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An Analog of Einstein's General Relativity Emerging from Classical Finite Elasticity Theory: Analytical and Computational Issues

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Abstract. The "analogue gravity formalism", an interdisciplinary theoretical scheme developed in the past for studying several non relativistic classical and quantum systems through effective relativistic curved space-times, is here applied to largely deformable elastic bodies described by the nonlinear theory of solid mechanics. Assuming the simplest nonlinear constitutive relation for the elastic material given by a Kirchhoff-St Venant strain-energy density function, it is possible to write for the perturbations an effective space-time metric if the deformation is purely longitudinal and depends on one spatial coordinate only. Theoretical and numerical studies of the corresponding dynamics are performed in selected cases and physical implications of the results obtained are finally discussed.

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1 Introduction

The mathematical structure behind Einstein's General Relativity (GR) is differential geometry. GR is a physical theory whose foundations lay in elegant variational principles for geometry and matter fields. Recently it has been found an analog of relativistic gravity manifested by several systems of non-relativistic condensed matter physics, mainly

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in the fields of classical and quantum fluids and in electromagnetism [1–3]. The origin of such an analogy has to be searched in the classical d'Alembert wave equation

$$\frac{\partial^2}{\partial t^2}\xi - v^2\nabla^2\xi = 0 \tag{1.1}$$

($\nabla^2 \equiv \Delta$ is the ordinary Laplacian operator in Euclidean space) which describes the propagation at constant speed v of a certain quantity ξ in an homogeneous and isotropic background medium or even in vacuum if v is the speed of light (so that such an equation results Lorentz invariant). As an example, in fluid dynamics, the quantity ξ describes the perturbations of pressure or of the velocity potential, in infinitesimal elasticity it appears as a transverse or longitudinal relative displacement vector and in electromagnetism it stands for the electric or magnetic field vectors. If the medium is not homogeneous, the corresponding wave equations become more complicated. The mathematical relations involved in fact contain now second order terms with mixed derivatives together with first order ones, all of these multiplied by time and/or space dependent coefficients. The equations for small elastic waves in anisotropic and inhomogeneous media are a typical example of such a situation [4, 5]. Because all the aforementioned mathematical expressions resemble very much the structure of second order wave equations in curved GR spacetimes, the question in the past naturally arose whether a possible connection between all of these Newtonian and Relativistic problems could be found. The first confirmation of such an hypothesis historically dates back to Unruh's work [6, 7] on perfect fluid perturbations, later extended by Visser and collaborators [8, 9]. In short, for a non relativistic, classical, perfect, irrotational, compressible and barotropic fluid, its linear perturbations can be rewritten as a real massless scalar field equation on a curved space-time characterized by an acoustic four dimensional metric tensor. Recently the authors have shown [10] the connection of the theory described above with the quasi-linear decoupled second order wave equation governing the velocity potential as derived by Von Mises [11]. Several complementary studies have been performed in the last decade then, focusing in particular on analogs of black/white holes and cosmological systems [12–28, 30]. The aim of this article is to clarify if the Analogue Gravity formulation just discussed could be applied also to the theory of nonlinear elasticity. Infinitesimal elasticity is a limiting case of the much more complicated theory of nonlinear solid mechanics, whose natural language is differential geometry. An appropriate introduction to linear (infinitesimal) elasticity can be found in Feynman's Lectures on Physics books [31] or in volume 7 of Landau-Lifshitz' Theoretical Physics course [5]. For the purposes of our study, in Section 2 we shall here introduce a short resume of nonlinear solid mechanics accounting however for appropriate references for this topic. The unknown quantities in continuum mechanics are the components of the relative displacement vector $\vec{u}$, a Lagrangian entity intrinsically three dimensional which maps the position of a material point initially located in $\vec{x}$ into a new point $\vec{x}' \equiv \vec{x} + \vec{u}$. A Lagrangian point of view is here adopted because in solid mechanics one deals with the complicated free boundary problem of locating the body's surface during the deformation [32], although an Eulerian formulation of the problem is also possible [42]. In contrast, for fluids, an Eulerian point of view is often more immediate in comparison with the mathematical complications of a Lagrangian formulation [33]. One may try to write the relative displacement vector $\vec{u}$ as a gradient of a scalar potential in analogy with fluid mechanics, where in the irrotational case, the Eulerian velocity of the fluid is written as the gradient of a velocity potential. If the perfect fluid is not irrotational, one can anyway adopt the so called Chlebsh potentials formalisms [9]. In fluid dynamics, this simplifying procedure leads to a quasi-linear second order wave equation for the latter scalar quantity (the aforementioned Von Mises' wave equation), which linearized leads to the Analogue Gravity formalism for a relativistic scalar field. In nonlinear solid mechanics however this procedure does not appear to work, except in a very peculiar case discussed in the Appendix. More explicitly, while the curved wave operator on a scalar field (a covariant equation) works fine both in Newtonian hydrodynamics and in GR, in non relativistic continuum mechanics, the curved wave operator would be applied to a three dimensional vector which is not a covariant quantity. In order to present a way-out, in section III we shall introduce the perturbation theory for a purely longitudinal elastic field and discuss its connections with the Analogue Gravity formalism. In section IV then we shall analyze the acoustic metric associated with two explicit background solutions, while in section V we discuss the results obtained.

2 Nonlinear theory of elasticity

Following Landau and Lifshitz [5], the starting point of nonlinear solid mechanics theory is the relation which links every point of (material) coordinate x_i (here $i=1,\cdots,3$ and similar for Latin indices, while Greek ones shall refer to four-dimensional quantities) of a material body prior to deformations caused by forces to the corresponding new position x'_i and related to the initial undeformed position by $x'_i = x_i + u_i$. The vector $u_i \equiv u_i(t,\vec{x})$ represents the deformation or relative displacement vector. We assume now, for the sake of simplicity, that the x_i are ordinary orthogonal Cartesian coordinates. We can introduce then the symmetric deformation tensor, known in the literature also as the Green-Lagrange strain tensor [34] (Einstein's summation convention is here adopted):

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}\right), \tag{2.1}$$

which is necessary to connect the distance between two points of the undeformed body with the correspective one in the deformed case through the relation

$$dl'^2 = dl^2 + 2u_{ik}dx_i dx_k \equiv C_{ik}dx^i dx^k \tag{2.2}$$

with $dl'^2 \equiv \sqrt{dx_1'^2 + dx_2'^2 + dx_3'^2}$ and $dl^2 \equiv \sqrt{dx_1^2 + dx_2^2 + dx_3^2}$. Here quantity $C_{ik} \equiv \delta_{ik} + 2u_{ik}$, having the role of a three dimensional metric tensor, is the right Cauchy-Green deformation tensor [35]. An action principle treatment of nonlinear elasticity (and even of nonlinear electro-elasticity [37]) is possible [36], although field equations can be derived also by standard balance laws arguments [5,35,36,38]. The fundamental set of relations for generic large deformations is given then by Newton's law:

$$\rho_R \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial P_{ki}}{\partial x_k} + \rho_R f_i^{(e)}, \tag{2.3}$$

where $\rho_R \equiv \rho_R(\vec{x})$ represents the time independent initial reference mass density, P_{ki} is the first Piola-Kirchhoff stress tensor and $f_i^{(e)}$ is an external body force in the material coordinates. Newton's equations are a system of coupled partial differential equations which, in order to be solved, must be supplemented by appropriate initial data and boundary conditions, which in general shall represent the possible tractions externally exerted, everywhere or in selected parts of the body's surface. For the aims of our analysis, we shall concentrate on a material of infinite extent, disregarding in this way such a complicated part of the nonlinear elasticity mathematical problem [42]. Let's assume now the simplest continuum mechanics stress choice (isotropic medium) i.e. the Kirchhoff -Saint Venant (KSV) [38,39] associated with the elastic strain-energy density

$$\psi = \frac{1}{2}\lambda u_{ii}^2 + \mu u_{ik} u_{ki}, \tag{2.4}$$

where μ and λ are the Lamé constants. The symmetric second Piola-Kirchhoff tensor can be obtained then by differentiation, i.e.

$$S_{ij} = \frac{\partial \psi}{\partial u_{ij}} \tag{2.5}$$

so that

$$S_{il} = 2\mu u_{il} + \lambda u_{mm} \delta_{il} \equiv \frac{E}{1+\kappa}\left(u_{il} + \frac{\kappa}{1-2\kappa} u_{mm} \delta_{il}\right), \tag{2.6}$$

where δ_{kl} is the Kroneker delta, κ is the Poisson coefficient and E is Young's modulus. We can finally write the non symmetric first Piola-Kirchhoff tensor as

$$P_{ki} = S_{kl} F_{il}, \qquad F_{kl} = \left(\delta_{kl} + \frac{\partial u_k}{\partial x_l}\right). \tag{2.7}$$

In the language of finite elasticity the tensor F_{kl} represents the deformation gradient, connected to the right Cauchy-Green tensor by relation $C_{rs} = F_{lr} F_{ls}$. Finally the density at a given time is connected to the reference material density by the relation

$$\rho(t,\vec{x}) = \frac{\rho_R(\vec{x})}{\det F_{ik}}. \tag{2.8}$$

The KSV model can be adopted for large displacement calculations assuming a material undergoing only small strains however. It describes a compressible material which becomes almost incompressible for very large Lamé constants [40]. This is the simplest nonlinear version of the continuum mechanics problem, although we must stress that many other different and more physically appropriate choices for the strain energy function are possible [41, 42]. Standard infinitesimal elasticity [5, 31] can be obtained by writing the relative displacement vector, reference density and body forces in Taylor series of a parameter ε around a constant or even vanishing background solution, inserting into Newton's Eq. (2.3), expanding in Taylor series for ε and keeping the first order terms in this parameter, obtaining for constant λ and μ, the standard formula for small perturbations [31]

$$\rho_R \frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda+\mu)\nabla(\nabla\cdot\vec{u}) + \mu\nabla^2\vec{u} + \rho_R \vec{f}^{(e)}. \tag{2.9}$$

What we shall do in this article instead will be to study more general perturbations of such a complicated nonlinear theory. Precisely, we shall analyse the infinitesimal perturbations of a nontrivial space and/or time dependent finite elasticity background solution (defined as a zero-th order quantity), i.e. $u_i = u_i^{(0)} + \varepsilon u_i^{(1)} + \cdots$ (together with a perturbative expansion of reference density ρ_R and body force $\vec{f}^{(e)}$). One would be tempted to try to introduce here a potentials formalism, in analogy with hydrodynamics. The assumption of an Helmholtz decomposition for the deformation vector $\vec{u}$ given by:

$$\vec{u} = \mathrm{grad}\,\Psi + \mathrm{curl}\,\vec{A}, \quad \mathrm{div}\,\vec{A} = 0, \tag{2.10}$$

leads in general to equations of order higher than second which clearly would be problematic for the implementation of an acoustic metric formalism. The gradient part in Eq. (2.10) is the so called *longitudinal* component field, while the curl term is the *transverse* one. Decoupled d'Alembert wave equations for Ψ and $\vec{A}$ can be obtained in infinitesimal theory for homogeneous bodies only [4, 43, 44]. In Appendix we show that it is still possible to obtain a totally nonlinear second order equation for an x-directed and x-dependent field only, provided a constant initial density everywhere.

3 Perturbation theory and analog geometry

As anticipated, we assume to deal with an infinite medium. In order to work with a scalar equation we require moreover $u_i = [u_1(t,x),0,0] \equiv [U(t,x),0,0]$ and $f_i^{(e)} \equiv [f_1^{(e)},0,0] = [\mathcal{F}(t,x),0,0]$ together with $\rho_R = \rho_R(x)$ i.e. a purely longitudinal nonlinear field in the x direction with the same value on each sectional plane orthogonal to the line of propagation. Regarding the density of the medium this assumption implies

$$\rho(t,\vec{x}) = \frac{\rho_R(\vec{x})}{\det F_{ik}} \Longleftrightarrow \rho(t,x) = \frac{\rho_R(x)}{\left(1+\frac{\partial U}{\partial x}\right)}. \tag{3.1}$$

Newton's Eq. (2.3) simplifies then to the unique relation

$$\rho_R \frac{\partial^2 U}{\partial t^2} - (2\mu+\lambda)\left[1+3\frac{\partial U}{\partial x}+\frac{3}{2}\left(\frac{\partial U}{\partial x}\right)^2\right]\frac{\partial^2 U}{\partial x^2} - \rho_R \mathcal{F} = 0. \tag{3.2}$$

Neglecting any nonlinear coupling (linearized theory) with zero external force, as expected we obtain from the relation just written the classical wave equation for longitudinal waves in an isotropic medium [5], i.e. $\partial^2 U/\partial t^2 - c_l^2 \partial^2 U/\partial x^2 = 0$ with longitudinal sound speed $c_l = \sqrt{(2\mu+\lambda)/\rho_R}$.

3.1 The hyperbolicity issue

Eq. (3.2) can be rewritten as

$$\rho_R \frac{\partial^2 U}{\partial t^2} - \frac{\partial}{\partial x} G - \rho_R \mathcal{F} = 0, \tag{3.3a}$$

$$G = \left\{ (2\mu+\lambda)\left[\frac{\partial U}{\partial x}+\frac{3}{2}\left(\frac{\partial U}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial U}{\partial x}\right)^3\right]\right\}, \tag{3.3b}$$

a relation which is useful to prove the strict hyperbolicity of the problem by taking into account also the relation

$$\frac{\partial G}{\partial x} = \left(\frac{\partial G}{\partial U_x}\right)\cdot\frac{\partial U_x}{\partial x}. \tag{3.4}$$

Let's introduce in fact

$$\frac{\partial U}{\partial t} = \frac{v(t,x)}{\rho_R(x)}, \quad \frac{\partial U}{\partial x} = w(t,x) \tag{3.5}$$

so that Eq. (3.3) can be rewritten as:

$$\frac{\partial w}{\partial t} - \frac{\partial}{\partial x}\left(\frac{1}{\rho_R} v\right) = 0, \tag{3.6a}$$

$$\frac{\partial v}{\partial t} - \frac{\partial}{\partial x}\left[(2\mu+\lambda)\left(w+\frac{3}{2}w^2+\frac{1}{2}w^3\right)\right] = \rho_R \mathcal{F} \tag{3.6b}$$

which is a nonlinear conservation law for an initially heterogeneous medium representing a nontrivial mathematical problem [45]. By taking into account Eq. (3.4), we can rewrite the previous relation in an equivalent vector form

$$\frac{\partial \vec{\Lambda}}{\partial t} + \hat{\mathcal{A}}\cdot\frac{\partial \vec{\Lambda}}{\partial x} + \hat{\mathcal{B}}\cdot\vec{\Lambda} = \vec{\mathcal{T}}, \tag{3.7}$$

where

$$\vec{\Lambda} = \begin{pmatrix} w \\ v \end{pmatrix}, \quad \hat{\mathcal{A}} = \begin{pmatrix} 0 & -\frac{1}{\rho_R} \\ -\frac{\partial G}{\partial w} & 0 \end{pmatrix}, \tag{3.8a}$$

$$\hat{\mathcal{B}} = \begin{pmatrix} 0 & \frac{1}{\rho_R^2}\frac{\partial \rho_R}{\partial x} \\ 0 & 0 \end{pmatrix}, \quad \vec{\mathcal{T}} = \begin{pmatrix} 0 \\ \rho_R \mathcal{F} \end{pmatrix}. \tag{3.8b}$$

If we assume a constant initial density ρ_R in order to simplify further the treatment, the system Eq. (3.7) is a conservation law (autonomous by neglecting the source term) in quasi-linear form [46] which can be analyzed through the method of the characteristics. In particular the eigenvalues of the matrix $\hat{\mathcal{A}}$ result in $\lambda_{1,2} = \pm\sqrt{\frac{1}{\rho_R}\frac{\partial G}{\partial w}}$. If the eigenvalues of this problem at a certain point are real, we have an hyperbolic system there. Moreover, if λ_1 and λ_2 remain real and distinct, the problem is strictly (or equivalently strongly) hyperbolic. Clearly assuming a positive density on physical grounds, this condition requires that $\frac{\partial G}{\partial w} < 0$ never. A strictly hyperbolic conservation law equations set is well posed in the sense of the Cauchy problem, although the solutions of our nonlinear problem can exist locally in time, but after a certain finite period it can happen that singularities would occur [47] (in hydrodynamics these would be shock waves or blowups, as an example). Consequently it appears physically important to have under control the breakup of hyperbolicity of our problem. We point out moreover that it is possible to perform a perturbative study in a series of a parameter ε of the conservation law in Eq. (3.7) around a certain nontrivial background solution

$$w = w^{(0)} + \varepsilon w^{(1)}, \quad v = v^{(0)} + \varepsilon v^{(1)}, \tag{3.9a}$$

$$\rho_R = \rho_R^{(0)} + \varepsilon \rho_R^{(1)}, \quad \mathcal{F} = \mathcal{F}^{(0)} + \varepsilon \mathcal{F}^{(1)}. \tag{3.9b}$$

By inserting all of these expansions in our equations, denoting in a compact way the expanded quantities as $\vec{\Lambda} = \vec{\Lambda}^{(0)} + \varepsilon \vec{\Lambda}^{(1)} + \cdots$ and similar for the other ones, we get at ε order schematically

$$\frac{\partial \vec{\Lambda}^{(1)}}{\partial t} + \hat{\mathcal{A}}^{(0)} \cdot \frac{\partial \vec{\Lambda}^{(1)}}{\partial x} + (\textit{non differential } \vec{\Lambda}^{(1)} \textit{ term}) = (\textit{source term}) \tag{3.10}$$

which again is strictly hyperbolic if the eigenvalues of the matrix $\hat{\mathcal{A}}$ computed on the background solution (here denoted by $\hat{\mathcal{A}}^{(0)}$), result real and distinct [48]. Again for a positive background reference density $\rho_R^{(0)}$, this occurs if $\frac{\partial G}{\partial w}|_{w=w^{(0)}}$ does not become negative. We perform now an equivalent perturbative study of Eq. (3.3) which will be helpful for an Analogue Gravity formulation of the problem.

3.2 Perturbation theory

The quasi-linear wave Eq. (3.2) can be perturbatively expanded by imposing

$$U(t,x) = U^{(0)}(t,x) + \varepsilon U^{(1)}(t,x) + \cdots, \tag{3.11a}$$

$$\rho_R(x) = \rho_R^{(0)}(x) + \varepsilon \rho_R^{(1)}(x) + \cdots, \tag{3.11b}$$

$$\mathcal{F}(t,x) = \mathcal{F}^{(0)}(t,x) + \varepsilon \mathcal{F}^{(1)}(t,x) + \cdots, \tag{3.11c}$$

inserting into Eq. (3.2) and equating to zero separately the various terms in powers of ε. At order zero we get the background nonlinear equation Eq. (3.2) for quantity $U^{(0)}$ with

$\rho_R^{(0)}$ and $\mathcal{F}^{(0)}$. At order one we get

$$\rho_R^{(0)}\frac{\partial^2 U^{(1)}}{\partial t^2}-\left[\left(\frac{2\mu+\lambda}{2}\right)\Xi^{(0)}\right]\frac{\partial^2 U^{(1)}}{\partial x^2} \\ -\left\{3(2\mu+\lambda)\left[\frac{\partial U^{(0)}}{\partial x}+1\right]\frac{\partial^2 U^{(0)}}{\partial x^2}\right\}\frac{\partial U^{(1)}}{\partial x}=\mathcal{S}^{(1)}(t,x) \tag{3.12}$$

with the dimensionless quantity

$$\Xi^{(0)}=\left[2+6\frac{\partial U^{(0)}}{\partial x}+3\left(\frac{\partial U^{(0)}}{\partial x}\right)^2\right]\equiv\left(\frac{2}{2\mu+\lambda}\right)\frac{\partial G}{\partial w}\bigg|_{w=w^{(0)}}, \tag{3.13}$$

and a source term given by

$$\mathcal{S}^{(1)}(t,x)=\left[\rho_R^{(0)}\mathcal{F}^{(1)}+\left(\mathcal{F}^{(0)}-\frac{\partial^2 U^{(0)}}{\partial t^2}\right)\rho_R^{(1)}\right]. \tag{3.14}$$

Eq. (3.12) can be geometrically rewritten as scalar field equation

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\partial_\nu\chi\right]=\sigma^{(1)} \tag{3.15}$$

with the requirement of dealing with a subclass of perturbations characterized by $\chi(t,\vec{x})\equiv U^{(1)}(t,x)$ and $\sigma^{(1)}(t,\vec{x})\equiv\mathcal{S}^{(1)}(t,x)$. In the previous relations we denoted $\partial_\mu=\partial/\partial x^\mu$ with $x^\mu=(t,\vec{x})$ and we have introduced a four dimensional acoustic metric tensor and associated inverse given by

$$g_{\mu\nu}=\begin{bmatrix}\frac{1}{\rho_R^{(0)}} & 0 & 0 & 0\\ 0 & -\frac{2}{(2\mu+\lambda)}\frac{1}{\Xi^{(0)}} & 0 & 0\\ 0 & 0 & -\frac{1}{K}\sqrt{\frac{\rho_R^{(0)}(2\mu+\lambda)\Xi^{(0)}}{2K}} & 0\\ 0 & 0 & 0 & -\frac{1}{K}\sqrt{\frac{\rho_R^{(0)}(2\mu+\lambda)\Xi^{(0)}}{2K}}\end{bmatrix}, \tag{3.16a}$$

$$g^{\mu\nu}=\begin{bmatrix}\rho_R^{(0)} & 0 & 0 & 0\\ 0 & -\frac{(2\mu+\lambda)}{2}\Xi^{(0)} & 0 & 0\\ 0 & 0 & -K\sqrt{\frac{2K}{\rho_R^{(0)}(2\mu+\lambda)\Xi^{(0)}}} & 0\\ 0 & 0 & 0 & -K\sqrt{\frac{2K}{\rho_R^{(0)}(2\mu+\lambda)\Xi^{(0)}}}\end{bmatrix}, \tag{3.16b}$$

where the constant K is here introduced in order to obtain consistent physical dimensions in the formulas but does not play any role in the subsequent analysis. We point out that the occurrence of a four dimensional space-time metric is here natural, even if we are dealing with purely plane symmetric configurations, because the elastic problem was initially posed in the Euclidean three dimensional space. At a first glance, as it happens in hydrodynamical analog geometry models, the acoustic metric results problematic if the reference density ρ_R goes to zero. This is not an unexpected result, because in this limit there is no matter to propagate waves anymore. Analytical solutions of quasi-linear Eq. (3.2) are non trivial to be found. As discussed in the next section, we shall analyze then some simplified configurations.

4 Explicit examples of elastic analog geometries

In this section we shall discuss two simple cases of analytical solutions for nonlinear Eq. (3.2), implementing the Analogue Gravity formalism in order to grasp the underlying physics codified in the weak perturbative fields on these background toy model solutions.

4.1 Inhomogeneous undeformed body

We adopt a background configuration characterized by $\mathcal{F}^{(0)} \equiv 0$ and $U^{(0)} = 0$ with $\rho_R^{(0)} = \rho_R^{(0)}(x)$ so that $\Xi^{(0)} \equiv 2$ and Eq. (3.12) results in

$$\rho_R^{(0)} \frac{\partial^2 U^{(1)}}{\partial t^2} - (2\mu + \lambda) \frac{\partial^2 U^{(1)}}{\partial x^2} = \rho_R^{(0)} \mathcal{F}^{(1)}, \tag{4.1}$$

where the explicit form of the perturbative force $\mathcal{F}^{(1)}$ is not important for the following analysis. In order to understand the light cone structure of the associated metric tensor (3.16), we study null geodesics, which in GR represent the massless test particles' trajectories. The four dimensional wave-vector $k^\alpha = dx^\alpha / d\tau$ (τ is the affine parameter) satisfies the geodesic equations

$$\frac{dk^\alpha}{d\lambda} + \Gamma^\alpha_{\sigma\delta} k^\sigma k^\delta = 0 \tag{4.2}$$

but this vector must be null i.e. $k^\alpha k_\alpha = 0$, so replacing $k_\alpha = \partial_\alpha S$ in the latter, we obtain the eikonal equation

$$g^{\alpha\beta} \partial_\alpha S \partial_\beta S = 0. \tag{4.3}$$

The very simple Killing vectors' nature of the acoustic metric in exam, whose fields depend on the x coordinate only, allows us to adopt standard Hamilton-Jacobi treatment of geodesics [49–51]. We choose, for null (i.e. massless) geodesics,

$$S(t,x,y,z) = \mathcal{E} t - f(x), \tag{4.4}$$

where $\mathcal{E}$ is a real constant. Inserting in the eikonal equation (4.3), we get finally the null geodesic four-vector

$$k^{\mu}_{\pm}=\left[\rho_R^{(0)}(x)\mathcal{E},\pm\sqrt{(2\mu+\lambda)\rho_R^{(0)}}\mathcal{E},0,0\right].$$

We can study now the relations

$$\frac{\left(\frac{dx}{d\tau}\right)}{\left(\frac{dt}{d\tau}\right)}\equiv\frac{dx}{dt}=\frac{k^x_{\pm}}{k^t_{\pm}}\quad\longleftrightarrow\quad\frac{dx}{dt}=\pm\sqrt{\frac{2\mu+\lambda}{\rho_R^{(0)}(x)}}\equiv\pm c_l(x),\tag{4.5}$$

where the affine parameter τ has disappeared and the local longitudinal sound speed

$$c_l(x)=\sqrt{(2\mu+\lambda)/\rho_R^{(0)}(x)}\tag{4.6}$$

has been introduced. This equation gives the coordinate velocity for null geodesic rays emanating at a certain initial (effective) space-time location. Integration of this differential equation on the other hand gives the space-time trajectory of a geodesic null ray in our metric, i.e.

$$\int_{x_0}^{x}\frac{1}{c_l(x')}dx'=\pm\int_{t_0}^{t}dt'.\tag{4.7}$$

The associated Kretschmann curvature invariant [52] $\mathcal{K}=R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ ($R_{\alpha\beta\gamma\delta}$ is the Riemann tensor) results in

$$\mathcal{K}=\frac{3(2\mu+\lambda)^2}{64\rho_R^{(0)}(x)^4}\left\{57\left[\frac{\partial\rho_R^{(0)}(x)}{\partial x}\right]^4+32\left[\rho_R^{(0)}(x)\right]^2\cdot\left[\frac{\partial^2\rho_R^{(0)}(x)}{\partial x^2}\right]^2\right.$$
$$\left.-80\rho_R^{(0)}(x)\left[\frac{\partial\rho_R^{(0)}(x)}{\partial x}\right]^2\cdot\left[\frac{\partial^2\rho_R^{(0)}(x)}{\partial x^2}\right]\right\}.\tag{4.8}$$

Assuming for the sake of simplicity the relation for the positive density, i.e. $\rho_R^{(0)}(x)\propto x^2$, the invariant results in $\mathcal{K}\propto 1/x^4$, i.e. where the density goes to zero, there acoustic curvature becomes infinite. As in the hydrodynamical cases studied in the past [26–28], curvature singularities develop where there is no medium to support waves anymore.

4.2 Initially homogeneous body periodically deformed

In this case we require for simplicity a classical separated uniform dilatation [38], i.e. $U^{(0)}(t,x)\equiv h_1(t)h_2(x)=Ax\sin\omega t$ where $A,\omega\in\mathcal{R}$. We obtain consequently, in order to have the background nonlinear problem to be satisfied, that $\mathcal{F}^{(0)}(t,x)=-A\omega^2x\sin\omega t$. Reference density's choice is arbitrary; here we take a constant one, i.e. $\rho_R^{(0)}(x)=r_0$ where $r_0>0$. From Eq. (3.1) we can write that

$$\rho^{(0)}(t,x)=\frac{r_0}{1+A\sin\omega t}\tag{4.9}$$

so that a positive density requires mandatorily to have $|A|<1$. In conclusion $x'=(1+A\sin\omega t)x$ so that for $t=0$ we have $x'\equiv x$ and $\mathcal{F}^{(0)}=0$ with $\rho^{(0)}\equiv\rho_R^{(0)}=r_0$, which confirms that the problem is physically well posed. Regarding the acoustic metric (3.16), in this case we have $\Xi^{(0)}=2+6A\sin\omega t+3A^2\sin^2\omega t$. A first look to the metric tensor in Eq. (3.16) shows that signature (and metric) breaks down when $\Xi^{(0)}\leq 0$. From the previous discussion on loss of hyperbolicity and from Eq. (3.13) we can see that this is exactly the case. Consequently simple inequalities show that, introducing $A_c=1-1/\sqrt{3}$, for $-A_c<A<A_c$ we have an always regular metric tensor while outside this range, pathologies are expected to occur because at certain finite time hyperbolicity gets lost. Regarding the null geodesic wave-vector, assuming a principal function dependence $S(t,\vec{x})=F(t)-\gamma x$ with $\gamma\in\mathcal{R}$, we obtain from Hamilton-Jacobi theory again that

$$k^\mu_\pm=\left[\pm\sqrt{\frac{(2\mu+\lambda)r_0\Xi^{(0)}}{2}}\gamma,\frac{(2\mu+\lambda)\Xi^{(0)}}{2}\gamma,0,0\right]. \tag{4.10}$$

We can study now the relations

$$\frac{\left(\frac{dx}{d\tau}\right)}{\left(\frac{dt}{d\tau}\right)}\equiv\frac{dx}{dt}=\frac{k^x_\pm}{k^t_\pm}\quad\longleftrightarrow\quad\frac{dx}{dt}=\pm\sqrt{\left(\frac{2\mu+\lambda}{2r_0}\right)\Xi^{(0)}(t)}, \tag{4.11}$$

where the affine parameter τ has disappeared. Direct integration of the latter gives the space-time trajectory of null geodesic rays, i.e.

$$x-x_0=\pm c_l\int_0^t\frac{1}{\sqrt{2}}\sqrt{\Xi^{(0)}(t')}dt' \tag{4.12}$$

expressible in terms of Elliptic functions. Here we have introduced the constant longitudinal wave speed at $t=0$ given by $c_l^2=\left(\frac{2\mu+\lambda}{r_0}\right)$. In Fig. 1 we plot the associated spacetime diagram for these null geodesics for the simplifying parameters' choice $\mu=\lambda=r_0=\omega=1$, numerically integrated by using XPPAUT software [29] via a fourth order Runge-Kutta scheme with $\delta t=5\cdot10^{-3}$, starting at $x=10$ at time $t=0$ for $A=0.25$ while in Fig. 2 the same plot is shown in the case $A=0.8$. While the latter case becomes problematic at a finite time because a geodesically incomplete behavior occurs (at a finite value of the affine parameter τ the integration breaks down), the former is regular and shows explicitly the effect of the periodic distortion of the light ray world-lines, which in absence of elasticity would look as straight lines. What it is happening here is that the acoustic perturbation is dragged by the oscillating behavior of the background matter. Clearly in order to compare the model's predictions with an experimental physical situation, other parameters' sets should be chosen; anyway in the corresponding dynamics, similar qualitative behaviors shall occur. We have confirmed these result for the full perturbative elastic problem by integrating the PDE in Eq. (3.12), assuming $\mathcal{S}^{(1)}\equiv0$ for the sake of

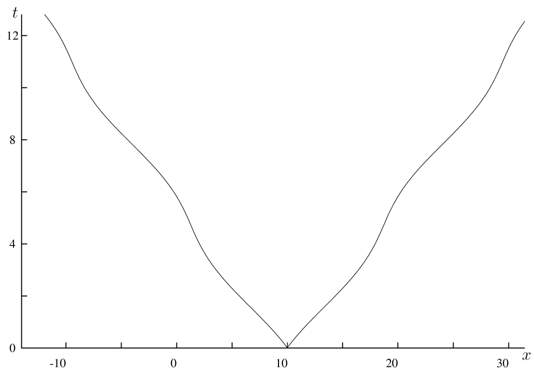

Figure 1: Space-time diagram in case $\mu=\lambda=r_0=\omega=1$ for null geodesics numerically integrated starting at $x=10$ at time $t=0$ for $A=0.25$. The dynamics is regular and shows explicitly the effect of the periodic deformation of the effective light ray world-line.

Figure 2: Space-time diagram in case $\mu=\lambda=r_0=\omega=1$ for null geodesics numerically integrated starting at $x=10$ at time $t=0$ for $A=0.8$. The trajectory becomes problematic at a finite time, in contrast with the $A=0.25$ case.

simplicity again. We have taken in particular a spatial domain $x\in[-23,43]$. The model parameters are $\mu=\lambda=r_0=\omega=1$ as before, but we have integrated the problem both for $A_0=0.25$ (simulation's results are shown in Fig. 3) and $A_0=0.8$ (numerical results are presented in Fig. 4) We have chosen a sharp Gaussian initial data $U(t,0)=P_0e^{-b(x-x_0)^2}$ together with $\partial_t U(t,0)=0$, selecting moreover $b=5$, $x_0=10$, $P_0=1$. The boundary conditions adopted on the domain borders are Neumann zero flux i.e. $\partial_x U(t,x)=0$ and numerical integration has been stopped before any boundary effect could interfere with the inner space evolution, in analogy with past Analogue Gravity studies in the context of hydrodynamics [16, 17, 20]. The partial differential equation has been numerically integrated by using finite elements methods via Comsol Multiphysics software. More in detail we have adopted a direct UMFPACK solver, with equally spaced Lagrange quadratic mesh

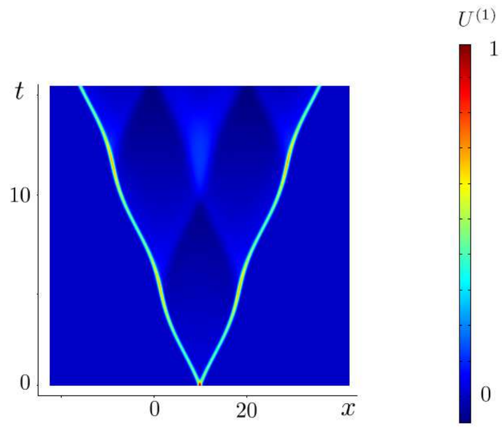

Figure 3: Space-time diagram generated by the numerical integration of the perturbative PDE in case $\mu=\lambda=r_0=\omega=1$ for $A=0.25$. The oscillations look the same as those of the null geodesic rays.

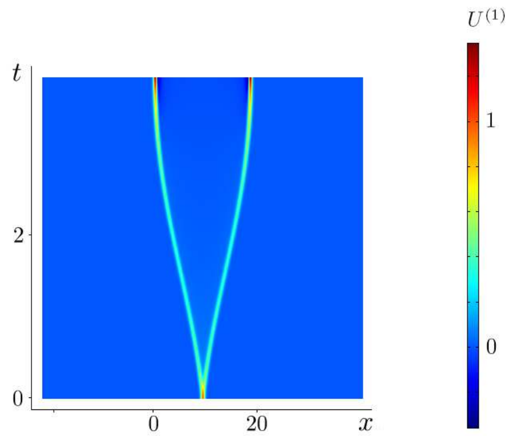

Figure 4: Space-time diagram generated by the numerical integration of the perturbative PDE in case $\mu=\lambda=r_0=\omega=1$ for $A=0.8$. The code manifests a problem at finite time, where hyperbolicity is lost and singularities are expected to occur.

elements of size $\Delta x=0.1$, while the time step has been chosen automatically by the software. Finally absolute and relative tolerances have been set to 10^{-6}. We have varied mesh sizes and tolerances is order to confirm the validity of our numerical simulations. The $A=0.25$ case, shown in Fig. 3, manifests an oscillating space-time behavior perfectly in agreement with the previously performed null geodesics study of Fig. 1. The case $A=0.8$ in Fig. 4 gives a diverging (blow up) behavior at a finite value of time, associated with the loss of hyperbolicity as expected from the study in Fig. 2. In conclusion, the integrated equation manifests the same light cone structure of the null rays previously discussed, confirming the appropriateness of the analog gravity formulation in studying also these types of continuum mechanics problems.

5 Discussion

Analogue Gravity formalisms appear to be applicable to a wide range of physical Newtonian systems where second order wave equations naturally occur in describing perturbative behaviors. While historically effective metrics were derived in the case of classical and quantum fluids and electromagnetic systems, we have here shown that such a type of analysis can be performed also in the case of the nonlinear theory of elasticity. Unfortunately the mathematical structure of this theory in Lagrangian form does not allow a useful reformulation of the nonlinear equations in terms of certain scalar potentials, as in the case of hydrodynamics. Moreover the decoupling of the field equations, well known in the infinitesimal theory of elasticity, in the exact regime is not immediate so that certain strong simplifying assumptions must be assumed, i.e. the request of one component only for the relative displacement vector $\vec{u}=(u_x,u_y,u_z)\equiv(u_x(t,x),0,0)$ and the dependence of the latter on one spatial coordinate only. In this way we have obtained an acoustic metric which describes purely longitudinal perturbations. We have analysed then two extremely simple background solutions, one initially inhomogeneous and static, the other not static but initially homogeneous and periodically forced, both to be perturbed. It has been shown in these cases that, as it happens for fluids, vanishing values of the reference density lead to infinite curvature singularities, an expected behavior because there is no medium on which acoustic perturbations can travel anymore. The null geodesics rays structure of the acoustic metric is immediately super-imposable to the behavior of the numerically integrated partial differential equation describing the Newtonian perturbative problem. In particular, where or when hyperbolicity breaks down, there geodesics manifest pathological behaviors. All of these phenomenologies constitute the main essence of the Analogue Gravity theory as studied in the past for other physical systems. Clearly our treatment can be physically applicable to planar configurations with purely longitudinal perturbations far from the boundaries, where the initial purely longitudinal assumption would break down. In this sense one could imagine to apply these results to study multilayered systems (composite materials [4]) in a wall-type configuration. The theory formulated may be interesting also for Quantum Mechanical behaviors. It is well known from standard Solid State Physics [53] in fact that in ionic crystals, the position of a certain (oscillating) ion is given by $\vec{x}'=\vec{x}+\vec{u}$. It is possible to write for this system an Hamiltonian for small oscillations with an harmonic total potential energy and then introduce quantization. When non small relative displacements are present, the potential has higher order terms in the relative displacement vectors and anharmonic behaviors occur [5]. One may look then at our previously analyzed systems in this limit, in search of possible experimental counterparts in the range where classical, or possibly semiclassical formulations could be applied. These future infinitesimal elasticity studies of "regular"materials may have some relevance if compared to a pre-existing literature of effective metric tensors mainly based on dislocation theory with a strain tensor in the linear approximation only (see Refs. [54–61] as an example). There, an effective emergent curvature (but also torsion, typical of alternative theories of gravitation and not present in General Relativity) in a crystal resulted to be linked to topological defects in the elastic media (i.e. dislocations and disclinations). We must stress however that because of our present assumptions of i) regular elastic materials (i.e. no defects) as standard in large part of Solid Mechanics literature ii) large deformations (contrasted to infinitesimal elasticity) iii) classical (i.e. non quantum) regimes and iv) the requirement of a a Riemannian four dimensional emergent effective geometry for perturbations as required by General Relativity and standard Analogue Gravity studies [1], our work addresses a different type of problem in comparison to the above cited important solid state physics studies. Finally, it would be interesting to check if a sort of black/white hole behavior could be observed in solid mechanics. In search of an analog of event horizon, it will be useful in the future to analyze the nonlinear elasticity problem in spherical coordinates by assuming purely radial longitudinal perturbations, i.e. $\vec{u}=(u_r,u_\theta,u_\phi)\equiv(u_r(t,r),0,0)$. Such a type of analysis could have implications for studying linear radial waves in spherical deformable physical systems as those encountered in geophysics [43]. This procedure has not been performed here because one should have started with the nonlinear solid mechanics equations in curvilinear coordinates, a problem which would have lead to more complicated mathematical relations than those here obtained working in Cartesian coordinates. For this reason, this important analysis is postponed to a future study.

Appendix

In this section we show how to obtain a second order fully nonlinear equation for purely longitudinal solutions. Let's introduce a body force potential first, i.e. $\vec{f}^{(e)}=\mathrm{grad}\Phi(\vec{x})$. We require now a dependence of every function on t and x only with a purely x-directed relative displacement $\vec{u}\equiv u_x\cdot\hat{x}$. In this situation we can write then in Eq. (2.10) $u_x=\partial_x\Psi$ ($\mathrm{curl}\vec{A}\equiv 0$ identically) so that Newton's Eq. (2.3) becomes

$$\rho_0(x)\frac{\partial^3\Psi}{\partial x\partial t^2}-(2\mu+\lambda)\frac{\partial^3\Psi}{\partial x^3}-\frac{3}{2}(\lambda+2\mu)\frac{\partial^3\Psi}{\partial x^3}\left(\frac{\partial^2\Psi}{\partial x^2}\right)^2 \\ -3(\lambda+2\mu)\frac{\partial^3\Psi}{\partial x^3}\cdot\left(\frac{\partial^2\Psi}{\partial x^2}\right)-\rho_0(x)\frac{\partial\Phi}{\partial x}=0. \tag{A.1}$$

If and only if $\rho_0(x)=\mathrm{constant}\equiv\rho_0$ (together with the already assumed constant elastic parameters λ and μ) we can reconstruct the *divergence* structure:

$$\frac{\partial}{\partial x}\left\{\rho_0\frac{\partial^2\Psi}{\partial t^2}-(\lambda+2\mu)\left[1+\frac{3}{2}\frac{\partial^2\Psi}{\partial x^2}+\frac{1}{2}\left(\frac{\partial^2\Psi}{\partial x^2}\right)^2\right]\frac{\partial^2\Psi}{\partial x^2}-\rho_0\Phi\right\}=0 \tag{A.2}$$

or equivalently

$$\rho_0\frac{\partial^2\Psi}{\partial t^2}-(\lambda+2\mu)\left[1+\frac{3}{2}\frac{\partial^2\Psi}{\partial x^2}+\frac{1}{2}\left(\frac{\partial^2\Psi}{\partial x^2}\right)^2\right]\frac{\partial^2\Psi}{\partial x^2}-\rho_0\Phi=C(t). \tag{A.3}$$

The constant $C(t)$ can be removed (or equivalently set equal to zero) then by rescaling one of the two potentials in this equation. We are left now with a second order equation for the potential Ψ which is fully nonlinear in contrast to quasi-linear equations normally met in continuum field theories. This equation too could be perturbed obtaining an effective metric formulation. Incidentally, if in Eq. (A.3) one neglects the nonlinear couplings, the wave equation for longitudinal $1D$ waves in an homogeneous medium is recovered. The constraint of constant initial density however makes this formulation too restrictive so that we have not continued further such an analysis in this paper. We point out that if the fields depend on more than one spatial coordinate, all the manipulations become extremely complicated and a decoupling of equations at nonlinear level does not appear to be feasible anymore.

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