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Quasi-Optimized Overlapping Schwarz Waveform Relaxation Algorithm for PDEs with Time-Delay

Shu-Lin Wu[1,2,*] and Ting-Zhu Huang[1]

[1] *School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan 610054, China.*
[2] *School of Science, Sichuan University of Science and Engineering, Zigong, Sichuan 643000, China.*

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Abstract. Schwarz waveform relaxation (SWR) algorithm has been investigated deeply and widely for regular time dependent problems. But for time delay problems, complete analysis of the algorithm is rare. In this paper, by using the reaction diffusion equations with a constant discrete delay as the underlying model problem, we investigate the convergence behavior of the overlapping SWR algorithm with Robin transmission condition. The key point of using this transmission condition is to determine a free parameter as better as possible and it is shown that the best choice of the parameter is determined by the solution of a min-max problem, which is more complex than the one arising for regular problems without delay. We propose new notion to solve the min-max problem and obtain a quasi-optimized choice of the parameter, which is shown efficient to accelerate the convergence of the SWR algorithm. Numerical results are provided to validate the theoretical conclusions.

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Key words: Schwarz method, waveform relaxation, time delay, min-max problem.

1 Introduction

In the past decade, Schwarz waveform relaxation (SWR) algorithm has received much attention by many authors. The algorithm is characterised by firstly partitioning the space domain into several overlapping subdomains, and then solving the subproblems simultaneously inside each subdomain through iterations; hence the algorithm is different from

*Corresponding author. *Email addresses:* `wushulin_ylp@163.com` (S.-L. Wu), `tzhuang@uestc.edu.cn` (T.-Z. Huang)

the classical domain decomposition method [2,3,30] and takes the form of waveform relaxation iteration (see [23–25,31] and references therein). We refer to [9,13,14] and [12] for the original idea of this algorithm.

Duo to the excellent capability in parallel computation for PDEs, the SWR algorithm is becoming more and more popular, particularly in the field of solving time dependent problems. It is a common point that the algorithm can be classified into two categories depending on the used transmission condition between the subdomains: the classical SWR algorithm and the optimized one. For the former, Dirichlet condition is used as transmission condition (see, e.g., [6,8,9,11–15,28]) and in this case the overlap between adjacent subdomains is essentially important to guarantee the convergence and the convergence rate can not be adjustable. It has been pointed out by Gander and Halpern [16] that Dirichlet condition is ineffective transmission condition. In particular, this transmission condition inhibits information exchange between subdomains and therefore the convergence rate of the classical SWR algorithm is slow.

To overcome this drawback, many authors utilize the transmission condition of Robin type on the artificial boundary interfaces (see, e.g., [1,4,16,19,21,22,26,27,32]). A free parameter, say p, is usually involved in this transmission condition and can be optimized technically to speed up the convergence of the algorithm. With the best p, the algorithm is called *optimized* SWR algorithm (for more details, we refer the interested reader to the systematic work by Gander and his colleagues [1,4,5,16–19,21,22,26,27,29], particularly to [1] and [16] for deep investigation of the optimization procedures for determining the best parameters). Nowadays, the optimized SWR algorithm is becoming more and more popular in the field of scientific and engineering computing and is adopted to solve complex problems, such as ferro-magnetics equations in the micro-magnetic model [10], wave equations [17,20], shallow-water problem [27,29] and Maxwell's equations [5], etc.

However, all of the aforementioned results are obtained for the regular PDEs without time delay. For PDEs with delay, the situation becomes very complex and the concrete results are rare. For example, the superlinear convergence of the classical SWR algorithm can be easily obtained for the regular reaction diffusion equations by using an existing results about inverse Fourier transform (see, e.g., [6,12,16]), while it is difficult and still unknown when time delay is taken into account. In the seminal paper [32], Vandewalle and Gander have shown that, the techniques proposed for the regular PDEs can not be straightforwardly applied to PDEs with time delay. In that paper, two representative model problems are studied: the heat equation with a constant discrete delay and one with a distributed delay. For the classical SWR algorithm, by using an elementary but very technical method they presented an estimate of the convergence rate. In our previous paper [34], we further analyzed the classical SWR algorithm for delay problems, where the reaction diffusion equations with a constant discrete delay were considered as the underlying model problems. We obtained a much sharper bound of the convergence rate and investigated the convergence behavior of the algorithm at semi-discrete level and in the case of arbitrary number of subdomains.

For the transmission condition of Robin type, the main difficulty arises from the complexity of the curve along which one needs to solve the min-max problems. To derive a proper parameter p involved in the transmission condition, Vandewalle and Gander [32] proposed a very technical idea: first, choosing a regular *box* which contains the complex curve, and then solving the min-max problem over the box, instead of along the curve. But they do not present the details about how to choose a proper box and how to solve the min-max problem over the box. Besides this, only the non-overlapping case was studied. In our previous paper [35], we investigated the convergence behavior of the Robin type SWR algorithm for the reaction diffusion equations with a constant discrete time delay and proposed a new idea to determine the involved parameter p. This idea is different from the one proposed in [32] and the key point lies in first choosing an upper bound of the convergence factor obtained in frequency domain, which is a function of the parameter p and the Fourier symbol ω, and then we solve the optimization problem with this upper bound in closed formulas. When the situation reduces to the heat equations with a constant discrete delay, it was shown that the quasi-optimized parameter derived by using this idea can result in much faster convergence compared to the parameter obtained in [32]. However, as in [32] we have only considered the non-overlapping situation in [35].

For delay PDEs, duo to the current status of the SWR algorithm with Robin transmission condition we believe that the convergence property has not been fully investigated yet, which is still open and remains challenging. This paper is a continuous work of [32] and [35]. Here, by using the same model problem as in [35] we continue to analyze the SWR algorithm with the Robin transmission condition. We consider the overlapping case and try to determine the parameter p as better as possible. The remainder of this paper is organized as follows. In Section 2, we introduce the model equations and the SWR algorithm with the Robin transmission condition. Our main results are presented in Section 3, where the min-max optimization problem is solved in great details. Section 4 provides several numerical examples to validate our theoretical results. Finally in Section 5, we finish this paper with some conclusion remarks.

2 Model problem and the SWR algorithm

Our guiding model is:

$$\begin{cases} \mathcal{L}u := \dfrac{\partial u}{\partial t} - \nu^2 \dfrac{\partial^2 u}{\partial x^2} + au(x,t) + du(x,t-\tau) = f(x,t), & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ u(x,t) = u_0(x,t), & (x,t) \in \mathbb{R} \times [-\tau, 0], \\ u(\pm\infty, t) = 0, & t \in \mathbb{R}^+, \end{cases} \tag{2.1}$$

where $\tau > 0$, $\nu > 0$ and a,d are constants with $a \neq 0$. This equation is also the basic model studied in the monograph [33]. For $\nu = 1$, $a = 0$ and $d\tau > 0$, (2.1) reduces to the one discussed in [32]. We decompose the space domain $\Omega = \mathbb{R}$ into two overlapping subdomains $\Omega_1 = (-\infty, L]$ and $\Omega_2 = [0, +\infty)$ with $L \geq 0$. The SWR algorithm then consists of solving iteratively subproblems on $\Omega_j\times\mathbb{R}^+ (j=1,\,2)$, using as a boundary condition at the interfaces $x=0$ and $x=L$ the values obtained from the previous iteration. The iterative scheme is thus for the k-th iteration given by

$$\begin{cases} \mathcal{L}u_j^k = f(x,t), & (x,t)\in\Omega_j\times\mathbb{R}^+,\\ u_j^k(x,t)=u_0(x,t), & (x,t)\in\Omega_j\times[-\tau,0],\\ \mathcal{B}_j u_j^k((2-j)L,t)=\mathcal{B}_j u_{3-j}^{k-1}((2-j)L,t), & t\in\mathbb{R}^+, \end{cases} \tag{2.2}$$

where $j=1,2$, u_1^0 and u_2^0 are initial guesses. The symbols $\mathcal{B}_1$ and $\mathcal{B}_2$ are given by

$$\mathcal{B}_1=\frac{\partial}{\partial x}+\frac{p}{\nu},\qquad \mathcal{B}_2=\frac{\partial}{\partial x}-\frac{p}{\nu}, \tag{2.3}$$

where p is a free parameter. The transmission condition defined by (2.3) is called Robin type. If we impose $\mathcal{B}_1=\mathcal{B}_2=\mathcal{I}$ in (2.2), the according algorithm is called *classical* SWR algorithm, where $\mathcal{I}$ is the identity operator.

From Theorem 2.1 given in [35], we know that for any $p>0$ each subproblem of (2.2) is well posed in the anisotropic Sobolev spaces $H^{r,s}(\Omega_j\times(0,T))=L^2(H^r(\Omega_j);(0,T))$ $\cap H^s(L^2(\Omega_j);\,(0,T))$, where $(0,T)$ denotes the time intervals. Duo to this, we only consider $p>0$ throughout this paper.

3 Towards the best choice of the parameter p

We denote the errors on subdomain Ω_j at iteration $k\geq 0$ by e_j^k, i.e.,

$$e_1^k=u|_{\Omega_1}-u_1^k,\qquad e_2^k=u|_{\Omega_2}-u_2^k.$$

We then have the following homogeneous equations

$$\begin{cases} \mathcal{L}e_j^k=0, & (x,t)\in\Omega_j\times\mathbb{R}^+,\\ e_j^k(x,t)=0, & (x,t)\in\Omega_j\times[-\tau,\,0],\\ \mathcal{B}_j e_j^k((2-j)L,t)=\mathcal{B}_j e_{3-j}^{k-1}((2-j)L,t), & t\in\mathbb{R}^+, \end{cases} \tag{3.1}$$

where $j=1,2$. We perform the Fourier transform in time of the error equations (3.1), and it leads to

$$\begin{cases} \dfrac{\partial^2\hat{e}_j^k(x,\omega)}{\partial x^2}-\dfrac{a+de^{-i\omega\tau}+i\omega}{\nu^2}\hat{e}_j^k(x,\omega)=0, & (x,\omega)\in\Omega_j\times\mathbb{R},\\ \mathcal{B}_j\hat{e}_j^k((2-j)L,\omega)=\mathcal{B}_j\hat{e}_{3-j}^{k-1}((2-j)L,\omega), & \omega\in\mathbb{R}, \end{cases} \tag{3.2}$$

where $\hat{e}_j^k(x,\omega)=\frac{1}{2\pi}\int_{\mathbb{R}}e_j^k(x,t)e^{-i\omega t}dt$ (we extend $e_j^k=0$ for $t<-\tau$ and denote the extension by e_j^k, too). We are thus led to solve an ordinary differential equation in each subdomain. The roots of the corresponding characteristic polynomial are

$$\lambda_+=\frac{1}{\nu}\sqrt{\lambda_c},\qquad \lambda_-=-\frac{1}{\nu}\sqrt{\lambda_c},\qquad \lambda_c=a+de^{-i\omega\tau}+i\omega, \tag{3.3}$$

where $\sqrt{\lambda_c}$ is the complex square root with positive real part. Routine calculation yields $\lambda_\pm = \pm\frac{\alpha(\omega)+i\delta(\omega)\beta(\omega)}{\nu}$, where $\delta(\omega)=\mathrm{sign}(\omega-d\sin(\omega\tau))$ and

$$\alpha(\omega)=\sqrt{\frac{a+d\cos(\omega\tau)+\sqrt{(a+d\cos(\omega\tau))^2+(\omega-d\sin(\omega\tau))^2}}{2}}, \tag{3.4a}$$

$$\beta(\omega)=\sqrt{\frac{-a-d\cos(\omega\tau)+\sqrt{(a+d\cos(\omega\tau))^2+(\omega-d\sin(\omega\tau))^2}}{2}}. \tag{3.4b}$$

Clearly, $\mathrm{Re}(\lambda^+)\geq 0$ and $\mathrm{Re}(\lambda^-)\leq 0$, and thus the solutions of (3.2) that do not increase exponentially at infinity are

$$\begin{cases} \hat{e}_1^k(x,\omega)=\alpha_k(\omega)e^{\lambda^+(x-L)}, & \text{for}\,(x,\omega)\in(-\infty,L)\times\mathbb{R}, \\ \hat{e}_2^k(x,\omega)=\beta_k(\omega)e^{\lambda^- x}, & \text{for}\,(x,\omega)\in(0,+\infty)\times\mathbb{R}, \end{cases} \tag{3.5}$$

where $\alpha_k(\omega)$ and $\beta_k(\omega)$ will be computed with the boundary conditions on $x=L$ and $x=0$:

$$\alpha_k(\omega)=\frac{\left(\lambda_-+\frac{p}{\nu}\right)e^{\lambda_- L}}{\lambda_++\frac{p}{\nu}}\beta_{k-1}(\omega), \quad \beta_k(\omega)=\frac{\left(\lambda_+-\frac{p}{\nu}\right)e^{-\lambda_+ L}}{\lambda_--\frac{p}{\nu}}\alpha_{k-1}(\omega). \tag{3.6}$$

Hence, the errors $\hat{e}_j^k(x,\omega)$ $(j=1,2)$ satisfy

$$\hat{e}_j^k(x,\omega)=\frac{\left(\lambda_-+\frac{p}{\nu}\right)\left(\lambda_+-\frac{p}{\nu}\right)}{\left(\lambda_++\frac{p}{\nu}\right)\left(\lambda_--\frac{p}{\nu}\right)}e^{(\lambda_--\lambda_+)L}\hat{e}_j^{k-2}(x,\omega), \quad j=1,2. \tag{3.7}$$

By using these relations and the well known Parseval-Plancherel identity we arrive at

$$\left\|e_j^k(x,\cdot)\right\|_{L^2(0,T)}\leq\rho(p,L)\left\|e_j^{k-2}(x,\cdot)\right\|_{L^2(0,T)}, \quad \forall x\in\Omega_j, \tag{3.8}$$

where $j=1,2$ and

$$\rho(p,L)=\max_{\omega\in\mathbb{R}}\left|\frac{\left(\lambda_-+\frac{p}{\nu}\right)\left(\lambda_+-\frac{p}{\nu}\right)}{\left(\lambda_++\frac{p}{\nu}\right)\left(\lambda_--\frac{p}{\nu}\right)}e^{(\lambda_--\lambda_+)L}\right|. \tag{3.9}$$

Definition 3.1. The quantity $\rho(p,\,L)$ defined by (3.9) is called the convergence factor of the SWR algorithm (2.2).

Clearly, the best constant p involved in the Robin transmission condition can be determined by the following min-max problem:

$$\min_{p>0}\max_{\omega\in\mathbb{R}}\left|\frac{\left(\lambda_-+\frac{p}{\nu}\right)\left(\lambda_+-\frac{p}{\nu}\right)}{\left(\lambda_++\frac{p}{\nu}\right)\left(\lambda_--\frac{p}{\nu}\right)}e^{(\lambda_--\lambda_+)L}\right|. \tag{3.10}$$

It is shown in [32] and [35] that the min-max problem (3.10) is essentially different from the one arising for regular PDEs without time delay and that it is usually impossible to obtain a closed formula for the solution. Duo to this, Vandewalle and Gander in [32] propose the so-called *box* technique to derive a *quasi-optimized* solution for the case $a=0$ and $L=0$. For $a=0$, the model problem (2.1) reduces to the heat equation with a constant discrete delay studied in [32]. In [35] we further consider the case $L=0$ and propose a new method to obtain quasi-optimized choice of the parameter. It is shown that by using the parameter obtained in [35] the algorithm converges faster.

In the sequel, we use the idea proposed in [35] to derive an approximate value of the solution of (3.10) with $L>0$. From the analysis given in Section 3 in [35], we know that the convergence factor $\rho(p,L)$ can be rewritten as

$$\rho(p,L)=\max_{\omega\in\mathbb{R}}\frac{(\xi(\omega)-p)^2+\xi^2(\omega)-a-d\cos(\omega\tau)}{(\xi(\omega)+p)^2+\xi^2(\omega)-a-d\cos(\omega\tau)}e^{-2\frac{\xi(\omega)}{\nu}L}, \tag{3.11}$$

where

$$\xi(\omega)=\sqrt{\frac{\sqrt{[a+d\cos(\omega\tau)]^2+[\omega-d\sin(\omega\tau)]^2}+a+d\cos(\omega\tau)}{2}}.$$

Let $\zeta_0=\min_{\omega\in\mathbb{R}}\xi(\omega)$ and then from [35] we have

$$\min_{p>0}\rho(p,L)\le\min_{p>0}\left(\max_{\zeta\ge\zeta_0}\frac{(\zeta-p)^2+\zeta^2-a+|d|}{(\zeta+p)^2+\zeta^2-a+|d|}e^{-2\frac{\zeta}{\nu}L}\right). \tag{3.12}$$

We remark that the approximation idea mentioned above leads to the upper line of the box, and with the minus sign one obtains the lower line. In the sequel, we try to solve the min-max problem in the right hand side of (3.12), which leads to an approximate value of the solution of the original min-max problem (3.10). To this end, we define the following arguments

$$\alpha=\left(\frac{2L}{\nu}\right)^2(a-|d|),\quad q=\frac{2L}{\nu}p,\quad y=\frac{2L}{\nu}\zeta,\quad y_0=\frac{2L}{\nu}\zeta_0, \tag{3.13a}$$

$$R(y,q,\alpha)=\frac{(y-q)^2+y^2-\alpha}{(y+q)^2+y^2-\alpha}e^{-y}, \tag{3.13b}$$

and then we get

$$\min_{p>0}\left(\max_{\zeta\ge\zeta_0}\frac{(\zeta-p)^2+\zeta^2-a+|d|}{(\zeta+p)^2+\zeta^2-a+|d|}e^{-2\frac{\zeta}{\nu}L}\right)=\min_{q>0}\left(\max_{y\ge y_0}R(y,q,\alpha)\right). \tag{3.14}$$

From Lemma 3.1 in [35] we know $\zeta_0^2\ge a-|d|$ and therefore the quantities y_0 and α defined by (3.13) satisfy $y_0^2\ge\alpha$.

Remark 3.1. For regular reaction diffusion equations, for example $d=0$ and $a>0$, it is easy to get $\zeta_0=\sqrt{a}$ and $\alpha=y_0^2$. In this situation, the min-max problem

$$\min_{q>0}\left(\max_{y\ge y_0}R(y,q,\alpha)\right)$$

can be solved in closed formulas [1,16]. However, if time delay occurs, i.e., $\tau>0$ and $d\neq 0$, we have $\alpha\leq y_0^2$ and this implies that we need to solve a more general min-max problem than the regular case.

Lemma 3.1. *For given $y\geq y_0>0$ and $q>0$, it holds*

$$\frac{\partial R(y,q,\alpha)}{\partial q}\left(q-\sqrt{2y^2-\alpha}\right)\geq 0.$$

Proof. A routine calculation yields

$$\frac{\partial R(y,q,\alpha)}{\partial q}=4y\frac{q^2-(2y^2-\alpha)}{[(y+q)^2+y^2-\alpha]^2}e^{-y}. \tag{3.15}$$

Hence, by using $y\geq y_0>0$ and $y_0^2\geq\alpha$ we have

$$\frac{\partial R(y,q,\alpha)}{\partial q}\left(q-\sqrt{2y^2-\alpha}\right)=4y\frac{q+\sqrt{2y^2-\alpha}}{[(y+q)^2+y^2-\alpha]^2}\left(q-\sqrt{2y^2-\alpha}\right)^2e^{-y}\geq 0,$$

and this completes the proof. □

Lemma 3.1 implies that the optimal parameter q which solves the min-max problem (3.14) should satisfy $q\geq\sqrt{2y_0^2-\alpha}$. Otherwise, for $y\geq y_0$ increasing q will decrease $R(y,q,\alpha)$.

Lemma 3.2. *For $\alpha\geq 0$, the cubic polynomial $S(\alpha,q)=-q^3-4q^2+q(4+2\alpha)+8\alpha$ has a unique positive root $q_1(\alpha)$ and with the argument $q_1(\alpha)$, we have*

1. *if $q\geq q_1(\alpha)$, the function $R(y,q,\alpha)$ is monotonously decreasing with respect to y for $y\in(0,+\infty)$;*
2. *if $0<q<q_1(\alpha)$, the function $R(y,q,\alpha)$ has a unique local maximum located at*

$$\bar{y}(\alpha,q)=\sqrt{\frac{2q+\alpha+\sqrt{qS(\alpha,q)}}{2}}. \tag{3.16}$$

If $\alpha\in(-\infty,\alpha_0]$, for any $q>0$ the function $R(y,q,\alpha)$ is monotonously decreasing with respect to y for $y\in(0,+\infty)$, where $\alpha_0=-0.10189181250394$.

Proof. We first note that $\frac{\partial S(\alpha,q)}{\partial q}=0$ has at most two roots $r_1=\frac{\sqrt{28+6\alpha}-4}{3}$ and $r_2=-\frac{\sqrt{28+6\alpha}+4}{3}$. Moreover, the larger one—r_1, must be a maximizer, since $S(\alpha,q)\to-\infty$ as $q\to+\infty$. Therefore, for $q\in(0,+\infty)$ there exist three cases for the distribution of the roots as shown in Fig. 1. Increasing α from $\alpha\ll 0$ to a positive quantity, we will first meet the left case, then the middle case, and finally the right case. Clearly, the right case occurs if and only if $\alpha\geq 0$, since $S(\alpha,0)=8\alpha$. Hence, if $\alpha\geq 0$ the cubic polynomial S has a unique positive root

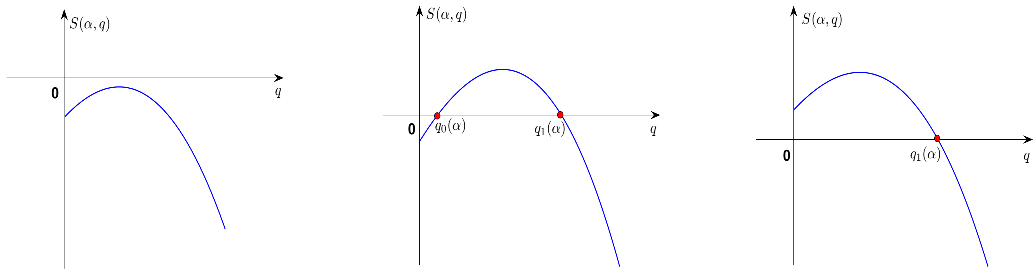

Figure 1: Sketch map of the distribution of the roots of $S(\alpha,q)$.

$q_1(\alpha)$. Moreover, we have $S(\alpha,q)>0$ if $0<q<q_1(\alpha)$ and $S(\alpha,q)<0$ if $q>q_1(\alpha)$. For $\alpha<0$, it is easy to know that there exist a critical value of α which distinguishes the left case and the middle case shown in Fig. 1. This critical value—denoted by α_0 here, is universal and can be determined by

$$S(\alpha_0,q)=0, \qquad \frac{\partial S(\alpha_0,q)}{\partial q}=0. \tag{3.17}$$

Solving (3.17) numerically, we get $\alpha_0=-0.10189181250394$. Hence, for $\alpha\le\alpha_0$ we have $S(\alpha,q)\le 0$ for $q>0$.

The cubic polynomial S is connected with the maximum of the function R with respect to y. Indeed, a partial derivative of R with respect to y leads to

$$\frac{\partial R(y,q,\alpha)}{\partial y}=-\frac{4y^4-4(2q+\alpha)y^2+(q^2+4q-\alpha)(q^2-\alpha)}{[(y+q)^2+y^2-\alpha]^2}e^{-y},$$

and therefore R has at most two local extrema

$$\bar{y}(\alpha,q)=\sqrt{\frac{2q+\alpha+\sqrt{qS(\alpha,q)}}{2}}, \qquad \underline{y}(\alpha,q)=\sqrt{\frac{2q+\alpha-\sqrt{qS(\alpha,q)}}{2}}, \tag{3.18}$$

where $qS(\alpha,q)$ is the discriminant. The larger quantity—$\bar{y}$, must be a maximizer, since $R\ge 0$ and R goes to 0 as $y\to+\infty$. We next consider the following three cases.

***Case 1**: $\alpha\ge 0$ and $q\ge q_1(\alpha)$.* For $\alpha\ge 0$ and $q>q_1(\alpha)$, we have $S(\alpha,q)<0$ for $q>q_1(\alpha)$, which implies that the function $R(y,q,\alpha)$ is monotonously decreasing with respect to y, since $R(y,q,\alpha)\to 0$ as $y\to+\infty$. For $q=q_1(\alpha)$, we have

$$\bar{y}(\alpha,q)=\underline{y}(\alpha,q)=\sqrt{\frac{2q_1(\alpha)+\alpha}{2}}>0.$$

Moreover, routine calculation yields

$$\left.\frac{\partial^2 R(y,q,\alpha)}{\partial y^2}\right|_{q=q_1(\alpha),\ y=\bar{y}}=-8\bar{y}\frac{2\bar{y}^2-(2q_1(\alpha)+\alpha)}{[(\bar{y}+q_1(\alpha))^2+\bar{y}^2-\alpha]^2}e^{-\bar{y}}=0, \tag{3.19a}$$

$$\left.\frac{\partial^3 R(y,q,\alpha)}{\partial y^3}\right|_{q=q_1(\alpha),\ y=\bar{y}}=-\frac{32\bar{y}^2}{[(\bar{y}+q_1(\alpha))^2+\bar{y}^2-\alpha]^2}e^{-\bar{y}}\neq 0. \tag{3.19b}$$

Therefore, we know that $y=\sqrt{\frac{2q_1(\alpha)+\alpha}{2}}$ is not a maximizer of $R(y,p_1(\alpha),\alpha)$. Hence, we get the conclusion that the function $R(y,q,\alpha)$ is monotonously decreasing with respect to y for $p\geq p_1(\alpha)$.

***Case 2**: $\alpha\geq 0$ and $q\in(0,q_1(\alpha))$.* In this case, we have $S(\alpha,q)>0$ and therefore $R(y,q,\alpha)$ has a unique local maximum located at $y=\bar{y}$.

***Case 3**: $\alpha\leq\alpha_0$.* If $\alpha=\alpha_0$ and $q=q_1(\alpha_0)$, we have

$$\bar{y}(\alpha_0,q_1(\alpha_0))=\underline{y}(\alpha_0,q_1(\alpha_0))=\sqrt{\frac{2q_1(\alpha_0)+\alpha_0}{2}}>0.$$

A treatment similar to (3.19) will show that $\bar{y}(\alpha_0,q_1(\alpha_0))$ is not a maximizer of R. For $\alpha<\alpha_0$, we have $S(\alpha,q)<0$ and this implies that the function R is monotonously decreasing with respect to y. It then follows by combing these two situations that the function $R(y,q,\alpha)$ is monotonously decreasing with respect to y for any $q>0$. □

Based on Lemma 3.2, we present the first result about the solution of the min-max problem

$$\min_{q>0}\max_{y\geq y_0}R(y,q,\alpha). \tag{3.20}$$

Theorem 3.1. *Let $L>0$, $\tilde{q}_0=\sqrt{2y_0^2-\alpha}$, $\alpha_0=-0.10189181250394$ and α, y_0 be defined by* (3.13). *Assume $\alpha\leq\alpha_0$ (i.e., $(a-|d|)(2L/\nu)^2\leq\alpha_0$) and $y_0>0$, then for the overlapping SWR algorithm* (2.2) *the parameter p involved in the Robin transmission condition* (2.3) *can be approximated by $p=p^*:=q^*\frac{\nu}{2L}$, where $q^*=\tilde{q}_0$ is the solution of the min-max problem* (3.20).

Proof. By using the conclusion concerning $\alpha\leq\alpha_0$ given by Lemma 3.2 we get

$$\min_{q>0}\max_{y\geq y_0}R(y,q,\alpha)=\min_{q>0}R(y_0,q,\alpha).$$

By Lemma 3.1, we know that $R(y_0,q,\alpha)$ decreases for $q\in(0,\tilde{q}_0)$ and increases for $q\in(\tilde{q}_0,+\infty)$. Moreover, from (3.15) we get $\partial_q R(y_0,\tilde{q}_0,\alpha)=0$ and therefore $R(y_0,q,\alpha)$ gets its global minimum at $q=\tilde{q}_0$. □

Lemma 3.3. *Let*

$$X(q)=-q^4-4q^3+2\alpha q^2+(4\alpha+8y_0^2)q-(\alpha-2y_0^2)^2.$$

Assume $y_0>0$ and $y_0^2\geq\alpha\geq 0$ and then $X(q)$ has at most two different positive roots. Moreover, if $X(q)$ has two different positive roots x_1 and x_2 ($x_1<x_2$) it holds $X(q)<0$ for $q\in(0,x_1)\cup(x_2,+\infty)$, $X(q)\geq 0$ for $q\in[x_1,x_2]$ and $x_1\leq\tilde{q}_0$, where $\tilde{q}_0=\sqrt{2y_0^2-\alpha}$.

Proof. Since $X'(q)=4(-q^3-3q^2+\alpha q+\alpha+2y_0^2)$, $X'(q)$ has at most three different positive roots. Because $X^{(3)}(q)=-24(q+1)<0$ for $q>0$, $X'(q)$ does not have local minimum for $q>0$. Since $X'(0)=4(\alpha+2y_0^2)>0$ and $X'(q)=-\infty$ as $q\to+\infty$, it is easy to know that $X'(q)$ has a unique positive root. This means that $X(q)$ has at most two different positive roots.

If $X(q)$ has two different positive roots x_1 and x_2 with $x_1<x_2$, it is easy to get $X(q)<0$ for $q\in(0,x_1)\cup(x_2,+\infty)$ and $X(q)\geq 0$ for $q\in[x_1,x_2]$, since $X(0)=-(\alpha-2y_0^2)^2<0$ and $X(q)=-\infty$ as $q\to+\infty$. It remains to prove $x_1\leq\tilde{q}_0$. We suppose by the contrary that $x_1>\tilde{q}_0$ and thus it holds $X(\tilde{q}_0)<0$ and $X'(\tilde{q}_0)>0$. This gives

$$-\tilde{q}_0^4-4\tilde{q}_0^3+2\alpha\tilde{q}_0^2+(4\alpha+8y_0^2)\tilde{q}_0-\tilde{q}_0^4<0,\qquad -\tilde{q}_0^3-3\tilde{q}_0^2+\alpha\tilde{q}_0+\alpha+2y_0^2>0,$$

i.e.,

$$-\tilde{q}_0^3-2\tilde{q}_0^2+\alpha\tilde{q}_0+2(\alpha+2y_0^2)<0,\qquad -\tilde{q}_0^3-3\tilde{q}_0^2+\alpha\tilde{q}_0+\alpha+2y_0^2>0. \tag{3.21}$$

We therefore get

$$-\tilde{q}_0^3-3\tilde{q}_0^2+\alpha\tilde{q}_0+\alpha+2y_0^2>-\tilde{q}_0^3-2\tilde{q}_0^2+\alpha\tilde{q}_0+2(\alpha+2y_0^2),$$

i.e., $-\tilde{p}_0^2-(\alpha+2y_0^2)>0$. Clearly, this is a contraction. $\square$

With the argument $q_1(\alpha)$ introduced in Lemma 3.2, we define

$$X_1=\begin{cases} x_1, & \text{if } X(q) \text{ has two positive roots } x_1 \text{ and } x_2\ (x_1<x_2),\\ -1, & \text{otherwise,}\end{cases} \tag{3.22a}$$

$$X_2=\begin{cases} x_2, & \text{if } X(q) \text{ has two positive roots } x_1 \text{ and } x_2\ (x_1<x_2),\\ 0, & \text{otherwise,}\end{cases} \tag{3.22b}$$

$$\tilde{X}_2=\begin{cases} X_2, & \text{if } \frac{\tilde{q}_0^2}{2}\geq q_1(\alpha),\\ q_1(\alpha), & \text{if } \frac{\tilde{q}_0^2}{2}<q_1(\alpha).\end{cases} \tag{3.22c}$$

Lemma 3.4. *Let $L>0$, $\alpha\geq 0$ and $Z(q)=q-\sqrt{2q+\sqrt{qS(\alpha,q)}}$ and $q\in(0,q_1(\alpha))$, where $S(\alpha,q)$ is the cubic polynomial defined in Lemma* 3.2 *and $q_1(\alpha)$ is the unique positive root of $S(\alpha,q)$. Then we have*

1. *if $q_1(\alpha)\in(0,2]$, it holds $Z(q)<0$ for $q\in(0,\ q_1(\alpha))$;*
2. *if $q_1(\alpha)>2$, the function $Z(q)$ has a unique positive root $z_0\in(0,\ q_1(\alpha))$. Moreover, it holds $Z(q)<0$ for $q\in(0,z_0)$ and $Z(q)>0$ for $q\in(z_0,\ q_1(\alpha))$.*

Proof. Let

$$\tilde{Z}(q)=Z(q)\left[q+\sqrt{2q+\sqrt{qS(\alpha,q)}}\right]=q^2-2q-\sqrt{qS(\alpha,q)}.$$

It is obvious that $\tilde{Z}(q)<0$, $\forall q\in(0,q_1(\alpha))$ if $q_1(\alpha)\in(0,2]$ and this gives $Z(q)<0$ for $q\in(0,q_1(\alpha))$ with $q_1(\alpha)\in(0,2]$. To discuss the case $q_1(\alpha)>2$, we first note that if $\alpha=0$ it holds $q_1(\alpha)=2(\sqrt{2}-1)<2$. Hence, $q_1(\alpha)>2$ implies $\alpha>0$. It is clear that $\tilde{Z}(q)<0$ for $q\in(0,2]$ and $\tilde{Z}(q_1(\alpha))=q_1(\alpha)[q_1(\alpha)-2]>0$. Therefore, $\tilde{Z}$ shall has at least one root z_0 which satisfies $z_0\in(2,\, q_1(\alpha))$. On the other hand, for $q\in(2,q_1(\alpha)]$ we have

$$\tilde{Z}(q)=0\Leftrightarrow (q^2-2q)=qS(\alpha,q) \;\Leftrightarrow\; q^3-\alpha q-4\alpha=0. \tag{3.23}$$

For $\alpha>0$, it is easy to show that the last equation $q^3-\alpha q-4\alpha=0$ has only one positive root. Therefore, the root z_0 of $Z(q)$ is unique. By using the continuity of $\tilde{Z}$, it is easy to get $\tilde{Z}(q)<0$ for $q\in(0,z_0)$ and $\tilde{Z}(q)>0$ for $q\in(z_0,q_1(\alpha)]$. $\square$

Theorem 3.2. *Let* $\tilde{q}_0=\sqrt{2y_0^2-\alpha}$ *and* α, y_0 *be defined by* (3.13). *Assume* $\alpha\geq 0$ *(i.e.,* $a\geq|d|$*) and* $y_0>0$. *Then the performance of the overlapping SWR algorithm* (2.2) *with Robin transmission condition* (2.3) *can be optimized for* $p=p^*:=q^*\frac{\nu}{2L}$, *where* q^*, *the solution of the min-max problem* (3.20), *is given by*

$$q^*=\begin{cases} q_1^*, & \text{if } Z(q_0^*)\geq 0,\\ q_0^*, & \text{otherwise},\end{cases} \tag{3.24a}$$

provided $(0,q_1(\alpha))\cap(X_1,X_2)\neq\varnothing$, $\tilde{X}_2>\tilde{q}_0$ *and* $R(y_0,\tilde{q}_0,\alpha)<R(\bar{y}(\alpha,\tilde{q}_0),\tilde{q}_0,\alpha)$; *otherwise,* $q^*=\tilde{q}_0$. *Here,*

$$Z(q)=q-\sqrt{2q+\sqrt{qS(\alpha,q)}},$$

X_1, X_2 *and* $\tilde{X}_2$ *are defined by* (3.22a)-(3.22c), $q_1(\alpha)$ *is the unique positive root of the cubic polynomial* $S(\alpha,q)$, q_1^* *is the unique solution of* $Z(q)=0$ *and* q_0^* *is the unique solution of the following equation*

$$R(\bar{y}(\alpha,q),q,\alpha)=R(y_0,q,\alpha). \tag{3.24b}$$

Note that, for $\alpha\geq 0$ and $q\in(0,\, q_1(\alpha))$ it follows by applying Lemma 3.2 that

$$\frac{dR(\bar{y}(\alpha,q),q,\alpha)}{dq}=\frac{\partial R(\bar{y},p,\alpha)}{\partial y}\times\frac{\partial\bar{y}(\alpha,q)}{\partial q}+\frac{\partial R(\bar{y}(\alpha,q),q,\alpha)}{\partial q}=\frac{\partial R(\bar{y}(\alpha,q),q,\alpha)}{\partial q},$$

since $\frac{\partial R(\bar{y},p,\alpha)}{\partial y}=0$ (because $\bar{y}$ is a local maximizer of R). Hence, by using Lemma 3.1 we have

$$\frac{dR(\bar{y}(\alpha,q),q,\alpha)}{dq}\left(q-\sqrt{2\bar{y}^2(\alpha,q)-\alpha}\right)\geq 0.$$

Now, keeping this in mind we begin our proof.

Proof. Since $\alpha\geq 0$, from the first conclusion of Lemma 3.2 we know

$$\min_{q>0}\left(\max_{y\geq y_0}R(y,q,\alpha)\right)=\min\left\{\min_{q\geq q_1(\alpha)}R(y_0,q,\alpha),\ \min_{0<q<q_1(\alpha)}\left(\max_{y\geq y_0}R(y,q,\alpha)\right)\right\}. \tag{3.25}$$

For $q\in(0,q_1(\alpha))$, we know from the second conclusion of Lemma 3.2 that the function R has a unique maximum at $y=\bar{y}(\alpha,q)$. Therefore, for $q\in(0,q_1(\alpha))$ it holds

$$\max_{y\geq y_0} R(y,q,\alpha)=\begin{cases} R(y_0,q,\alpha), & \text{if } \bar{y}(\alpha,q)\leq y_0,\\ \max\{R(y_0,q,\alpha),R(\bar{y}(\alpha,q),q,\alpha)\}, & \text{otherwise.}\end{cases} \tag{3.26}$$

Hence, we first need to know under what condition it holds $\bar{y}(\alpha,q)>y_0$. To this end, we note

$$\bar{y}(\alpha,q)>y_0 \Leftrightarrow \sqrt{qS(\alpha,q)}>2\left(\frac{\tilde{q}_0^2}{2}-q\right). \tag{3.27}$$

Clearly, if $\frac{\tilde{q}_0^2}{2}\leq q<q_1(\alpha)$ $\big($of course, this requires $\frac{\tilde{q}_0^2}{2}<q_1(\alpha)\big)$, we have $\bar{y}(\alpha,q)>y_0$. If $q\leq\frac{\tilde{q}_0^2}{2}$, routine calculation yields

$$\bar{y}(\alpha,q)>y_0 \Leftrightarrow \sqrt{qS(\alpha,q)}>2\left(\frac{\tilde{q}_0^2}{2}-q\right) \Leftrightarrow qS(\alpha,q)>4\left(\frac{\tilde{q}_0^2}{2}-q\right)^2 \Leftrightarrow X(q)>0, \tag{3.28}$$

where $X(q)$ is the quartic polynomial defined in Lemma 3.3. To determine the best parameter which solves the min-max problem (3.20), we consider the following two cases.

Case A*: $(0,q_1(\alpha))\cap(X_1,X_2)\neq\emptyset$, i.e.,* $0<X_1<q_1(\alpha)$. In this case, we know that the quartic polynomial $X(q)$ has two different positive roots x_1 and x_2 and from (3.22a)-(3.22b) we have $X_1=x_1$ and $X_2=x_2$. We claim

$$\bar{y}(\alpha,q)>y_0, \text{ if and only if } q\in\big(X_1,\ \tilde{X}_2\big), \tag{3.29}$$

where $\tilde{X}_2$ is defined by (3.22c). The proof of (3.29) is divided into two cases and depends on $\frac{\tilde{q}_0^2}{2}\geq q_1(\alpha)$ or not.

If $\frac{\tilde{q}_0^2}{2}\geq q_1(\alpha)$, by using (3.28) and Lemma 3.3, we have

$$\bar{y}(\alpha,q)>y_0, \text{ if and only if } q\in(X_1,\min\{X_2,q_1(\alpha)\}). \tag{3.30}$$

Suppose $q_1(\alpha)<X_2$ and then we have $q_1(\alpha)\in(X_1,X_2)$ and this gives $X(q_1(\alpha))>0$. By using (3.28), this however implies

$$\sqrt{q_1(\alpha)S(\alpha,q_1(\alpha))}>2\left(\frac{\tilde{q}_0^2}{2}-q_1(\alpha)\right),$$

i.e., $0>2\big(\frac{\tilde{q}_0^2}{2}-q_1(\alpha)\big)$, which is a contradiction since $\frac{\tilde{q}_0^2}{2}\geq q_1(\alpha)$. Hence, it holds $q_1(\alpha)\geq X_2$ and this together with (3.30) gives

$$\bar{y}(\alpha,q)>y_0, \text{ if and only if } q\in(X_1,X_2). \tag{3.31}$$

If $\frac{\tilde{q}_0^2}{2}<q_1(\alpha)$, from (3.27)-(3.28) and Lemma 3.3 we get

$$\bar{y}(\alpha,q)>y_0, \text{ if and only if } q\in\left\{\left(0,\frac{\tilde{q}_0^2}{2}\right)\cap(X_1,X_2)\right\}\cup\left[\frac{\tilde{q}_0^2}{2},\ q_1(\alpha)\right). \tag{3.32}$$

Since $\frac{\tilde{q}_0^2}{2}<q_1(\alpha)$, we get

$$\left[\sqrt{qS(\alpha,q)}-2\left(\frac{\tilde{q}_0^2}{2}-q\right)\right]_{q=\tilde{q}_0^2/2}>0.$$

Hence, by using (3.28) and Lemma 3.3 we have $X\left(\frac{\tilde{q}_0^2}{2}\right)>0$ and this implies $\frac{\tilde{q}_0^2}{2}\in(X_1,X_2)$. Therefore, from (3.32) we get

$$\bar{y}(\alpha,q)>y_0,\ \text{ if and only if } q\in(X_1,\ q_1(\alpha)). \tag{3.33}$$

Now, using the argument $\tilde{X}_2$ defined by (3.22c), we get (3.29) by combining (3.31) and (3.33). Moreover, from the aforementioned analysis we know

$$\tilde{X}_2\leq q_1(\alpha). \tag{3.34}$$

Hence, for $0<X_1<q_1(\alpha)$ it follows by using (3.25), (3.26) and (3.29) that

$$\begin{aligned}&\min_{q>0}\max_{y\geq y_0}R(y,q,\alpha)\\ =&\min\left\{\min_{q\in(X_1,\tilde{X}_2)}(\max\{R(y_0,q,\alpha),R(\bar{y}(\alpha,q),q,\alpha)\}),\min_{q\in(0,X_1]\cup[\tilde{X}_2,+\infty)}R(y_0,q,\alpha)\right\}.\end{aligned} \tag{3.35}$$

We next claim

$$R(y_0,\tilde{X}_2,\alpha)\geq R(\bar{y}(\alpha,\tilde{X}_2),\tilde{X}_2,\alpha). \tag{3.36}$$

If $\tilde{X}_2=X_2$, we have $X(X_2)=0$, since $X_2=x_2$. By routine calculation we get $\bar{y}(\alpha,X_2)=y_0$, which gives (3.36) in the sense of "=". If $\tilde{X}_2=q_1(\alpha)$, by using the first conclusion in Lemma 3.2 we know $R(y,q_1(\alpha),\alpha)\leq R(y_0,q_1(\alpha),\alpha)$ for all $y\geq y_0$. Beginning with (3.34)-(3.36), we consider in the sequel two cases which depend on $\tilde{X}_2\leq\tilde{q}_0$ or not.

ⓐ $\tilde{X}_2\leq\tilde{q}_0$. In this case, for any $q\in(X_1,\tilde{X}_2)$ we know from Lemma 3.1 that $\frac{\partial R(y_0,q,\alpha)}{\partial q}\leq 0$. Moreover, since $\bar{y}(\alpha,q)>y_0$ holds for any $q\in(X_1,\tilde{X}_2)$, we get $q-\sqrt{2\bar{y}^2-\alpha}<q-\tilde{q}_0<0$ and this together with Lemma 3.1 gives $\frac{\partial R(\bar{y},q,\alpha)}{\partial q}\leq 0$. Hence, we know that both $R(y_0,q,\alpha)$ and $R(\bar{y}(\alpha,q),q,\alpha)$ decrease monotonically for $q\in(X_1,\tilde{X}_2)$. We thus get

$$\begin{aligned}&\min_{q\in(X_1,\tilde{X}_2)}(\max\{R(y_0,q,\alpha),R(\bar{y}(\alpha,q),q,\alpha)\})\\ =&\max\{R(y_0,\tilde{X}_2,\alpha),R(\bar{y}(\alpha,\tilde{X}_2),\tilde{X}_2,\alpha)\}=R(y_0,\tilde{X}_2,\alpha),\end{aligned} \tag{3.37}$$

where in the last equality we have used (3.36). For $q\in(0,X_1]\cup[\tilde{X}_2,+\infty)$, by using Lemma 3.1 we have

$$\min_{q\in(0,X_1]\cup[\tilde{X}_2,+\infty)}R(y_0,q,\alpha)=R(y_0,\tilde{q}_0,\alpha), \tag{3.38}$$

since $\tilde{X}_2 \le \tilde{q}_0$. Then, by combing (3.37) and (3.38) it follows by using (3.35) that

$$\min_{q>0} \max_{y\ge y_0} R(y,q,\alpha) = \min\{R(y_0,\tilde{X}_2,\alpha), R(y_0,\tilde{q}_0,\alpha)\} = R(y_0,\tilde{q}_0,\alpha), \tag{3.39}$$

where in the last equality we have used $\tilde{X}_2 \le \tilde{q}_0$ and Lemma 3.1 again.

ⓑ $\tilde{X}_2 > \tilde{q}_0$. In this case, we know from Lemma 3.3 that $\tilde{q}_0 \in (X_1, \tilde{X}_2)$. Since $\tilde{X}_2 \le q_1(\alpha)$ (see (3.34)) and $\bar{y}(\alpha,q) > y_0$ for $q \in (X_1,\tilde{X}_2)$, we know from Lemma 3.1 that both $R(y_0,q,\alpha)$ and $R(\bar{y},q,\alpha)$ decrease monotonically for $q \in (X_1, \tilde{q}_0]$. Besides this, we also know that the function $R(y_0,q,\alpha)$ is increasing for $q \ge \tilde{q}_0$ and that increasing q from $\tilde{q}_0$ decreases $R(\bar{y},q,\alpha)$ provided $q < \sqrt{2\bar{y}^2(\alpha,q) - \alpha}$, after which $R(\bar{y},q,\alpha)$ will increase. We next consider the following two situations for $q \in [\tilde{q}_0, \tilde{X}_2]$.

① $R(y_0,\tilde{q}_0,\alpha) \ge R(\bar{y}(\alpha,\tilde{q}_0),\tilde{q}_0,\alpha)$. In this case, it is easy to get

$$\min_{q\in(X_1,\tilde{X}_2)} (\max\{R(y_0,q,\alpha), R(\bar{y}(\alpha,q),q,\alpha)\}) = R(y_0,\tilde{q}_0,\alpha), \tag{3.40}$$

since $R(y_0,q,\alpha)$ is increasing for $q \ge \tilde{q}_0$; see the illustration shown in Fig. 2. Moreover, since $\tilde{q}_0 \in (X_1,\tilde{X}_2)$ we have

$$\min_{q\in(0,X_1]\cup[\tilde{X}_2,+\infty)} R(y_0,q,\alpha) \ge R(y_0,\tilde{q}_0,\alpha).$$

Hence, by using (3.35) we get

$$\min_{q>0} \max_{y\ge y_0} R(y,q,\alpha) = R(y_0,\tilde{q}_0,\alpha).$$

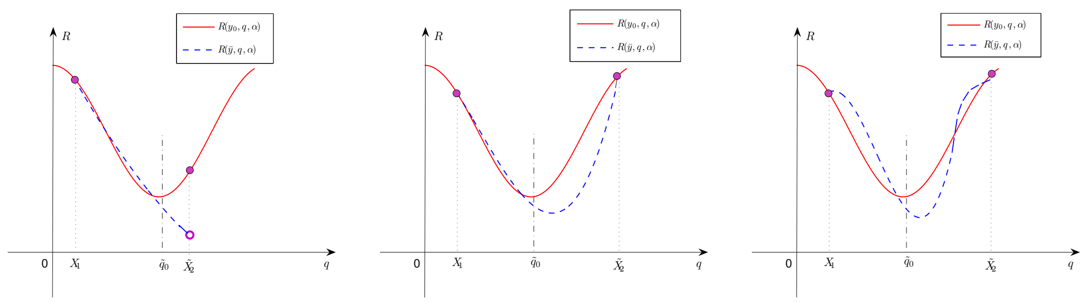

Figure 2: Illustration of three representative relationship between the two functions $R(y_0,q,\alpha)$ (solid line) and $R(\bar{y},q,\alpha)$ (dash line) for $R(y_0,\tilde{q}_0,\alpha) \ge R(\bar{y}(\alpha,\tilde{q}_0),\tilde{q}_0,\alpha)$.

② $R(y_0,\tilde{q}_0,\alpha) < R(\bar{y}(\alpha,\tilde{q}_0),\tilde{q}_0,\alpha)$. In this case, by using (3.36) and the continuity of $R(\bar{y},q,\alpha)$, we know that $R(y_0,q,\alpha)$ and $R(\bar{y}(\alpha,q),q,\alpha)$ intersect for $q \in (\tilde{q}_0, \tilde{X}_2]$. Moreover, duo to the monotonicity of these two functions the following equation has a unique solution for $\tilde{q}_0 < q < q_1^*$:

$$R(\bar{y}(\alpha,q),q,\alpha) = R(y_0,q,\alpha), \tag{3.41}$$

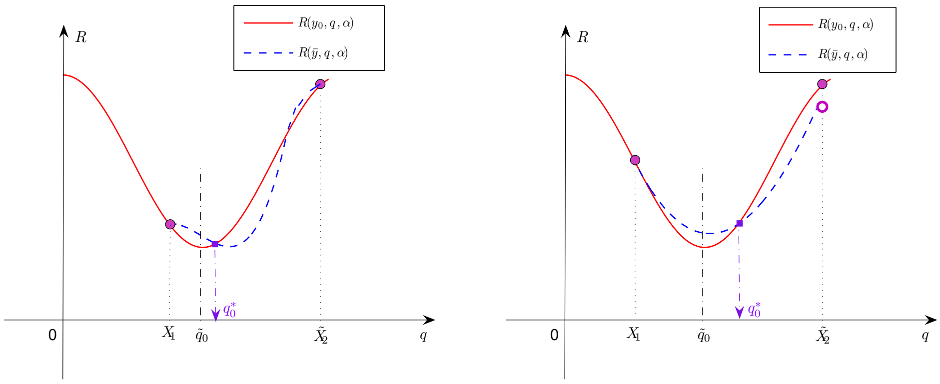

Figure 3: Illustration of two representative relationship between the two functions $R(y_0,q,\alpha)$ (solid line) and $R(\bar{y},q,\alpha)$ (dash line) for $R(y_0,\tilde{q}_0,\alpha)<R(\bar{y}(\alpha,\tilde{q}_0),\tilde{q}_0,\alpha)$. Left: $Z(q_0^*)<0$; Right: $Z(q_0^*)\geq 0$.

where q_1^* is the unique positive solution of $Z(q)=0$ (the uniqueness of q_1^* is given by Lemma 3.4). We denote the unique solution of (3.41) by q_0^*. Then, by using Lemma 3.1 we know that if $q_0^*<\sqrt{2\bar{y}^2(\alpha,q_0^*)-\alpha}$, i.e., $Z(q_0^*)<0$, the maximum of R is minimized for $q=q_0^*$; see the illustration shown in Fig. 3 on the left. Otherwise, the best parameter q^* is determined by $q^*=q_1^*$, i.e., $q^*=\sqrt{2\bar{y}^2(\alpha,q^*)-\alpha}$. The situation $Z(q_0^*)\geq 0$ is shown in Fig. 3 on the right.

***Case B**:* $(0,\ q_1(\alpha))\cap(X_1,\ X_2)=\varnothing$. In this case, we first claim $(0,\ q_1(\alpha))\cap\big[\frac{\tilde{q}_0^2}{2},\ +\infty\big)=\varnothing$. By the contrary, we shall have $q_1(\alpha)>\frac{\tilde{q}_0^2}{2}$ and this gives

$$\left[\sqrt{qS(\alpha,q)}-2\left(\frac{\tilde{q}_0^2}{2}-q\right)\right]_{q=\tilde{q}_0^2/2}>0.$$

This together with (3.28) gives $X\big(\frac{\tilde{q}_0^2}{2}\big)>0$. By using Lemma 3.3 we know $\frac{\tilde{q}_0^2}{2}\in(X_1,\ X_2)$. Hence, we get a contradiction, since $q_1(\alpha)>\frac{\tilde{q}_0^2}{2}$ and $\frac{\tilde{q}_0^2}{2}\in(X_1,\ X_2)$ implies $(0,\ q_1(\alpha))\cap(X_1,\ X_2)\neq\varnothing$.

Now, by using $(0,q_1(\alpha))\cap\big[\frac{\tilde{q}_0^2}{2},+\infty\big)=\varnothing$, we have $\frac{\tilde{q}_0^2}{2}\geq q_1(\alpha)$. Moreover, by using $(0,q_1(\alpha))\cap(X_1,X_2)=\varnothing$, we know $X_2=0$ or $q_1(\alpha)\leq X_1$. Therefore, by using (3.28) and Lemma 3.3 we get $\bar{y}(\alpha,q)\leq y_0$ for $q\in(0,\ q_1(\alpha))$. Hence, from (3.25) and (3.26) we get

$$\min_{q>0}\left(\max_{y\geq y_0}R(y,q,\alpha)\right)=\min_{q>0}R(y_0,q,\alpha)=R(y_0,\tilde{q}_0,\alpha), \tag{3.42}$$

where in the last equality we have used Lemma 3.1. □

For the case $\alpha_0<\alpha<0$ which corresponds to the middle case shown in Fig. 1, the equality (3.25) does not hold and we need to consider more cases in the right hand side of (3.25). Therefore, in this case the analysis of finding the solution of (3.20) is more complex than the case $\alpha \geq 0$. A complete analysis is tedious and does not give any new insight compared to the proof of Theorem 3.2. Hence, we do not plan to analyze this case in this paper.

Remark 3.2. The proof of Theorem 3.2 is performed *case by case* and is complicated. It is however can be shown that each case can occur by choosing special values of α and y_0. The following values are selected for this purpose:

Case A-ⓐ: $\alpha=1$, $y_0=1.5$. For this choice, we have

$$\begin{aligned}&\tilde{q}_0=1.87082869338697, \quad q_1(\alpha)=1.80044140002619,\\&X_1=0.56524881023725, \quad X_2=1.80013328000739, \quad \tilde{X}_2=1.80044140002619,\end{aligned}$$

which implies $(0,q_1(\alpha))\cap(X_1,X_2)\neq\varnothing$ and $\tilde{X}_2\leq\tilde{q}_0$;

Case A-ⓑ-①: $\alpha=0.8$, $y_0=1.336$. For this choice, we have

$$\begin{aligned}&\tilde{q}_0=1.66426920899234, \quad q_1(\alpha)=1.66620550558771,\\&X_1=0.44304599273716, \quad X_2=1.65514565316201, \quad \tilde{X}_2=1.66620550558771,\\&R(y_0,\tilde{q}_0,\alpha)=0.02876421298830, \quad R(\bar{y}(\alpha,y_0),\tilde{q}_0,\alpha)=0.02875253464464.\end{aligned}$$

Hence, we get $(0,q_1(\alpha))\cap(X_1,X_2)\neq\varnothing$, $\tilde{X}_2>\tilde{q}_0$ and $R(y_0,\tilde{q}_0,\alpha)\geq R(\bar{y}(\alpha,y_0),\tilde{q}_0,\alpha)$;

Case A-ⓑ-②: $\alpha=1$, $y_0=1.25$. With this choice, we have

$$\begin{aligned}&\tilde{q}_0=1.45773797371133, \quad q_1(\alpha)=1.80044140002619,\\&X_1=0.26993212134574, \quad X_2=1.74032740687594, \quad \tilde{X}_2=1.80044140002619,\\&R(y_0,\tilde{q}_0,\alpha)=0.02198068148992, \quad R(\bar{y}(\alpha,y_0),\tilde{q}_0,\alpha)=0.03032604729804.\end{aligned}$$

Hence, we get $(0,q_1(\alpha))\cap(X_1,X_2)\neq\varnothing$, $\tilde{X}_2>\tilde{q}_0$ and $R(y_0,\tilde{q}_0,\alpha)<R(\bar{y}(\alpha,y_0),\tilde{q}_0,\alpha)$;

Case B: $\alpha=\frac{16}{15}$, $y_0=1.941$. Under this choice, we have $\tilde{q}_0=2.54328435951101$ and $q_1(\alpha)=1.84249172393323$. Moreover, it is easy to verify that the quartic polynomial $X(q)$ defined in Lemma 3.3 does not have positive roots. Hence, $(0,q_1(\alpha))\cap[\frac{\tilde{q}_0^2}{2},+\infty)=\varnothing$ and $(0,q_1(\alpha))\cap(X_1,X_2)=\varnothing$.

Remark 3.3. We have stated in Remark 3.1 that, for regular reaction diffusion equations without time delay the quantity α equals to y_0^2. For $\alpha=y_0^2$, we get $\tilde{q}_0=y_0$ and

$$S(y_0,y_0)=y_0^3+4y_0^2+4y_0>0, \quad X(y_0)=8y_0^3>0, \quad \bar{y}(y_0,y_0)=\sqrt{y_0^2+2y_0}, \tag{3.43}$$

which gives $X_2>y_0>X_1$, $q_1(y_0^2)>y_0$ and $\bar{y}>y_0$. Hence $\tilde{X}_2>y_0$ and $q_1(y_0^2)>X_1$. Moreover, it is easy to get

$$\frac{\partial R(y,y_0,y_0^2)}{\partial y}=-\frac{4y^4-4(2y_0+y_0^2)y^2}{[(y+y_0)^2+y^2-y_0^2]^2}e^{-y}=-4y^2\frac{y^2-2y_0-y_0^2}{[(y+y_0)^2+y^2-y_0^2]^2}e^{-y}, \tag{3.44}$$

which implies that $R(y,y_0,y_0^2)$ is monotonically increasing for $y\in[y_0,\sqrt{y_0^2+2y_0}]$. Clearly, the conditions $(X_1,X_2)\cap(0,q_1(\alpha))\neq\varnothing$, $\tilde{X}_2>\tilde{q}_0$ and $R(y_0,\tilde{q}_0,\alpha)<R(\bar{y}(\alpha,\tilde{q}_0),\tilde{q}_0,\alpha)$ are fully satisfied. Therefore, from Theorem 3.2 we know that the solution q^* of the min-max problem (3.20) will be only determined by (3.24a). This implies that the proof of Theorem 3.2 generalizes the optimization procedure given by Gander and Halpern [16].

4 Numerical results

We do in this section several numerical experiments to measure the effectiveness of the quasi-optimized parameter obtained in this paper. We use the model problem (2.1) with $x\in(0,4)$ (i.e., $\Omega=(0,4)$) and $t\in(0,10)$. We impose homogeneous boundary condition, $u(0,t)=u(4,t)=0$, and use various source function $f(x,t)$ and initial condition $u_0(x,t)$ for $(x,t)\in\Omega\times[-\tau,0]$.

We show results of numerical experiments for only the algorithm with overlap, since with overlap we can compare the results to the classical SWR algorithm (i.e., $\mathcal{B}_1=\mathcal{B}_2=\mathcal{I}$ in (2.2), where $\mathcal{I}$ is the identity operator). From [35], we know that the classical SWR algorithm does not converge without overlap. We discretize the SWR algorithm (2.2) by using the central finite difference scheme in space with mesh parameter Δx and a backward Euler method in time with time step $\Delta t=0.02$.

Example 4.1. (The case of two subdomains) We choose the parameters $\nu=1$, $a=4$, $d=-1.5$, $\tau=3$, $\Delta x=0.05$ and $L=6\Delta x$. Then, by using Theorem 3.2 we know that the quasi-optimized parameter is $p^*=2.73529031016621$. In this first set of experiments, the initial condition and the source function are chosen as

$$u_0(x,t)=1+\cos\left(e^{t\sin(\pi x)}\pi\right),\quad f(x,t)=(x-1)(x-3)\sin\left(xt^2\right). \tag{4.1}$$

In Fig. 4 on the left, we show the convergence rate of the SWR algorithm with Dirichlet transmission condition (dot line) and Robin transmission condition (solid line). We see clearly in this panel that, compared to the Dirichlet transmission condition the Robin transmission condition with $p=p^*$ can significantly speed up the convergence of the SWR algorithm. We next verify to what degree the choice of the parameter derived in this paper corresponds to the best choice one can make in the fully discretized algorithm. To this end, in Fig. 4 on the right we show the errors obtained after running the SWR algorithm with Robin transmission condition for 5 iterations using various values for the free parameter p. The choice $p=p^*$ is indicated by a star. One can find in this panel that the quasi-optimized parameter $p=p^*$ predicts the best one well.

Example 4.2. (The case of many subdomains) We now show experiments which indicate that the results obtained for two subdomains are also relevant for many subdomains. In this second set of experiments, the initial condition and the source function are chosen as

$$u_0(x,t)=\frac{1}{4}\sin(4\pi x),\quad f(x,t)=100e^{\cos(5t)}\sin(5\pi x). \tag{4.2}$$

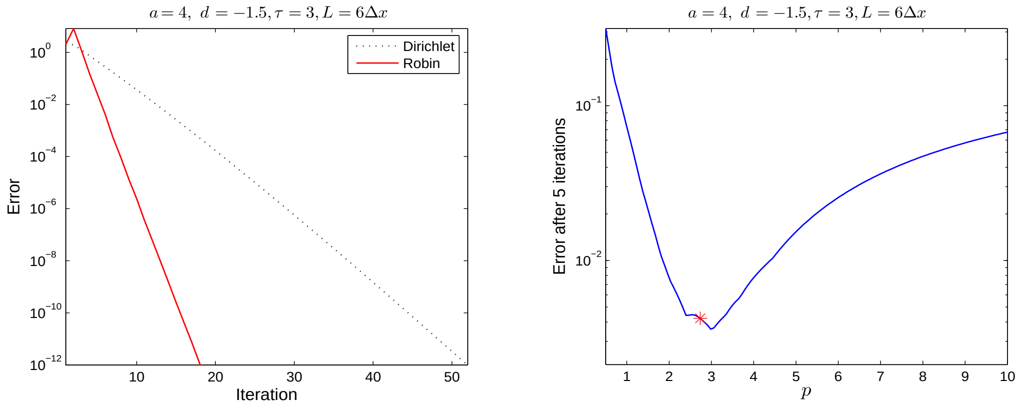

Figure 4: Left: measured convergence rate of the classical SWR algorithm (dot line) and the Robin type SWR algorithm with parameter $p=p^*$ (solid line). Right: the errors obtained by running the Robin type SWR algorithm after 5 iterations and various choices of the free parameters p, and indicated by a star the choice $p=p^*$.

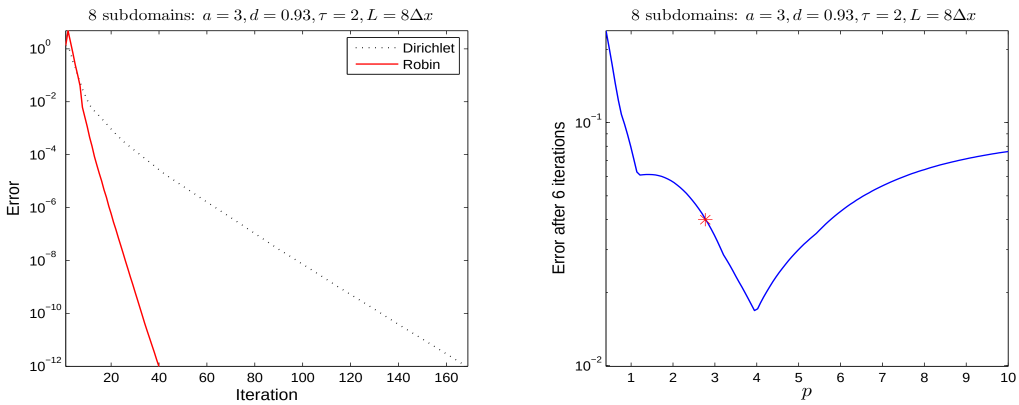

Figure 5: Left: measured convergence rate of the classical SWR algorithm (dot line) and the Robin type SWR algorithm with parameter $p=p^*$ (solid line). Right: the errors obtained by running the Robin type SWR algorithm after 6 iterations and various choices of the free parameters p, and indicated by a star the choice $p=p^*$.

We choose for problem parameters $\nu=1$, $a=3$, $d=0.93$, $\tau=2$ and for discretization parameters $\Delta x=0.025$ and $L=8\Delta x$. Then, by using Theorem 3.2 we know that the quasi-optimized parameter is $p^*=2.77013082323719$. We now decompose the whole space domain $\Omega=[0,4]$ into eight subdomains and then we show in Fig. 5 on the left the convergence rates of the SWR algorithm with Dirichlet transmission condition (dot line) and Robin transmission condition (solid line). We see clearly in this panel that, the convergence rate of SWR algorithm with the optimized Robin transmission condition is remarkable sharper than the algorithm with Dirichlet transmission condition. We next verify to what degree the choice of the parameter $p=p^*$ derived in two subdomain case corresponds to the best choice one can make in eight subdomain case and in the fully discretized algorithm. To this end, we show in Fig. 5 on the right the errors obtained after running the Robin type SWR algorithm for 6 iterations using various values for the free parameter p. The choice $p=p^*$ is also indicated by a star. It is interesting to see in this panel that the quasi-optimized parameter $p=p^*$ can not predict the best one very well. In particular, compared to the case of two subdomains, the quasi-optimized parameter $p=p^*$ goes away from the best one.

5 Conclusions

Schwarz waveform relaxation algorithm has been investigated deeply and widely for regular PDEs without time delay, while there is only a few experience of this algorithm for delay PDEs. In this paper, we focus on investigating the convergence behavior of the algorithm with Robin type transmission condition in the overlapping case. The analysis towards determining the parameter involved in the Robin transmission condition is deeper and more technical than the non-overlapping case studied in [35]. It is shown that the obtained parameter is a reliable approximation of the best choice and that by using this quasi-optimized parameter the Robin transmission condition can remarkably outperform the Dirichlet transmission condition.

There are still some important problems that need to be answered. Firstly, we only considered the case of two subdomains, which is very special in the field of SWR algorithm. For the case of arbitrary number of subdomains, we have shown in Fig. 5 on the right that the parameter $p=p^*$ analyzed in two subdomain case can not predict the best one very well. Therefore, the generation of the work in this paper to the case of many subdomains is meaningful. Second, for practical applications results in higher spatial dimensions would be needed. We intend to extend our analysis to the 2D case in a close future.

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