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The Approximate Solutions of FPK Equations in High Dimensions for Some Nonlinear Stochastic Dynamic Systems

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Abstract. The probabilistic solutions of the nonlinear stochastic dynamic (NSD) systems with polynomial type of nonlinearity are investigated with the subspace-EPC method. The space of the state variables of large-scale nonlinear stochastic dynamic system excited by white noises is separated into two subspaces. Both sides of the Fokker-Planck-Kolmogorov (FPK) equation corresponding to the NSD system is then integrated over one of the subspaces. The FPK equation for the joint probability density function of the state variables in another subspace is formulated. Therefore, the FPK equation in low dimensions is obtained from the original FPK equation in high dimensions and it makes the problem of obtaining the probabilistic solutions of large-scale NSD systems solvable with the exponential polynomial closure method. Examples about the NSD systems with polynomial type of nonlinearity are given to show the effectiveness of the subspace-EPC method in these cases.

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Key words: Nonlinear stochastic dynamic systems, large-scale systems, probability density function, Fokker-Planck-Kolmogorov equation, subspace.

1 Introduction

FPK equations have been widely used in statistical mechanics and other areas of science and engineering since it was formulated almost one century ago [1–4]. It is known that practical problems are frequently described as multi-degree-of-freedom (MDOF) or high-dimensional systems with random excitations. Therefore, the probabilistic solutions of

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nonlinear stochastic dynamic MDOF systems are needed in many areas of science and engineering. However, only in some limited cases, the exact probabilistic solutions of two-degree-of-freedom systems are obtainable [4, 5]. It is well known that even the solutions of nonlinear single-degree-of-freedom (SDOF) systems attracted much attention in the last decades [6–24]. The challenge in analyzing the nonlinear stochastic dynamic (NSD) systems lies in the difficulties in obtaining the probability density function (PDF) of the system responses. Even if the probabilistic solution of a NSD system is governed by FPK equation, it is still difficult to obtain the exact solution if the MDOF system is nonlinear. Therefore, some methods were proposed for the approximate solutions of FPK equations. The most frequently employed approximation method is the equivalent linearization (EQL) procedure [3, 6, 7]. The advantage of the EQL method is that it can be used for analyzing large-scale NSD systems, but it is considered unsuitable when the system is highly nonlinear or multiplicative random excitations are present, because in either case the probability distribution of the system response is usually far from being Gaussian. To improve the accuracy of the approximate solution, various approximation methods were proposed, such as non-Gaussian closure method with Gram-Charlier series or Hermite-polynomial approximation [8, 9], maximum entropy method [10, 11], stochastic average method [12, 13], perturbation method [14–16], equivalent nonlinear system method [17–19], finite element method [20, 21], finite difference method [21, 22], and exponential polynomial closure (EPC) method [23–25]. It is well known that all these methods are only suitable for analyzing the one, two or at most few-degree-of-freedom systems. Monte Carlo simulation (MCS) is versatile [26, 27], but the amount of computation with it is usually unacceptable for estimating the PDF solutions of the responses of large-scale NSD systems, especially for small probability problems. The numerical convergence and stability are also challenge problems for analyzing nonlinear stochastic dynamic systems with MCS. It is seen that the problem of obtaining the PDF solutions of large-scale NSD systems has been a challenge in this area for decades. There is no effective methods available for obtaining the acceptable approximate PDF solutions of large-scale NSD systems with high nonlinearity. Recently, a new method named subspace method was proposed for the approximate PDF solutions of large-scale nonlinear stochastic dynamic systems [28]. In this paper, the subspace method is applied to analyze the nonlinear stochastic dynamic systems with polynomial types of nonlinearities to further examine the effectiveness of the subspace method in these cases. With subspace method, the problem of solving the FPK equation in high-dimensions is reduced to the problem of solving some FPK equations in low-dimensions. Thereafter, the EPC method can be employed to solve the FPK equation in low-dimensions. Hence the whole solution procedure is named subspace-EPC method. The solution procedure is presented and numerical examples are given for the systems with various system nonlinearities. The numerical results obtained with the subspace-EPC method are compared with those from MCS and EQL to show the effectiveness of the subspace-EPC method in analyzing various highly nonlinear systems.

2 Problem formulation

In the following discussion, the summation convention applies unless stated otherwise. The random state variable or vector is denoted with capital letter and the corresponding deterministic state variable or vector is denoted with the same letter in low case.

A lot of problems in science and engineering can be described with the following stochastic dynamic system:

$$\frac{d}{dt}X_i = f_i(\mathbf{X}) + g_{ij}W_j(t), \quad i=1,2,\cdots,n_x;\ j=1,2,\cdots,m \tag{2.1}$$

in Ito's form, where $\mathbf{X}\in\Re^{n_\mathbf{x}}$; X_i $(i=1,2,\cdots,n_\mathbf{x})$, are components of the state vector process $\mathbf{X}$; $f_i(\mathbf{X}):\Re^{n_\mathbf{x}}\to\Re$. Function $f_i(\mathbf{X})$ are generally nonlinear and their functional forms are assumed to be deterministic. In this paper, we consider the case that $f_i(\mathbf{X})$ are the polynomial in the state variables $\mathbf{X}$. The excitations $W_j(t)$ are the white noises with zero mean and cross-correlation

$$E\left[W_j(t)W_k(t+\tau)\right] = S_{jk}\delta(\tau), \tag{2.2}$$

where $\delta(\tau)$ is Dirac function and S_{jk} are constants representing the cross-spectral density of W_j and W_k.

The state vector process $\mathbf{X}$ is Markovian and the PDF $p(\mathbf{x},t)$ of the Markov vector is governed by FPK equation. Without loss of generality, consider the case when the white noises are Gaussian. In this case, the stationary PDF $p(\mathbf{x})$ of the Markov vector is governed by the following reduced FPK equation [1]:

$$\frac{\partial}{\partial x_j}\left[f_j(\mathbf{x})p(\mathbf{x})\right] - \frac{1}{2}\frac{\partial^2}{\partial x_i\partial x_j}\left[G_{ij}p(\mathbf{x})\right] = 0, \tag{2.3}$$

where $\mathbf{x}$ is the deterministic state vector, $\mathbf{x}\in\Re^{n_\mathbf{x}}$, and $G_{ij} = S_{ls}g_{il}g_{js}$.

It is assumed that the solution of Eq. (2.3) fulfills the following conditions:

$$\lim_{x_i\to\pm\infty} f_i(\mathbf{x})p(\mathbf{x}) = 0 \quad \text{and} \quad \lim_{x_i\to\pm\infty}\frac{\partial p(\mathbf{x})}{\partial x_i} = 0, \quad i=1,2,\cdots,n_\mathbf{x}. \tag{2.4}$$

3 Subspace method

Separate the state vector $\mathbf{X}$ into two parts $\mathbf{X}_1\in\Re^{n_{\mathbf{x}_1}}$ and $\mathbf{X}_2\in\Re^{n_{\mathbf{x}_2}}$, i.e., $\mathbf{X}=\{\mathbf{X}_1,\mathbf{X}_2\}\in\Re^{n_\mathbf{x}} = \Re^{n_{\mathbf{x}_1}}\times\Re^{n_{\mathbf{x}_2}}$.

Denote the PDF of $\mathbf{X}_1$ as $p_1(\mathbf{x}_1)$. In order to obtain $p_1(\mathbf{x}_1)$, integrating both sides of Eq. (2.3) over $\Re^{n_{\mathbf{x}_2}}$ gives

$$\int_{\Re^{n_{\mathbf{x}_2}}}\frac{\partial}{\partial x_j}\left[f_j(\mathbf{x})p(\mathbf{x})\right]d\mathbf{x}_2 - \frac{1}{2}\int_{\Re^{n_{\mathbf{x}_2}}}\frac{\partial^2}{\partial x_i\partial x_j}\left[G_{ij}p(\mathbf{x})\right]d\mathbf{x}_2 = 0. \tag{3.1}$$

Because of Eqs. (2.4), we have

$$\int_{\Re^{n_{\mathbf{x}_2}}} \frac{\partial}{\partial x_j}\left[f_j(\mathbf{x})p(\mathbf{x})\right]d\mathbf{x}_2=0, \qquad x_j\in\Re^{n_{\mathbf{x}_2}}, \tag{3.2a}$$

$$\int_{\Re^{n_{\mathbf{x}_2}}} \frac{\partial^2}{\partial x_i\partial x_j}\left[G_{ij}p(\mathbf{x})\right]d\mathbf{x}_2=0, \qquad x_i \text{ or } x_j\in\Re^{n_{\mathbf{x}_2}}. \tag{3.2b}$$

Eq. (3.1) can then be expressed as

$$\int_{\Re^{n_{\mathbf{x}_2}}} \frac{\partial}{\partial x_j}\left[f_j(\mathbf{x})p(\mathbf{x})\right]d\mathbf{x}_2-\frac{1}{2}\int_{\Re^{n_{\mathbf{x}_2}}} \frac{\partial^2}{\partial x_i\partial x_j}\left[G_{ij}p(\mathbf{x})\right]d\mathbf{x}_2=0, \quad x_i,x_j\in\Re^{n_{\mathbf{x}_1}}, \tag{3.3}$$

which can be further expressed as

$$\frac{\partial}{\partial x_j}\left[\int_{\Re^{n_{\mathbf{x}_2}}} f_j(\mathbf{x})p(\mathbf{x})d\mathbf{x}_2\right]-\frac{1}{2}\frac{\partial^2}{\partial x_i\partial x_j}\left[\int_{\Re^{n_{\mathbf{x}_2}}} G_{ij}p(\mathbf{x})d\mathbf{x}_2\right]=0, \quad x_i,x_j\in\Re^{n_{\mathbf{x}_1}}. \tag{3.4}$$

Separate $f_j(\mathbf{x})$ into two parts as

$$f_j(\mathbf{x})=f_j^I(\mathbf{x}_1)+f_j^{II}(\mathbf{x}). \tag{3.5}$$

Substituting Eq. (3.5) into Eq. (3.4) gives

$$\frac{\partial}{\partial x_j}\left[f_j^I(\mathbf{x}_1)p_1(\mathbf{x}_1)+\int_{\Re^{n_{\mathbf{x}_2}}} f_j^{II}(\mathbf{x})p(\mathbf{x})d\mathbf{x}_2\right]-\frac{1}{2}\frac{\partial^2}{\partial x_i\partial x_j}\left[\int_{\Re^{n_{\mathbf{x}_2}}} G_{ij}p(\mathbf{x})d\mathbf{x}_2\right]=0, \quad x_i,x_j\in\Re^{n_{\mathbf{x}_1}}. \tag{3.6}$$

Denote $f_j^{II}(\mathbf{x})=\sum_k f_j^{II}(\mathbf{x}_1,\mathbf{z}_k)$, in which $\mathbf{z}_k\in\Re^{n_{\mathbf{z}_k}}\subset\Re^{n_{\mathbf{x}_2}}$. $n_{\mathbf{z}_k}$ is the number of the state variables in $\mathbf{z}_k$. Therefore, Eq. (3.6) can be expressed as

$$\frac{\partial}{\partial x_j}\left[f_j^I(\mathbf{x}_1)p_1(\mathbf{x}_1)+\sum_k\int_{\Re^{n_{\mathbf{z}_k}}} f_j^{II}(\mathbf{x}_1,\mathbf{z}_k)p_k(\mathbf{x}_1,\mathbf{z}_k)d\mathbf{z}_k\right]-\frac{1}{2}\frac{\partial^2\left[G_{ij}p_1(\mathbf{x}_1)\right]}{\partial x_i\partial x_j}=0, \quad x_i,x_j\in\Re^{n_{\mathbf{x}_1}}, \tag{3.7}$$

in which $p_k(\mathbf{x}_1,\mathbf{z}_k)$ denotes the joint PDF of $\{\mathbf{X}_1,\mathbf{Z}_k\}$. The summation convention not applies on the indexes k in Eq. (3.7) and in the following discussions.

From Eq. (3.7), it is seen that the coupling of $\mathbf{X}_1$ and $\mathbf{X}_2$ comes from $f_j^{II}(\mathbf{x}_1,\mathbf{z}_k)p_k(\mathbf{x}_1,\mathbf{z}_k)$. Express $p_k(\mathbf{x}_1,\mathbf{z}_k)$ as

$$p_k(\mathbf{x}_1,\mathbf{z}_k)=p_1(\mathbf{x}_1)q_k(\mathbf{z}_k;\mathbf{x}_1), \tag{3.8}$$

where $q_k(\mathbf{z}_k;\mathbf{x}_1)$ is the conditional PDF of $\mathbf{Z}_k$ for given $\mathbf{X}_1=\mathbf{x}_1$.

Substituting Eq. (3.8) into Eq. (3.7) gives

$$\frac{\partial}{\partial x_j}\left\{\left[f_j^I(\mathbf{x}_1)+\sum_k\int_{\Re^{n_{\mathbf{z}_k}}} f_j^{II}(\mathbf{x}_1,\mathbf{z}_k)q_k(\mathbf{z}_k;\mathbf{x}_1)d\mathbf{z}_k\right]p_1(\mathbf{x}_1)\right\}-\frac{1}{2}\frac{\partial^2\left[G_{ij}p_1(\mathbf{x}_1)\right]}{\partial x_i\partial x_j}=0, \quad x_i,x_j\in\Re^{n_{\mathbf{x}_1}}. \tag{3.9}$$

Approximately replace the conditional PDF $q_k(\mathbf{z}_k;\mathbf{x}_1)$ by that from EQL, then Eq. (3.9) is expressed as

$$\frac{\partial}{\partial x_j}\Big\{\Big[f_j^I(\mathbf{x}_1)+\sum_k\int_{\Re^{n_{\mathbf{z}_k}}} f_j^{II}(\mathbf{x}_1,\mathbf{z}_k)\overline{q}_k(\mathbf{z}_k;\mathbf{x}_1)d\mathbf{z}_k\Big]\tilde{p}_1(\mathbf{x}_1)\Big\}-\frac{1}{2}\frac{\partial^2[G_{ij}\tilde{p}_1(\mathbf{x}_1)]}{\partial x_i\partial x_j}=0,\ \ x_i,x_j\in\Re^{n_{\mathbf{x}_1}}, \tag{3.10}$$

where $\overline{q}_k(\mathbf{z}_k;\mathbf{x}_1)$ is the conditional PDF of $\mathbf{Z}_k$ from EQL for given $\mathbf{X}_1=\mathbf{x}_1$ and $\tilde{p}_1(\mathbf{x}_1)$ is the approximate PDF of $\mathbf{X_1}$. Denote

$$\tilde{f}_j(\mathbf{x}_1)=f_j^I(\mathbf{x}_1)+\sum_k\int_{\Re^{n_{\mathbf{z}_k}}} f_j^{II}(\mathbf{x}_1,\mathbf{z}_k)\overline{q}_k(\mathbf{z}_k;\mathbf{x}_1)d\mathbf{z}_k. \tag{3.11}$$

Then Eq. (3.11) can be expressed as

$$\frac{\partial}{\partial x_j}[\tilde{f}_j(\mathbf{x}_1)\tilde{p}_1(\mathbf{x}_1)]-\frac{1}{2}\frac{\partial^2}{\partial x_i\partial x_j}[G_{ij}\tilde{p}_1(\mathbf{x}_1)]=0,\quad x_i,x_j\in\Re^{n_{\mathbf{x}_1}}, \tag{3.12}$$

which is the approximate FPK equation for the joint PDF of the state variables in the sub state space $\Re^{n_{\mathbf{x}_1}}$.

If $\mathbf{X}_1$ only contains few state variables, the EPC method can be employed to solve Eq. (3.12) (see [23]). Therefore, the whole solution procedure is named Subspace-EPC method in the following discussions.

4 Examples and numerical analysis

From the above discussion, it is seen that the subspace-EPC method is not limited by the number of state variables in the system. Four systems are analyzed with the above solution procedure in the following numerical analysis. The first example is about a NSD system with 10-DOF and nonlinear terms in displacements. There are 20 state variables for this system. The second example is same as the first one except that the nonlinear terms in displacements are replaced by the nonlinear terms in velocities. The third example is also same as the first one except that the nonlinear terms are in both displacements and velocities. The fourth example is about the random vibration of a flexural beam supported by nonlinear springs and excited by dynamic force. The results obtained with the subspace-EPC method are compared with those from MCS and EQL to verify the effectiveness of the presented solution procedure in these cases. The sample size is 10^7 in MCS. In the presented figures, the results corresponding to "EPC $n=4$" are the results obtained from subspace-EPC when the polynomial order equals 4 in the EPC solution procedure, σ_{y_i} denotes the standard deviation of Y_i from EQL, and $\sigma_{\dot{y}_i}$ denotes the standard deviation of $\dot{Y}_i$ from EQL.

4.1 System with nonlinear terms in displacements

Consider the following 10-degree-of-freedom system with highly nonlinear terms in displacements.

$$\ddot{\mathbf{Y}}(t)+\mathbf{C}\dot{\mathbf{Y}}(t)+\mathbf{K}\mathbf{Y}(t)+\mathbf{H}(\mathbf{Y})=\mathbf{F}(t), \tag{4.1}$$

in which

$$\mathbf{Y}(t)=\left\{Y_1,Y_2,\cdots,Y_{10}\right\}^t,\quad \mathbf{H}(\mathbf{Y})=0.3\left\{Y_1^3,Y_2^3,\cdots,Y_{10}^3\right\}^t,\quad \mathbf{F}(t)=\left\{1,1,\cdots,1\right\}^t W(t),$$

$W(t)$ is Gaussian white noise with power spectral density being 0.04; and

$$\mathbf{K}=\begin{bmatrix} 1.50 & -0.50 & 0.25 & -0.12 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.50 & 1.40 & -0.50 & 0.25 & -0.12 & 0 & 0 & 0 & 0 & 0 \\ 0.25 & -0.50 & 1.30 & -0.50 & 0.25 & -0.12 & 0 & 0 & 0 & 0 \\ -0.12 & 0.25 & -0.50 & 1.20 & -0.50 & 0.25 & -0.12 & 0 & 0 & 0 \\ 0 & -0.12 & 0.25 & -0.50 & 1.00 & -0.50 & 0.25 & -0.12 & 0 & 0 \\ 0 & 0 & -0.12 & 0.25 & -0.50 & 1.00 & -0.50 & 0.25 & -0.12 & 0 \\ 0 & 0 & 0 & -0.12 & 0.25 & -0.50 & 1.20 & -0.50 & 0.25 & -0.12 \\ 0 & 0 & 0 & 0 & -0.12 & 0.25 & -0.50 & 1.30 & -0.50 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & -0.12 & 0.25 & -0.50 & 1.40 & -0.50 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.12 & 0.25 & -0.50 & 1.50 \end{bmatrix}, \tag{4.2a}$$

$$\mathbf{C}=\begin{bmatrix} 0.30 & -0.03 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.03 & 0.30 & -0.03 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.03 & 0.30 & -0.03 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.03 & 0.30 & -0.03 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.03 & 0.30 & -0.03 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.03 & 0.30 & -0.03 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.03 & 0.30 & -0.03 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.03 & 0.30 & -0.03 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.03 & 0.30 & -0.03 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.03 & 0.30 \end{bmatrix}. \tag{4.2b}$$

The stationary PDFs obtained with the subspace-EPC method, MCS and EQL methods are compared in order to show the effectiveness of the subspace-EPC method in analyzing the large-scale NSD systems with high nonlinearity in displacements. Because there are 10 degrees of freedom or 20 state variables, it is not possible to present all the results in this paper. Only the PDFs and logarithmic PDFs of Y_2, $\dot{Y}_2$, Y_3, $\dot{Y}_3$, Y_5, and $\dot{Y}_5$, are shown and compared in Figs. 1-3 in view that the system is symmetric. With the subspace-EPC method, the stationary PDFs $\tilde{p}_1(\mathbf{X}_{1i})$ are obtained by taking $\mathbf{X}_{1i}=\{Y_i,\dot{Y}_i\}$, $(i=2,3,5)$. It is seen that the PDFs and the tails of the PDFs of Y_i and $\dot{Y}_i$, $(i=2,3,5)$, obtained with subspace-EPC, are close to MCS while the PDFs of displacements from EQL method deviate much from simulation. Similar behavior of the PDFs and the tails of the PDFs of other state variables can also be observed without being presented here. It is also observed that the displacements of this system are much non-Gaussian while the velocities of this system are close to Gaussian in this case. If the state variables, such as the velocities in this example, are close to Gaussian, EQL method can also give the results close to simulation.

4.2 System with nonlinear terms in velocities

Consider the following 10-degree-of-freedom system with highly nonlinear terms in velocities:

$$\ddot{\mathbf{Y}}(t)+\mathbf{C}\dot{\mathbf{Y}}(t)+\mathbf{K}\mathbf{Y}(t)+\mathbf{H}(\dot{\mathbf{Y}})=\mathbf{F}(t). \tag{4.3}$$

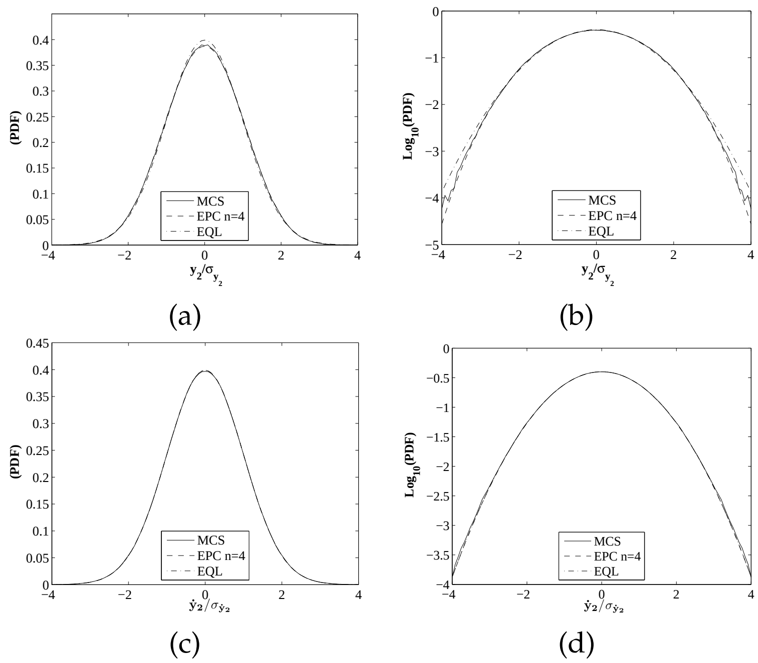

Figure 1: Comparison of the PDFs and logarithmic PDFs in example 1: (a) PDFs of displacement Y_2; (b) Logarithmic PDFs of displacement Y_2; (c) PDFs of velocity $\dot{Y}_2$; (d) Logarithmic PDFs of velocity $\dot{Y}_2$.

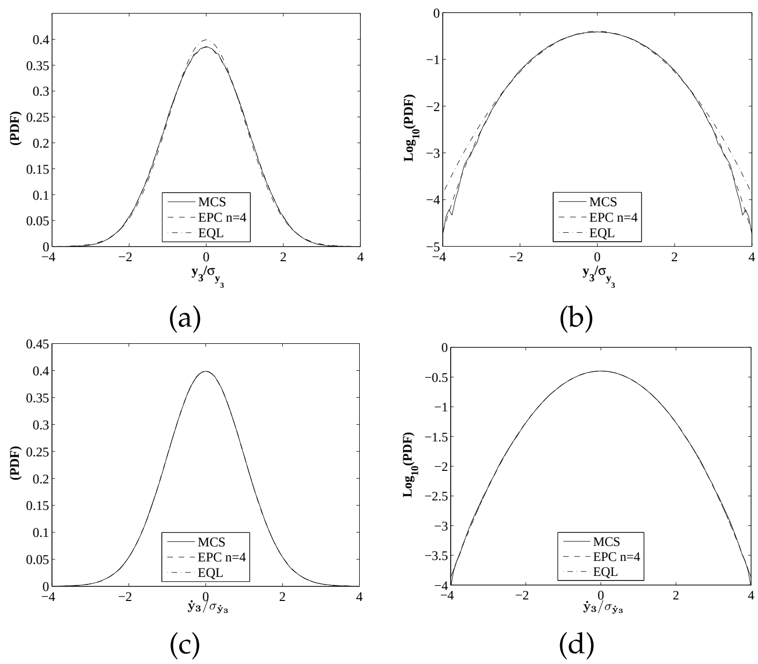

Figure 2: Comparison of the PDFs and logarithmic PDFs in example 1: (a) PDFs of displacement Y_3; (b) Logarithmic PDFs of displacement Y_3; (c) PDFs of velocity $\dot{Y}_3$; (d) Logarithmic PDFs of velocity $\dot{Y}_3$.

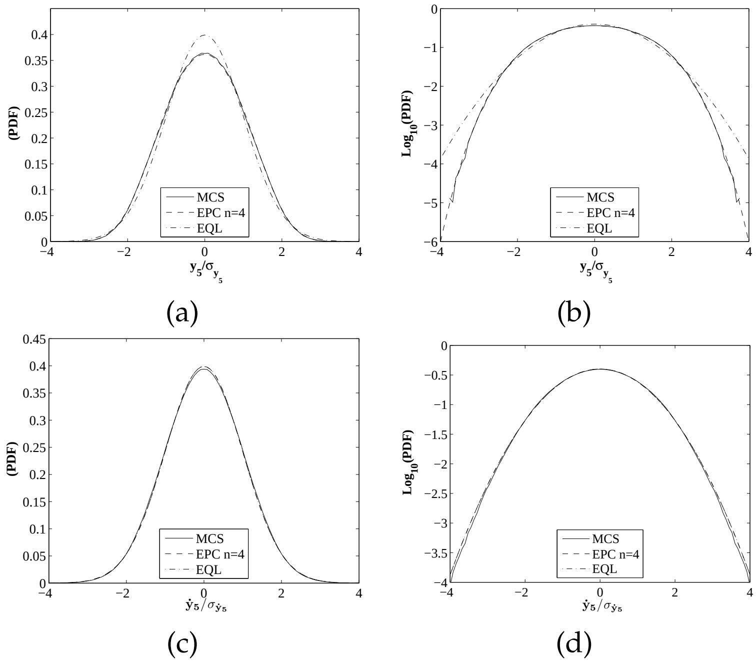

Figure 3: Comparison of the PDFs and logarithmic PDFs in example 1: (a) PDFs of displacement Y_5; (b) Logarithmic PDFs of displacement Y_5; (c) PDFs of velocity $\dot{Y}_5$; (d) Logarithmic PDFs of velocity $\dot{Y}_5$.

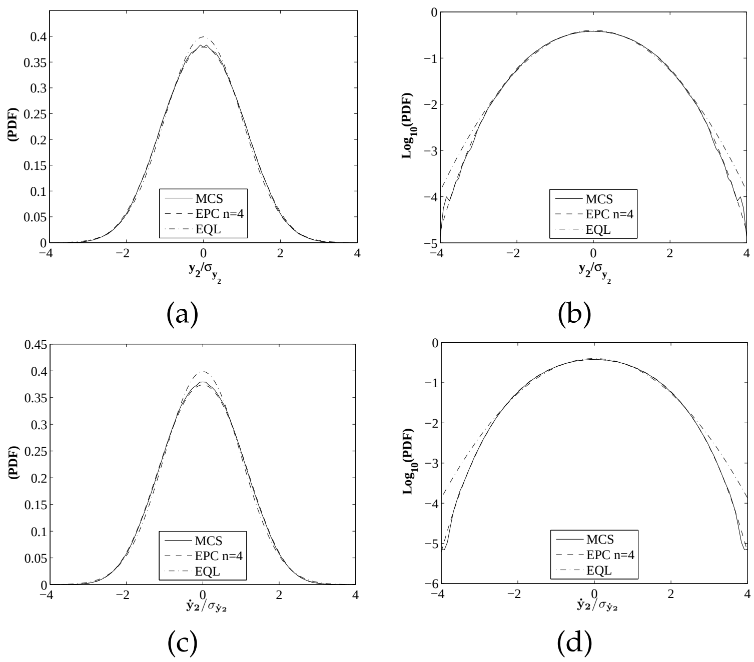

Figure 4: Comparison of the PDFs and logarithmic PDFs in example 2: (a) PDFs of displacement Y_2; (b) Logarithmic PDFs of displacement Y_2; (c) PDFs of velocity $\dot{Y}_2$; (d) Logarithmic PDFs of velocity $\dot{Y}_2$.

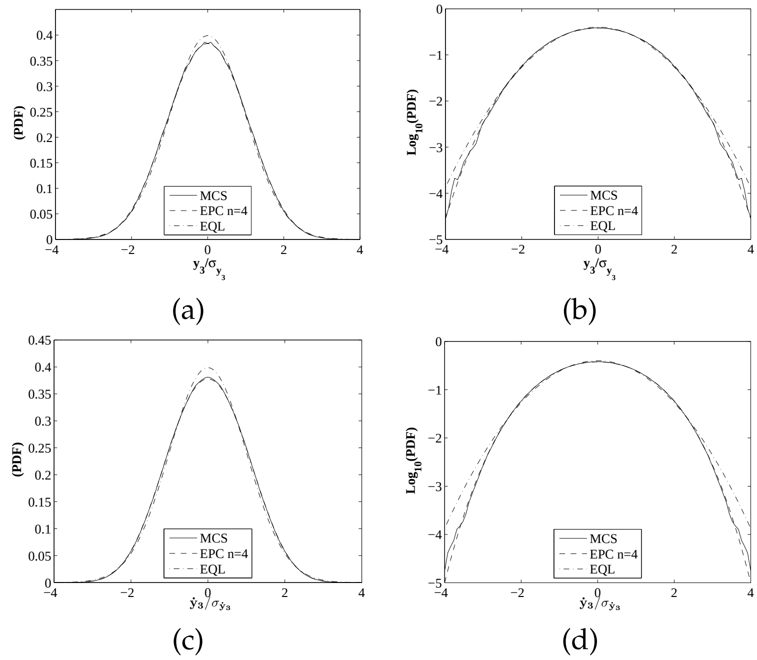

Figure 5: Comparison of the PDFs and logarithmic PDFs in example 2: (a) PDFs of displacement Y_3; (b) Logarithmic PDFs of displacement Y_3; (c) PDFs of velocity $\dot{Y}_3$; (d) Logarithmic PDFs of velocity $\dot{Y}_3$.

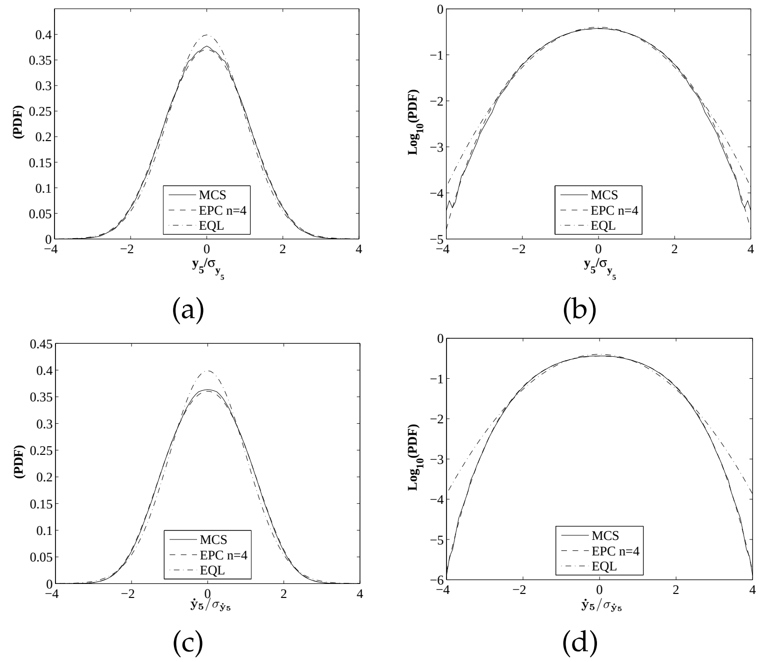

Figure 6: Comparison of the PDFs and logarithmic PDFs in example 2: (a) PDFs of displacement Y_5; (b) Logarithmic PDFs of displacement Y_5; (c) PDFs of velocity $\dot{Y}_5$; (d) Logarithmic PDFs of velocity $\dot{Y}_5$.

This system is same as that in the above example except $\mathbf{H}(\dot{\mathbf{Y}})=0.5\{\dot{Y}_1^3,\dot{Y}_2^3,\cdots,\dot{Y}_{10}^3\}^t$.

The stationary PDFs obtained with the subspace-EPC, MCS, and EQL are compared in order to further show the effectiveness of the subspace-EPC method in analyzing the large-scale NSD systems with nonlinear terms in velocities. Similar to the above example, it is not possible to present all the results here because there are too many state variables in this system. Only the PDFs and logarithmic PDFs of Y_2, $\dot{Y}_2$, Y_3, $\dot{Y}_3$, Y_5, and $\dot{Y}_5$ are shown and compared in Figs. 4-6. With the subspace-EPC method, the PDF solutions are also obtained by taking $\mathbf{X}_{1i}=\{Y_i,\dot{Y}_i\}$, $(i=2,3,5)$, in order to obtain the PDFs $\tilde{p}_1(x_{1i})$, $(i=2,3,5)$. It is seen from Figs. 4-6 that the PDFs and the tails of the PDFs of Y_2, $\dot{Y}_2$, Y_3, $\dot{Y}_3$, Y_5, and $\dot{Y}_5$, obtained from subspace-EPC when the polynomial order equals 4 in the EPC procedure are still close to MCS. On the other hand, the PDFs from EQL method deviate much from simulation. Similar behavior of the PDFs and the tails of the PDFs of other state variables can also be observed without being presented here. It is seen that the displacements of this system are close to Gaussian though not being Gaussian while the velocities of this system are much non-Gaussian.

4.3 System with nonlinear terms in both displacements and velocities

Consider the following 10-degree-of-freedom system with highly nonlinear terms in both displacements and velocities:

$$\ddot{\mathbf{Y}}(t)+\mathbf{C}\dot{\mathbf{Y}}(t)+\mathbf{K}\mathbf{Y}(t)+\mathbf{H}(\mathbf{Y},\dot{\mathbf{Y}})=\mathbf{F}(t). \tag{4.4}$$

This system is same as that in the above example except $\mathbf{H}(\mathbf{Y},\dot{\mathbf{Y}})=0.3\{Y_1^3,Y_2^3,\cdots,Y_{10}^3\}^t+0.5\{\dot{Y}_1^3,\dot{Y}_2^3,\cdots,\dot{Y}_{10}^3\}^t$.

The stationary PDFs obtained with the subspace-EPC, MCS, and EQL are compared in order to further show the effectiveness of the subspace-EPC method in analyzing the large-scale NSD systems with the nonlinear terms in both displacements and velocities. Similar to the above examples, it is not possible to present all the results here. Only the PDFs and logarithmic PDFs of Y_2, $\dot{Y}_2$, Y_3, $\dot{Y}_3$, Y_5, and $\dot{Y}_5$ are shown and compared in Figs. 7-9. With the subspace-EPC method, the PDF solutions are also obtained by taking $\mathbf{X}_{1i}=\{Y_i,\dot{Y}_i\}$, $(i=2,3,5)$, in order to obtain the PDFs $\tilde{p}_1(x_{1i})$, $(i=2,3,5)$. It is seen from Figs. 7-9 that the PDFs and the tails of the PDFs of Y_2, $\dot{Y}_2$, Y_3, $\dot{Y}_3$, Y_5, and $\dot{Y}_5$, obtained from subspace-EPC method when the polynomial order equals 4 in the EPC procedure are still close to MCS. On the other hand, the PDFs from EQL method deviate much from simulation. Similar behavior of the PDFs and the tails of the PDFs of other state variables can also be observed without being presented here. It is also observed that both the displacements and velocities of the system are much non-Gaussian.

4.4 Nonlinear random vibration of a flexural beam

Consider a flexural steel beam with pin support at one end and roller support at another, supported by nonlinear springs, and excited by uniformly distributed load being Gaus-

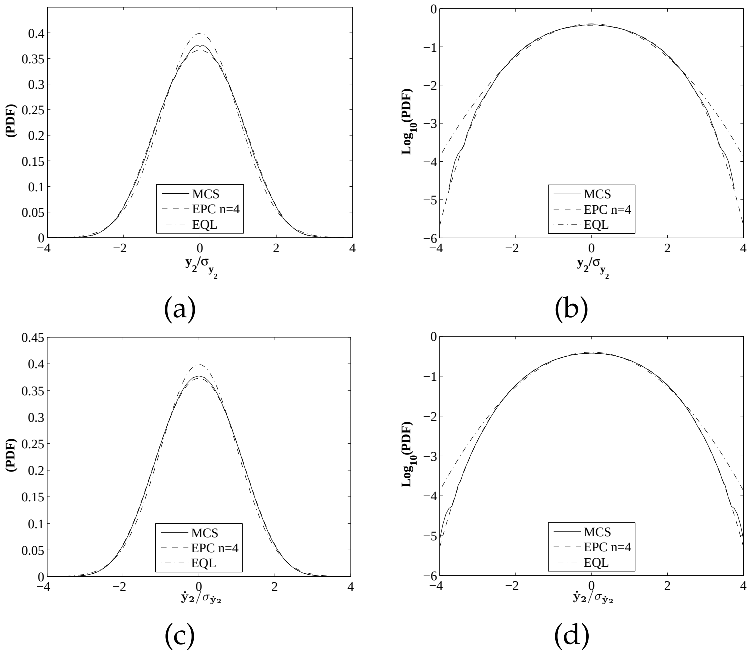

Figure 7: Comparison of the PDFs and logarithmic PDFs in example 3: (a) PDFs of displacement Y_2; (b) Logarithmic PDFs of displacement Y_2; (c) PDFs of velocity $\dot{Y}_2$; (d) Logarithmic PDFs of velocity $\dot{Y}_2$.

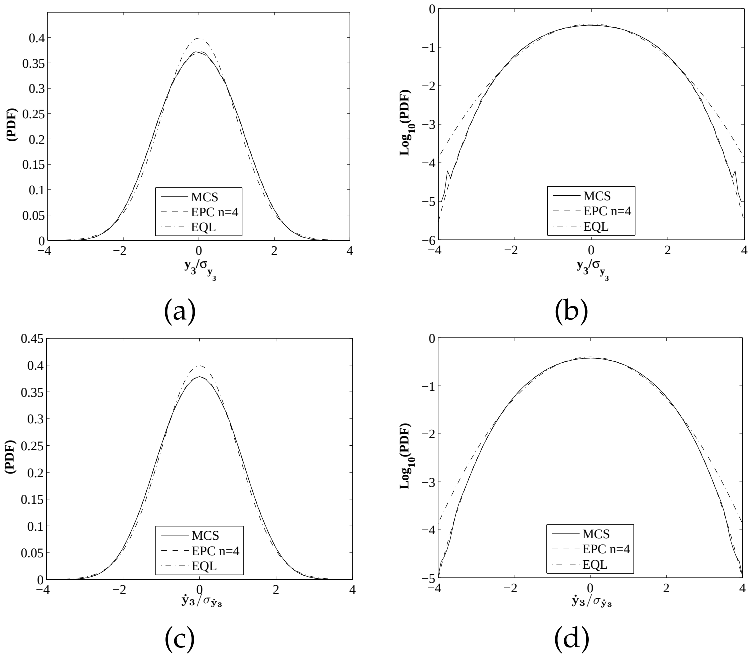

Figure 8: Comparison of the PDFs and logarithmic PDFs in example 3: (a) PDFs of displacement Y_3; (b) Logarithmic PDFs of displacement Y_3; (c) PDFs of velocity $\dot{Y}_3$; (d) Logarithmic PDFs of velocity $\dot{Y}_3$.

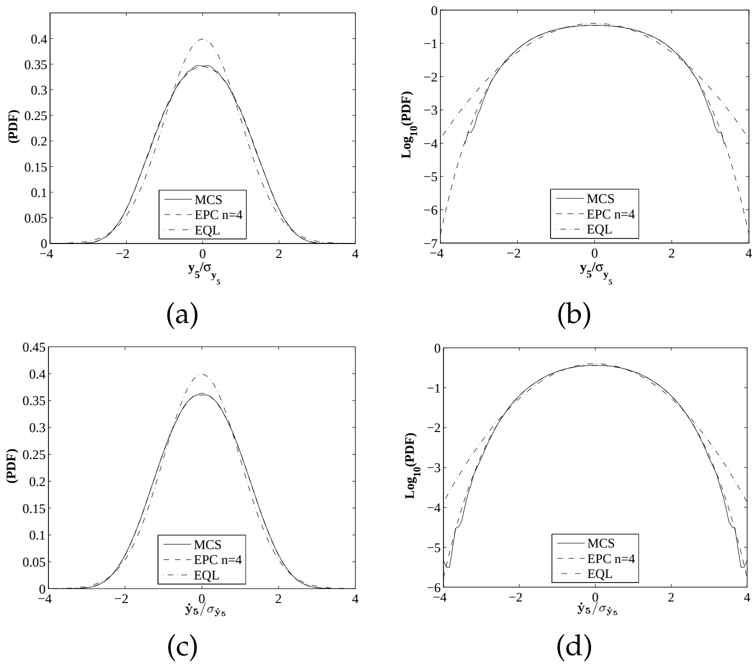

Figure 9: Comparison of the PDFs and logarithmic PDFs in example 3: (a) PDFs of displacement Y_5; (b) Logarithmic PDFs of displacement Y_5; (c) PDFs of velocity $\dot{Y}_5$; (d) Logarithmic PDFs of velocity $\dot{Y}_5$.

sian white noise. The beam is shown in Fig. 10 and the mechanical model for vibrational analysis of the beam is shown in Fig. 11.

The vertical displacement of mass m_i is denoted as Y_i. The beam length is 5m, The young's modulus is 2.1×10^{11}Pa, The area of the cross section of the beam is $8.61\times10^{-3}\text{m}^2$, the moment inertia of the cross section is $2.17\times10^{-4}\text{m}^4$, The mass density of the beam

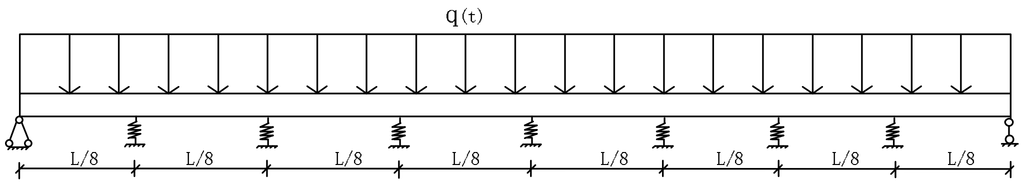

Figure 10: Flexural beam supported by nonlinear springs.

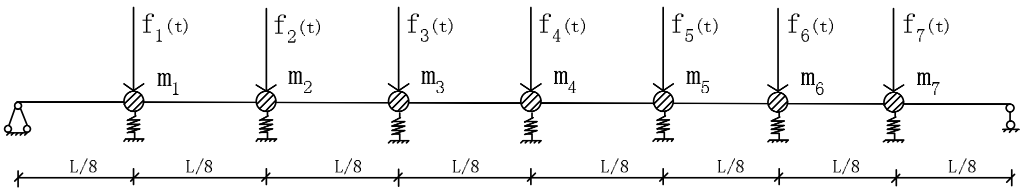

Figure 11: Model for dynamic analysis of the flexural beam.

material is 7850kg/m^2, and the uniformly distributed load $q(t)=5\times 10^3 W(t)$kg/m where $W(t)$ is Gaussian white noise with unit power spectral density. Then the governing equations for the vertical displacements can be obtained with flexibility method as follows:

$$\ddot{\mathbf{Y}}(t)+\mathbf{C}\dot{\mathbf{Y}}(t)+\mathbf{K}\mathbf{Y}+\mathbf{H}(\mathbf{Y})=\mathbf{F}(t), \tag{4.5}$$

in which $\mathbf{Y}(t)=\{Y_1,Y_2,\cdots,Y_7\}^t$; $\mathbf{H}(\mathbf{Y})=2\times 10^6\{Y_1^3,Y_2^3,\cdots,Y_7^3\}^t$; $c_{ii}=1000$ $(i=1,2,\cdots,7)$ and $c_{ij}=0$ $(i\neq j;\ i,j=1,2,\cdots,7)$ in $\mathbf{C}$; and

$$\mathbf{K}=\begin{bmatrix} 4363.7 & -4200.7 & 1835.8 & -491.90 & 131.76 & -35.136 & 8.7839 \\ -4200.7 & 61995. & -4692.6 & 1967.6 & -527.04 & 140.54 & -35.136 \\ 1835.8 & -4692.6 & 6331.3 & -4727.7 & 1976.4 & -527.04 & 131.76 \\ -491.90 & 1967.6 & -4727.7 & 6340.1 & -4727.7 & 1967.6 & -491.90 \\ 131.76 & -527.04 & 1976.4 & -4727.7 & 6331.3 & -4692.6 & 1835.8 \\ -35.136 & 140.54 & -527.04 & 1967.6 & -4692.6 & 6199.5 & -4200.7 \\ 8.7839 & -35.136 & 131.76 & -491.90 & 1835.8 & -4200.7 & 4363.7 \end{bmatrix}\times 10^4, \tag{4.6a}$$

$$\mathbf{F}(\mathbf{t})=\begin{bmatrix} 449.45 \\ 770.49 \\ 963.11 \\ 1027.3 \\ 963.11 \\ 770.49 \\ 449.45 \end{bmatrix}\times W(t). \tag{4.6b}$$

This system is analyzed with the above subspace-EPC procedure, EQL, and Monte Carlo simulation for comparison. Under the action of the excitations, the maximum vertical displacement of the beam is Y_4 which is concerned in structural design. The PDF and the logarithmic PDFs of Y_4 are shown in Figs. 12(a) and (b), respectively. It is observed that the PDFs and the tails of the PDFs of Y_4 obtained from subspace-EPC when the polynomial order equals 4 in the EPC procedure are also close to MCS for this system, specially in the tails of the PDFs as shown by the logarithmic PDFs. On the other hand, the PDFs and the tails of the PDFs obtained from EQL deviate much from simulation. In practice, the mean up-crossing rate (MCR) of the structural response is frequently used for structural reliability analysis. The MCR at $Y_i=y_i$ is defined as

$$\nu^+(y_i)=\int_0^{+\infty}\dot{y}_i p(y_i,\dot{y}_i)d\dot{y}_i, \tag{4.7}$$

where $p(y_i,\dot{y}_i)$ denotes the joint PDF of Y_i and $\dot{Y}_i$, and $\nu^+(y_i)$ denotes the MCR at $Y_i=y_i$.

The MCRs and the logarithmic MCRs of Y_4 are shown and compared in Figs. 12(c) and (d), respectively. It is still observed that the MCRs and the tails of the MCRs of Y_4 obtained from subspace-EPC when the polynomial order equals 4 in the EPC procedure are close to MCS, specially in the tails of the MCRs as shown by the logarithmic MCRs, which is important for structural reliability analysis. On the other hand, the MCRs and the tails of the MCRs obtained from EQL deviate much from simulation.

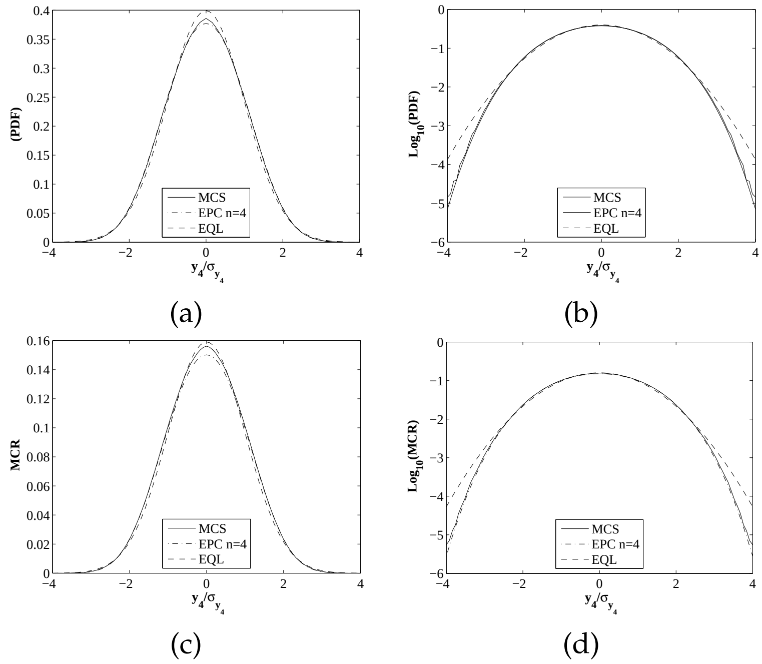

Figure 12: Comparison of PDFs, logarithmic PDFs, MCRs, and logarithmic MCRs in example 4: (a) PDFs of displacement Y_4; (b) Logarithmic PDFs of displacement Y_4; (c) MCRs at $Y_4=y_4$; (d) Logarithmic MCRs at $Y_4=y_4$.

The solution procedure and the numerical results presented in the above four examples have demonstrated that the subspace-EPC method is not limited by the number of state variables, the level of system nonlinearity, and the presence of nonlinear terms in either displacement or velocity. Because the problem of solving the FPK equation in high-dimensional state space is reduced to the problem of solving some FPK equations in low-dimensional state spaces, the required computational effort is very small compared to MCS. Hence it may provide an analytical tool that makes the probabilistic solutions of some MDOF or large-scale NSD systems obtainable accurately and efficiently.

5 Conclusions

The nonlinear stochastic dynamic systems with various system nonlinearities are analyzed with a new method named subspace-EPC method. With the subspace-EPC method, the space of state variables is split into two subspaces. The FPK equation is integrated over one of the subspaces and the FPK equation in another subspace is derived which governs the approximate joint PDF of the state variables at choices. Therefore, the problem of solving the FPK equation in high-dimensions is reduced to the problem of solving some FPK equations in low-dimensions. The EPC method can then be employed to solve the FPK equation in low-dimensions. This method is not limited by the number of state variables of the systems. Therefore it can be used for analyzing the PDF solutions of large-scale NSD systems. The effectiveness of the subspace-EPC method is investigated with the numerical results about the nonlinear systems with nonlinear terms in either displacements or velocities, in both displacements and velocities, and the nonlinear random vibration of a flexural beam. Numerical results have shown that the PDFs and logarithmic PDFs of the system responses obtained with the subspace-EPC method are close to Monte Carlo simulation even if the systems are highly nonlinear. The tails of the PDFs obtained from the subspace-EPC method also behave well, which is a challenge problem with other methods. Hence it can be concluded that the subspace-EPC method is effective in analyzing the systems with polynomial type of nonlinearity in displacements and velocities and it provides an effective tool for obtaining the probabilistic solutions of some practical NSD systems in science and engineering.

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