basis functions, which allows exact evaluation. This requires fewer degrees of freedom compared to suboptimal numerical quadrature.

In this paper we demonstrate that in fact the Spectral Difference Scheme for nonlinear hyperbolic conservation laws and a particular nodal (quadrature-free) Discontinuous Galerkin scheme are equivalent to each other under certain well-defined conditions. More specifically, the Spectral Difference Scheme is obtained from previously documented nodal DG schemes [8–10] by using the numerical flux function in the quadrature-free discretization of the volume integrals, whereas in the traditional nodal DG approach only analytical flux function evaluations are used for that purpose. Furthermore, the Spectral Difference Scheme is identified as a particulary efficient scheme among the class of quadrature-free DG schemes. It is hoped that establishing the variational formulation of the Spectral Difference Scheme may be useful for further theoretical analysis, perhaps allowing previously established results for nodal DG schemes to be re-used.

Unifying treatment for discretizations using locally discontinuous polynomial approximations has recently been advanced in a more general context by Wang et al. in their formulation of lifting collocation penalty methods [11]. The quadrature-free paradigm for DG methods used here may be the necessary ingredient of extending such a unifying formulation in a clean way to nonlinear equations.

The paper is organized as follows: we briefly and rather informally recall the definition of the Spectral Difference Scheme in the classical derivation from the strong form of the governing equations in Section 2. Subsequently we demonstrate in Section 3 the essential steps for the derivation from the variational formulation for a nonlinear one-dimensional conservation law. It remains to incorporate the more complicated metric structure in the multi-dimensional case, which is considered in Section 4.

2 The Spectral Difference Scheme

Consider the scalar hyperbolic conservation law

$$\frac{\partial u}{\partial t}+\nabla\cdot f(u)=0, \tag{2.1}$$

on some domain $\Omega\subset\mathbb{R}^d$, subject to suitable initial and boundary conditions, where $f(u)=(f^{(1)}(u),\cdots,f^{(d)}(u))^T$ is a smooth nonlinear flux function. Consider a triangulation $\mathcal{T}_h=\{T_i\}$, such that $\overline{\Omega}_h=\bigcup\overline{T_i}$. By default we consider simplex elements, while occasionally pointing out obvious extensions to other element types. Assume that there exist for each cell i a smooth invertible mapping $\Phi_i:\xi\to x$ with Jacobian $J_i=\partial x/\partial\xi$, such that each element in the triangulation can be mapped to a reference domain $\hat{T}$, and $\Phi_i(\hat{T})=T_i$. Traditionally the Spectral Difference Scheme has been derived from the strong form of the governing equations via a projection of the form

$$J_M\Big(\frac{\partial u}{\partial t}+\nabla\cdot\big(I_N f(u)\big)\Big)=0, \tag{2.2}$$

where J_M and I_N are nodal interpolation operators defined on the reference domain. The number of degrees of freedom M and N depend on the local topology, and whether multivariate or tensor-product-based interpolation is used. For simplices one may let I_N be given by multivariate interpolation using polynomials of degree $m+1$, while for quadrilateral or hexahedral elements one may use tensor products of one-dimensional interpolation. In any case, let $N=N_{m+1}$. Assuming straight sided elements, i.e., $|J_i|=$ const, the divergence is written in coordinates local to the reference domain

$$\nabla^x\cdot f=\nabla^\xi\cdot\widetilde{f}, \tag{2.3}$$

where $\widetilde{f}=J^{-1}f$. Applying the divergence to the projection $(I_N\widetilde{f})$ leads to polynomials of order m. Hence, choosing the interpolation J_M, to be exact for such polynomials, i.e., $M=N_m$, the divergence is evaluated exactly at the corresponding nodes. This leads to the scheme

$$\frac{du_{i,j}}{dt}+\sum_{k=1}^{N_{m+1}}(\nabla^\xi\hat{L}_k)\Big|_{\xi_j}\cdot\widetilde{f}_k=0,\qquad\forall T_i\in\mathcal{T}_h,\quad j=1,\cdots,N_m, \tag{2.4}$$

where $u_{i,j}$ is the nodal value at ξ_j, which belongs to the set of interpolation nodes of J_M, and $\hat{L}_j(\xi)$ are the Lagrange fundamental polynomials corresponding to the interpolation operator I_N. Coupling between mesh elements is introduced by evaluating the flux function as

$$\widetilde{f}_k=\begin{cases} J^{-1}f(u_{i,k}), & \hat{\xi}_k\in\hat{T},\\ \widetilde{f}^{num}, & \hat{\xi}_k\in\partial\hat{T},\end{cases} \tag{2.5}$$

where $\hat{\xi}_k$ belongs to the set of interpolation nodes corresponding to I_N. The coefficients $\widetilde{f}^{num}$ are chosen such that $f^{num}\cdot n=h$, where n is the outward pointing normal on ∂T, and h is a standard numerical flux function.

The scheme (2.4) with condition (2.5) is the Spectral Difference Scheme in semi-discrete form, which may be treated with a suitable time integration scheme. To verify

discrete conservation, enough nodes should be placed on the boundary of elements to support exact numerical integration of the flux interpolation polynomials. Obviously, this does not specify the node placement on the boundary uniquely. A few remarks are in order.

Remark 2.1. The condition $f^{num}\cdot n=h$ does not uniquely define the local coefficient $\widetilde{f}_k$, as only the normal flux is specified. The tangential component is also needed for the scheme, but its definition is not crucial. For instance, the tangential component may be averaged, or taken from one element only.

Remark 2.2. For points that are shared by more than one $(d-1)$-dimensional face, e.g., vertices in 2D, or edges and vertices in 3D, the condition $f^{num}\cdot n=h$ is not well-defined. As an additional constraint one often requires that it be satisfied for all faces that share the node. For simplices this never leads to an overspecified problem, since at most d faces of dimension $(d-1)$ meet anywhere on ∂T.

It will become apparent that these conditions are enough to allow one to establish equivalency with a variational formulation.

3 Variational formulation of the Spectral Difference Scheme in one dimension

Consider the scalar one-dimensional conservation law

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0, \tag{3.1}$$

where $f(u)$ is a smooth nonlinear flux function. For the following analysis boundary conditions are immaterial, so we assume either periodic boundary conditions or a pure initial-value problem, with respectively suitable initial conditions. Consider a partition of the real line into subintervals $I_i=(x_{i-1/2},x_{i+1/2})$ with midpoints $x_i=(x_{i+1/2}+x_{i-1/2})/2$ and volumes $\Delta x_i=x_{i+1/2}-x_{i-1/2}$. Without loss of generality we may assume constant cell volumes. Consider the finite-dimensional space $\mathcal{V}_h^m$ of bounded functions v for which $v|_{I_i}\in\mathcal{P}^m(I_i)$, where $\mathcal{P}^m$ is the space of polynomials of maximum degree m. The DG discretization of the variational formulation is to find $u_h\in\mathcal{V}_h^m$, such that for each cell i,

$$\int_{I_i}(u_h)_t\phi dx-\int_{I_i}f(u_h)\phi' dx+h_{i+\frac{1}{2}}\phi|_{x_{i+\frac{1}{2}}}-h_{i-\frac{1}{2}}\phi|_{x_{i-\frac{1}{2}}}=0,\quad\forall\phi\in\mathcal{P}^m(I_i), \tag{3.2}$$

where $h_{i+1/2}$ is a numerical flux function consistent with f(u) at $x_{i+1/2}$, which one usually chooses in the class of Lipschitz-continuous monotone flux functions [1] for easy incorporation of standard TVD stability theory [12].

It is convenient to formulate a Discontinuous Galerkin discretization of (3.1) in local coordinates, defined by a linear map $\Phi:\mathcal{I}=(-1,1)\to I_i$ with $\Phi(\xi)=x_i+\xi\Delta x/2$. Let

$\mathcal{S}_m=\{\xi_k;\ k=0,\cdots,m\}$ be any nodal set allowing unique Lagrangian interpolation on $\mathcal{I}$. The corresponding fundamental polynomials $l_k(\xi)$, $k=0,\cdots,m$ satisfying $l_k(\xi_j)=\delta_{kj}$ for all $\xi_j\in\mathcal{S}_m$ form a basis for $\mathcal{P}^m$. Define a projection onto this space using polynomial interpolation

$$u_h|_{I_i}:=\sum_{k=0}^{m}u_{i,k}l_k(\xi), \tag{3.3}$$

where $u_{i,k}=u(\Phi_i(\xi_k))$. (Henceforth we often omit the cell index i whenever reference to a particular cell is clear.) The Discontinuous Galerkin Scheme (3.2) may be written for each cell

$$\frac{\Delta x}{2}\sum_{k=0}^{m}\dot{u}_k\int_{-1}^{1}l_jl_kd\xi-\int_{-1}^{1}f(u_h)l_j'd\xi+h_{i+\frac{1}{2}}l_j(1)-h_{i-\frac{1}{2}}l_j(-1)=0,\quad j=0,\cdots,m. \tag{3.4}$$

Consider the quadrature-free DG method, as introduced by Atkins and Shu [7], which has the defining characteristic that the nonlinear flux $f(u_h)$ is projected onto a finite-dimensional space using polynomials of degree $m+1$. Here we use an alternate nodal set $\mathcal{Q}_{m+1}=\{\xi_k;\ k=0,\cdots,m+1\}$ with corresponding fundamental polynomials $\hat{l}_k$. We impose the additional restriction that the end points $|\xi|=1$ be included in the set, and we may suppose that we have $\xi_0=-1$, and $\xi_{m+1}=1$. An approximation for the flux function is thus written

$$f_h|_{I_i}:=\sum_{k=0}^{m+1}f_{i,k}\hat{l}_k(\xi), \tag{3.5}$$

where the crucial step is to choose the degrees of freedom f_k such that the numerical flux function is incorporated into the interpolation:

$$f_k=\begin{cases}h_{i-\frac{1}{2}}, & k=0,\\ f(u_h(\xi_k)), & 0<k<m+1,\\ h_{i+\frac{1}{2}}, & k=m+1.\end{cases} \tag{3.6}$$

Substituting (3.5) into Eq. (3.4), and integrating by parts, leads to

$$\frac{\Delta x}{2}\sum_{k=0}^{m}\dot{u}_k\int_{-1}^{1}l_jl_kd\xi+\sum_{k=0}^{m+1}f_k\int_{-1}^{1}l_j\hat{l}_k'd\xi=0,\quad j=0,\cdots,m. \tag{3.7}$$

Note that the boundary terms vanish. Define a local solution vector for the degrees of freedom in each cell by $u_i=(u_{i,0},\cdots,u_{i,m})^T$, and likewise a vector $f_i=(f_{i,0},\cdots,f_{i,m+1})^T$ for the flux coefficients (3.6). The scheme becomes

$$M\dot{u}_i+Sf_i=0,\quad \forall i, \tag{3.8}$$

where entries of the matrices M and S are defined

$$m_{jk} = \int_{-1}^{1} l_k l_j dx, \qquad j,k = 0,\cdots,m, \tag{3.9a}$$

$$s_{jk} = \frac{2}{\Delta x}\int_{-1}^{1} \hat{l}'_k l_j dx, \qquad j = 0,\cdots,m,\ k = 0,\cdots,m+1. \tag{3.9b}$$

Note that M is symmetric positive definite [13], and thus has an inverse, which may be used to define $D := M^{-1}S$. The entries of D are given by $d_{ij} = (2/\Delta x)\hat{l}'_j|_{\xi_i}$, which is easily verified by direct calculation (cf. [13]):

$$\sum_{k=0}^{m} m_{jk} d_{kn} = \frac{2}{\Delta x}\int_{-1}^{1} l_j \sum_{k=0}^{m} l_k \hat{l}'_n|_{\xi_k} d\xi = \frac{2}{\Delta x}\int_{-1}^{1} l_j \hat{l}'_n d\xi = s_{jn}, \tag{3.10}$$

where the second equality holds because $\hat{l}'_j$ is by definition a polynomial of degree m, and is hence interpolated exactly on the set $\mathcal{S}_m$. The matrix D is a standard differentiation matrix, and Eq. (3.8) can be written in terms of the degrees of freedom as

$$\dot{u}_{i,j} + \sum_{k=0}^{m+1} l'_{k|_{\xi_j}} \widetilde{f}_{i,k} = 0, \quad \forall i, \quad j = 0,\cdots,m, \tag{3.11}$$

where $\widetilde{f} = (\Phi')^{-1} f = 2/(\Delta x) f$, and the flux coefficients f_k are evaluated as in (3.6). This by definition is the 1D Spectral Difference Scheme.

4 The variational formulation of the Spectral Difference Scheme: the multidimensional case

Consider the scalar hyperbolic conservation law (2.1) on a triangulated domain Ω_h, with each mesh element mapped to a reference domain, as discussed in Section 2. We restrict ourselves to straight-sided simplex elements. Consider the finite-dimensional space

$$\mathcal{V}_h^m = \left\{ v \in L^2(\Omega_h) : v|_{T_i} \circ \Phi_i \in \mathcal{P}^m(\hat{T}) \right\},$$

where

$$\mathcal{P}^m(\hat{T}) = \operatorname{span}\left\{ \xi^\alpha :\ \xi \in \hat{T},\ \alpha_i \geq 0,\ |\alpha| \leq m \right\},$$

and $\alpha \in \mathbb{N}^d$ is a multiindex. The dimension of the space $\mathcal{P}^m$ is given by

$$N_m^{(d)} = \frac{\Pi_{k=1}^{d}(m+k)}{d!}. \tag{4.1}$$

In the following we often suppress the superscript d. As a particular basis for $\mathcal{P}^m(\hat{T})$ consider the fundamental polynomials L_j of multivariate Lagrangian interpolation, corresponding to a nodal set $\mathcal{S}_m = \{\xi_j,\ j = 1,\cdots,N_m\}$, i.e.,

$$u_h|_{T_i} := \sum_{j=1}^{N_m} u_{i,j} L_j(\xi), \tag{4.2}$$

where $u_j = u(\Phi(\xi_j))$. Since $\Phi' =: J_i = \text{const}$ for straight sided elements, this leads to the DG discretization

$$\sum_{j=1}^{N_m} \dot{u}_j \int_{\hat{T}} L_j L_k d\xi - \int_{\hat{T}} \nabla^\xi L_k \cdot \widetilde{f}(u_h) d\xi + \int_{\partial\hat{T}} \widetilde{h} L_k ds = 0, \quad k = 1, \cdots, N_m, \tag{4.3}$$

where the cell index i has been omitted.

In the multidimensional case the quadrature-free DG approximation necessitates a projection of both the flux function and the numerical flux onto a finite-dimensional space using polynomials of maximum total degree $m+1$. Here we use multivariate interpolation on a nodal set $\mathcal{Q}_{m+1} = \{\xi_j,\ j = 1, \cdots, N_{m+1}\}$ with $\hat{L}_j$ the corresponding fundamental polynomials. This leads to

$$\widetilde{f}_h|_{T_i} := \sum_{j=1}^{N_{m+1}} \widetilde{f}_j \hat{L}_j(\xi), \tag{4.4}$$

where the $\widetilde{f}_j = J^{-1}(f_j^{(1)}, \cdots, f_j^{(d)})^T$ are vector-valued coefficients to be determined shortly. Substituting this into Eq. (4.3) and integrating by parts leads to

$$\sum_{j=1}^{N_m} \dot{u}_j \int_{\hat{T}} L_j L_k d\xi + \sum_{j=1}^{N_{m+1}} \widetilde{f}_j \cdot \int_{\hat{T}} L_k \nabla^\xi \hat{L}_j d\xi + \int_{\partial\hat{T}} (\widetilde{h} - \widetilde{f}_h \cdot n^\xi) \hat{L}_k ds = 0. \tag{4.5}$$

It remains to produce a projection of the numerical flux $\widetilde{h}$ onto a finite-dimensional space. We make the assumption that the nodal set $\mathcal{Q}_{m+1}$ supports interpolation of order $m+1$ not only in the interior of an element, but that additionally on each face, the restriction of the nodal set allows interpolation of the same order. Formally stated we say

Assumption 4.1. Let ζ be a parametrization of a $(d-1)$-dimensional face $e \in \partial\hat{T}$, The restriction of the d-dimensional nodal set $\mathcal{Q}_{m+1}$ to the face e supports a unique $(d-1)$-dimensional interpolation of order $m+1$ with corresponding Lagrangian fundamental polynomials $l_k(\zeta)$. We assume that a subset of exactly $N_{m+1}^{(d-1)}$ points is located on each $e \in \partial\hat{T}$.

Interpolation nodes that satisfy Assumption 4.1 are actually quite common, including popular choices with low Lebesgue constants, such as Hesthaven's electrostatics nodes, and Fekete points [9, 14–18]. Identify the subset

$$\{\xi_{j(k)},\ k = 1, \cdots, N_{m+1}^{(d-1)}\} \subset \mathcal{Q}_{m+1}$$

located on face $e \in \partial\hat{T}$. The uniqueness of Lagrangian interpolation (cf. [8]) ensures that for the interpolation of some function g

$$\sum_{j=1}^{N_{m+1}^{(d)}} g(\xi_j) \hat{L}_j(\xi^e) = \sum_{k=1}^{N_{m+1}^{(d-1)}} g(\xi_{j(k)}) l_k(\zeta), \tag{4.6}$$

where $\xi_j \in \mathcal{Q}_{m+1}$, and ξ^e is the restriction of the d-dimensional coordinate ξ on the face e. A consequence of this fact is that if $\widetilde{f}\cdot n^\xi = \widetilde{h}$ at the interpolation nodes $\xi_{j(k)}$, the surface integral in (4.5) identically vanishes. Note that since for any scalar function ϕ, and vector-valued function v the identity

$$\int_{\partial\hat{T}} \phi(\xi)\widetilde{v}\cdot n^\xi ds^\xi = \int_{\partial T} \phi(x(\xi))v\cdot n ds \tag{4.7}$$

holds, this is equivalent with requiring $f\cdot n = h$ in untransformed space. This is actually accomplished if the degrees of freedom in (4.4) are chosen as in Eq. (2.5) in Section 2 for the Spectral Difference Scheme. With this modification in the DG scheme, the surface integral in Eq. (4.5) vanishes.

Define a local vector $u_i = (u_{i,1},\cdots,u_{i,N_m})^T$, and similarly for the flux coefficients $\widetilde{f}^{(l)} = (\widetilde{f}_1^{(l)},\cdots,\widetilde{f}_{N_{m+1}}^{(l)})^T$ for $l=1,\cdots,d$. Upon defining the matrices M with

$$m_{ij} = \int L_i L_j d\xi,$$

and $S^{(l)}$ with

$$s_{ij}^{(l)} = \int L_i \frac{\partial \hat{L}_j}{\partial \xi^{(l)}} d\xi,$$

the scheme may be written

$$M\dot{u}_i + \sum_{l=1}^{d} S^{(l)}\widetilde{f}^{(l)} = 0, \quad i=1,\cdots,N_{elem}, \tag{4.8}$$

where N_{elem} is the number of mesh elements.

As in the one-dimensional case, there holds $M^{-1}S^{(l)} = D^{(l)}$, where the entries of $D^{(l)}$are given by

$$d_{ij}^{(l)} = \frac{\partial \hat{L}_j}{\partial \xi^{(l)}}\Big|_{\xi_i},$$

which can be verified by noting that $\hat{L}_j(\xi)$ is a polynomial of degree $m+1$, and hence

$$(MD^{(l)})_{ij} = \sum_{k=1}^{N_m} \int_{\hat{T}} L_i L_k \frac{\partial \hat{L}_j}{\partial \xi^{(l)}}\Big|_{\xi_k} d\xi = \int_{\hat{T}} L_i \frac{\partial \hat{L}_j}{\partial \xi^{(l)}} d\xi = s_{ij}^{(l)}. \tag{4.9}$$

Using this identity the scheme can be written in terms of the local degrees of freedom, i.e.,

$$\frac{du_{i,j}}{dt} + \sum_{k=1}^{N_{m+1}} (\nabla^\xi \hat{L}_k)\Big|_{\xi_j}\cdot \widetilde{f}_k = 0, \quad i=1,\cdots,N_{elem},\ j=1,\cdots,N_m. \tag{4.10}$$

This recovers the Spectral Difference Scheme (2.4). Compared with the standard Spectral Difference Scheme, as outlined in Section 2, there is an additional constraint regarding the

placement of nodes on the boundary of elements (cf. Assumption 4.1). The derivation from the strong form of the equations does not include this constraint, and may thus be viewed as more general. The present derivation identifies a subset of the Spectral Difference Scheme that is equivalent to a nodal DG scheme, and perhaps also amenable to techniques for analysis established for the variational formulation.

5 Conclusions

We may identify three ingredients that allow the derivation of the Spectral Difference Scheme in from a variational DG formulation:

1. A nodal (Lagrange) basis.
2. The quadrature-free paradigm.
3. Use of the numerical flux in the quadrature-free discretization of the volume integral.

It is worth pointing out that the first two ingredients define standard nodal DG schemes [8–10] as a subset of the general variational DG formulation. The third requirement, i.e., the use of the *numerical* flux in the interpolation of the nonlinear flux function, leads to elimination of the surface integral, and hence completes the transition to a Spectral Difference Scheme. Collocating only the *analytical* flux function at all nodes retains the surface integral, which may be multiplied with a penalty term, simplifying stability analysis [8,9]. A more thorough investigation into the similarities and differences of these approaches should prove interesting. This is left for future publication.

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