A Priori and a Posteriori Error Estimates for H(div)-Elliptic Problem with Interior Penalty Method

Yuping Zeng and Jinru Chen[*]

Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China.

Abstract. In this paper, we propose and analyze the interior penalty discontinuous Galerkin method for H(div)-elliptic problem. An optimal a priori error estimate in the energy norm is proved. In addition, a residual-based a posteriori error estimator is obtained. The estimator is proved to be both reliable and efficient in the energy norm. Some numerical testes are presented to demonstrate the effectiveness of our method.

AMS subject classifications: 65N15, 65N30

Key words: Discontinuous Galerkin method, H(div)-elliptic problem, a priori error estimate, a posteriori error estimate.

1 Introduction

We are concerned with solving the **H**(div)-elliptic model problem

$$-\mathbf{grad}(\mathrm{div}\,\mathbf{u})+\mathbf{u}=\mathbf{f} \quad \text{in } \Omega, \tag{1.1a}$$

$$\mathbf{u}\cdot\mathbf{n}=0 \quad \text{on } \Gamma, \tag{1.1b}$$

where Ω is a bounded polyhedral domain in $R^d(d=2,3)$ with boundary $\Gamma=\partial\Omega$, $\mathbf{n}$ is its unit outward normal vector, and $\mathbf{f}\in(L^2(\Omega))^d$.

The weak formulation of (1.1) is to find $\mathbf{u}\in H_0(\mathrm{div};\Omega)$ such that

$$a(\mathbf{u},\mathbf{v}):=\int_\Omega(\mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\mathbf{v}+\mathbf{u}\cdot\mathbf{v})dx=\int_\Omega\mathbf{f}\cdot\mathbf{v}dx, \quad \forall\mathbf{v}\in H_0(\mathrm{div};\Omega). \tag{1.2}$$

[*]Corresponding author. *Email addresses:* `yuping_zeng@163.com` (Y. Zeng), `jrchen@njnu.edu.cn` (J. Chen)

H(div)-elliptic problem (1.1) is ubiquitous in solid and fluid mechanics [11, 18]. It may arises from, the first-order system least-squares formulation of H^1-elliptic problem [12], the implementation of the sequential regularization method for the nonstationary incompressible Navier-Stokes equations [24], the mixed methods with augmented Lagrangians [13], or the stabilized formulations of the Stokes equations [37]. For more background on **H**(div)-elliptic problem and its applications, please see [5] for details. As we know, in two dimensions, conforming finite element methods for **H**(div)-elliptic problem can be treated by Raviart-Thomas(RT) element [30] or Brezzi-Douglas-Marini (BDM) element [10]. The extensions of RT element and BDM element to three dimensions were given by Nédélec in [26] and [27], respectively. Sometimes they are referred to as the first kind **H**(div)-conforming element and the second kind **H**(div)-conforming element.

Recently, there has been increased interest in the discontinuous Galerkin(DG) method due to its suitability for *hp*-adaptive techniques. For the applications of this method to a wide variety of problems, we can see the book [17] for details. An overview and a priori error analysis of DG for elliptic problems in H^1 were provided in [4]. For more details of the a priori error estimates for H^1-elliptic problem, please refer to [31]. A posterior error estimates of conforming finite element methods have been extensively studied, and we can refer to a series of monographs [2, 6, 28, 35] for the comprehensive analysis of such methods for elliptic problems in H^1, see also [13] for **H**(div)-conforming finite element method and [8] for **H**(**curl**)-conforming finite element method. However, a posteriori error estimates for DG have gained interest only in recent years, see [1, 7, 22, 23, 32, 34] for the analysis of elliptic problem in H^1, and see [21] for elliptic problem in **H**(**curl**).

In this paper, we consider the interior penalty(IP) DG method for **H**(div)-elliptic problem (1.1), and provide a priori error estimate and a posteriori error estimate of such method. The analysis for the a posteriori error estimator is largely based on the reference [21]. To the best of our knowledge, there exists no work on DG for **H**(div)-elliptic problem, here we make an initial work on this direction.

This paper is organized as follows. In Section 2, a discontinuous Galerkin method for the problem (1.1) is introduced. An optimal a priori error estimate of the DG method in the energy norm is proved in Section 3. In Section 4, we provide a residual-based a posteriori error estimator for the DG method. And both the upper bound and lower bound analysis are proved for the error estimator in the energy norm. Finally, some numerical experiments are given in Section 5.

2   Discontinuous Galerkin formulation

In this section, we introduce the interior penalty discontinuous Galerkin method for the problem (1.1). For convenience, we assume that the domain is in R^3. Before discussion, we first give some notations: for a bounded domain $\mathcal{D}$ in R^3, we denote by $H^s(\mathcal{D})$ the standard Sobolev space of functions with regularity exponent $s \geq 0$ and norm $||\cdot||_{s,\mathcal{D}}$ and seminorm $|\cdot|_{s,\mathcal{D}}$. For $s=0$, $H^0(\mathcal{D})$ is written by $L^2(\mathcal{D})$. When $\mathcal{D}=\Omega$, the norm $||\cdot||_{s,\Omega}$

is simply written by $||\cdot||_s$. $H_0^s(\mathcal{D})$ is the subspace of $H^s(\mathcal{D})$ with vanishing trace on $\partial\mathcal{D}$. For the space $H^s(\mathcal{D})^3$, the norm is also denoted by $||\cdot||_{s,\mathcal{D}}$. In addition, we define the following Sobolev spaces

$$\mathbf{H}(\mathrm{div};\Omega) = \{\mathbf{v}\in L^2(\Omega)^3 : \mathrm{div}\,\mathbf{v}\in L^2(\Omega)\},$$
$$\mathbf{H}_0(\mathrm{div};\Omega) = \{\mathbf{v}\in\mathbf{H}(\mathrm{div};\Omega) : \mathbf{v}\cdot\mathbf{n}=0 \ \text{ on } \ \partial\Omega\},$$
$$\mathbf{H}(\mathbf{curl};\Omega) = \{\mathbf{v}\in L^2(\Omega)^3 : \mathbf{curl}\,\mathbf{v}\in L^2(\Omega)^3\},$$

here and hereafter, we use boldface letter to denote vectors and vector spaces. The corresponding norms in $\mathbf{H}(\mathrm{div};\Omega)$ and $\mathbf{H}(\mathbf{curl};\Omega)$ are denoted by $||\mathbf{v}||_{\mathrm{div}} = \left(||\mathbf{v}||_0^2 + ||\mathrm{div}\,\mathbf{v}||_0^2\right)^{\frac{1}{2}}$ and $||\mathbf{v}||_{\mathbf{curl}} = \left(||\mathbf{v}||_0^2 + ||\mathbf{curl}\,\mathbf{v}||_0^2\right)^{\frac{1}{2}}$ respectively. Finally, we denote the standard inner product in $L^2(\Omega)$ or $L^2(\Omega)^3$ by $(\cdot,\cdot)$.

Let $\mathcal{T}_h$ be a regular family of decompositions of Ω into tetrahedra $\{K\}$, h_K denote the diameter of K, and

$$h = \max_{K\in\mathcal{T}_h} h_K.$$

$\mathcal{F}_h^0$ denotes the set of interior faces of elements in $\mathcal{T}_h$, and $\mathcal{F}_h^\partial$ denotes the set of boundary faces. Set $\mathcal{F}_h = \mathcal{F}_h^0\cup\mathcal{F}_h^\partial$. Let F be an interior face in $\mathcal{F}_h^0$ shared by element K_1 and K_2, and define the unit normal vectors $\mathbf{n}^1$ and $\mathbf{n}^2$ on face F pointing exterior to K_1 and K_2, respectively. The diameter of the face F is denoted by h_F. We assume that the elements of $\mathcal{T}_h$ satisfy the minimum angle condition. This means that there exists a constant $\theta_0 > 0$ such that $h_K/\rho_K \geq \theta_0$, where ρ_K denotes the diameter of the inscribed ball of K. For a scalar piecewise smooth function φ, with $\varphi^i = \varphi|_{K_i}$, we define the following average and jump by

$$\{\{\varphi\}\} = \frac{1}{2}(\varphi^1 + \varphi^2), \quad [\![\varphi]\!] = \varphi^1\mathbf{n}^1 + \varphi^2\mathbf{n}^2 \quad \text{on} \ F\in\mathcal{F}_h^0.$$

For a vector valued piecewise smooth function $\mathbf{v}$, with $\mathbf{v}^i = \mathbf{v}|_{K_i}$, we set

$$\{\{\mathbf{v}\}\} = \frac{1}{2}(\mathbf{v}^1 + \mathbf{v}^2),$$
$$[\![\mathbf{v}]\!]_N = \mathbf{v}^1\cdot\mathbf{n}^1 + \mathbf{v}^2\cdot\mathbf{n}^2, \quad [\![\mathbf{v}]\!]_T = \mathbf{v}^1\times\mathbf{n}^1 + \mathbf{v}^2\times\mathbf{n}^2.$$

For a boundary face $F\in\mathcal{F}_h^\partial$, we set

$$[\![\mathbf{v}]\!]_N = \mathbf{v}\cdot\mathbf{n}.$$

Next, with any $l\geq 1$, we associate the discontinuous Galerkin finite element space for vector valued functions

$$\mathbf{V}_h^l = \left\{\mathbf{v}\in L^2(\Omega)^3 : \mathbf{v}|_K\in P_l(K)^3, \ \forall K\in\mathcal{T}_h\right\}, \tag{2.1}$$

where $P_l(K)$ is the space of polynomials of total degree $\leq l$. Similarly, with any $j \geq 0$, we introduce the following space for scalar functions

$$S_h^j = \left\{ v \in L^2(\Omega) : v|_K \in P_j(K), \ \forall K \in \mathcal{T}_h \right\}. \tag{2.2}$$

Thereby, for the problem (1.1), we propose the following DG method: find $u_h \in \mathbf{V}_h^l$ such that

$$a_h(\mathbf{u}_h, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_h^l, \tag{2.3}$$

where the bilinear form $a_h(\mathbf{u}_h, \mathbf{v}_h) : \mathbf{V}_h^l \times \mathbf{V}_h^l \to \mathbb{R}$ is defined as

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = \sum_{K \in \mathcal{T}_h} \int_K \left(\operatorname{div} \mathbf{u}_h \operatorname{div} \mathbf{v}_h + \mathbf{u}_h \cdot \mathbf{v}_h \right) dx - \sum_{F \in \mathcal{F}_h} \int_F [\![\mathbf{u}_h]\!]_N \{\!\{\operatorname{div} \mathbf{v}_h\}\!\} ds$$

$$- \sum_{F \in \mathcal{F}_h} \int_F [\![\mathbf{v}_h]\!]_N \{\!\{\operatorname{div} \mathbf{u}_h\}\!\} ds + \sum_{F \in \mathcal{F}_h} \int_F \beta [\![\mathbf{u}_h]\!]_N [\![\mathbf{v}_h]\!]_N ds, \tag{2.4}$$

where $\beta = \theta h_F^{-1}$, here θ is the interior penalty parameter which is to be defined to guarantee the coercivity of bilinear form a_h.

For the DG method in (2.3), we can easily obtain the following Galerkin orthogonality

$$a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{V}_h^l. \tag{2.5}$$

3 A priori error analysis

Define the mesh-dependent norm $\|\cdot\|_h$ by

$$\|\mathbf{v}\|_h = \left(\sum_{K \in \mathcal{T}_h} \|\operatorname{div} \mathbf{v}\|_{0,K}^2 + \|\mathbf{v}\|_0^2 + \theta^{-1} \sum_{F \in \mathcal{F}_h} h_F \|\{\!\{\operatorname{div} \mathbf{v}\}\!\}\|_{0,F}^2 + \theta \sum_{F \in \mathcal{F}_h} h_F^{-1} \|[\![\mathbf{v}]\!]_N\|_{0,F}^2 \right)^{\frac{1}{2}}. \tag{3.1}$$

Let $\mathbf{V}(h) = H_0(\operatorname{div}; \Omega) + \mathbf{V}_h^l$. It follows from Cauchy-Schwarz inequality that

$$|a_h(\mathbf{u}, \mathbf{v})| \leq \|\mathbf{u}\|_h \|\mathbf{v}\|_h, \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}(h). \tag{3.2}$$

To obtain the coercivity of the bilinear form $a_h(\cdot, \cdot)$ on $\mathbf{V}_h^l$, we introduce another mesh-dependent energy norm $\|\!|\cdot|\!\|_h$ defined by

$$\|\!|\mathbf{v}|\!\|_h = \left(\sum_{K \in \mathcal{T}_h} \|\operatorname{div} \mathbf{v}\|_{0,K}^2 + \|\mathbf{v}\|_0^2 + \theta \sum_{F \in \mathcal{F}_h} h_F^{-1} \|[\![\mathbf{v}]\!]_N\|_{0,F}^2 \right)^{\frac{1}{2}}. \tag{3.3}$$

Obviously, we have

$$\|\!|\mathbf{v}|\!\|_h \leq \|\mathbf{v}\|_h, \quad \forall v \in \mathbf{V}(h). \tag{3.4}$$

On the other hand, we have the following result.

Lemma 3.1. *There exists a constant $C > 0$ depending on the minimum angle of $\mathcal{T}_h$ and the degree of the polynomial l such that*

$$\|v\|_h \le C(1+\theta^{-1})^{\frac{1}{2}} \|\|v\|\|_h, \quad \forall v \in V_h^l. \tag{3.5}$$

Proof. First, we have the following inequality [3]

$$\|\phi\|_{0,\partial K}^2 \le Ch_K^{-1}\|\phi\|_{0,K}^2, \quad \forall \phi \in S_h^j. \tag{3.6}$$

Observing $\mathbf{v} \in V_h^l$, then $\operatorname{div}\mathbf{v} \in S_h^{l-1}$, thus from the above inequality we obtain that

$$\|\operatorname{div}\mathbf{v}\|_{0,F}^2 \le Ch_K^{-1}\|\operatorname{div}\mathbf{v}\|_{0,K}^2, \quad \forall \mathbf{v} \in V_h^l, \tag{3.7}$$

where $F \in \partial K$. Therefore, for $\mathbf{v} \in V_h^l$, by the definition of $\{\{\operatorname{div}\mathbf{v}\}\}$, we have that

$$\sum_{F \in \mathcal{F}_h} h_F \|\{\{\operatorname{div}\mathbf{v}\}\}\|_{0,F}^2 \le C \sum_{K \in \mathcal{T}_h} \|\operatorname{div}\mathbf{v}\|_{0,K}^2, \tag{3.8}$$

where $C > 0$ is a constant depending on the minimum angle of $\mathcal{T}_h$ and the degree of the polynomial l. Then the estimate (3.5) follows from (3.1), (3.3) and (3.8). $\square$

Provided the interior penalty parameter θ is sufficient large, the bilinear $a_h(\cdot,\cdot)$ is coercive with respect to $\|\|\cdot\|\|_h$, which is showed by the following lemma.

Lemma 3.2. *There exists a constant $\theta^* > 0$ depending on the minimum angle of $\mathcal{T}_h$ and the degree of the polynomial l such that*

$$a_h(v,v) \ge \frac{1}{2}\|\|v\|\|_h^2, \quad \forall v \in V_h^l, \ \theta > \theta^*. \tag{3.9}$$

Proof. For any $\varepsilon > 0$ and $\mathbf{v} \in V_h^l$, it follows from Cauchy-Schwarz inequality and (3.8) that

$$\sum_{F \in \mathcal{F}_h} \int_F \{\{\operatorname{div}\mathbf{v}\}\} [\![\mathbf{v}]\!]_N ds \le \sum_{F \in \mathcal{F}_h} h_F^{1/2}\|\{\{\operatorname{div}\mathbf{v}\}\}\|_{0,F} h_F^{-1/2}\|[\![\mathbf{v}]\!]_N\|_{0,F}$$

$$\le \left(\sum_{F \in \mathcal{F}_h} h_F \|\{\{\operatorname{div}\mathbf{v}\}\}\|_{0,F}^2 \right)^{1/2} \left(\sum_{F \in \mathcal{F}_h} h_F^{-1}\|[\![\mathbf{v}]\!]_N\|_{0,F}^2 \right)^{1/2}$$

$$\le \left(C \sum_{K \in \mathcal{T}_h} \|\operatorname{div}\mathbf{v}\|_{0,K}^2 \right)^{1/2} \left(\sum_{F \in \mathcal{F}_h} h_F^{-1}\|[\![\mathbf{v}]\!]_N\|_{0,F}^2 \right)^{1/2}$$

$$\le \frac{\varepsilon C}{2} \sum_{K \in \mathcal{T}_h} \|\operatorname{div}\mathbf{v}\|_{0,K}^2 + \frac{1}{2\varepsilon} \sum_{F \in \mathcal{F}_h} h_F^{-1}\|[\![\mathbf{v}]\!]_N\|_{0,F}^2,$$

where $C > 0$ is a constant depending on the minimum angle of $\mathcal{T}_h$ and the degree of the polynomial l. Thus, from (2.4) we obtain

$$a_h(\mathbf{v},\mathbf{v}) \ge (1-\varepsilon C) \sum_{K \in \mathcal{T}_h} \|\operatorname{div}\mathbf{v}\|_{0,K}^2 + \|\mathbf{v}\|_0^2 + \left(\theta - \frac{1}{\varepsilon}\right) \sum_{F \in \mathcal{F}_h} h_F^{-1}\|[\![\mathbf{v}_h]\!]_N\|_{0,F}^2.$$

Then we can choose $\theta^* = \frac{2}{\varepsilon}$ and $\varepsilon = \frac{1}{2C}$ to make inequality (3.9) be valid. $\square$

The following lemma provides an abstract error estimate for our DG method in the mesh-dependent norm $||\cdot||_h$.

Lemma 3.3. *Let u denote the solution of the problem* (1.1)*, and u_h denote the numerical solution of the DG method in* (2.3)*. There exists a constant $C > 0$ depending on the minimum angle of $\mathcal{T}_h$ and the degree of the polynomial l such that*

$$||u-u_h||_h \leq C(1+\theta^{-1}) \inf_{v \in V_h^l} ||u-v||_h. \tag{3.10}$$

Proof. From the Galerkin orthogonality (2.5), we have

$$a_h(\mathbf{u}-\mathbf{u}_h,\mathbf{v})=0, \quad \forall \mathbf{v} \in \mathbf{V}_h^l.$$

Thus, for any $\mathbf{v} \in \mathbf{V}_h^l$, based on (3.2),(3.5),(3.9) and the above equality we have

$$\begin{aligned}
||\mathbf{u}-\mathbf{u}_h||_h &\leq ||\mathbf{u}-\mathbf{v}||_h+||\mathbf{u}_h-\mathbf{v}||_h \\
&\leq ||\mathbf{u}-\mathbf{v}||_h+C(1+\theta^{-1})^{\frac{1}{2}}|||\mathbf{u}_h-\mathbf{v}|||_h \\
&\leq ||\mathbf{u}-\mathbf{v}||_h+C(1+\theta^{-1})^{\frac{1}{2}} \sup_{\mathbf{w} \in V_h^l \backslash \{0\}} \frac{|a_h(\mathbf{u}_h-\mathbf{v},\mathbf{w})|}{|||\mathbf{w}|||_h} \\
&= ||\mathbf{u}-\mathbf{v}||_h+C(1+\theta^{-1})^{\frac{1}{2}} \sup_{\mathbf{w} \in V_h^l \backslash \{0\}} \frac{|a_h(\mathbf{u}-\mathbf{v},\mathbf{w})|}{|||\mathbf{w}|||_h} \\
&\leq C(1+\theta^{-1}) \inf_{\mathbf{v} \in V_h^l} ||\mathbf{u}-\mathbf{v}||_h,
\end{aligned}$$

and the lemma follows.   □

To derive concrete error estimate based on the above lemma, we choose $\mathbf{v} = \mathcal{I}_h\mathbf{u}$ in (3.10). Here $\mathcal{I}_h\mathbf{u}$ is the Lagrange interpolation defined by

$$(\mathcal{I}_h\mathbf{u})|_K=\mathcal{I}_K(\mathbf{u}|_K),$$

where $\mathcal{I}_K(\mathbf{u}|_K)$ is the unique function in $P_l(T)$ which interpolates $\mathbf{u}|_K$ (componentwise) at the $(l+1)(l+2)(l+3)/6$ points of K with barycentric coordinates in $\{0,1/l,2/l,\cdots,1\}$. Then we have the following approximation property [9,15]

$$||\mathbf{u}-\mathcal{I}_h\mathbf{u}||_{i,K}^2 \leq Ch^{2(j-i)}|\mathbf{u}|_{j,K}^2, \quad 0 \leq i \leq j \leq l+1, \ j \geq 2, \tag{3.11}$$

thus, we obtain

$$||\mathrm{div}(\mathbf{u}-\mathcal{I}_h\mathbf{u})||_{0,K}^2 \leq Ch^{2(j-1)}|\mathbf{u}|_{j,K}^2, \quad 2 \leq j \leq l+1, \tag{3.12a}$$

$$||\mathrm{div}(\mathbf{u}-\mathcal{I}_h\mathbf{u})||_{1,K}^2 \leq Ch^{2(j-2)}|\mathbf{u}|_{j,K}^2, \quad 2 \leq j \leq l+1. \tag{3.12b}$$

We recall the trace inequality [3]

$$\|\phi\|_{0,F}^2 \le C\left(h_F^{-1}\|\phi\|_{0,K}^2 + h_F|\phi|_{1,K}^2\right), \tag{3.13}$$

where ϕ can be either scalar or vector valued function. It follows from (3.11), (3.12a), (3.12b) and (3.13) that

$$\sum_{F\in\mathcal{F}_h} h_F\|\{\{\mathrm{div}\,(\mathbf{u}-\mathcal{I}_h\mathbf{u})\}\}\|_{0,F}^2 \le C\sum_{K\in\mathcal{T}_h} h_K^{2(j-1)}|\mathbf{u}|_{j,K}^2, \quad 2\le j\le l+1, \tag{3.14a}$$

$$\sum_{F\in\mathcal{F}_h} h_F^{-1}\|[\![\mathbf{u}-\mathcal{I}_h\mathbf{u}]\!]_N\|_{0,F}^2 \le C\sum_{K\in\mathcal{T}_h} h_K^{2(j-1)}|\mathbf{u}|_{j,K}^2, \qquad 2\le j\le l+1. \tag{3.14b}$$

In the following theorem, we give a concrete a priori error estimate for the DG method in (2.3).

Theorem 3.1. *Let u be the solution of the problem (1.1), and u_h be the numerical solution of DG method in (2.3). Assume that θ is greater than or equal to the constant θ^* in Lemma 3.2. Then there exists a constant $C>0$ depending on the minimum angle of $\mathcal{T}_h$ and the degree of the polynomial l such that*

$$\|u-u_h\|_h \le C(1+\theta+\theta^{-1}+\theta^{-2}+\theta^{-3})^{\frac{1}{2}}\sum_{K\in\mathcal{T}_h} h_K^{j-1}|u|_{j,K}, \quad 2\le j\le l+1. \tag{3.15}$$

Proof. Set $\mathbf{v}=\mathcal{I}_h\mathbf{u}$ in (3.10), we obtain

$$\|\mathbf{u}-\mathbf{u}_h\|_h^2 \le C(1+\theta^{-1})^2 \inf_{\mathbf{v}\in V_h^l} \|\mathbf{u}-\mathcal{I}_h\mathbf{u}\|_h^2. \tag{3.16}$$

The theorem follows by the definition of $\|\cdot\|_h$, (3.11), (3.12a), (3.14a), (3.14b) and (3.16).
$\square$

4 A posteriori error analysis

Recalling that $\mathbf{V}(h) = H_0(\mathrm{div};\Omega) + \mathbf{V}_h^l$, we introduce an auxiliary bilinear form $\tilde{a}_h(\cdot,\cdot) : \mathbf{V}(h)\times\mathbf{V}(h) \to \mathbb{R}$ defined by

$$\tilde{a}_h(\mathbf{u},\mathbf{v}) = \sum_{K\in\mathcal{T}_h}\int_K (\mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\mathbf{v}+\mathbf{u}\cdot\mathbf{v})dx - \sum_{K\in\mathcal{T}_h}\int_K \mathcal{L}(\mathbf{u})(\mathrm{div}\,\mathbf{v})ds$$
$$- \sum_{K\in\mathcal{T}_h}\int_K \mathcal{L}(\mathbf{v})(\mathrm{div}\,\mathbf{u})ds + \sum_{F\in\mathcal{F}_h}\int_F \beta[\![\mathbf{u}]\!]_N[\![\mathbf{v}]\!]_N ds,$$

where the lifting operator $\mathcal{L} : \mathbf{V}(h)\to S_h^j$ is defined as

$$\int_\Omega \mathcal{L}(\mathbf{v})\varphi dx = \sum_{F\in\mathcal{F}_h}\int_F [\![\mathbf{v}]\!]_N\{\{\varphi\}\}ds, \quad \forall\varphi\in S_h^j. \tag{4.1}$$

The lifting operator $\mathcal{L}(\mathbf{v})$ can be bounded by [29]

$$\|\mathcal{L}(\mathbf{v})\|_0^2 \leq \theta^{-1} C_{\text{lift}} \sum_{F \in \mathcal{F}_h} \|\beta^{\frac{1}{2}} [\![\mathbf{v}]\!]_N\|_{0,F}^2. \tag{4.2}$$

We may observe that $\tilde{a}_h = a_h$ on $\mathbf{V}_h^l \times \mathbf{V}_h^l$ and $\tilde{a}_h = a$ on $H_0(\text{div};\Omega) \times H_0(\text{div};\Omega)$, then the DG method in (2.3) is equivalent to: find $u_h \in \mathbf{V}_h^l$ such that

$$\tilde{a}_h(\mathbf{u}_h, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_h^l. \tag{4.3}$$

Moreover, we have

$$\tilde{a}_h(\mathbf{v}, \mathbf{v}) = \|\mathbf{v}\|_{\text{div}}^2 = \|\|\mathbf{v}\|\|_h^2, \quad \forall \mathbf{v} \in H_0(\text{div};\Omega). \tag{4.4}$$

Furthermore, we can prove that $\tilde{a}_h$ is continuous on $\mathbf{V}(h)$ [4,21].

Lemma 4.1. *There exists a constant $C_{cont} > 0$ depending on the minimum angle of $\mathcal{T}_h$ and the degree of the polynomial l such that*

$$|\tilde{a}_h(\boldsymbol{u}, \boldsymbol{v})| \leq C_{\text{cont}} \|\|\boldsymbol{u}\|\|_h \|\|\boldsymbol{v}\|\|_h, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V(h). \tag{4.5}$$

Proof. It follows from Cauchy-Schwarz inequality and (4.2) that

$$|\tilde{a}_h(\mathbf{u}, \mathbf{v})| \leq \sum_{K \in \mathcal{T}_h} \Big(\|\text{div}\,\mathbf{u}\|_{0,K} \|\text{div}\,\mathbf{v}\|_{0,K} + \|\mathbf{u}\|_{0,K} \|\mathbf{v}\|_{0,K} + \|\mathcal{L}(\mathbf{u})\|_{0,K} \|\text{div}\,\mathbf{v}\|_{0,K}$$

$$+ \|\text{div}\,\mathbf{u}\|_{0,K} \|\mathcal{L}(\mathbf{v})\|_{0,K} \Big) + \sum_{F \in \mathcal{F}_h} \|\beta^{\frac{1}{2}} [\![\mathbf{u}]\!]_N\|_{0,F} \|\beta^{\frac{1}{2}} [\![\mathbf{v}]\!]_N\|_{0,F}$$

$$\leq \Big(\sum_{K \in \mathcal{T}_h} \|\text{div}\,\mathbf{u}\|_{0,K}^2 \Big)^{\frac{1}{2}} \Big(\sum_{K \in \mathcal{T}_h} \|\text{div}\,\mathbf{v}\|_{0,K}^2 \Big)^{\frac{1}{2}} + \Big(\sum_{K \in \mathcal{T}_h} \|\mathbf{u}\|_{0,K}^2 \Big)^{\frac{1}{2}} \Big(\sum_{K \in \mathcal{T}_h} \|\mathbf{v}\|_{0,K}^2 \Big)^{\frac{1}{2}}$$

$$+ \Big(\sum_{K \in \mathcal{T}_h} \|\mathcal{L}(\mathbf{u})\|_{0,K}^2 \Big)^{\frac{1}{2}} \Big(\sum_{K \in \mathcal{T}_h} \|\text{div}\,\mathbf{v}\|_{0,K}^2 \Big)^{\frac{1}{2}} + \Big(\sum_{K \in \mathcal{T}_h} \|\text{div}\,\mathbf{u}\|_{0,K}^2 \Big)^{\frac{1}{2}} \Big(\sum_{K \in \mathcal{T}_h} \|\mathcal{L}(\mathbf{v})\|_{0,K}^2 \Big)^{\frac{1}{2}}$$

$$+ \Big(\sum_{F \in \mathcal{F}_h} \|\beta^{\frac{1}{2}} [\![\mathbf{u}]\!]_N\|_{0,F}^2 \Big)^{\frac{1}{2}} \Big(\sum_{F \in \mathcal{F}_h} \|\beta^{\frac{1}{2}} [\![\mathbf{v}]\!]_N\|_{0,F}^2 \Big)^{\frac{1}{2}}$$

$$\leq \Big(\sum_{K \in \mathcal{T}_h} \|\text{div}\,\mathbf{u}\|_{0,K}^2 \Big)^{\frac{1}{2}} \Big(\sum_{K \in \mathcal{T}_h} \|\text{div}\,\mathbf{v}\|_{0,K}^2 \Big)^{\frac{1}{2}} + \Big(\sum_{K \in \mathcal{T}_h} \|\mathbf{u}\|_{0,K}^2 \Big)^{\frac{1}{2}} \Big(\sum_{K \in \mathcal{T}_h} \|\mathbf{v}\|_{0,K}^2 \Big)^{\frac{1}{2}}$$

$$+ \theta^{-\frac{1}{2}} C_{\text{lift}}^{\frac{1}{2}} \Big(\sum_{F \in \mathcal{F}_h} \|\beta^{\frac{1}{2}} [\![\mathbf{u}]\!]_N\|_{0,F}^2 \Big)^{\frac{1}{2}} \Big(\sum_{K \in \mathcal{T}_h} \|\text{div}\,\mathbf{v}\|_{0,K}^2 \Big)^{\frac{1}{2}}$$

$$+ \theta^{-\frac{1}{2}} C_{\text{lift}}^{\frac{1}{2}} \Big(\sum_{K \in \mathcal{T}_h} \|\text{div}\,\mathbf{u}\|_{0,K}^2 \Big)^{\frac{1}{2}} \Big(\sum_{F \in \mathcal{F}_h} \|\beta^{\frac{1}{2}} [\![\mathbf{v}]\!]_N\|_{0,F}^2 \Big)^{\frac{1}{2}}$$

$$+ \Big(\sum_{F \in \mathcal{F}_h} \|\beta^{\frac{1}{2}} [\![\mathbf{u}]\!]_N\|_{0,F}^2 \Big)^{\frac{1}{2}} \Big(\sum_{F \in \mathcal{F}_h} \|\beta^{\frac{1}{2}} [\![\mathbf{u}]\!]_N\|_{0,F}^2 \Big)^{\frac{1}{2}}$$

$$\leq C_{\text{cont}} \left(\sum_{K \in \mathcal{T}_h} \left(||\text{div}\,\mathbf{u}||_{0,K}^2 + ||\mathbf{u}||_{0,K}^2 \right) + \sum_{F \in \mathcal{F}_h} ||[\![\beta^{\frac{1}{2}}\mathbf{u}]\!]_N||_{0,F}^2 \right)^{\frac{1}{2}}$$

$$\cdot \left(\sum_{K \in \mathcal{T}_h} \left(||\text{div}\,\mathbf{v}||_{0,K}^2 + ||\mathbf{v}||_{0,K}^2 \right) + \sum_{F \in \mathcal{F}_h} ||[\![\beta^{\frac{1}{2}}\mathbf{v}]\!]_N||_{0,F}^2 \right)^{\frac{1}{2}}$$

$$= C_{\text{cont}} |||\mathbf{u}|||_h |||\mathbf{v}|||_h,$$

and the lemma follows. □

The main idea behind our error analysis is that any discontinuous function can be approximated by a conforming finite element one. Thus, we define the conforming finite element space

$$\mathbf{V}_h^c = \mathbf{V}_h^l \cap H_0(\text{div};\Omega), \tag{4.6}$$

in fact, $\mathbf{V}_h^c$ is the second kind $\mathbf{H}(\text{div})$-conforming Nédélec element [27]. Similar to Proposition 4.1 in [21](see also Theorem 2.2 in [23]), we can obtain the following approximation property.

Lemma 4.2. *Let $v \in V_h^l$, then there exists a conforming finite element approximation $v^c \in V_h^c$ such that*

$$|||v - v^c|||_h^2 \leq \left(2\theta^{-1}C_{\text{app}} + 1 \right) \sum_{F \in \mathcal{F}_h} ||\beta^{\frac{1}{2}}[\![v]\!]_N||_{0,F}^2, \tag{4.7}$$

where $C_{\text{app}} > 0$ depends on the minimum angle of $\mathcal{T}_h$ and the degree of the polynomial l.

Proof. The proof we provide is constructive. Given $\mathbf{v} \in \mathbf{V}_h^l$, we construct a function $\mathbf{v}^c \in \mathbf{V}_h^c$ as follows: At every node of the mesh $\mathcal{T}_h$ corresponding to a degree of freedom for $\mathbf{V}_h^c$, the value of $\mathbf{v}^c$ is set to be the average of the values of $\mathbf{v}$ at that node.

First, we introduce the second kind $\mathbf{H}(\text{div})$-conforming Nédélec element. For a face f of K, let $\{q_f^j\}_{j=1}^{N_f}$ denote a basis of $P_l(f)$, and $\{\mathbf{q}_K^j\}_{j=1}^{N_b}$ a basis of $\mathcal{R}_{l-1}(K)$ for element K, here $\mathcal{R}_l = P_{l-1}^3 \oplus \mathcal{S}_l$, $\mathcal{S}_l = \{\mathbf{p} \in (\tilde{P}_l)^3 | \mathbf{x} \cdot \mathbf{p} = 0\}$, and $\tilde{P}_l$ denote the homogeneous polynomials of degree l. For a fixed $K \in \mathcal{T}_h$ and let $\mathbf{v} \in P^l(K)^3$, we define the following degrees of freedom:

$$M_k^f(\mathbf{v}) = \left\{ \int_f (\mathbf{v} \cdot \mathbf{n}_f) q_f^j ds \colon j = 1,2,\cdots,N_f \right\}, \text{ for any face } f \text{ of } K,$$

$$M_k^b(\mathbf{v}) = \left\{ \int_K \mathbf{v} \cdot \mathbf{q}_K^j dx \colon j = 1,2,\cdots,N_b \right\}.$$

It is well known that above degrees of freedom uniquely define the polynomial $\mathbf{v} \in P^l(K)^3$, see [25, 27]. And for a face f of K, the normal trace $\mathbf{v} \cdot \mathbf{n}_f$ is uniquely determined by the degrees of freedom M_k^f, see [25, 27]. Thus, any $\mathbf{v} \in P^l(K)^3$ can be written by

$$\mathbf{v} = \sum_{f \in \mathcal{F}(K)} \sum_{j=1}^{N_f} v_{K,f}^j \phi_{K,f}^j + \sum_{j=1}^{N_b} v_{K,b}^j \phi_{K,b}^j, \tag{4.8}$$

where $\mathcal{F}(K)$ denotes the set of faces of K. The functions $\{\phi^j_{K,f}\}$ and $\{\phi^j_{K,b}\}$ are lagrange basis functions on $P^l(K)^3$ with respect to the degrees of freedom defined above.

For each $K \in \mathcal{T}_h$, let $G_K = \{x^j_K, j=1,2,\cdots,|\mathcal{F}(K)|\cdot|N_f|\}$ be the face node set to formally match the corresponding local basic functions of faces $\phi_{K,f} = \{\phi^j_{K,f}, j=1,2,\cdots,|\mathcal{F}(K)|\cdot |N_f|\}$, here and in the following $|\cdot|$ denotes the cardinality of corresponding set. Let $H_K = \{x^j_K, j=1,2,\cdots,|N_b|\}$ be the element node set to formally match the corresponding local basic functions of element interior $\phi_{K,b} = \{\phi^j_{K,b}, j=1,2,\cdots,|N_b|\}$. Set $\phi_K = \phi_{K,f} \cup \phi_{K,b}$, $G = \cup_{K \in \mathcal{T}_h} G_K$. Let G be the union of two disjoint parts:

$$G_0 = \{v \in G : v \in f \in \mathcal{F}^0_h\},$$
$$G_\partial = \{v \in G : v \in f \in \mathcal{F}^\partial_h\}.$$

For each node $v \in G \cup H$, let $\delta_v = \{K \in \mathcal{T}_h | v \in K\}$ and its cardinality is denoted by $|\delta_v|$. We note that, if $v \in H$, then $|\delta_v| = 1$, if $v \in G_\partial$, then $|\delta_v| = 1$, and if $v \in G_0$, then $|\delta_v| = 2$. Then we associate each node v with a basis function ϕ^v defined by

$$\text{supp } \phi^v = \bigcup_{K \in \delta_v} K, \quad \phi^v|_K = \phi^v_K, \quad x^j_K = v.$$

Now, given $\mathbf{v} \in \mathbf{V}^l_h$, assume

$$\mathbf{v} = \sum_{K \in \mathcal{T}_h} \left(\sum_{f \in \mathcal{F}(K)} \sum_{j=1}^{N_f} v^j_{K,f} \phi^j_{K,f} + \sum_{j=1}^{N_b} v^j_{K,b} \phi^j_{K,b} \right),$$

we construct the function $\mathbf{v}^c \in \mathbf{V}^c_h$ by

$$\mathbf{v}^c = \sum_{v \in G \cup H} \gamma^v \phi^v, \quad \text{where } \gamma^v = \begin{cases} \dfrac{1}{|\delta_v|} \sum_{x^j_K = v} v^j_{K,b}, & \text{if } v \in H, \\[2ex] \dfrac{1}{|\delta_v|} \sum_{x^j_K = v} v^j_{K,f}, & \text{if } v \in G_0, \\[2ex] 0, & \text{if } v \in G_\partial. \end{cases} \tag{4.9}$$

Set $\gamma^j_K = \gamma^j$ whenever $x^j_K = v$.

By application of the Piola transformation (see [25])

$$B\hat{\mathbf{v}}(\hat{\mathbf{x}}) = \det(B)\mathbf{v}(\mathbf{x}), \quad \mathbf{x} = B\hat{\mathbf{x}} + \mathbf{t}, \quad B \in R^{3,3}, \quad \mathbf{t} \in R^3, \tag{4.10}$$

and through a scaling argument, we obtain that $\|\text{div}\,\phi^j_{K,f}\|^2_{0,K} \le ch_K^{-3}$. Since the degrees of

freedom of $\mathbf{v}-\mathbf{v}^c$ corresponding to the interior nodes of elements is zero, we get

$$\sum_{K\in\mathcal{T}_h}||\mathrm{div}\,(\mathbf{v}-\mathbf{v}^c)||^2_{0,K}\leq C|\mathcal{F}(K)||N_f|\sum_{K\in\mathcal{T}_h}h_K^{-3}\sum_{f\in\mathcal{F}(K)}\sum_{j=1}^{N_f}|v^j_{K,f}-\gamma^j_K|^2$$

$$\leq\sum_{v\in G_0}h_v^{-3}\sum_{x_K^j=v}|v^j_{K,f}-\gamma^j|^2+\sum_{v\in G_\partial}h_v^{-3}\sum_{x_K^j=v}|v^j_{K,f}|^2\quad\left(h_v=\max_{K\in\delta_v}h_K\right).\tag{4.11}$$

Note that

$$\sum_{x_K^j=v}|v^j_{K,f}-\gamma^j|^2\leq c|v^{j^+}_{K^+,f}-v^{j^-}_{K^-,f}|^2,\tag{4.12}$$

where K^+ and K^- denote the elements which share the face f, we obtain

$$\sum_{K\in\mathcal{T}_h}||\mathrm{div}\,(\mathbf{v}-\mathbf{v}^c)||^2_{0,K}\leq C\sum_{f\in\mathcal{F}_h^0}\sum_{v\in f}h_v^{-3}|v^{j_v^+}_{K^+,f}-v^{j_v^-}_{K^-,f}|^2+C\sum_{f\in\mathcal{F}_h^\partial}\sum_{v\in f}h_v^{-3}|v^j_{K,f}|^2.\tag{4.13}$$

By application of the Piola transformation in (4.10), and through a scaling argument, we have

$$\sum_{v\in f}h_v^{-3}|v^{j_v^+}_{K^+,f}-v^{j_v^-}_{K^-,f}|^2\leq Ch_f^{-1}||[\![\mathbf{v}]\!]_N||^2_{0,f}.\tag{4.14}$$

Similarly, for $v\in f\in G_\partial$, we have

$$\sum_{v\in f}h_v^{-3}|v^j_{K,f}|^2\leq Ch_f^{-1}||\mathbf{v}\cdot\mathbf{n}_f||^2_{0,f}.\tag{4.15}$$

Thus, from (4.13)-(4.15), replacing f by F in the above inequalities, we get

$$\sum_{K\in\mathcal{T}_h}||\mathrm{div}\,(\mathbf{v}-\mathbf{v}^c)||^2_{0,K}\leq C_{\mathrm{app}}\sum_{F\in\mathcal{F}_h}h_F^{-1}||[\![\mathbf{v}]\!]_N||^2_{0,F}.\tag{4.16}$$

On the other hand, by similar arguments, we can obtain

$$\sum_{K\in\mathcal{T}_h}||\mathbf{v}-\mathbf{v}^c||^2_{0,K}\leq C_{\mathrm{app}}\sum_{F\in\mathcal{F}_h}h_F||[\![\mathbf{v}]\!]_N||^2_{0,F}.\tag{4.17}$$

Then the lemma follows by the definition of $|||\cdot|||_h$, (4.16) and (4.17).   □

Before stating the main result, we first introduce some local error indicators. Set

$$\eta^2_{R_K}=h_K^2||\mathbf{f}_h+\mathbf{grad}(\mathrm{div}\,\mathbf{u}_h)-\mathbf{u}_h||^2_{0,K},\tag{4.18}$$

where $\mathbf{f}_h\in\mathbf{V}_h^l$ is an approximation of $\mathbf{f}$, here $\mathbf{f}_h\in\mathbf{V}_h^l$ may be chosen to be, e.g. the L^2 projection of $\mathbf{f}$, for more details see Remark 4.1. This term measures the residual of the governing partial differential equation (1.1). To measure the error by **curl** operator, we introduce η_{C_K} denoted by

$$\eta^2_{C_K}=h_K^2||\mathbf{curl}(\mathbf{f}_h-\mathbf{u}_h)||^2_{0,K}.\tag{4.19}$$

The face residual with respect to the jump of $\operatorname{div}\mathbf{u}_h$ is denoted by

$$\eta_{D_K}^2 = \frac{1}{2}\sum_{F\in\partial K\backslash\Gamma} h_K \|[\![\operatorname{div}\mathbf{u}_h]\!]\|_{0,F}^2. \tag{4.20}$$

We also introduce

$$\eta_{T_K}^2 = \frac{1}{2}\sum_{F\in\partial K\backslash\Gamma} h_K \|[\![\,\mathbf{f}_h-\mathbf{u}_h]\!]_T\|_{0,F}^2 \tag{4.21}$$

to measure the tangential jump $\mathbf{f}_h-\mathbf{u}_h$ over interior face. Furthermore, to measure the normal jump of numerical solution $\mathbf{u}_h$, we introduce η_{N_K} defined by

$$\eta_{N_K}^2 = \frac{1}{2}\sum_{F\in\partial K\backslash\Gamma} \|\beta^{\frac{1}{2}}[\![\mathbf{u}_h]\!]_N\|_{0,F}^2 + \sum_{F\in\partial K\cap\Gamma} \|\beta^{\frac{1}{2}}(\mathbf{u}_h\cdot\mathbf{n})\|_{0,F}^2. \tag{4.22}$$

Then the sum of the above five local error indicators is denoted by η_K,

$$\eta_K^2 = \eta_{R_K}^2 + \eta_{C_K}^2 + \eta_{D_K}^2 + \eta_{T_K}^2 + \eta_{N_K}^2. \tag{4.23}$$

4.1 Reliability

The aim in this subsection is to provide an upper bound for the total error $|\!|\!|\mathbf{u}-\mathbf{u}_h|\!|\!|_h$, which is showed in the following theorem.

Theorem 4.1. *Let $\mathbf{u}$ denote the solution of the problem (1.1), and $\mathbf{u}_h$ denote the numerical solution of the DG method in (2.3). Assume that θ is greater than or equal to the constant θ^* in Lemma 3.2. Then there exist constants $C_R > 0, C_P > 0$ depending on the minimum angle of $\mathcal{T}_h$ and the degree of the polynomial l, such that*

$$|\!|\!|\mathbf{u}-\mathbf{u}_h|\!|\!|_h \leq C_R\left(\sum_{K\in\mathcal{T}_h}\eta_K^2\right)^{\frac{1}{2}} + C_P\|f-f_h\|_0, \tag{4.24}$$

where $f_h\in V_h^l$ is an approximation of f, $f_h\in V_h^l$ may be chosen to be, e.g. the L^2 projection of f.

Before proving the above theorem, we make some preparations. To begin, we can obtain the following result by using similar ideas to Lemma 4.3 in [21].

Lemma 4.3. *The following error bound holds*

$$|\!|\!|\mathbf{u}-\mathbf{u}_h|\!|\!|_h \leq \sup_{\mathbf{w}\in H_0(\operatorname{div};\Omega)} \mathcal{A}(\mathbf{w}) + (1+C_{\operatorname{cont}})|\!|\!|\mathbf{u}_h-\mathbf{u}_h^c|\!|\!|_h, \tag{4.25}$$

where $\mathbf{u}_h^c$ is the conforming approximation of $\mathbf{u}_h$ from Lemma 4.2, and $\mathcal{A}(\cdot)$ is defined as

$$\mathcal{A}(\mathbf{w}) = \inf_{\mathbf{w}_h\in V_h^l} \frac{|\int_\Omega f\cdot(\mathbf{w}-\mathbf{w}_h)dx - \tilde{a}_h(\mathbf{u}_h,\mathbf{w}-\mathbf{w}_h)|}{|\!|\!|\mathbf{w}|\!|\!|_h}, \quad \forall\mathbf{w}\in H_0(\operatorname{div};\Omega). \tag{4.26}$$

Proof. Using the triangle inequality, we get

$$|||\mathbf{u}-\mathbf{u}_h|||_h \le |||\mathbf{u}-\mathbf{u}_h^c|||_h + |||\mathbf{u}_h-\mathbf{u}_h^c|||_h, \tag{4.27}$$

where $\mathbf{u}_h^c$ is the conforming approximation of $\mathbf{u}_h$ from Lemma 4.2. Letting $\mathbf{w}=\mathbf{u}-\mathbf{u}_h^c \in H_0(\mathrm{div};\Omega)$ and using the coercivity property in (4.4), we obtain

$$|||\mathbf{w}|||_h^2 = \tilde{a}_h(\mathbf{w},\mathbf{w}) = \tilde{a}_h(\mathbf{u},\mathbf{w}) - \tilde{a}_h(\mathbf{u}_h,\mathbf{w}) + \tilde{a}_h(\mathbf{u}_h-\mathbf{u}_h^c,\mathbf{w}).$$

Since $\mathbf{u},\mathbf{w} \in H_0(\mathrm{div};\Omega)$, we get

$$\tilde{a}_h(\mathbf{u},\mathbf{w}) = a(\mathbf{u},\mathbf{w}) = \int_\Omega \mathbf{f}\cdot\mathbf{w}dx.$$

Using (4.3), we have

$$\tilde{a}_h(\mathbf{u}_h,\mathbf{w}_h) = \int_\Omega \mathbf{f}\cdot\mathbf{w}_h dx, \quad \forall \mathbf{w}_h \in \mathbf{V}_h^l.$$

Thus, we obtain that

$$|||\mathbf{w}|||_h^2 = \int_\Omega \mathbf{f}\cdot(\mathbf{w}-\mathbf{w}_h)dx - \tilde{a}_h(\mathbf{u}_h,\mathbf{w}-\mathbf{w}_h) + \tilde{a}_h(\mathbf{u}_h-\mathbf{u}_h^c,\mathbf{w})$$

for any $\mathbf{w}_h \in \mathbf{V}_h^l$. Using the continuity of $\tilde{a}_h(\cdot,\cdot)$ in Lemma 4.1, we have

$$|||\mathbf{w}|||_h^2 \le |\int_\Omega \mathbf{f}\cdot(\mathbf{w}-\mathbf{w}_h)dx - \tilde{a}_h(\mathbf{u}_h,\mathbf{w}-\mathbf{w}_h)| + C_{\mathrm{cont}}|||\mathbf{u}_h-\mathbf{u}_h^c|||_h|||\mathbf{w}|||_h$$

for any $\mathbf{w}_h \in \mathbf{V}_h^l$. Then we get

$$|||\mathbf{u}-\mathbf{u}_h^c|||_h \le \mathcal{A}(\mathbf{u}-\mathbf{u}_h^c) + C_{\mathrm{cont}}|||\mathbf{u}_h-\mathbf{u}_h^c|||_h.$$

Noting that $\mathcal{A}(\mathbf{u}-\mathbf{u}_h^c) \le \sup_{\mathbf{w}\in H_0(\mathrm{div};\Omega)}\mathcal{A}(\mathbf{w})$ and using the triangle inequality in (4.27), we complete the proof. $\qquad\square$

Noting that $|||\mathbf{u}_h-\mathbf{u}_h^c|||_h$ can be bounded as follows

$$|||\mathbf{u}_h-\mathbf{u}_h^c|||_h \le (2\theta^{-1}C_{\mathrm{app}}+1)\sum_{F\in\mathcal{F}_h}||\beta^{\frac{1}{2}}[\![\mathbf{u}_h]\!]_N||_{0,F}^2 = (2\theta^{-1}C_{\mathrm{app}}+1)\sum_{K\in\mathcal{T}_h}\eta_{N_K}^2, \tag{4.28}$$

it leaves us to give an upper bound for $\mathcal{A}(\mathbf{w})$. To proceed, we need the following regular decomposition [13,19].

Lemma 4.4. *Suppose that $\Omega \in R^3$ is a bounded Lipschitz domain which is topology equivalent to a ball, for any $\mathbf{P} \in H_0(\mathrm{div};\Omega)$, there exist Q, $\boldsymbol{\phi} \in H_0^1(\Omega)^3$ such that*

$$\mathbf{P} = Q + \mathbf{curl}\,\boldsymbol{\phi}. \tag{4.29}$$

Moreover, there exists a constant $C_{\mathrm{dec}}>0$ depending only on Ω such that

$$||Q||_1 \le C_{\mathrm{dec}}||\mathbf{P}||_{\mathrm{div}}, \quad ||\boldsymbol{\phi}||_1 \le C_{\mathrm{dec}}||\mathbf{P}||_{\mathrm{div}}. \tag{4.30}$$

We shall also take advantage of the approximation property of the following (quasi)-interpolation [13].

Lemma 4.5. *For any $w \in H_0(\mathrm{div};\Omega) \cap H^1(\Omega)^3$, there exists a (quasi)-interpolation $w_h \in V_h^c$ such that*

$$\sum_{K \in \mathcal{T}_h} \left(\|\mathrm{div}\,(w - w_h)\|_{0,K}^2 + h_K^{-2}\|w - w_h\|_{0,K}^2 + h_K^{-1}\|w - w_h\|_{0,\partial K}^2 \right) \leq C_{\mathrm{int}}^2 \|w\|_1^2, \qquad (4.31)$$

where $C_{\mathrm{int}} > 0$ is a constant depending on the minimum angle of $\mathcal{T}_h$ and the degree of the polynomial l.

Furthermore, we need the following result by using Clément or Scott-Zhang interpolation [16,33].

Lemma 4.6. *For any $\boldsymbol{\varphi} \in H_0^1(\Omega)^3$, there exists a piecewise linear approximation $\boldsymbol{\varphi}_h \in H_0^1(\Omega)^3 \cap S_h^1$ such that*

$$\sum_{K \in \mathcal{T}_h} \left(|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h|_{1,K}^2 + h_K^{-2}\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,K}^2 + h_K^{-1}\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{0,\partial K}^2 \right) \leq C_{\mathrm{cle}}^2 \|\boldsymbol{\varphi}\|_1^2, \qquad (4.32)$$

where $C_{\mathrm{cle}} > 0$ is a constant depending only on minimum angle of $\mathcal{T}_h$.

With the above preparations, we can prove the following result.

Lemma 4.7. *For any $w \in H_0(\mathrm{div};\Omega)$, the following bound for $\mathcal{A}(w)$ holds*

$$\mathcal{A}(w) \leq C \left(\sum_{K \in \mathcal{T}_h} \eta_K^2 \right)^{\frac{1}{2}} + C\|f - f_h\|_0^2, \qquad (4.33)$$

where $C > 0$ is a constant depending on the minimum angle of $\mathcal{T}_h$ and the degree of the polynomial l.

Proof. For any $\mathbf{w} \in H_0(\mathrm{div};\Omega)$, noting that $\|\|\mathbf{w}\|\|_h = \|\mathbf{w}\|_{\mathbf{div}}$, it follows the definition of $\mathcal{A}(\mathbf{w})$ that

$$\mathcal{A}(\mathbf{w}) \leq \frac{|\int_\Omega \mathbf{f} \cdot (\mathbf{w} - \mathbf{w}_h^c)dx - \tilde{a}_h(\mathbf{u}_h, \mathbf{w} - \mathbf{w}_h^c)|}{\|\mathbf{w}\|_{\mathbf{div}}}, \qquad (4.34)$$

for any $\mathbf{w}_h^c \in \mathbf{V}_h^c$. In terms of Lemma 4.4, we obtain the regular decomposition of $\mathbf{w}$ as

$$\mathbf{w} = \mathbf{w}^0 + \mathbf{curl}\boldsymbol{\phi}.$$

In (4.34), we choose $\mathbf{w}_h^c \in \mathbf{V}_h^c$ to be

$$\mathbf{w}_h^c = \mathbf{w}_h^0 + \mathbf{curl}\boldsymbol{\phi}_h,$$

where $\mathbf{w}_h^0$ is defined by the (quasi)-interpolation in Lemma 4.5 and $\boldsymbol{\phi}_h$ is defined by the Clément interpolation in Lemma 4.6. For the reason of $\mathbf{curl}\boldsymbol{\phi}_h \in \mathbf{V}_h^c$, please refer to ([13], page 1876). Hence we obtain

$$\int_\Omega \mathbf{f}\cdot(\mathbf{w}-\mathbf{w}_h^c)dx - \tilde{a}_h(\mathbf{u}_h,\mathbf{w}-\mathbf{w}_h^c) \equiv \mathcal{B}_1+\mathcal{B}_2,$$

with

$$\mathcal{B}_1 = \int_\Omega \mathbf{f}\cdot(\mathbf{w}^0-\mathbf{w}_h^0)dx - \tilde{a}_h(\mathbf{u}_h,\mathbf{w}^0-\mathbf{w}_h^0), \tag{4.35a}$$

$$\mathcal{B}_2 = \int_\Omega (\mathbf{f}-\mathbf{u}_h)\cdot\mathbf{curl}(\boldsymbol{\phi}-\boldsymbol{\phi}_h)dx. \tag{4.35b}$$

We next establish bounds for $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. For $\mathcal{B}_1$, by the definition of $\tilde{a}_h(\cdot,\cdot)$ we obtain

$$\mathcal{B}_1 = \int_\Omega (\mathbf{f}_h-\mathbf{u}_h)\cdot(\mathbf{w}^0-\mathbf{w}_h^0)dx - \sum_{K\in\mathcal{T}_h}\int_K \mathrm{div}\,\mathbf{u}_h\mathrm{div}\,(\mathbf{w}^0-\mathbf{w}_h^0)dx$$

$$- \sum_{K\in\mathcal{T}_h}\int_K \mathcal{L}(\mathbf{u}_h)\mathrm{div}\,(\mathbf{w}^0-\mathbf{w}_h^0)dx + \sum_{K\in\mathcal{T}_h}\int_K (\mathbf{f}-\mathbf{f}_h)\cdot(\mathbf{w}^0-\mathbf{w}_h^0)dx$$

with $\mathbf{f}_h\in\mathbf{V}_h^l$. For the second term on the right hand of the above equality, integrating by parts, and using the conformity of $\mathbf{w}^0-\mathbf{w}_h^0$, we see that

$$-\sum_{K\in\mathcal{T}_h}\int_K \mathrm{div}\,\mathbf{u}_h\mathrm{div}\,(\mathbf{w}^0-\mathbf{w}_h^0)dx$$

$$= \sum_{K\in\mathcal{T}_h}\int_K \big(\mathbf{grad}(\mathrm{div}\,\mathbf{u}_h)\big)\cdot(\mathbf{w}^0-\mathbf{w}_h^0)dx - \sum_{K\in\mathcal{T}_h}\int_{\partial K} \mathrm{div}\,\mathbf{u}_h\big((\mathbf{w}^0-\mathbf{w}_h^0)\cdot\mathbf{n}_K\big)ds$$

$$= \sum_{K\in\mathcal{T}_h}\int_K \big(\mathbf{grad}(\mathrm{div}\,\mathbf{u}_h)\big)\cdot(\mathbf{w}^0-\mathbf{w}_h^0)dx - \sum_{K\in\mathcal{T}_h}\sum_{F\in\partial K\backslash\Gamma}\frac{1}{2}\int_F [\![\mathrm{div}\,\mathbf{u}_h]\!]\cdot(\mathbf{w}^0-\mathbf{w}_h^0)ds,$$

where $\mathbf{n}_k$ denotes the outward unit normal vector on ∂K. Thus,

$$\mathcal{B}_1 = \sum_{K\in\mathcal{T}_h}\int_K \big(\mathbf{f}_h+\mathbf{grad}(\mathrm{div}\,\mathbf{u}_h)-\mathbf{u}_h\big)\cdot(\mathbf{w}^0-\mathbf{w}_h^0)dx - \sum_{K\in\mathcal{T}_h}\int_K \mathcal{L}(\mathbf{u}_h)\mathrm{div}\,(\mathbf{w}^0-\mathbf{w}_h^0)dx$$

$$- \sum_{K\in\mathcal{T}_h}\sum_{F\in\partial K\backslash\Gamma}\frac{1}{2}\int_F [\![\mathrm{div}\,\mathbf{u}_h]\!]\cdot(\mathbf{w}^0-\mathbf{w}_h^0)ds + \sum_{K\in\mathcal{T}_h}\int_K (\mathbf{f}-\mathbf{f}_h)\cdot(\mathbf{w}^0-\mathbf{w}_h^0)dx$$

$$\equiv \mathcal{B}_{11}+\mathcal{B}_{12}+\mathcal{B}_{13}+\mathcal{B}_{14}.$$

Clearly, we have that

$$\mathcal{B}_{11} \le \sum_{K\in\mathcal{T}_h}\eta_{R_K}h_K^{-1}\|\mathbf{w}^0-\mathbf{w}_h^0\|_{0,K}, \tag{4.36}$$

where η_{R_K} is the residual defined by (4.18).

By Cauchy-Schwarz inequality and the boundness of the lifting operator in (4.2), we obtain

$$
\begin{aligned}
\mathcal{B}_{12} &\leq \Big(\sum_{K \in \mathcal{T}_h} \|\mathcal{L}(\mathbf{u}_h)\|_{0,K}^2 \Big)^{\frac{1}{2}} \Big(\sum_{K \in \mathcal{T}_h} \|\mathrm{div}\,(\mathbf{w}^0 - \mathbf{w}_h^0)\|_{0,K}^2 \Big)^{\frac{1}{2}} \\
&\leq \theta^{-\frac{1}{2}} C_{\mathrm{lift}}^{\frac{1}{2}} \Big(\sum_{F \in \mathcal{F}_h} \|\beta^{\frac{1}{2}} [\![\mathbf{u}_h]\!]_N\|_{0,F}^2 \Big)^{\frac{1}{2}} \Big(\sum_{K \in \mathcal{T}_h} \|\mathrm{div}\,(\mathbf{w}^0 - \mathbf{w}_h^0)\|_{0,K}^2 \Big)^{\frac{1}{2}} \\
&= \theta^{-\frac{1}{2}} C_{\mathrm{lift}}^{\frac{1}{2}} \Big(\sum_{K \in \mathcal{T}_h} \eta_{N_K}^2 \Big)^{\frac{1}{2}} \Big(\sum_{K \in \mathcal{T}_h} \|\mathrm{div}\,(\mathbf{w}^0 - \mathbf{w}_h^0)\|_{0,K}^2 \Big)^{\frac{1}{2}}
\end{aligned}
\tag{4.37}
$$

with η_{N_K} the jump residual defined in (4.22).

Also, using Cauchy-Schwarz inequality, we obtain that

$$
\begin{aligned}
\mathcal{B}_{13} &\leq \sum_{K \in \mathcal{T}_h} \Big(\sum_{F \in \partial K \backslash \Gamma} \frac{1}{2} h_K \|[\![\mathrm{div}\,\mathbf{u}_h]\!]\|_{0,F}^2 \Big)^{\frac{1}{2}} \Big(\sum_{F \in \partial K} \frac{1}{2} h_K^{-1} \|\mathbf{w}^0 - \mathbf{w}_h^0\|_{0,F}^2 \Big)^{\frac{1}{2}} \\
&\leq \sum_{K \in \mathcal{T}_h} \eta_{D_K} h_K^{-\frac{1}{2}} \|\mathbf{w}^0 - \mathbf{w}_h^0\|_{0,\partial K},
\end{aligned}
\tag{4.38}
$$

where η_{D_K} is the jump residual defined in (4.20). Similarly, we have

$$
\mathcal{B}_{14} \leq \sum_{K \in \mathcal{T}_h} h_K \|\mathbf{f} - \mathbf{f}_h\|_{0,K} h_K^{-1} \|\mathbf{w}^0 - \mathbf{w}_h^0\|_{0,K}.
\tag{4.39}
$$

Combining (4.36), (4.37), (4.38) and (4.39) together, using Cauchy-Schwarz inequality and approximation property in (4.31), we see that

$$
\mathcal{B}_1 \leq C \Big(\sum_{K \in \mathcal{T}_h} (\eta_{R_K}^2 + \eta_{N_K}^2 + \eta_{D_K}^2) \Big)^{\frac{1}{2}} \|\mathbf{w}^0\|_1 + C h_K \|\mathbf{f} - \mathbf{f}_h\|_0 \|\mathbf{w}^0\|_1.
\tag{4.40}
$$

For $\mathcal{B}_2$, integrating by parts, we obtain

$$
\begin{aligned}
\mathcal{B}_2 &= \int_\Omega (\mathbf{f} - \mathbf{f}_h) \cdot \mathbf{curl}(\boldsymbol{\phi} - \boldsymbol{\phi}_h) dx + \int_\Omega (\mathbf{f}_h - \mathbf{u}_h) \cdot \mathbf{curl}(\boldsymbol{\phi} - \boldsymbol{\phi}_h) dx \\
&= \int_\Omega (\mathbf{f} - \mathbf{f}_h) \cdot \mathbf{curl}(\boldsymbol{\phi} - \boldsymbol{\phi}_h) dx + \sum_{K \in \mathcal{T}_h} \int_K \mathbf{curl}(\mathbf{f}_h - \mathbf{u}_h) \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h) ds \\
&\quad - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \big((\mathbf{f}_h - \mathbf{u}_h) \times \mathbf{n}_K \big) \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h) ds \\
&= \int_\Omega (\mathbf{f} - \mathbf{f}_h) \cdot \mathbf{curl}(\boldsymbol{\phi} - \boldsymbol{\phi}_h) dx + \sum_{K \in \mathcal{T}_h} \int_K \mathbf{curl}(\mathbf{f}_h - \mathbf{u}_h) \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h) ds \\
&\quad - \sum_{K \in \mathcal{T}_h} \sum_{F \in \partial K \backslash \Gamma} \frac{1}{2} \int_F [\![\mathbf{f}_h - \mathbf{u}_h]\!]_T \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h) ds \\
&\equiv \mathcal{B}_{21} + \mathcal{B}_{22} + \mathcal{B}_{23}.
\end{aligned}
$$

Obviously, we have

$$\mathcal{B}_{21} \leq \|\mathbf{f} - \mathbf{f}_h\|_0 \|\mathbf{curl}(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\|_0. \tag{4.41}$$

By Cauchy-Schwarz inequality, we get that

$$\begin{aligned}
\mathcal{B}_{22} &\leq \sum_{K \in \mathcal{T}_h} h_K \|\mathbf{curl}(\mathbf{f}_h - \mathbf{u}_h)\|_{0,K} h_K^{-1} \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{0,K} \\
&= \sum_{K \in \mathcal{T}_h} \eta_{C_K} h_K^{-1} \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{0,K},
\end{aligned} \tag{4.42}$$

where η_{C_K} is the residual defined in (4.19).

For $\mathcal{B}_{23}$, we have

$$\begin{aligned}
\mathcal{B}_{23} &\leq \sum_{K \in \mathcal{T}_h} \left(\sum_{F \in \partial K \backslash \Gamma} \frac{1}{2} h_K \|[\![\mathbf{f}_h - \mathbf{u}_h]\!]_T\|_{0,F}^2 \right)^{\frac{1}{2}} \left(\sum_{F \in \partial K} \frac{1}{2} h_K^{-1} \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{0,F}^2 \right)^{\frac{1}{2}} \\
&\leq \sum_{K \in \mathcal{T}_h} \eta_{T_K} h_K^{-\frac{1}{2}} \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{0,\partial K},
\end{aligned} \tag{4.43}$$

with η_{T_K} defined in (4.21).

Combining (4.41), (4.42), and (4.43) together, using Cauchy-Schwarz inequality and approximation property in (4.32), we see that

$$\mathcal{B}_2 \leq C \left(\sum_{K \in \mathcal{T}_h} (\eta_{C_K}^2 + \eta_{T_K}^2) \right)^{\frac{1}{2}} \|\boldsymbol{\phi}\|_1 + C \|\mathbf{f} - \mathbf{f}_h\|_0 \|\boldsymbol{\phi}\|_1. \tag{4.44}$$

Finally, based on (4.40) and (4.44), we conclude that

$$\mathcal{B}_1 + \mathcal{B}_2 \leq C \left(\sum_{K \in \mathcal{T}_h} \eta_K^2 \right)^{\frac{1}{2}} \left(\|\boldsymbol{\phi}\|_1^2 + \|\mathbf{w}^0\|_1^2 \right)^{\frac{1}{2}} + C \|\mathbf{f} - \mathbf{f}_h\|_0 \left(\|\boldsymbol{\phi}\|_1^2 + \|\mathbf{w}^0\|_1^2 \right)^{\frac{1}{2}}. \tag{4.45}$$

The desired result (4.33) follows from the above estimate and the stability bounds of regular decomposition in (4.30). □

From the above mentioned, we can obtain Theorem 4.1.

Remark 4.1. We observe that Theorem 4.1 is valid for any $\mathbf{f}_h \in \mathbf{V}_h^l$. To ensure that the data approximation term $\|\mathbf{f} - \mathbf{f}_h\|_0$ does not dominate the overall a posteriori error bound stated in (4.24), we should choose $\mathbf{f}_h$ to make $\|\mathbf{f} - \mathbf{f}_h\|_0$ tends to zero, at least, has the same convergence rate as the first term on the right-hand side of (4.24) (and thus also, at least, the same rate as $\|\|\mathbf{u} - \mathbf{u}_h\|\|_h$, see Theorem 4.2 in Subsection 4.2) as the mesh is refined. We can choose $\mathbf{f}_h \in \mathbf{V}_h^l$, e.g. the L^2 projection of $\mathbf{f}$ to satisfy the above property.

Remark 4.2. In the case when the source term $\mathbf{f}$ belongs to $\mathbf{H}(\mathbf{curl};\Omega)$, the data approximation term $\|\mathbf{f}-\mathbf{f}_h\|_0$ in Theorem 4.1 may be replaced by

$$\left(\sum_{K\in\mathcal{T}_h} h_K^2 \left(\|\mathbf{f}-\mathbf{f}_h\|_{0,K} + \|\mathbf{curl}(\mathbf{f}-\mathbf{f}_h)\|_{0,K}\right)\right)^{\frac{1}{2}}. \tag{4.46}$$

Indeed, integrating by parts, $\mathcal{B}_{21}$ can be bounded by $h_K\|\mathbf{curl}(\mathbf{f}-\mathbf{f}_h)\|_0 h_K^{-1}\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_0$, and all other terms in Theorem 4.1 remain unchanged. We then get the data approximation term defined by (4.46). In this case, the tangential jump error indicator η_{T_K} is defined by

$$\eta_{T_K}^2 = \frac{1}{2}\sum_{F\in\partial K\backslash\Gamma} h_K \|[\![\mathbf{u}_h]\!]_T\|_{0,F}^2, \tag{4.47}$$

which indicates the fact that the tangential components of $\mathbf{f}$ across the element faces are continuous.

4.2 Efficiency

In this subsection, we shall discuss the efficiency of the error estimator. To prove the efficiency bound, we take advantage of bubble function technique introduced by Verfürth [35]. Let b_K be the standard polynomial bubble function on element K, and b_F the standard polynomial bubble function on an interior face F, shared by two elements K and K'. Then we have the following results [21,36].

Lemma 4.8. *For any vector valued polynomial function v, there exists a constant $C>0$ depending on the minimum angle of $\mathcal{T}_h$ and the degree of the polynomial l such that*

$$\|b_K v\|_{0,K} \leq C\|v\|_{0,K}, \tag{4.48a}$$

$$\|v\|_{0,K} \leq C\|b_K^{\frac{1}{2}}v\|_{0,K}, \tag{4.48b}$$

$$\|\mathrm{div}(b_K v)\|_{0,K} \leq Ch_K^{-1}\|v\|_{0,K}, \tag{4.48c}$$

$$\|\mathbf{curl}(b_K v)\|_{0,K} \leq Ch_K^{-1}\|v\|_{0,K}. \tag{4.48d}$$

Similarly, for any vector valued polynomial function w on interior face F, there exists a constant $C>0$ depending on the minimum angle of $\mathcal{T}_h$ and the degree of the polynomial l such that

$$\|w\|_{0,F} \leq C\|b_F^{\frac{1}{2}}w\|_{0,F}. \tag{4.49}$$

Furthermore, there exists an extension $W_b\in H_0^1(\bar{K}\cup\bar{K'})^3$ of $b_F w$ such that $W_b|_F = b_F w$ and

$$\|W_b\|_{0,K} \leq Ch_F^{\frac{1}{2}}\|w\|_{0,F}, \tag{4.50a}$$

$$\|\mathrm{div}\,W_b\|_{0,K} \leq Ch_F^{-\frac{1}{2}}\|w\|_{0,F}, \tag{4.50b}$$

$$\|\mathbf{curl}\,W_b\|_{0,K} \leq Ch_F^{-\frac{1}{2}}\|w\|_{0,F}, \tag{4.50c}$$

where $C > 0$ are some constants depending on the minimum angle of $\mathcal{T}_h$ and the degree of the polynomial l.

To begin, we obtain the following local bounds.

Lemma 4.9. *Let $\boldsymbol{u}$ be the solution of the problem* (1.1), *and $\boldsymbol{u}_h$ be the numerical solution of the DG method in* (2.3). *Assume that θ is greater than or equal to the constant θ^* in Lemma* 3.2. *Then the following local bounds hold:*

(i) *For any $K \in \mathcal{T}_h$, we have*

$$\eta_{R_K} \leq C \left(||\text{div}(\boldsymbol{u} - \boldsymbol{u}_h)||_{0,K} + h_K ||\boldsymbol{u} - \boldsymbol{u}_h||_{0,K} + h_K ||\boldsymbol{f} - \boldsymbol{f}_h||_{0,K} \right). \tag{4.51}$$

(ii) *For any $K \in \mathcal{T}_h$, we have*

$$\eta_{C_K} \leq C \left(||\boldsymbol{u} - \boldsymbol{u}_h||_{0,K} + ||\boldsymbol{f} - \boldsymbol{f}_h||_{0,K} \right). \tag{4.52}$$

(iii) *For any interior face $F \in \mathcal{F}_h$ which belongs to two elements K and K', we obtain*

$$h_F^{\frac{1}{2}} ||[\![\text{div}\,\boldsymbol{u}_h]\!]||_{0,F} \leq \sum_{K \in U_F} \left(||\text{div}(\boldsymbol{u} - \boldsymbol{u}_h)||_{0,K} + h_K ||\boldsymbol{u} - \boldsymbol{u}_h||_{0,K} + h_K ||\boldsymbol{f} - \boldsymbol{f}_h||_{0,K} \right). \tag{4.53}$$

with $U_F = \{K, K'\}$.

(iv) *For any interior face $F \in \mathcal{F}_h$ which belongs to two elements K and K', we obtain*

$$h_F^{\frac{1}{2}} ||[\![\boldsymbol{f}_h - \boldsymbol{u}_h]\!]_N||_{0,F} \leq \sum_{K \in U_F} \left(||\boldsymbol{u}_h - \boldsymbol{u}_h||_{0,K} + ||\boldsymbol{f} - \boldsymbol{f}_h||_{0,K} \right). \tag{4.54}$$

(v) *For any interior face F, we have*

$$||\beta^{\frac{1}{2}} [\![\boldsymbol{u}_h]\!]_N||_{0,F} = ||\beta^{\frac{1}{2}} [\![\boldsymbol{u} - \boldsymbol{u}_h]\!]_N||_{0,F}. \tag{4.55}$$

For any boundary face F, we have

$$||\beta^{\frac{1}{2}} (\boldsymbol{u}_h \cdot \boldsymbol{n})||_{0,F} = ||\beta^{\frac{1}{2}} ((\boldsymbol{u} - \boldsymbol{u}_h) \cdot \boldsymbol{n})||_{0,F}. \tag{4.56}$$

All the constants $C > 0$ appear in the above inequalities depend on the minimum angle of $\mathcal{T}_h$ and the degree of the polynomial l.

Proof. (i) Let $\mathbf{v}_h = \mathbf{f}_h + \mathbf{grad}(\text{div}\,\mathbf{u}_h) - \mathbf{u}_h$, and $\mathbf{v}_b = b_K \mathbf{v}_h$. Noting that $-\mathbf{grad}(\text{div}\,\mathbf{u}) + \mathbf{u} = \mathbf{f}$ in $L^2(K)^3$, we have

$$
\begin{aligned}
||b_K^{\frac{1}{2}} \mathbf{v}_h||_{0,K}^2 &= \int_K \left(\mathbf{f}_h + \mathbf{grad}(\text{div}\,\mathbf{u}_h) - \mathbf{u}_h \right) \cdot \mathbf{v}_b dx \\
&= \int_K \left(\mathbf{f} + \mathbf{grad}(\text{div}\,\mathbf{u}_h) - \mathbf{u}_h \right) \cdot \mathbf{v}_b dx + \int_K (\mathbf{f}_h - \mathbf{f}) \cdot \mathbf{v}_b dx \\
&= \int_K \left(-\mathbf{grad}(\text{div}(\mathbf{u} - \mathbf{u}_h)) + (\mathbf{u} - \mathbf{u}_h) \right) \cdot \mathbf{v}_b dx + \int_K (\mathbf{f}_h - \mathbf{f}) \cdot \mathbf{v}_b dx \\
&= \int_K \text{div}(\mathbf{u} - \mathbf{u}_h) \text{div}\,\mathbf{v}_b dx + \int_K (\mathbf{u} - \mathbf{u}_h) \cdot \mathbf{v}_b dx + \int_K (\mathbf{f}_h - \mathbf{f}) \cdot \mathbf{v}_b dx,
\end{aligned}
$$

where in the last step we used integration by parts and the fact that $\mathbf{v}_b = 0$ on ∂K. Then from Cauchy-Schwarz inequality we obtain

$$\|\mathbf{v}_h\|_{0,K}^2 \leq \left(\|\text{div}(\mathbf{u} - \mathbf{u}_h)\|_{0,K}\|\text{div}\,\mathbf{v}_b\|_{0,K} + \|\mathbf{u} - \mathbf{u}_h\|_{0,K}\|\mathbf{v}_b\|_{0,K} + \|\mathbf{f} - \mathbf{f}_h\|_{0,K}\|\mathbf{v}_b\|_{0,K}\right).$$

Moreover, using (4.48c) and (4.48a), we get

$$\|\mathbf{v}_h\|_{0,K} \leq C\left(h_K^{-1}\|\text{div}(\mathbf{u} - \mathbf{u}_h)\|_{0,K} + \|\mathbf{u} - \mathbf{u}_h\|_{0,K} + \|\mathbf{f} - \mathbf{f}_h\|_{0,K}\right).$$

Observing that $\eta_{R_K} = h_K\|\mathbf{v}_h\|_{0,K}$, the above inequality gives (i).

(ii) Let $\mathbf{v}_h = \mathbf{curl}(\mathbf{f}_h - \mathbf{u}_h)$, and $\mathbf{v}_b = b_K\mathbf{v}_h$. Then from (4.48a) we have

$$\|\mathbf{v}_h\|_{0,K}^2 \leq C\|b_K^{\frac{1}{2}}\mathbf{v}_h\|_{0,K}^2 \leq C\int_K \mathbf{curl}(\mathbf{f}_h - \mathbf{u}_h)\cdot\mathbf{v}_b\,dx.$$

Moreover, noting that $\mathbf{curl}(\mathbf{f} - \mathbf{u}) = 0$ in $L^2(K)^3$, we obtain

$$\|\mathbf{v}_h\|_{0,K}^2 \leq C\int_K \mathbf{curl}\left((\mathbf{f}_h - \mathbf{f}) + (\mathbf{u} - \mathbf{u}_h)\right)\cdot\mathbf{v}_b\,dx.$$

Integrating by parts and combining Cauchy-Schwarz inequality with (4.48d), we have

$$\|\mathbf{v}_h\|_{0,K} \leq C(h_K^{-1}\|\mathbf{f} - \mathbf{f}_h\|_{0,K} + h_K^{-1}\|\mathbf{u} - \mathbf{u}_h\|_{0,K}).$$

Noting that $\eta_{C_K} = h_K\|\mathbf{v}_h\|_{0,K}$, the above inequality yields (ii).

(iii) Let $\mathbf{w}_h = [\![\text{div}\,\mathbf{u}_h]\!]$, $\mathbf{w}_b = b_F\mathbf{w}_h$. Defining $\mathbf{W}_b \in H_0^1(\bar{K}\cup\bar{K}')^3$ be the extension of $\mathbf{w}_b$ which satisfies (4.50a), (4.50b) and (4.50c). Using the fact that $[\![\text{div}\,\mathbf{u}]\!] = 0$, we obtain

$$\begin{aligned}
\|b_F^{\frac{1}{2}}\mathbf{w}_h\|_{0,F}^2 &= \int_F [\![\text{div}\,\mathbf{u}_h]\!]\cdot\mathbf{w}_b\,ds \\
&= \int_F [\![\text{div}(\mathbf{u}_h - \mathbf{u})]\!]\cdot\mathbf{w}_b\,ds \\
&= \sum_{K\in U_F}\int_K \left(\mathbf{grad}(\text{div})(\mathbf{u}_h - \mathbf{u})\right)\cdot\mathbf{W}_b\,dx + \int_K \text{div}(\mathbf{u}_h - \mathbf{u})\text{div}\,\mathbf{W}_b\,dx \\
&= \sum_{K\in U_F}\int_K \left((\mathbf{f}_h - \mathbf{f}) + \left(\mathbf{grad}(\text{div})(\mathbf{u}_h - \mathbf{u})\right) + (\mathbf{u} - \mathbf{u}_h)\right)\cdot\mathbf{W}_b\,dx \\
&\quad + \int_K \text{div}(\mathbf{u}_h - \mathbf{u})\text{div}\,\mathbf{W}_b\,dx - \int_K (\mathbf{f}_h - \mathbf{f})\cdot\mathbf{W}_b\,dx - \int_K (\mathbf{u} - \mathbf{u}_h)\cdot\mathbf{W}_b\,dx.
\end{aligned}$$

Since $\mathbf{f} + \mathbf{grad}(\text{div}\,\mathbf{u}) - \mathbf{u} = 0$ in $L^2(K)^3$, in view of (4.49), (4.50a) and (4.50b), we obtain

$$\|\mathbf{w}_h\|_{0,F} \leq C\sum_{K\in U_F}\left(h_F^{\frac{1}{2}}\|\mathbf{f}_h + \mathbf{grad}(\text{div}\,\mathbf{u}_h) - \mathbf{u}_h\|_{0,K} + h_F^{-\frac{1}{2}}\|\text{div}(\mathbf{u} - \mathbf{u}_h)\|_{0,K} \right.$$

$$\left. + h_F^{\frac{1}{2}}\|\mathbf{f} - \mathbf{f}_h\|_{0,K} + h_F^{\frac{1}{2}}\|\mathbf{u} - \mathbf{u}_h\|_{0,K}\right).$$

Making use of the bound for η_{R_K} and the shape-regularity of the mesh, we get

$$h_F^{\frac{1}{2}}||[\![\mathrm{div}\,\mathbf{u}_h]\!]||_{0,F} \leq C \sum_{K\in U_F} \left(||\mathrm{div}(\mathbf{u}-\mathbf{u}_h)||_{0,K} + h_K||\mathbf{f}-\mathbf{f}_h||_{0,K} + h_K||\mathbf{u}-\mathbf{u}_h||_{0,K} \right).$$

This yields (iii).

(iv) Let $\mathbf{w}_h = [\![\mathbf{f}_h-\mathbf{u}_h]\!]_T$, $\mathbf{w}_b = b_F\mathbf{w}_h$, and $\mathbf{W}_b \in H_0^1(\bar{K}\cup\bar{K}')^3$ be the extension of $\mathbf{w}_b$ which satisfies (4.50a), (4.50b) and (4.50c). Noting that $[\![\mathbf{f}-\mathbf{u}]\!]_T = 0$ in $L^2(K)^3$, we have

$$\begin{aligned}
||b_F^{\frac{1}{2}}\mathbf{w}_h||_{0,F}^2 &= \int_F [\![\mathbf{f}_h-\mathbf{u}_h]\!]_T\cdot\mathbf{w}_b ds \\
&= \int_F [\![\mathbf{f}_h-\mathbf{u}_h-\mathbf{f}+\mathbf{u}]\!]_T\cdot\mathbf{w}_b ds \\
&= \sum_{K\in U_F} \left(\int_K \mathbf{curl}(\mathbf{f}_h-\mathbf{u}_h)\cdot\mathbf{W}_b dx + \int_K (\mathbf{f}_h-\mathbf{u}_h-\mathbf{f}+\mathbf{u})\cdot\mathbf{curl}\,\mathbf{W}_b dx \right).
\end{aligned}$$

In view of (4.49), (4.50a) and (4.50c), we obtain

$$||\mathbf{w}_h||_{0,F} \leq C \sum_{K\in U_F} \left(h_F^{\frac{1}{2}}||\mathbf{curl}(\mathbf{f}_h-\mathbf{u}_h)||_{0,K} + h_F^{-\frac{1}{2}}||\mathbf{f}_h-\mathbf{f}||_{0,K} + h_F^{-\frac{1}{2}}||\mathbf{u}-\mathbf{u}_h||_{0,K} \right).$$

Using the bound for η_{C_K} and the shape-regularity of the mesh, we obtain

$$h_F^{\frac{1}{2}}||[\![\mathbf{f}_h-\mathbf{u}_h]\!]_T||_{0,F} \leq C \sum_{K\in U_F} \left(||\mathbf{u}-\mathbf{u}_h||_{0,K} + ||\mathbf{f}-\mathbf{f}_h||_{0,K} \right).$$

This gives (iv).

(v) Since $[\![\mathbf{u}]\!]_N = 0$ on interior faces and $\mathbf{u}\cdot\mathbf{n} = 0$ on the boundary faces, we can immediately obtain (4.55)-(4.56). □

We formulate the main result of this subsection in the following theorem, which follows immediately from the above lemma.

Theorem 4.2. *Let $\boldsymbol{u}$ denote the solution of the problem* (1.1), *and $\boldsymbol{u}_h$ denote the numerical solution of DG method in* (2.3). *Assume that θ is greater than or equal to the constant θ^* in Lemma* 3.2. *Then there exists a constant C_{EFF} depending on the minimum angle of $\mathcal{T}_h$ and the degree of the polynomial l such that*

$$\left(\sum_{K\in\mathcal{T}_h} \eta_K^2 \right)^{\frac{1}{2}} \leq C_{\mathrm{EFF}}\left(|||\boldsymbol{u}-\boldsymbol{u}_h|||_h + ||\boldsymbol{f}-\boldsymbol{f}_h||_0 \right). \tag{4.57}$$

Remark 4.3. In the case when the source term $\mathbf{f}$ belongs to $\mathbf{H}(\mathbf{curl};\Omega)$, similar efficiency bound may also be derived. In fact, bounds (i), (iii) and (v) in Lemma 4.9 remain unchanged. On the right-hand side of bounds in (ii) and (iv), the term $||\mathbf{f}-\mathbf{f}_h||_{0,K}$ may

be replaced by $h_K||\mathbf{curl}(\mathbf{f}-\mathbf{f}_h)||_{0,K}$. In the latter case, the term $h_F^{\frac{1}{2}}||[\![\mathbf{f}_h-\mathbf{u}_h]\!]_N||_{0,F}$ on the left-hand side of the inequality (4.54) must be replaced by $h_F^{\frac{1}{2}}||[\![\mathbf{u}_h]\!]_N||_{0,F}$. The data approximation term $||\mathbf{f}-\mathbf{f}_h||_0$ in Theorem 4.2 may be replaced by

$$\left(\sum_{K\in\mathcal{T}_h} h_K^2\left(||\mathbf{f}-\mathbf{f}_h||_{0,K}+||\mathbf{curl}(\mathbf{f}-\mathbf{f}_h)||_{0,K}\right)\right)^{\frac{1}{2}}. \tag{4.58}$$

5 Numerical experiments

In this section, we report some numerical experiments using MATLAB. We only choose the linear discontinuous finite element to show the numerical results. In each adaptive finite element procedure, we refine the marked triangles by the bisection algorithm, which derives from the *AFEM@Matlab* implementation [14].

Example 5.1. The test problem is two-dimensional equation of (1.1) in $\Omega=(0,1)\times 0,1)$. We set the right-hand side function side so that the exact solution is given by

$$\mathbf{u}(x,y)=\begin{pmatrix} u_1(x,y)\\ u_2(x,y) \end{pmatrix}=\begin{pmatrix} \sin(\pi x)\\ \sin(\pi y) \end{pmatrix}.$$

First, we present the a priori results for the penalty parameter $\theta=15$. Fig. 1 describes the energy errors $|||\mathbf{u}-\mathbf{u}_h|||_h$ with respect to the mesh size h in logarithmic scale. We can see that the slope is 1.0052, these results confirm Theorem 3.1. Moreover, we provide the error between the exact solution $u_1(x,y)=\sin(\pi x)$ and its numerical solution in Fig. 2.

We also provide the a posteriori results for this example. The true error $|||\mathbf{u}-\mathbf{u}_h|||_h$ and the error estimator

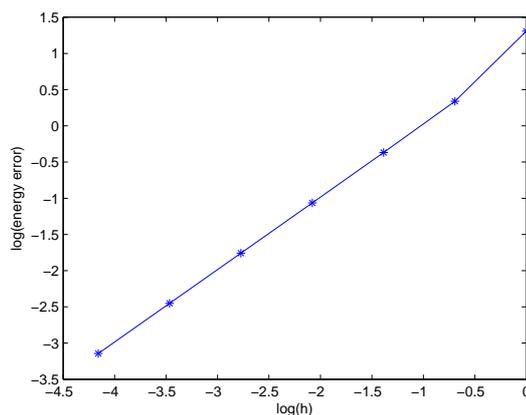$$\eta=\left(\sum_{K\in\mathcal{T}_h}\eta_K^2\right)^{\frac{1}{2}}$$

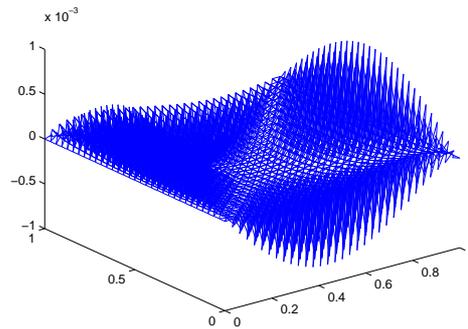Figure 1: The convergence rate for linear discontinuous finite element.

Figure 2: The error between the exact solution $u_1(x,y) = sin(\pi x)$ and its numerical solution for the $h = \frac{1}{32}$.

are computed on a sequence of adaptive meshes as functions of number of degrees of freedom and then showed in Fig. 3. The effectivity index given by $\frac{\eta}{\||\mathbf{u} - \mathbf{u}_h\||_h}$ is computed as a function of the number of degrees of freedom and then plotted in Fig. 4. It is between
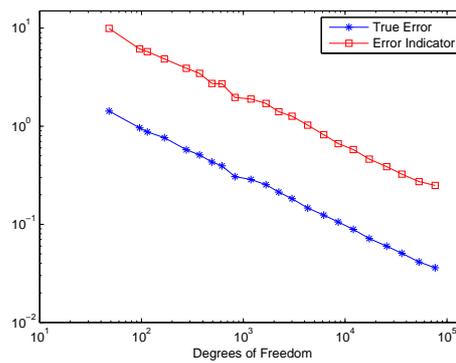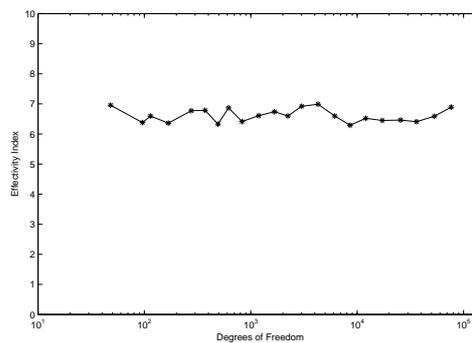

Figure 3: Performance of the indicator for Example 5.1.

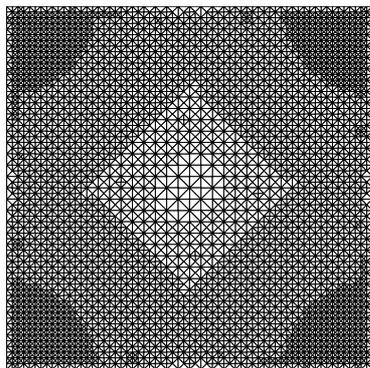Figure 4: Efficiency index for Example 5.1.

Figure 5: Adaptive mesh of level 20 for Example 5.1.

6 and 7. These results agree with the Theorem 4.1 and Theorem 4.2. Finally, in Fig. 5, we show the adaptive mesh of 20 level used in the computation. From Fig. 3, we observe the quasi-optimality of the adaptive algorithm in the sense that $|||\mathbf{u}-\mathbf{u}_h|||_h \approx CN^{-1/2}$ asymptotically, where N is the number of degrees of freedom.

Example 5.2. We consider the problem of (1.1) defined on the L-shaped domain $\Omega = (-1,1)^2 \backslash ([0,1] \times [-1,0])$ with the exact solution given by $\mathbf{u} = \mathrm{grad}(r^{\frac{2}{3}} \sin(\frac{2}{3}\theta))$ (in cylindrical coordinates).

As in Example 5.1, Fig. 6 displays actual errors $|||\mathbf{u}-\mathbf{u}_h|||_h$ and the error estimator η as functions of the number degrees of freedom. And the effectivity index as a function of the number of degrees of freedom is plotted in Fig. 7. It is between 5 and 6. These results confirm the Theorem 4.1 and Theorem 4.2. Furthermore, we can see from Fig. 8 that the singularity of the solution $\mathbf{u}$ at the re-entrant corner is captured by the error estimator. From Fig. 6, we observe that the adaptive algorithm have the quasi-optimality
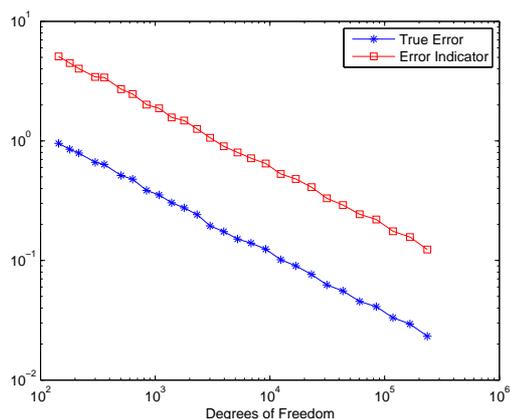

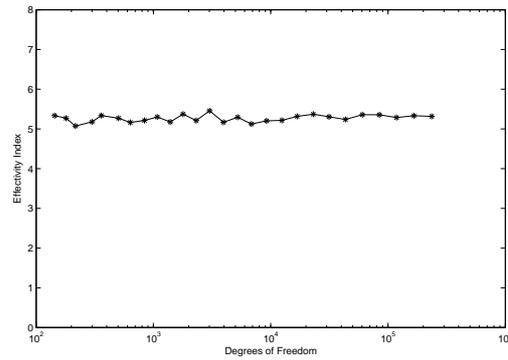Figure 6: Performance of the indicator for Example 5.2.

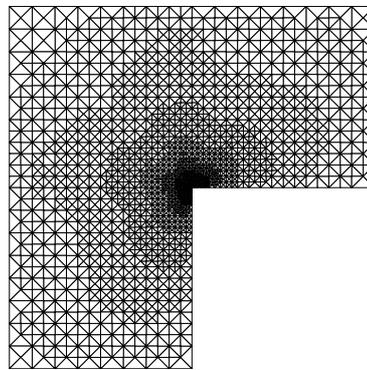Figure 7: Efficiency index for Example 5.2.

Figure 8: Adaptive mesh of level 18 for Example 5.2.

in the sense that $|||\mathbf{u}-\mathbf{u}_h|||_h \approx CN^{-1/2}$ asymptotically, where N is the number of degrees of freedom.

Acknowledgments

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