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Optimal L^2 Error Estimates for the Interior Penalty DG Method for Maxwell's Equations in Cold Plasma

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Abstract. In this paper, we consider an interior penalty discontinuous Galerkin (DG) method for the time-dependent Maxwell's equations in cold plasma. In Huang and Li (J. Sci. Comput., 42 (2009), 321–340), for both semi and fully discrete DG schemes, we proved error estimates which are optimal in the energy norm, but sub-optimal in the L^2-norm. Here by filling this gap, we show that these schemes are optimally convergent in the L^2-norm on quasi-uniform tetrahedral meshes if the solution is sufficiently smooth.

AMS subject classifications: 65N30, 35L15, 78-08

Key words: Maxwell's equations, cold plasma, discontinuous Galerkin method.

1 Introduction

Recently, there is a growing interest in the finite element modeling and analysis of Maxwell's equations (see books [7, 14, 21] and references cited therein). However, most work are still limited to the simple medium (such as vacuum) case. On the other hand, dispersive media (whose physical parameters are wavelength dependent) are ubiquitous. Examples include human tissue, soil, snow, ice, plasma, optical fibers and radar-absorbing materials. Hence the study of how electromagnetic waves interacting with dispersive media becomes an important subject.

Though the original discontinuous Galerkin (DG) method has been known since its introduction in 1973 by Reed and Hill, it was only recently that DG regained its popularity in solving various differential equations. It is known that the DG method offers great flexibility in the mesh construction by allowing different types of elements, non-matching grids, and even varying polynomial orders. Due to the imposition of weak continuity across element interfaces, the DG method is easy for parallel implementation.

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A detailed overview of the evolution of the DG methods from 1973 to 1999 is provided by Cockburn et al. [6]. More details and early references on DG can be found in [2,6].

Some DG methods have been developed for Maxwell's equations in the simple medium case [4, 5, 8, 9, 13, 15, 22] in the past decade. We like to remark that most of the DG methods are based on writing the Maxwell's equations in first-order hyperbolic systems; while [9,15] treated the Maxwell's equations in second order vector wave equation. Some most recent developments of DG methods for wave problems can be found in the Proceedings of Waves 2009 [3]. However, the study of DG method for Maxwell's equations in dispersive media is quite limited. In 2004, a time-domain DG method was investigated in [20] for solving the first-order Maxwell's equations in dispersive media. In 2009, we [16] initiated the analysis of the interior penalty DG method for Maxwell's equations in dispersive media. However, the error estimates obtained there is optimal in the energy norm, but sub-optimal in the L^2-norm. In this paper, by borrowing many ideas from [9,11,12,15] originally developed for the curl-curl operator, we manage to prove the optimal error estimates in the L^2-norm for both semi and fully discrete schemes. Note that our proof is slightly different from [9, 11, 12, 15] by considering that our problem is a differential-integral equation instead of the standard vector wave equation. For simplicity, we only consider the cold plasma model here, since analysis of other dispersive media models [17] can be carried out similarly.

By introducing $c_v=(\sqrt{\epsilon_0\mu_0})^{-1}$ as the speed of wave propagation in vacuum, we can rewrite the governing equation for the isotropic nonmagnetized cold electron plasma model [16, Eq. (1)] as

$$\boldsymbol{E}_{tt}+\nabla\times\left(c_v^2\nabla\times\boldsymbol{E}\right)+\omega_p^2\boldsymbol{E}-\boldsymbol{J}(\boldsymbol{E})=0, \quad \text{in } \Omega\times I, \tag{1.1}$$

where $\boldsymbol{E}$ is the electric field, ω_p is the plasma frequency, and $\boldsymbol{J}$ is the polarization current density represented as

$$\boldsymbol{J}(\boldsymbol{x},t;\boldsymbol{E})\equiv\boldsymbol{J}(\boldsymbol{E})=\nu\omega_p^2\int_0^t e^{-\nu(t-s)}\boldsymbol{E}(\boldsymbol{x},s)ds, \tag{1.2}$$

here $\nu\geq 0$ is the electron-neutral collision frequency. In c_v, ϵ_0 and μ_0 represent the permittivity and permeability in vacuum, respectively. Here $I=(0,T)$ is a finite time interval and Ω is a bounded Lipschitz polyhedron in R^3.

Moreover, we assume that the boundary of Ω is a perfect conductor so that

$$\boldsymbol{n}\times\boldsymbol{E}=0, \quad \text{on } \partial\Omega\times I, \tag{1.3}$$

where $\boldsymbol{n}$ denotes the unit outward normal of $\partial\Omega$. Furthermore, we assume that the initial conditions for (1.1) are given as

$$\boldsymbol{E}(\boldsymbol{x},0)=\boldsymbol{E}_0(\boldsymbol{x}) \quad \text{and} \quad \boldsymbol{E}_t(\boldsymbol{x},0)=\boldsymbol{E}_1(\boldsymbol{x}), \tag{1.4}$$

where $\boldsymbol{E}_0(\boldsymbol{x})$ and $\boldsymbol{E}_1(\boldsymbol{x})$ are some given functions.

The rest of the paper is organized as follows. In Section 2, we describe the semi-discrete DG formulation for the plasma model, and state the optimal error estimate in the L^2-norm. Detailed proof of the error estimate is given in Section 2.1. In Section 3, a fully discrete DG scheme is constructed, and the optimal error estimate in the L^2-norm.

In this paper, C (sometimes with subindex) denotes a generic constant which is independent of both the time step τ and the mesh size h. Here we denote $H^\alpha(\Omega)^3$ for the standard Sobolev space equipped with the norm $\|\cdot\|_{\alpha,\Omega}$. When $\alpha=0$, we just denote $\|\cdot\|_0$ for the $L^2(\Omega)^3$ norm. For a time-dependent solution $\boldsymbol{u}(\boldsymbol{x},t)$, we need the Bochner space

$$L^p\big(0,T;H^\alpha(\Omega)^3\big)=\Big\{\boldsymbol{u}:(0,T)\to H^\alpha(\Omega)^3;\ \Big(\int_0^T\|\boldsymbol{u}(\cdot,t)\|^p_{\alpha,\Omega}dt\Big)^{\frac{1}{p}}<\infty\Big\},\quad 1\le p<\infty,$$

endowed with norm

$$\|\boldsymbol{u}\|_{L^p(0,T;H^\alpha(\Omega)^3)}=\Big(\int_0^T\|\boldsymbol{u}(\cdot,t)\|^p_{\alpha,\Omega}dt\Big)^{\frac{1}{p}}.$$

When $p=\infty$, we define the space $L^\infty(0,T;H^\alpha(\Omega)^3)$ equipped with norm

$$\|\boldsymbol{u}\|_{L^\infty(0,T;H^\alpha(\Omega)^3)}=\max_{0\le t\le T}\|\boldsymbol{u}(\cdot,t)\|_{H^\alpha(\Omega)^3}.$$

2 Semi-discrete DG scheme

We consider a shape-regular mesh T_h that partitions the domain Ω into disjoint tetrahedral elements $\{K\}$, such that $\overline{\Omega}=\bigcup_{K\in T_h}K$. We denote the diameter of K by h_K, and the mesh size h by $h=\max_{K\in T_h}h_K$. Furthermore, we denote the set of all interior faces by F_h^I, the set of all boundary faces by F_h^B, and the set of all faces by $F_h=F_h^I\cup F_h^B$. We want to remark that our optimal L^2-norm error estimate is based on a duality argument and inverse estimate, hence we need to assume that the mesh be quasi-uniform and the domain Ω be convex.

We assume that the finite element space is given by

$$\boldsymbol{V}_h=\Big\{\boldsymbol{v}\in L^2(\Omega)^3:\ \boldsymbol{v}|_K\in(P_l(K))^3,\ K\in T_h\Big\},\quad l\ge 1,\tag{2.1}$$

where $P_l(K)$ denotes the space of polynomials of total degree at most l on K.

A semi-discrete DG scheme can be formed for (1.1): For any $t\in(0,T)$, find $\boldsymbol{E}^h(\cdot,t)\in\boldsymbol{V}_h$ such that

$$(\boldsymbol{E}^h_{tt},\boldsymbol{\phi})+a_h(\boldsymbol{E}^h,\boldsymbol{\phi})+\omega_p^2(\boldsymbol{E}^h,\boldsymbol{\phi})-(\boldsymbol{J}(\boldsymbol{E}^h),\boldsymbol{\phi})=0,\quad\forall\boldsymbol{\phi}\in\boldsymbol{V}_h,\tag{2.2}$$

subject to the initial conditions

$$\boldsymbol{E}^h|_{t=0}=\Pi_2\boldsymbol{E}_0,\quad \boldsymbol{E}^h_t|_{t=0}=\Pi_2\boldsymbol{E}_1,\tag{2.3}$$

where Π_2 denotes the standard L_2- projection onto $\boldsymbol{V}_h$. Moreover, the bilinear form a_h is defined on $\boldsymbol{V}_h\times\boldsymbol{V}_h$ as

$$\begin{aligned}a_h(\boldsymbol{u},\boldsymbol{v})=&\sum_{K\in T_h}\int_K c_v^2\nabla\times\boldsymbol{u}\cdot\nabla\times\boldsymbol{v}d\boldsymbol{x}-\sum_{f\in F_h}\int_f[\![\boldsymbol{u}]\!]_T\cdot\{\{c_v^2\nabla\times\boldsymbol{v}\}\}dA\\&-\sum_{f\in F_h}\int_f[\![\boldsymbol{v}]\!]_T\cdot\{\{c_v^2\nabla\times\boldsymbol{u}\}\}dA+\sum_{f\in F_h}\int_f a[\![\boldsymbol{u}]\!]_T\cdot[\![\boldsymbol{v}]\!]_T dA.\end{aligned}$$

Here $[\![\boldsymbol{v}]\!]_T$ and $\{\{\boldsymbol{v}\}\}$ are the standard notation for the tangential jumps and averages of $\boldsymbol{v}$ across an interior face $f=\partial K^+\cap\partial K^-$ between two neighboring elements K^+ and K^-:

$$[\![\boldsymbol{v}]\!]_T=\boldsymbol{n}^+\times\boldsymbol{v}^++\boldsymbol{n}^-\times\boldsymbol{v}^-,\qquad \{\{\boldsymbol{v}\}\}=\frac{(\boldsymbol{v}^++\boldsymbol{v}^-)}{2},\tag{2.4}$$

where $\boldsymbol{v}^\pm$ denote the traces of $\boldsymbol{v}$ from within $K^\pm$, and $\boldsymbol{n}^\pm$ denote the unit outward normal vectors on the boundaries $\partial K^\pm$, respectively. While on a boundary face $f=\partial K\cap\partial\Omega$, we define $[\![\boldsymbol{v}]\!]_T=\boldsymbol{n}\times\boldsymbol{v}$ and $\{\{\boldsymbol{v}\}\}=\boldsymbol{v}$. Finally, a is a penalty function, which is defined on each face $f\in F_h$ as:

$$a|_f=\gamma c_v^2\hbar^{-1},$$

where $\hbar|_f=\min\{h_{K^+},h_{K^-}\}$ for an interior face $f=\partial K^+\cap\partial K^-$ and $\hbar|_f=h_K$ for a boundary face $f=\partial K\cap\partial\Omega$. The penalty parameter γ is a positive constant and has to be chosen sufficiently large, which might negatively affect the CFL condition given in (3.6).

Furthermore, we denote the space $\boldsymbol{V}(h)=H_0(\text{curl};\Omega)+\boldsymbol{V}_h$ and define the semi-norm

$$|\boldsymbol{v}|_h^2=\sum_{K\in F_h}\|c_v\nabla\times\boldsymbol{v}\|_{0,K}^2+\sum_{f\in F_h}\left\|a^{\frac{1}{2}}[\![\boldsymbol{v}]\!]_T\right\|_{0,f}^2$$

and the DG energy norm by

$$\|\boldsymbol{v}\|_h^2=\|\omega_p\boldsymbol{v}\|_{0,\Omega}^2+|\boldsymbol{v}|_h^2.$$

In order to carry out the error analysis, we introduce an auxiliary bilinear form $\tilde{a}_h$ on $\boldsymbol{V}(h)\times\boldsymbol{V}(h)$ defined as [11,15]

$$\begin{aligned}\tilde{a}_h(\boldsymbol{u},\boldsymbol{v})=&\sum_{K\in T_h}\int_K c_v^2\nabla\times\boldsymbol{u}\cdot\nabla\times\boldsymbol{v}d\boldsymbol{x}-\sum_{f\in F_h}\int_f[\![\boldsymbol{u}]\!]_T\cdot\{\{c_v^2\Pi_2(\nabla\times\boldsymbol{v})\}\}dA\\&-\sum_{f\in F_h}\int_f[\![\boldsymbol{v}]\!]_T\cdot\{\{c_v^2\Pi_2(\nabla\times\boldsymbol{u})\}\}dA+\sum_{f\in F_h}\int_f a[\![\boldsymbol{u}]\!]_T\cdot[\![\boldsymbol{v}]\!]_T dA,\end{aligned}$$

where Π_2 is the L_2-projection onto $\boldsymbol{V}_h$. Note that $\tilde{a}_h$ equals a_h on $\boldsymbol{V}_h\times\boldsymbol{V}_h$ and is well defined on $H_0(\text{curl};\Omega)\times H_0(\text{curl};\Omega)$. To prove the error estimate for our DG scheme, we need some basic results developed for the curl-curl operator.

Lemma 2.1. (*see* [9, Lemma 5]) *For γ larger than a positive constant $\gamma_{\min}$, independent of the local mesh sizes, we have*

$$|\tilde{a}_h(\boldsymbol{u},\boldsymbol{v})| \le C_{cont}|\boldsymbol{u}|_h|\boldsymbol{v}|_h, \quad \tilde{a}_h(\boldsymbol{v},\boldsymbol{v}) \ge C_{coer}|\boldsymbol{v}|_h^2, \quad \boldsymbol{u},\boldsymbol{v}\in \boldsymbol{V}(h),$$

where $C_{cont}=\sqrt{2}$ and $C_{coer}=1/2$.

For an element K and any $\boldsymbol{u}\in(P_l(K))^3$, we have the standard inverse estimate

$$\|\nabla\times\boldsymbol{u}\|_{0,K}\le Ch_K^{-1}\|\boldsymbol{u}\|_{0,K}$$

and the trace estimate

$$\|\boldsymbol{u}\|_{0,\partial K}\le Ch_K^{-\frac{1}{2}}\|\boldsymbol{u}\|_{0,K},$$

which, along with Lemma 2.1, yields the following lemma.

Lemma 2.2. *For a quasi-uniform mesh T_h, there holds*

$$|\tilde{a}_h(\boldsymbol{u},\boldsymbol{u})|\le C_b h^{-2}\|\boldsymbol{u}\|_0^2, \quad \boldsymbol{u}\in\boldsymbol{V}_h,$$

where the constant $C_b>0$ depends on the quasi-uniformity constant of the mesh and polynomial degree l, but is independent of the mesh size h.

For the standard L^2-projection, it is known that

Lemma 2.3. *For any $\boldsymbol{v}\in H^\alpha(K)^3$, $\alpha\ge 0$, $K\in T_h$, there holds*

$$\|\boldsymbol{v}-\Pi_2\boldsymbol{v}\|_{0,K}\le Ch_K^{\min\{\alpha,l+1\}}\|\boldsymbol{v}\|_{\alpha,K}.$$

Our proof for the optimal L^2 error estimates depends on a Galerkin-type projection Π_h introduced by Grote et al. [11]. Let $\boldsymbol{u}\in H^\alpha(\Omega)^3$ with $\nabla\times\boldsymbol{u}\in H^\alpha(\Omega)^3$, for $\alpha>1/2$, then the projection $\Pi_h\boldsymbol{u}\in\boldsymbol{V}_h$ is the solution of

$$\tilde{a}_h(\Pi_h\boldsymbol{u}-\boldsymbol{u},\boldsymbol{v})+\omega_p^2(\Pi_h\boldsymbol{u}-\boldsymbol{u},\boldsymbol{v})=-r_h(\boldsymbol{u};\boldsymbol{v}), \quad \forall\boldsymbol{v}\in\boldsymbol{V}_h. \tag{2.5}$$

Here the operator r_h was introduced by Houston et al. [15] as follows: For any $\boldsymbol{v}\in\boldsymbol{V}(h)$ and any $\boldsymbol{u}$ with $\nabla\times\boldsymbol{u}\in H^\alpha(\Omega)^3$, $\alpha>1/2$,

$$r_h(\boldsymbol{u};\boldsymbol{v})=\sum_{f\in F_h}\int_f [\![\boldsymbol{v}]\!]_T\cdot\{\{\mu_0^{-1}\nabla\times\boldsymbol{u}-\mu_0^{-1}\Pi_2(\nabla\times\boldsymbol{u})\}\}dA. \tag{2.6}$$

Lemma 2.4. (*see* [11, Lemma 5.3]) *For any $\boldsymbol{u}\in H^{\alpha+\sigma_E}(\Omega)^3$ with $\nabla\times\boldsymbol{u}\in H^\alpha(\Omega)^3$ for $\alpha>1/2$, where $\sigma_E\in(1/2,1]$ is closely related to the regularity properties of the Laplacian in polyhedra* [1]. *Then we have the optimal projection error*

$$\|\boldsymbol{u}-\Pi_h\boldsymbol{u}\|_0\le Ch^{\min(\alpha,l)+\sigma_E}\big(\|\boldsymbol{u}\|_{\alpha+\sigma_E}+\|\nabla\times\boldsymbol{u}\|_\alpha\big),$$

where the constant $C>0$ is independent of the mesh size h. Furthermore, $\sigma_E=1$ when Ω is convex.

Theorem 2.1. *Let* $\boldsymbol{E}$ *and* $\boldsymbol{E}^h$ *be the solutions of* (1.1) *and* (2.2)*, respectively. Then under the following regularity assumptions*

$$\boldsymbol{E},\boldsymbol{E}_t\in L^\infty(0,T;H^{\alpha+\sigma_E}(\Omega)^3),\quad \nabla\times\boldsymbol{E},\nabla\times\boldsymbol{E}_t\in L^\infty(0,T;H^\alpha(\Omega)^3),$$

for $\alpha>1/2$*, there holds*

$$\|\boldsymbol{E}-\boldsymbol{E}^h\|_{L^\infty(0,T;L^2(\Omega)^3)}\le Ch^{\min(\alpha,l)+\sigma_E},$$

where $l\ge 1$ *is the degree of the polynomial function in the finite element space* (2.1)*, and the constant* $C>0$ *is independent of* h*.*

Remark 2.1. For smooth solutions on convex domains (so $\sigma_E=1$), Theorem 2.1 gives optimal error estimate in the L^2-norm:

$$\|\boldsymbol{E}-\boldsymbol{E}^h\|_{L^\infty(0,T;L^2(\Omega)^3)}\le Ch^{l+1}.$$

2.1 Proof of Theorem 2.1

For the analytical solution $\boldsymbol{E}$ of (1.1), using integration by parts and the fact that $[\![\boldsymbol{E}]\!]_T=0$ across all faces, it is easy to see that [9, pp. 381]:

$$\tilde{a}_h(\boldsymbol{E},\boldsymbol{v})=\big(\nabla\times(c_v^2\nabla\times\boldsymbol{E}),\boldsymbol{v}\big)+r_h(\boldsymbol{E};\boldsymbol{v}),\quad \forall\boldsymbol{v}\in\boldsymbol{V}(h).$$

Hence by (1.1), we obtain

$$(\boldsymbol{E}_{tt},\boldsymbol{v})+\tilde{a}_h(\boldsymbol{E},\boldsymbol{v})+\omega_p^2(\boldsymbol{E},\boldsymbol{v})=(\boldsymbol{J}(\boldsymbol{E}),\boldsymbol{v})+r_h(\boldsymbol{E};\boldsymbol{v}),\quad \forall\boldsymbol{v}\in\boldsymbol{V}(h). \tag{2.7}$$

Denote the error $e=\boldsymbol{E}-\boldsymbol{E}^h$. Subtracting (2.2) from (2.7), and using the fact that $\tilde{a}_h$ coincides with a_h on $\boldsymbol{V}_h\times\boldsymbol{V}_h$, we can obtain the error equation

$$(e_{tt},\boldsymbol{v})+\tilde{a}_h(e,\boldsymbol{v})+\omega_p^2(e,\boldsymbol{v})=(\boldsymbol{J}(e),\boldsymbol{v})+r_h(\boldsymbol{E};\boldsymbol{v}),\quad \forall\boldsymbol{v}\in\boldsymbol{V}_h, \tag{2.8}$$

which leads to

$$r_h(\boldsymbol{E};\boldsymbol{v})-\tilde{a}_h(\boldsymbol{E}-\boldsymbol{E}^h,\boldsymbol{v})=(e_{tt},\boldsymbol{v})+\omega_p^2(e,\boldsymbol{v})-(\boldsymbol{J}(e),\boldsymbol{v}),\quad \forall\boldsymbol{v}\in\boldsymbol{V}(h). \tag{2.9}$$

Using (2.9) and the definition of Π_h, we obtain

$$\begin{aligned}
&((\boldsymbol{E}^h-\Pi_h\boldsymbol{E})_{tt},\boldsymbol{v})+\tilde{a}_h(\boldsymbol{E}^h-\Pi_h\boldsymbol{E},\boldsymbol{v})\\
=&((\boldsymbol{E}^h-\Pi_h\boldsymbol{E})_{tt},\boldsymbol{v})+\tilde{a}_h(\boldsymbol{E}-\Pi_h\boldsymbol{E},\boldsymbol{v})-\tilde{a}_h(\boldsymbol{E}-\boldsymbol{E}^h,\boldsymbol{v})\\
=&((\boldsymbol{E}^h-\Pi_h\boldsymbol{E})_{tt},\boldsymbol{v})-\omega_p^2(\boldsymbol{E}-\Pi_h\boldsymbol{E},\boldsymbol{v})+r_h(\boldsymbol{E};\boldsymbol{v})-\tilde{a}_h(\boldsymbol{E}-\boldsymbol{E}^h,\boldsymbol{v})\\
=&((\boldsymbol{E}^h-\Pi_h\boldsymbol{E})_{tt},\boldsymbol{v})-\omega_p^2(\boldsymbol{E}-\Pi_h\boldsymbol{E},\boldsymbol{v})+(e_{tt},\boldsymbol{v})+\omega_p^2(e,\boldsymbol{v})-(\boldsymbol{J}(e),\boldsymbol{v})\\
=&((\boldsymbol{E}-\Pi_h\boldsymbol{E})_{tt},\boldsymbol{v})+\omega_p^2(\Pi_h\boldsymbol{E}-\boldsymbol{E}^h,\boldsymbol{v})-(\boldsymbol{J}(e),\boldsymbol{v}).
\end{aligned} \tag{2.10}$$

Integration of (2.10) over $[0,t]$ yields

$$\begin{aligned}&((\boldsymbol{E}^h-\Pi_h\boldsymbol{E})_t,\boldsymbol{v})-((\boldsymbol{E}^h-\Pi_h\boldsymbol{E})_t(0),\boldsymbol{v})+\int_0^t\tilde{a}_h(\boldsymbol{E}^h-\Pi_h\boldsymbol{E},\boldsymbol{v})dt\\&=((\boldsymbol{E}-\Pi_h\boldsymbol{E})_t,\boldsymbol{v})-((\boldsymbol{E}-\Pi_h\boldsymbol{E})_t(0),\boldsymbol{v})+\omega_p^2\int_0^t(\Pi_h\boldsymbol{E}-\boldsymbol{E}^h,\boldsymbol{v})dt-\int_0^t(\boldsymbol{J}(e),\boldsymbol{v})dt.\end{aligned}\tag{2.11}$$

Choosing $\boldsymbol{v}=\boldsymbol{E}^h-\Pi_h\boldsymbol{E}$ in (2.11) and using the fact that $((\boldsymbol{E}-\boldsymbol{E}^h)_t(0),\boldsymbol{v})=0$, we can rewrite (2.11) as

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{E}^h-\Pi_h\boldsymbol{E}\|_0^2+\int_0^t\tilde{a}_h(\boldsymbol{E}^h-\Pi_h\boldsymbol{E},\boldsymbol{E}^h-\Pi_h\boldsymbol{E})dt\\&=((\boldsymbol{E}-\Pi_h\boldsymbol{E})_t,\boldsymbol{E}^h-\Pi_h\boldsymbol{E})-\omega_p^2\int_0^t(\boldsymbol{E}^h-\Pi_h\boldsymbol{E},\boldsymbol{E}^h-\Pi_h\boldsymbol{E})dt-\int_0^t(\boldsymbol{J}(e),\boldsymbol{E}^h-\Pi_h\boldsymbol{E})dt.\end{aligned}$$

Integrating the previous identity over $[0,t]$, and using the facts that

$$\tilde{a}_h\big(\boldsymbol{E}^h-\Pi_h\boldsymbol{E},\boldsymbol{E}^h-\Pi_h\boldsymbol{E}\big)\ge 0,\quad \big(\boldsymbol{E}^h-\Pi_h\boldsymbol{E},\boldsymbol{E}^h-\Pi_h\boldsymbol{E}\big)\ge 0,$$

we have

$$\begin{aligned}&\|(\boldsymbol{E}^h-\Pi_h\boldsymbol{E})(t)\|_0^2-\|(\boldsymbol{E}^h-\Pi_h\boldsymbol{E})(0)\|_0^2\\&\le 2\int_0^t((\boldsymbol{E}-\Pi_h\boldsymbol{E})_t,\boldsymbol{E}^h-\Pi_h\boldsymbol{E})dt-2\int_0^t(\boldsymbol{J}(e),\boldsymbol{E}^h-\Pi_h\boldsymbol{E})dt\\&=Err_1+Err_2.\end{aligned}\tag{2.12}$$

By Cauchy-Schwarz inequality and Lemma 2.4, we have

$$\begin{aligned}Err_1\le&\int_0^t\|(\boldsymbol{E}-\Pi_h\boldsymbol{E})_t\|_0^2dt+\int_0^t\|\boldsymbol{E}^h-\Pi_h\boldsymbol{E}\|_0^2dt\\\le& Ch^{2(\min(\alpha,l)+\sigma_E)}\big(\|\boldsymbol{E}_t\|^2_{L^2(I;H^{\alpha+\sigma_E}(\Omega)^3)}+\|\nabla\times\boldsymbol{E}_t\|^2_{L^2(I;H^{\alpha}(\Omega)^3)}\big)+\int_0^t\|\boldsymbol{E}^h-\Pi_h\boldsymbol{E}\|_0^2dt.\end{aligned}\tag{2.13}$$

By the definition of $\boldsymbol{J}$, we have

$$\begin{aligned}\|\boldsymbol{J}(e(t))\|_0^2=&\nu^2\omega_p^4\int_\Omega\Big|\int_0^t e^{-\nu(t-s)}e(s)ds\Big|^2d\Omega\\\le&\nu^2\omega_p^4\int_\Omega\Big(\int_0^t e^{-2\nu(t-s)}ds\Big)\Big(\int_0^t|e(s)|^2ds\Big)d\Omega\\\le&\frac{\nu\omega_p^4}{2}\int_0^t\|e(s)\|_0^2ds,\end{aligned}$$

using which, along with the Cauchy-Schwarz inequality and Lemma 2.4, we have

$$\begin{aligned}Err_2\le&\int_0^t\|\boldsymbol{J}(e)\|_0^2dt+\int_0^t\|\boldsymbol{E}^h-\Pi_h\boldsymbol{E}\|_0^2dt\\\le&\frac{\nu\omega_p^4T}{2}\int_0^t\|\boldsymbol{E}-\Pi_h\boldsymbol{E}+\Pi_h\boldsymbol{E}-\boldsymbol{E}^h\|_0^2dt+\int_0^t\|\boldsymbol{E}^h-\Pi_h\boldsymbol{E}\|_0^2dt\end{aligned}$$

$$\begin{aligned}&\leq \nu\omega_p^4 T\cdot Ch^{2(\min(\alpha,l)+\sigma_E)}\big(\|\boldsymbol{E}\|^2_{L^2(I;H^{\alpha+\sigma_E}(\Omega)^3)}+\|\nabla\times\boldsymbol{E}\|^2_{L^2(I;H^{\alpha}(\Omega)^3)}\big)\\&\quad+(\nu\omega_p^4T+1)\int_0^t\|\boldsymbol{E}^h-\Pi_h\boldsymbol{E}\|_0^2dt. \qquad (2.14)\end{aligned}$$

Substituting (2.13) and (2.14) into (2.12) and using Gronwall inequality, we obtain

$$\|(\boldsymbol{E}^h-\Pi_h\boldsymbol{E})(t)\|_0^2\leq C\|(\boldsymbol{E}^h-\Pi_h\boldsymbol{E})(0)\|_0^2+Ch^{2(\min(\alpha,l)+\sigma_E)},$$

which, along with the triangle inequality, Lemmas 2.3 and 2.4, and the estimate

$$\begin{aligned}&\|(\boldsymbol{E}^h-\Pi_h\boldsymbol{E})(0)\|_0^2\leq 2(\|(\boldsymbol{E}^h-\boldsymbol{E})(0)\|_0^2+\|(\boldsymbol{E}-\Pi_h\boldsymbol{E})(0)\|_0^2)\\&\leq Ch^{2(\min(\alpha,l+1)}\|\boldsymbol{E}_0\|_\alpha+Ch^{2(\min(\alpha,l)+\sigma_E)}(\|\boldsymbol{E}_0\|^2_{\alpha+\sigma_E}+\|\nabla\times\boldsymbol{E}_0\|^2_\alpha), \qquad (2.15)\end{aligned}$$

which completes the proof.

2.2 Existence and uniqueness

In this subsection, we prove the existence and uniqueness of solutions for both the continuous model (1.1) and the discrete model (2.2).

First, let us consider the discrete model. Choosing $\phi=\boldsymbol{E}_t^h$ in (2.2), we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\boldsymbol{E}_t^h\|_0^2+a_h(\boldsymbol{E}^h,\boldsymbol{E}^h)+\omega_p^2\|\boldsymbol{E}^h\|_0^2)-(\boldsymbol{J}(\boldsymbol{E}^h),\boldsymbol{E}_t^h)=0.$$

Integrating the above, we have

$$\begin{aligned}&\|\boldsymbol{E}_t^h(t)\|_0^2+a_h(\boldsymbol{E}^h(t),\boldsymbol{E}^h(t))+\omega_p^2\|\boldsymbol{E}^h(t)\|_0^2\\&=\|\boldsymbol{E}_t^h(0)\|_0^2+a_h(\boldsymbol{E}^h(0),\boldsymbol{E}^h(0))+\omega_p^2\|\boldsymbol{E}^h(0)\|_0^2+2\int_0^t(\boldsymbol{J}(\boldsymbol{E}^h),\boldsymbol{E}_t^h)dt. \qquad (2.16)\end{aligned}$$

Using the Cauchy-Schwarz inequality and the definition of $\boldsymbol{J}(\boldsymbol{E})$, we have

$$\begin{aligned}2\int_0^t(\boldsymbol{J}(\boldsymbol{E}^h),\boldsymbol{E}_t^h)dt&\leq\int_0^t\|\boldsymbol{J}(\boldsymbol{E}^h(t))\|_0^2dt+\int_0^t\|\boldsymbol{E}^h(t)\|_0^2dt\\&\leq\int_0^t\frac{\nu\omega_p^4}{2}\int_0^t\|\boldsymbol{E}^h(s)\|_0^2dsdt+\int_0^t\|\boldsymbol{E}^h(t)\|_0^2dt\\&\leq\frac{\nu\omega_p^4t}{2}\int_0^t\|\boldsymbol{E}^h(t)\|_0^2dt+\int_0^t\|\boldsymbol{E}^h(t)\|_0^2dt.\end{aligned}$$

Substituting the above estimate to (2.16), then using Lemma 2.1 and the discrete Gronwall inequality, we have the following stability

$$\|\boldsymbol{E}_t^h(t)\|_0^2+|\boldsymbol{E}^h(t)|_h^2+\|\boldsymbol{E}^h(t)\|_0^2\leq\|\boldsymbol{E}_t^h(0)\|_0^2+|\boldsymbol{E}^h(0)|_h^2+\|\boldsymbol{E}^h(0)\|_0^2,$$

which implies the uniqueness of solution for (2.2). Since (2.2) is a finite dimensional linear system, the uniqueness of solution gives the existence immediately.

Now we prove the existence for (1.1). Taking the Laplace transformation of (1.1) and denoting $\tilde{\boldsymbol{E}}(s)$ as the Laplace transformation of $\boldsymbol{E}(t)$, we have

$$s^2\tilde{\boldsymbol{E}}-s\boldsymbol{E}(0)-\boldsymbol{E}_t(0)+\nabla\times(c_v^2\nabla\times\tilde{\boldsymbol{E}})+\omega_p^2\tilde{\boldsymbol{E}}-\nu\omega_p^2\frac{1}{s+\nu}\tilde{\boldsymbol{E}}=0,$$

which can be rewritten as

$$s(s^2+\nu s+\omega_p^2)\tilde{\boldsymbol{E}}+(s+\nu)\nabla\times(c_v^2\nabla\times\tilde{\boldsymbol{E}})=s(s+\nu)\boldsymbol{E}(0)+(s+\nu)\boldsymbol{E}_t(0). \tag{2.17}$$

The weak formulation of (2.17) can be formulated as: Find $\tilde{\boldsymbol{E}}\in H_0(\text{curl};\Omega)$ such that

$$s(s^2+\nu s+\omega_p^2)(\tilde{\boldsymbol{E}},\boldsymbol{\phi})+(s+\nu)(c_v^2\nabla\times\tilde{\boldsymbol{E}},\nabla\times\boldsymbol{\phi})=\big(s(s+\nu)\boldsymbol{E}(0)+(s+\nu)\boldsymbol{E}_t(0),\boldsymbol{\phi}\big) \tag{2.18}$$

holds true for any $\boldsymbol{\phi}\in H_0(\text{curl};\Omega)$. The existence of a unique weak solution $\tilde{\boldsymbol{E}}$ is guaranteed by the Lax-Milgram lemma. Taking the inverse Laplace transformation of $\tilde{\boldsymbol{E}}$ leads to the existence of a unique solution $\boldsymbol{E}$ for (1.1).

3 Fully discrete DG scheme

To define a fully discrete scheme, we divide the time interval $(0,T)$ into N uniform subintervals by points $0=t_0<t_1<\cdots<t_N=T$, where $t_k=k\tau$, and denote the k-th subinterval by $I^k=[t_k,t_{k+1}]$. Moreover, we define $\boldsymbol{u}^k=\boldsymbol{u}(\cdot,k\tau)$ for $0\le k\le N$ and denote the second-order central difference operator:

$$\delta^2\boldsymbol{u}^k=\frac{(\boldsymbol{u}^{k+1}-2\boldsymbol{u}^k+\boldsymbol{u}^{k-1})}{\tau^2}.$$

Now we can formulate a fully explicit scheme for (1.1): For any $1\le n\le N-1$, find $\boldsymbol{E}_h^{n+1}\in\boldsymbol{V}_h$ such that

$$(\delta_\tau^2\boldsymbol{E}_h^n,\boldsymbol{v})+a_h(\boldsymbol{E}_h^n,\boldsymbol{v})+\omega_p^2(\boldsymbol{E}_h^n,\boldsymbol{v})-(\boldsymbol{J}_h^n,\boldsymbol{v})=0,\quad\forall\boldsymbol{v}\in\boldsymbol{V}_h \tag{3.1}$$

subject to the initial approximation

$$\boldsymbol{E}_h^0=\Pi_2\boldsymbol{E}_0,\qquad \boldsymbol{E}_h^1=\boldsymbol{E}_h^0+\tau\Pi_2\boldsymbol{E}_1+\frac{\tau^2}{2}\Pi_2\boldsymbol{E}_{tt}(0), \tag{3.2}$$

where $\boldsymbol{E}_{tt}(0)=-[\nabla\times(c_v^2\nabla\times\boldsymbol{E}_0)+\omega_p^2\boldsymbol{E}_0]$ is obtained by setting $t=0$ in the governing equation (1.1). Furthermore, $\boldsymbol{J}_h^n$ is obtained from the following recursive formula

$$\boldsymbol{J}_h^0=0,\quad \boldsymbol{J}_h^n=e^{-\nu\tau}\boldsymbol{J}_h^{n-1}+\frac{\nu\omega_p^2}{2}\tau(e^{-\nu\tau}\boldsymbol{E}_h^{n-1}+\boldsymbol{E}_h^n),\quad n\ge1, \tag{3.3}$$

where $\boldsymbol{E}_h^n$ denotes the finite element solution of $\boldsymbol{E}$ at time $t=t_n$.

Lemma 3.1. *Let* $\boldsymbol{J}^k\equiv\boldsymbol{J}(\boldsymbol{E}(\cdot,t_k))$ *be defined by* (1.2) *and* $\boldsymbol{J}_h^k$ *be defined by* (3.3). *Then for any* $1\le k\le N$*, we have*

$$\begin{aligned}\|\boldsymbol{J}^k-\boldsymbol{J}_h^k\|_0^2\le& C\tau\sum_{j=0}^k\|\boldsymbol{E}_h^j-\Pi_h\boldsymbol{E}^j\|_0^2+Ch^{2(\min(\alpha,l)+\sigma_E)}\big(\|\boldsymbol{E}\|^2_{L^\infty(I;H^{\alpha+\sigma_E}(\Omega)^3)}\\&+\|\nabla\times\boldsymbol{E}\|^2_{L^\infty(I;H^{\alpha}(\Omega)^3)}\big)+C\tau^4\int_0^{t_k}(\|\boldsymbol{E}\|_0^2+\|\boldsymbol{E}_t\|_0^2+\|\boldsymbol{E}_{tt}\|_0^2)dt,\end{aligned}\tag{3.4}$$

where the constant $C>0$ *is independent of both* h *and* τ.

Proof. From [18, Lemma 3.3], we have

$$|\boldsymbol{J}^k-\boldsymbol{J}_h^k|\le C\tau\sum_{j=0}^k|\boldsymbol{E}_h^j-\boldsymbol{E}^j|+C\tau^2\int_0^{t_k}(|\boldsymbol{E}|+|\boldsymbol{E}_t|+|\boldsymbol{E}_{tt}|)dt,$$

which easily leads to

$$\begin{aligned}\|\boldsymbol{J}^k-\boldsymbol{J}_h^k\|_0^2\le& C\tau^2\Big(\sum_{j=0}^k1^2\Big)\sum_{j=0}^k\big(\|\boldsymbol{E}_h^j-\Pi_h\boldsymbol{E}^j\|_0^2+\|\Pi_h\boldsymbol{E}^j-\boldsymbol{E}^j\|_0^2\big)\\&+C\tau^4\Big(\int_0^{t_k}1^2dt\Big)\int_0^{t_k}\big(\|\boldsymbol{E}\|_0^2+\|\boldsymbol{E}_t\|_0^2+\|\boldsymbol{E}_{tt}\|_0^2\big)dt\\\le& CT\tau\sum_{j=0}^k\|\boldsymbol{E}_h^j-\Pi_h\boldsymbol{E}^j\|_0^2+CT^2h^{2(\min(\alpha,l)+\sigma_E)}\big(\|\boldsymbol{E}\|^2_{L^\infty(I;H^{\alpha+\sigma_E}(\Omega)^3)}\\&+\|\nabla\times\boldsymbol{E}\|^2_{L^\infty(I;H^{\alpha}(\Omega)^3)}\big)+CT\tau^4\int_0^{t_k}(\|\boldsymbol{E}\|_0^2+\|\boldsymbol{E}_t\|_0^2+\|\boldsymbol{E}_{tt}\|_0^2)dt,\end{aligned}\tag{3.5}$$

where we used the fact that $k\tau\le T$ and Lemma 2.4. $\square$

Theorem 3.1. *Let* $\boldsymbol{E}$ *and* $\boldsymbol{E}_h^n$ *be the solutions of the problem* (1.1) *and the finite element scheme* (3.1)-(3.3) *at time* t *and* t^n*, respectively. Under the CFL condition*

$$\tau<2h\Big(\sqrt{C_b+\omega_p^2h^2}\Big)^{-1},\tag{3.6}$$

where C_b *is the constant of Lemma 2.2. Furthermore, we assume that*

$$\begin{aligned}&\boldsymbol{E},\boldsymbol{E}_t\in L^\infty(0,T;H^{\alpha+\sigma_E}(\Omega)^3), &&\nabla\times\boldsymbol{E},\nabla\times\boldsymbol{E}_t\in L^\infty(0,T;H^\alpha(\Omega)^3),\\&\boldsymbol{E}_t,\boldsymbol{E}_{t^2},\boldsymbol{E}_{t^3}\in L^\infty(0,T;L^2(\Omega)^3), &&\boldsymbol{E}_{t^4}\in L^2(0,T;L^2(\Omega)^3).\end{aligned}$$

Then there is a constant $C>0$*, independent of both the time step* τ *and mesh size* h*, such that*

$$\max_{1\le n\le N}\|\boldsymbol{E}_h^n-\boldsymbol{E}^n\|_0\le C(\tau^2+h^{\min(\alpha,l)+\sigma_E}),\quad l\ge1.$$

Remark 3.1. For smooth solutions on convex domain, we have the optimal L^2 error estimate:

$$\max_{1\le n\le N}\|\boldsymbol{E}_h^n-\boldsymbol{E}^n\|_0\le C(\tau^2+h^{l+1}),\quad l\ge1.$$

3.1 Proof of Theorem 3.1

Denote the error

$$e^n = \boldsymbol{E}^n - \boldsymbol{E}_h^n = (\boldsymbol{E}^n - \Pi_h \boldsymbol{E}^n) + (\Pi_h \boldsymbol{E}^n - \boldsymbol{E}_h^n) = \eta^n + \phi^n,$$

where $\Pi_h \boldsymbol{E}$ is the Galerkin projection of $\boldsymbol{E}$ defined by (2.5).

Using the equivalence of a_h and $\tilde{a}_h$ on $\boldsymbol{V}_h \times \boldsymbol{V}_h$, and subtracting (3.1) from (2.7) at $t=t_n$, we obtain

$$\begin{aligned}&(\boldsymbol{E}_{tt}^n - \delta^2 \Pi_h \boldsymbol{E}^n + \delta^2 \Pi_h \boldsymbol{E}^n - \delta^2 \boldsymbol{E}_h^n, \boldsymbol{v}) + \tilde{a}_h(\boldsymbol{E}^n - \Pi_h \boldsymbol{E}^n + \Pi_h \boldsymbol{E}^n - \boldsymbol{E}_h^n, \boldsymbol{v}) \\ &\quad + \omega_p^2 (\boldsymbol{E}^n - \Pi_h \boldsymbol{E}^n + \Pi_h \boldsymbol{E}^n - \boldsymbol{E}_h^n, \boldsymbol{v}) - (\boldsymbol{J}^n - \boldsymbol{J}_h^n, \boldsymbol{v}) = r_h(\boldsymbol{E}^n; \boldsymbol{v}),\end{aligned}$$

from which and the definition of the Galerkin projection (2.5), we have

$$(\delta^2 \phi^n, \boldsymbol{v}) + \tilde{a}_h(\phi^n, \boldsymbol{v}) + \omega_p^2 (\phi^n, \boldsymbol{v}) = (r^n, \boldsymbol{v}) + (\boldsymbol{J}^n - \boldsymbol{J}_h^n, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h, \tag{3.7}$$

where we denote

$$r^n = \delta^2 (\Pi_h \boldsymbol{E}^n) - \boldsymbol{E}_{tt}^n, \quad 1 \le n \le N-1.$$

Summing (3.7) from $n=1$ to $n=m$, $1 \le m \le N-1$, and multiplying the resultant by τ, we have

$$\begin{aligned}&\Big(\frac{\phi^{m+1} - \phi^m}{\tau}, \boldsymbol{v}\Big) - \Big(\frac{\phi^1 - \phi^0}{\tau}, \boldsymbol{v}\Big) + \tau \sum_{n=1}^m \tilde{a}_h(\phi^n, \boldsymbol{v}) + \tau \omega_p^2 \sum_{n=1}^m (\phi^n, \boldsymbol{v}) \\ =&\tau \sum_{n=1}^m (r^n, \boldsymbol{v}) + \tau \sum_{n=1}^m (\boldsymbol{J}^n - \boldsymbol{J}_h^n, \boldsymbol{v}),\end{aligned}$$

which can be rewritten as

$$\Big(\frac{\phi^{m+1} - \phi^m}{\tau}, \boldsymbol{v}\Big) + \tilde{a}_h(\Phi^m, \boldsymbol{v}) + \omega_p^2 (\Phi^m, \boldsymbol{v}) = (R^m, \boldsymbol{v}) + \tau \sum_{n=1}^m (\boldsymbol{J}^n - \boldsymbol{J}_h^n, \boldsymbol{v}), \tag{3.8}$$

where we introduced notations

$$\Phi^0 = 0, \qquad \Phi^m = \tau \sum_{n=1}^m \phi^n, \qquad R^m = \tau \sum_{n=0}^m r^n, \qquad r^0 = \frac{(\phi^1 - \phi^0)}{\tau^2}. \tag{3.9}$$

Choosing $\boldsymbol{v} = \tau(\phi^m + \phi^{m+1}) \in \boldsymbol{V}_h$ in (3.8), and summing the resultant from $m=0$ to $m=n-1$, $1 \le n \le N$, we have

$$\begin{aligned}&\|\phi^n\|_0^2 - \|\phi^0\|_0^2 + \tau \sum_{m=0}^{n-1} \tilde{a}_h(\Phi^m, \phi^m + \phi^{m+1}) + \tau \omega_p^2 \sum_{m=0}^{n-1} (\Phi^m, \phi^m + \phi^{m+1}) \\ =&\tau \sum_{m=0}^{n-1} (R^m, \phi^m + \phi^{m+1}) + \tau^2 \sum_{m=0}^{n-1} \sum_{k=1}^m (\boldsymbol{J}^k - \boldsymbol{J}_h^k, \phi^m + \phi^{m+1}).\end{aligned} \tag{3.10}$$

From (3.9), we see that

$$\tau(\phi^m+\phi^{m+1})=\Phi^{m+1}-\Phi^{m-1},$$

using which and the definition $\Phi^0=0$, we have

$$\begin{aligned}\tau\sum_{m=0}^{n-1}(\Phi^m,\phi^m+\phi^{m+1})&=\sum_{m=0}^{n-1}\left[(\Phi^m,\Phi^{m+1})-(\Phi^{m-1},\Phi^m)\right]\\&=(\Phi^{n-1},\Phi^n)-(\Phi^{-1},\Phi^0)=(\Phi^{n-1},\Phi^n)\\&=\frac{1}{4}\left[(\Phi^{n-1}+\Phi^n,\Phi^{n-1}+\Phi^n)-(\Phi^n-\Phi^{n-1},\Phi^n-\Phi^{n-1})\right]\\&\geq-\frac{1}{4}(\Phi^n-\Phi^{n-1},\Phi^n-\Phi^{n-1})=-\frac{\tau^2}{4}\|\phi^n\|_0^2.\end{aligned}\tag{3.11}$$

Similarly, we have

$$\tau\sum_{m=0}^{n-1}\tilde{a}_h(\Phi^m,\phi^m+\phi^{m+1})\geq-\frac{\tau^2}{4}\tilde{a}_h(\phi^n,\phi^n),$$

substituting which and (3.11) into (3.10), we obtain

$$\begin{aligned}&\|\phi^n\|_0^2-\frac{\tau^2}{4}\tilde{a}_h(\phi^n,\phi^n)-\frac{\tau^2}{4}\omega_p^2\|\phi^n\|_0^2\\\leq&\|\phi^0\|_0^2+\tau\sum_{m=0}^{n-1}(R^m,\phi^m+\phi^{m+1})+\tau^2\sum_{m=0}^{n-1}\sum_{k=1}^{m}(\boldsymbol{J}^k-\boldsymbol{J}_h^k,\phi^m+\phi^{m+1}).\end{aligned}\tag{3.12}$$

Before we continue the proof, we need two lemmas for estimating r^0 and r^n, $n\geq 1$.

Lemma 3.2. *For* $1\leq n\leq N-1$, *we have*

$$\|r^n\|_0\leq\frac{C}{\tau}\left[h^{\min(\alpha,l)+\sigma_E}\int_{t_{n-1}}^{t_{n+1}}\left(\|\boldsymbol{E}\|_{H^{\alpha+\sigma_E}(\Omega)^3}+\|\nabla\times\boldsymbol{E}\|_{H^\alpha(\Omega)^3}\right)dt+\tau^2\int_{t_{n-1}}^{t_{n+1}}\|\boldsymbol{E}_{t^4}\|_0dt\right],$$

where the constant $C>0$ *is independent of* h *and* τ.

Proof. By the definition of r^n, $1\leq n\leq N-1$, we have

$$\|r^n\|_0=\|\delta^2(\Pi_h\boldsymbol{E}^n)-\boldsymbol{E}_{tt}^n\|_0\leq\|\delta^2(\Pi_h-I)\boldsymbol{E}^n\|_0+\|\delta^2\boldsymbol{E}^n-\boldsymbol{E}_{tt}^n\|_0.\tag{3.13}$$

By the following inequality [19, Lemma 4.1]

$$\|\delta^2\boldsymbol{u}^n\|_0\leq\frac{1}{\tau}\int_{t_{n-1}}^{t_{n+1}}\|\boldsymbol{u}_{tt}(t)\|_0dt,\quad\forall\boldsymbol{u}\in H^2([0,T];L^2(\Omega)^3),$$

we have

$$\begin{aligned}\|\delta^2(\Pi_h-I)\boldsymbol{E}^n\|_0&\leq\frac{1}{\tau}\int_{t_{n-1}}^{t_{n+1}}\|(\Pi_h-I)\boldsymbol{E}_{tt}\|_0dt\\&\leq\frac{Ch^{\min(\alpha,l)+\sigma_E}}{\tau}\int_{t_{n-1}}^{t_{n+1}}\left(\|\boldsymbol{E}\|_{H^{\alpha+\sigma_E}(\Omega)^3}+\|\nabla\times\boldsymbol{E}\|_{H^\alpha(\Omega)^3}\right)dt.\end{aligned}\tag{3.14}$$

Using the identity [19, Eq. (20)]:

$$\delta^2 \boldsymbol{u}^n - \boldsymbol{u}_{tt}^n = \frac{1}{6\tau^2}\Big[\int_{t_{n-1}}^{t_n}(t-t_{n-1})^3\boldsymbol{u}_{t^4}dt + \int_{t_n}^{t_{n+1}}(t_{n+1}-t)^3\boldsymbol{u}_{t^4}dt\Big],$$

we deduce that

$$\|\delta^2\boldsymbol{E}^n - \boldsymbol{E}_{tt}^n\|_0 \le \frac{\tau}{6}\int_{t_{n-1}}^{t_{n+1}}\|\boldsymbol{E}_{t^4}\|_0 dt. \tag{3.15}$$

The proof completes by substituting the estimates (3.14) and (3.15) into (3.13). ☐

Lemma 3.3. *There holds*

$$\|r^0\|_0 \le \frac{C}{\tau}\Big[h^{\min(\alpha,l)+\sigma_E}\big(\|\boldsymbol{E}_t\|_{L^\infty(I;H^{\alpha+\sigma_E}(\Omega)^3)} + \|\nabla\times\boldsymbol{E}_t\|_{L^\infty(I;H^\alpha(\Omega)^3)}\big) + \tau^2\|\boldsymbol{E}_{t^3}\|_{L^\infty(I;L^2(\Omega)^3)}\Big],$$

where the constant $C>0$ is independent of h and τ.

Proof. For any $\boldsymbol{v}\in\boldsymbol{V}^h$, using the fact that $(\boldsymbol{E}^0-\boldsymbol{E}_h^0,\boldsymbol{v})=0$, we obtain

$$\begin{aligned}(\phi^1-\phi^0,\boldsymbol{v}) &= (\Pi_h\boldsymbol{E}^1-\boldsymbol{E}_h^1,\boldsymbol{v}) - (\Pi_h\boldsymbol{E}^0-\boldsymbol{E}_h^0,\boldsymbol{v})\\ &= (\Pi_h\boldsymbol{E}^1-\boldsymbol{E}^1+\boldsymbol{E}^1-\boldsymbol{E}_h^1,\boldsymbol{v}) - (\Pi_h\boldsymbol{E}^0-\boldsymbol{E}^0+\boldsymbol{E}^0-\boldsymbol{E}_h^0,\boldsymbol{v})\\ &= ((\Pi_h-I)(\boldsymbol{E}^1-\boldsymbol{E}^0),\boldsymbol{v}) + (\boldsymbol{E}^1-\boldsymbol{E}_h^1,\boldsymbol{v}).\end{aligned} \tag{3.16}$$

By Lemma 2.4, we have

$$\begin{aligned}&\big|((\Pi_h-I)(\boldsymbol{E}^1-\boldsymbol{E}^0),\boldsymbol{v})\big| = \Big|\int_0^{t_1}(\partial_t(\Pi_h-I)\boldsymbol{E}(\cdot,s),\boldsymbol{v})ds\Big|\\ \le& C\tau h^{\min(\alpha,l)+\sigma_E}\big(\|\boldsymbol{E}_t\|_{L^\infty(I;H^{\alpha+\sigma_E}(\Omega)^3)} + \|\nabla\times\boldsymbol{E}_t\|_{L^\infty(I;H^\alpha(\Omega)^3)}\big)\|\boldsymbol{v}\|_0.\end{aligned} \tag{3.17}$$

Using Taylor expansion

$$\boldsymbol{E}^1 = \boldsymbol{E}_0 + \tau\boldsymbol{E}_t(0) + \frac{\tau^2}{2}\boldsymbol{E}_{tt}(0) + \frac{1}{2}\int_0^{t_1}(\tau-s)^2\boldsymbol{E}_{ttt}(\cdot,s)ds$$

and the definition of $\boldsymbol{E}_h^1$, we have

$$|(\boldsymbol{E}^1-\boldsymbol{E}_h^1,\boldsymbol{v})| \le \frac{1}{2}\int_0^{t_1}(\tau-s)^2|(\boldsymbol{E}_{ttt}(\cdot,s),\boldsymbol{v})|ds \le \frac{1}{2}\tau^3\|\boldsymbol{E}_{t^3}\|_{L^\infty(I;L^2(\Omega)^3)}\|\boldsymbol{v}\|_0. \tag{3.18}$$

Choosing $\boldsymbol{v}=\phi^1-\phi^0\in\boldsymbol{V}^h$ in (3.16) and using the estimates (3.17) and (3.18), we obtain

$$\begin{aligned}\|r^0\|_0 &= \frac{1}{\tau^2}\|\phi^1-\phi^0\|_0\\ &\le \frac{C}{\tau}\Big[h^{\min(\alpha,l)+\sigma_E}\big(\|\boldsymbol{E}_t\|_{L^\infty(I;H^{\alpha+\sigma_E}(\Omega)^3)} + \|\nabla\times\boldsymbol{E}_t\|_{L^\infty(I;H^\alpha(\Omega)^3)}\big) + \tau^2\|\boldsymbol{E}_{t^3}\|_{L^\infty(I;L^2(\Omega)^3)}\Big],\end{aligned}$$

which completes the proof. ☐

Now we can continue our error estimate. From Lemmas 3.2 and 3.3, we have

$$\|R^m\|_0 \le \tau\|r^0\|_0 + \tau\sum_{n=1}^{m}\|r^n\|_0 \le C\tau^2 + Ch^{\min(\alpha,l)+\sigma_E},$$

from which and the arithmetic-geometric mean inequality, we yield

$$\begin{aligned}
&\tau\sum_{m=0}^{n-1}(R^m,\phi^m+\phi^{m+1})\\
\le&\tau\sum_{m=0}^{n-1}\|R^m\|_0\|\phi^m+\phi^{m+1}\|_0\\
\le&\tau\delta_1\sum_{m=0}^{n-1}\|\phi^m+\phi^{m+1}\|_0^2+\frac{\tau}{4\delta_1}\sum_{m=0}^{n-1}\|R^m\|_0^2\\
\le&\tau\delta_1\cdot 2\sum_{m=0}^{n-1}(\|\phi^m\|_0^2+\|\phi^{m+1}\|_0^2)+\frac{\tau}{4\delta_1}\sum_{m=0}^{n-1}C(\tau^2+h^{\min(\alpha,l)+\sigma_E})^2\\
\le&\tau\delta_1\Big(\sum_{m=0}^{n-1}4\|\phi^m\|_0^2+2\|\phi^n\|_0^2\Big)+\frac{CT}{4\delta_1}\cdot(\tau^4+h^{2(\min(\alpha,l)+\sigma_E)}).
\end{aligned}\tag{3.19}$$

Similarly, from Lemma 3.1, we have

$$\begin{aligned}
&\tau^2\sum_{m=0}^{n-1}\sum_{k=1}^{m}(\boldsymbol{J}^k-\boldsymbol{J}_h^k,\phi^m+\phi^{m+1})\\
\le&\tau^2\sum_{m=0}^{n-1}\sum_{k=1}^{m}\Big(\frac{1}{4\delta_2}\|\boldsymbol{J}^k-\boldsymbol{J}_h^k\|_0^2+\delta_2\|\phi^m+\phi^{m+1}\|_0^2\Big)\\
\le&\frac{CT^2}{4\delta_2}\Big[\tau\sum_{m=0}^{n-1}\|\boldsymbol{E}_h^m-\Pi_h\boldsymbol{E}^m\|_0^2+h^{2(\min(\alpha,l)+\sigma_E)}+\tau^4\Big]\\
&\quad+\tau\delta_2 T\Big(\sum_{m=0}^{n-1}4\|\phi^m\|_0^2+2\|\phi^n\|_0^2\Big).
\end{aligned}\tag{3.20}$$

Using estimate (2.15), we have

$$\|\phi^0\|_0 \le \|\Pi_h\boldsymbol{E}^0-\boldsymbol{E}^0\|_0+\|\boldsymbol{E}^0-\boldsymbol{E}_h^0\|_0 \le Ch^{\min(\alpha,l)+\sigma_E}.\tag{3.21}$$

Substituting (3.19)-(3.21) into (3.12), using Lemma 2.2, the CFL condition (3.6) and using the discrete Gronwall inequality, we have

$$\|\phi^n\|_0^2 \le Ch^{2(\min(\alpha,l)+\sigma_E)}+C\tau^4,$$

which, along with the triangle inequality and Lemma 2.4, completes the proof of Theorem 3.1.

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