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ERROR ESTIMATES FOR AN OPTIMAL CONTROL PROBLEM GOVERNED BY THE HEAT EQUATION WITH STATE AND CONTROL CONSTRAINTS

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Abstract. In this work, we study priori error estimates for the numerical approximation of an optimal control problem governed by the heat equation with certain control constraint and ending point state constraint. By making use of the classical space-time discretization scheme, namely, finite element method with the space variable and backward Euler discretization for the time variable, we first project the original optimal control problem into a semi-discrete control and state constrained optimal control problem governed by an ordinary differential equation, and then project the aforementioned semi-discrete problem into a fully discrete optimization problem with constraints. With the help of Pontryagin's maximum principle, we obtain, under a certain reasonable condition of Slater style, not only an error estimate between the optimal controls for the original problem and the semi-discrete problem, but also an error estimate between the solutions of the semi-discrete problem and the fully discrete problem, which leads to an error estimate between the solutions of the original problem and the fully discrete problem. By making use of the aforementioned result, we also establish an numerical approximation for the exactly null controllability of the internally controlled heat equation.

Key Words. Finite element approximation, optimal control problem, the heat equation, ending point state constraint, error estimate.

1. Introduction

In this paper, we shall study error analysis for the discretization of an optimal control problem governed by the heat equation with certain control constraint and ending point (in time) state constraint, which will be introduced as follows. Let Ω be a bounded domain in $\mathbb{R}^d(d \leq 3)$ with a smooth boundary $\partial\Omega$, ω be an open subset of Ω and T be a positive number. We denote by Q the product set $\Omega \times (0,T)$ and by χ_ω the characteristic function of the subset ω. Write

$$\mathcal{K} = \{v \in L^2(0,T;L^2(\Omega)) \; ; \; \|v(t)\| \leq 1, \text{ for a.e. } t \in [0,T]\}$$

and

$$K = \{w \in L^2(\Omega) \; ; \; \|w\| \leq 1\}.$$

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Here and in what follows, we shall use $\|\cdot\|$ and $(\cdot,\cdot)$ to denote the usual norm and the inner product of the space $L^2(\Omega)$. The optimal control problem which we shall study reads

$$(P) \qquad \min_{u\in\mathcal{K}}\left\{\frac{1}{2}\int_0^T\int_\Omega (y-y_d)^2dxdt+\frac{1}{2}\int_0^T\int_\Omega u^2dxdt\right\}$$

subject to

$$\begin{cases} \partial_t y-\triangle y=\chi_\omega u & \text{in } \Omega\times(0,T),\\ y=0 & \text{on } \partial\Omega\times(0,T),\\ y(0)=y_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

and the ending point state constraint

$$y(T)\in K.$$

Here, $y_d\in L^2(Q)$ is a given target function and $y_0\in H_0^1(\Omega)$ is a given initial data. Throughout the paper, the notation $y(t)$ stands for the value of the function $y:[0,T]\to L^2(\Omega)$ at the time t. As a matter of convenience, we shall often omit the notation t in functions of t and the notation (x,t) in functions of (x,t) whenever no confusion is possible.

We are going first set up a semi-discrete optimal control problem (P_h) projected by the original problem (P) in the sense of finite element, which is an optimal control problem governed by a system of linear ordinary differential equations with the ending point state constraint and a certain control constraint, and then establish a fully discrete optimal control problem $(P_{h\tau})$ projected by the aforementioned semi-discrete problem according to the classical backward Euler discretization scheme for the time variable. The problem $(P_{h\tau})$ is an optimal control problem governed by a system of linear algebraic equations with certain state and control constraints, and can be viewed as a problem of minimization of a quadratic function with convex constraints in a finite dimensional space, which, we assume and believe, can be solved numerically.

The purpose of this work is to obtain a convergence order for $L^2(Q)-$error between the optimal control for the original problem (P) and the solution of the fully discrete problem $(P_{h\tau})$. There should be several ways to reach such an aim. We shall make use of Pontryagin's maximum principle of the original problem (P), the semi-discrete problem (P_h) and the fully discrete problem (P_h) to establish first an error estimate between the solutions of the problems (P) and (P_h), and then an error estimate between the solutions for the problems (P_h) and $(P_{h\tau})$. The Pontryagin maximum principle of the problem (P) (also for the problems (P_h) and (P_h)) consists in a state equation, an adjoint equation, a transversality condition and a connection between the optimal control and the adjoint state, namely, the solution of the adjoint equation, through a variational inequality. The advantage that can be taken from the Pontryagin maximum principle in dealing with such error estimates is that one can expect quantitative expressions of the optimal controls via the adjoint states. Such relationships are helpful for us to get the desired error estimates in many cases.

However, due to the involvement of ending point state constraint, there will be a pair of multipliers in the space $\mathbb{R}\times L^2(\Omega)$ and appeared in the Pontryagin maximum principle for each problem among the problems (P), (P_h) and $(P_{h\tau})$. We denote them by (λ,μ), (λ_h,μ_h) and $(\lambda_{h\tau},\mu_{h\tau})$ for the problems (P), (P_h) and $(P_{h\tau})$, respectively. The multipliers λ, λ_h and $\lambda_{h\tau}$ appear in both variational inequalities and adjoint state equations, while the multipliers μ, μ_h and $\mu_{h\tau}$ arise in the initial data of adjoint state equations. These multipliers are big trouble makers in dealing with such problems. First, we must make sure that the multipliers λ, λ_h and $\lambda_{h\tau}$ are not zero such that the corresponding Potryagin maximum principle are qualified, only which could give us valuable information for the optimal controls from the adjoint states. Secondly, after we guarantee that the qualified Pontryagin maximum principles hold for the aforementioned three problems, namely, the multipliers λ, λ_h and $\lambda_{h\tau}$ can be taken as number 1 and the multipliers $-\bar{\mu}$, $-\bar{\mu}_h$ and $-\bar{\mu}_{h\tau}$ are exactly the initial values for the adjoint equations corresponding to the problems (P), (P_h) and $(P_{h\tau})$, respectively, we still lack enough quantitative information for the multipliers $-\bar{\mu}$,$-\bar{\mu}_h$ and $-\bar{\mu}_{h\tau}$ to get error estimates among them, but only know that they stay in certain given subsets. Thus, we shall face with a real challenge in getting the error estimates among the solutions of the adjoint state equations, which plays an important role in dealing with the error estimate between the optimal controls of the problems (P) and $(P_{h\tau})$.

To overcome the first difficulty, we shall first make the following assumption on the problem (P) throughout the paper.

(A) : *There exists a control function $u_0 \in \mathcal{K}$ such that the corresponding solution $y(\cdot)$ of the equation (1.1) with $u = u_0$ reaches the interior of the set K at the time T, namely, $y(T) \in intK$.*

Then, we prove that such kind of property holds for the projected problems (P_h) and $(P_{h\tau})$ provided that (A) is assumed. Based on these, we establish the qualified Pontryagin maximum principle for the problems (P), (P_h) and $(P_{h\tau})$.

The condition (A) is called Slater condition, which ont only helps us to get the qualified Pontryagin maximum principle for the problems (P), (P_h) and $(P_{h\tau})$, but also guarantees the existence of the optimal controls for the original problem and projected problems. Moreover, it plays an important role when we prove that the families $\{\mu_h\}$ and $\{\mu_{h\tau}\}$ are bounded in certain norms for sufficiently small h and τ. We would like explain the reasonableness of the assumption (A) for the problem (P) by the following fact. Since the state equation of the problem (P) is the internally controlled heat equation, the condition (A) holds automatically provided that either the initial data y_0 has a smaller $L^2(\Omega)-$norm or the ending time T is bigger enough.

To surmount the second difficulty, we first establish explicit expressions for the optimal controls via the adjoint states for the problems (P), (P_h) and $(P_{h\tau})$, respectively, which are new to our best knowledge, and then we obtain the $H^1(0, T; L^2(\Omega))-$ regularity for the optimal controls $\bar{u}$ and $\bar{u}_h$ of the problems (P) and (P_h), respectively, which is not trivial in the optimal control problems involving both state and control constraints. After these, we show that both families $\{\bar{\mu}_h\}_{0<h\leq\bar{h}}$ and $\{\bar{\mu}_{h\tau}\}_{0<h\leq h_0, 0<\tau\leq\bar{\tau}}$ are bounded in the space $H_0^1(\Omega)$, where $\bar{h}$ and $\bar{\tau}$ are two given positive numbers. Based on the above results, we prove that the optimal controls for the problems (P_h) and $(P_{h\tau})$ are uniformly bounded in certain sense with respect to sufficiently small h and τ. Finally, by making use of all aforementioned results and according to all information, in particular, the transversality conditions, provided by the Pontryagin maximum principle, we establish an error estimate between optimal controls for the problems (P) and $(P_{h\tau})$ in the absence of error estimates among the quantities $-\bar{\mu}$,$-\bar{\mu}_h$ and $-\bar{\mu}_{h\tau}$.

We would like to mention that the Pontryagin maximum principle for the problem (P) provides us a necessary and sufficient condition for the optimal control. However, because of the involvement of unclear quantity $\bar{\mu}$ in the maximum principle, we can not expect to compute the solution by projecting the necessary and sufficient condition into a discrete form. Instead of it, we project the problem (P)

into a discrete problem, which, as we mentioned earlier, can be viewed as a problem of minimization of a quadratic function with certain convex constraints in a finite dimensional space. On the other hand, when we deal with the error estimate between the solutions of the problem (P) and (P_h), weaker conditions will be put on the optimal controls, the initial data y_0 and the target function y_d, and lower regularity for the multipliers $\bar{\mu}$ and $\bar{\mu}_h$ are called for, while as we study the error estimate between the solutions of the problems (P_h) and $(P_{h\tau})$, stronger conditions on the above quantities and higher regularity on the multipliers $\bar{\mu}_h$ and $\bar{\mu}_{h\tau}$ are required. This is why we make two steps to project the problem (P).

Next, we shall roughly state the main result obtained in this paper. Let h be the mesh size of a given finite element triangulation of Ω associated with the problem (P_h) and let τ be the uniform time step of the partition of the interval $[0,T]$ associated with the problem $(P_{h\tau})$. If we denote by $\bar{u}$ and $\bar{U}_{h\tau} = (\bar{U}_h^1, \bar{U}_h^2, \cdots, \bar{U}_h^N)$ the optimal controls of the problems (P) and $(P_{h\tau})$, respectively, then it holds that

$$\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \|\bar{u} - \bar{U}_h^i\|^2 dt \leq C(h^2 + \tau)$$

for all numbers h and τ with $0 < h \leq \bar{h}$ and $0 < \tau \leq \bar{\tau}$, where $\bar{h}$ and $\bar{\tau}$ are two given positive numbers.

As the development of the theory of optimal controls for partial differential equations, the related theoretic results are expected to be used to fields of applied sciences. Thus, people are getting more and more interesting in problems on the numerical approximations of optimal controls for partial differential equations. The error analysis plays an important role in such studies. In most related works, people do not consider any state constraint. We mention the works [2, 7, 16, 26] and the works [17, 21, 22] on priori error estimates for elliptic optimal control problems and parabolic optimal control problems respectively. We also quote the papers [4, 5, 18] on posteriori error estimates for the optimal control problems of partial differential equations. Due to the significance of state constraint in views of both mathematical theory and applied sciences, the numerical approximations of optimal controls for partial differential equations involving state constraints are very important but much more difficult to be studied. We would like to mention the works [8], [9] [12] and [23] on the error estimates and numerical approximations for optimal control problems governed by elliptic differential equations with certain state constraints. However, to our best knowledge, the investigations on numerical approximations for optimal control problems governed by parabolic equations with state constraints are quite few. In 1996, D. Tiba and F. Tröltzsch [27] investigated error estimates for the discretization of state constrained convex control problems, where the state equation is an abstract parabolic-like equation. They successfully proved, by making use of Pontryagin's maximum principle, that the norm of the difference between the optimal controls of the original problem and the corresponding discrete problem is governed by the norm of difference of adjoint states. Moreover, they claimed that the later includes the disretization error for linear parabolic equation. However, as what we mentioned earlier, due to the involvement of the state constraint, to get the error estimate between the adjoint states of the original problem and the discrete problem is not a trivial job. This is one of the main reasons that the work [27] does not contain any order of the estimates. In [1, 20, 28], the authors studied error estimates of optimal controls between the optimal control problems of parabolic equations with certain state constraints and corresponding semi-discrete problems. They mainly used the properties of optimizations of the cost functions and state equations for both original problems and the corresponding semi-discrete problems. This is a method often used in investigating the numerical approximations for inverse problems. The weakness to use such a way in dealing with the error estimates for optimal control problems, compared with the method to make use of Pontryagin's maximum principle, is that one may lose certain chances to gain more valuable information provided by Pontryagin's maximum principle, such as the connection between the control and the adjoint state and the relationship between the adjoint states for original problem and the discrete problem. Consequently, the orders for error estimates obtained by making use of such a way may be worse than those provided by Pontryagin's maximum principle. For instance, by making use of the method provided in [1, 20] to our problem (P), one can only have the order $h^{\frac{1}{2}}$ for the error estimate between the optimal controls for the problems (P) and the problem (P_h).

The rest of the paper is organized as follows. In section 2, we investigate the original problem (P) including the Pontryagin maximum principle, an explicit expression of the optimal control via the adjoint state and the regularity of the optimal control. In section 3, we set up a semi-discrete finite element approximation problem (P_h) for the original problem and discuss the similar subjects as those in the previous section. In section 4, we obtain an error estimate between the solutions of the problems (P) and (P_h). In section 5, we establish a fully discrete optimal control problem $(P_{h\tau})$ projected by the semi-discrete problem (P_h) and study the similar projects as those in section 2. In section 6, we derive an error estimate between the solutions of the problem (P_h) and the problem $(P_{h\tau})$, which leads to an error estimate between the solutions of the original problem (P) and the fully discrete problem $(P_{h\tau})$. In the last section, we establish, by making use of the main result in the paper, a numerical approximation for the exactly null controllability for the internally controlled heat equation.

2. Optimality conditions for solution of the problem (P)

In this section, we shall discuss certain properties for the optimal control of the problem (P). First of all, we quote from [13] the following well known result, which will be used frequently in this paper.

Lemma 2.1. *Let $y_0 \in H_0^1(\Omega)$. Then, for any $u \in L^2(Q)$, the equation (1.1) has a unique solution $y \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$. Moreover, there exists a positive constant C independent of y_0 and u such that the following estimate holds:*

$$\sup_{t\in[0,T]} \|y(t)\|_1^2 + \|y\|^2_{L^2(0,T;H^2(\Omega))} + \|\partial_t y\|^2_{L^2(Q)} \le C(\|y_0\|_1^2 + \|u\|^2_{L^2(Q)}).$$

Here and in what follows, the notation $\|\cdot\|_1$ stands for the usual norm of the space $H_0^1(\Omega)$. Then, we give the following theorem which contains the existence and uniqueness of the optimal control and the Pontryagin maximum principle for the problem (P).

Theorem 2.1. *The problem (P) has a unique optimal control. Moreover, a function $\bar{u} \in \mathcal{K}$ is the optimal control for the problem (P) if and only if there exist functions $\bar{\mu} \in H_0^1(\Omega)$ and $\bar{y}, \bar{\varphi} \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ enjoying the following properties:*

$$\bar{y}(T) \in K, \quad (\bar{\mu}, z - \bar{y}(T)) \le 0, \quad \forall\, z \in K, \tag{2.1}$$

$$
\begin{cases} \partial_t \bar{y} - \triangle \bar{y} = \chi_\omega \bar{u} & \textit{in } \Omega \times (0,T), \\ \bar{y} = 0 & \textit{on } \partial\Omega \times (0,T), \\ \bar{y}(0) = y_0 & \textit{in } \Omega, \end{cases} \tag{2.2}
$$

$$
\begin{cases} \partial_t \bar{\varphi} + \triangle \bar{\varphi} = \bar{y} - y_d & \textit{in } \Omega \times (0,T), \\ \bar{\varphi} = 0 & \textit{on } \partial\Omega \times (0,T), \\ \bar{\varphi}(x,T) = -\bar{\mu} & \textit{in } \Omega, \end{cases} \tag{2.3}
$$

$$
\bar{u} \in \mathcal{K}, \quad \int_0^T \int_\Omega (\bar{u} - \chi_\omega \bar{\varphi})(u - \bar{u}) dxdt \geq 0, \qquad \forall\, u \in \mathcal{K}. \tag{2.4}
$$

Remark 2.1. *Theorem 2.1 gives us a qualified Pontryagin maximum principle for the problem (P) under the Slater condition (A). We shall further show that such kind of Slater-type conditions hold for the discrete problems with sufficiently small mesh sizes if the condition (A) is assumed to be true. Thus we can derive the qualified Pontryagin maximum principle for the discrete problems under assumption (A).*

Proof of Theorem 2.1. Write y for the solution of the equation (1.1) corresponding to the control u. We define a functional J over $L^2(Q)$ by setting

$$
J(u) = \begin{cases} \frac{1}{2}\int_0^T \int_\Omega (y - y_d)^2 dxdt + \frac{1}{2}\int_0^T \int_\Omega u^2 dxdt & \text{if } u \in \mathcal{K}, y(T) \in K, \\ +\infty & \text{otherwise.} \end{cases}
$$

One can check that the functional J is convex and lower semi-continuous. Moreover, it is strictly convex in its effective domain. Thus, the existence and uniqueness of the optimal control for the problem (P) can be obtained easily from the Slater condition (A).

Now, we are on the position to prove the Potryagin maximum principle for the problem (P). We start from the necessity. Let $\bar{u}$ be the optimal control and $\bar{y}$ be the optimal trajectory for the problem (P). Then, by the assumption (A), it follows directly from Theorem 5.2 in [6] that there exist a function $\bar{\mu} \in L^2(\Omega)$ and a function $\bar{\varphi}$ such that (2.1)-(2.4) hold. Obviously, the condition (2.1) is equivalent to $\bar{\mu} \in N_K(\bar{y}(T))$, where $N_K(\bar{y}(T))$ is the normal cone of K at $\bar{y}(T)$. However,

$$
N_K(\bar{y}(T)) = \begin{cases} 0 & \text{if } \|\bar{y}(T)\| < 1, \\ \bigcup_{k \geq 0} k\bar{y}(T) & \text{if } \|\bar{y}(T)\| = 1. \end{cases}
$$

Thus, we must have

$$
\bar{\mu} = k\bar{y}(T), \tag{2.5}
$$

where

$$
k = \begin{cases} 0 & \text{if } \|\bar{y}(T)\| < 1, \\ \geq 0 & \text{if } \|\bar{y}(T)\| = 1. \end{cases} \tag{2.6}
$$

On the other hand, it follows from Lemma 2.1 that

$$
\bar{y} \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0,T; L^2(\Omega)) \subset C([0,T], H_0^1(\Omega)).
$$

This, together with (2.5), shows $\bar{\mu} \in H_0^1(\Omega)$ and $\bar{\varphi} \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0,T; L^2(\Omega))$.

Next, we show the sufficiency. Let functions $\bar{u}$, $\bar{y}$, $\bar{\varphi}$ and $\bar{\mu}$ satisfy (2.1)-(2.4). Let $u \in \mathcal{K}$ be a control function such that the corresponding solution y of the equation (1.1) has the property that $y(T) \in K$. We first notice the following identity:

$$
J(u) - J(\bar{u}) + \int_0^T \int_\Omega (\bar{y} - y_d)(\bar{y} - y) dxdt + \int_0^T \int_\Omega \bar{u}(\bar{u} - u) dxdt
$$

$$= \frac{1}{2}\int_0^T\int_\Omega (\bar{y}-y)^2 dxdt + \frac{1}{2}\int_0^T\int_\Omega (\bar{u}-u)^2 dxdt \geq 0.$$

This, together with (2.1)-(2.4) gives us the following inequality:

$$\begin{aligned}
J(\bar{u}) &\leq J(u) + \int_0^T\int_\Omega (\bar{y}-y_d)(\bar{y}-y)dxdt + \int_0^T\int_\Omega \bar{u}(\bar{u}-u)dxdt \\
&= J(u) + (\bar{\varphi}(T), \bar{y}(T)-y(T)) - (\bar{\varphi}(0), \bar{y}(0)-y(0)) \\
&\quad - \int_0^T\int_\Omega [\bar{\varphi}\partial_t(\bar{y}-y) + \nabla\bar{\varphi}\cdot\nabla(\bar{y}-y)]dxdt + \int_0^T\int_\Omega \bar{u}(\bar{u}-u)dxdt \\
&\leq J(u) - \int_0^T\int_\Omega \bar{\varphi}\chi_\omega(\bar{u}-u)dxdt + \int_0^T\int_\Omega \bar{u}(\bar{u}-u)dxdt \\
&= J(u) + \int_0^T\int_\Omega (\bar{u}-\chi_\omega\bar{\varphi})(\bar{u}-u)dxdt \\
&\leq J(u),
\end{aligned}$$

which shows that $\bar{u}$ is the optimal control for the problem (P). This completes the proof. $\square$

Next, we shall give an explicit expression for the optimal control of the problem (P) via the adjoint state, which plays an important role in our work.

Proposition 2.1. *If $\bar{u}$ is the optimal control of the problem (P), then it holds that for almost every $t \in [0,T]$,*

$$\bar{u}(t) = \frac{\chi_\omega\bar{\varphi}(t)}{1+k(t)}, \tag{2.7}$$

where

$$k(t) = \begin{cases} 0, & \textit{for a.e. } t\in[0,T] \textit{ if } \|\chi_\omega\bar{\varphi}(t)\| < 1, \\ \|\chi_\omega\bar{\varphi}(t)\| - 1, & \textit{for a.e. } t\in[0,T] \textit{ if } \|\chi_\omega\bar{\varphi}(t)\| \geq 1. \end{cases} \tag{2.8}$$

Proof. Let E be a measurable subset of the interval $[0,T]$. Let v_0 be a function in the space $L^2(\Omega)$ such that $\|v_0\| \leq 1$. We define a function $v(\cdot) : [0,T] \to L^2(\Omega)$ by setting

$$v(t) = \begin{cases} v_0, & \text{for all } t \in E, \\ \bar{u}(t), & \text{for all } t \in [0,T]\setminus E. \end{cases}$$

It is clear that $v \in \mathcal{K}$. By (2.4), we get

$$-\int_E (\bar{u}(t) - \chi_\omega\bar{\varphi}(t), v_0 - \bar{u}(t))dt \leq 0.$$

Since the set E in the above inequality can be taken arbitrarily from the family of measurable subsets of the interval $[0,T]$, we can apply Lebesgue's differentiation theorem to the aforementioned inequality to get that for any $v_0 \in L^2(\Omega)$ with $\|v_0\| \leq 1$,

$$-(\bar{u}(t) - \chi_\omega\bar{\varphi}(t), v_0 - \bar{u}(t)) \leq 0, \quad \text{for a.e. } t \in [0,T].$$

Thus, it holds that

$$-(\bar{u}(t) - \chi_\omega\bar{\varphi}(t)) \in N_K(\bar{u}(t)), \quad \text{for a.e. } t \in [0,T].$$

Similar to (2.5) and (2.6), we have the following equation:

$$-(\bar{u}(t) - \chi_\omega\bar{\varphi}(t)) = k(t)\bar{u}(t), \qquad \text{for a.e. } t \in [0,T], \tag{2.9}$$

where

$$k(t)\begin{cases} = 0 & \text{for a.e. } t\in[0,T] \text{ if } \|\bar u(t)\|<1, \\ \geq 0 & \text{for a.e. } t\in[0,T] \text{ if } \|\bar u(t)\|=1, \end{cases} \tag{2.10}$$

which implies (2.7) with $k(t)$ given by (2.10).

Next, we shall prove that the above function $k(t)$ has the form (2.8). To this end, we first claim

$$\|\bar u(t)\|<1 \text{ if and only if } \|\chi_\omega\bar\varphi(t)\|<1. \tag{2.11}$$

Indeed, if $\|\bar u(t)\|<1$, then it follows from (2.10) that $k(t)=0$. This, together with (2.9), shows $\chi_\omega\bar\varphi(t)=\bar u(t)$, from which it follows that $\|\chi_\omega\bar\varphi(t)\|<1$. Conversely, if $\|\chi_\omega\bar\varphi(t)\|<1$, then, by making use of the equation (2.7) where the function $k(t)$ is given by (2.10), we get $\|\bar u(t)\|<1$. Thus, we have proved (2.11).

Then we claim that

$$\|\bar u(t)\|=1 \quad \text{if and only if } \|\chi_\omega\bar\varphi(t)\|\geq 1. \tag{2.12}$$

Indeed, if $\|\bar u(t)\|=1$, then, by taking the $L^2(\Omega)$-norm on the both sides of the equation (2.7), we get that $0\leq k(t)=\|\chi_\omega\varphi(t)\|-1$. Thus, we have $\|\chi_\omega\varphi(t)\|\geq 1$. Conversely, if $\|\chi_\omega\varphi(t)\|\geq 1$, then it follows from (2.11) that $\|\bar u(t)\|\geq 1$. However, since $\bar u\in\mathcal{K}$, namely, $\|\bar u(t)\|\leq 1$, we necessarily have $\|\bar u(t)\|=1$. Hence, (2.12) holds.

Finally, by making use of (2.10), (2.11), and (2.12), and according to the fact that $0\leq k(t)=\|\chi_\omega\varphi(t)\|-1$ provided $\|\bar u(t)\|=1$, we obtain the equation (2.8). This completes the proof. □

Now we turn to consider the regularity for the optimal control of the problem (P). The following lemma is quoted from [3] and will be used later.

Lemma 2.2. *Let X be a reflexive Banach space and $f\in L^2(0,T;X)$. Then $f\in H^1(0,T;X)$ if and only if there exists a positive constant M such that*

$$\int_0^{T-\alpha}\|f(t+\alpha)-f(t)\|^2dt\leq M\alpha^2, \quad \forall\, \alpha\in(0,T).$$

Proposition 2.2. *Let $\bar u$ be the optimal control for the problem (P). Then it holds that $\bar u\in H^1(0,T;L^2(\Omega))$. Moreover, the control $\bar u$ enjoys the following property:*

$$\|\bar u\|_{H^1(0,T;L^2(\Omega))}\leq C(1+\|\bar\varphi\|_{C([0,T],L^2(\Omega))})\|\bar\varphi\|_{H^1(0,T;L^2(\Omega))},$$

where $\bar\varphi$ is the adjoint state given by Theorem 2.1 and C is a positive constant.

Proof. Let $k(\cdot)$ be the function given in Proposition 2.1. We first claim that the function $k(\cdot)$ is in the space $H^1(0,T)$ and satisfies the estimate:

$$|k'(t)|\leq|f'(t)|\leq\|\partial_t\bar\varphi(t)\|, \quad \text{for a.e. } t\in[0,T]. \tag{2.13}$$

Here is the argument. Let $f(t)=\|\chi_\omega\bar\varphi(t)\|-1$. Then it holds that $k(t)=f^+(t)$. Since $\bar\varphi\in H^1(0,T;L^2(\Omega))$, it follows from Lemma 2.2 that there exists a positive constant M independent of α such that

$$\begin{aligned}\int_0^{T-\alpha}|f(t+\alpha)-f(t)|^2dt &= \int_0^{T-\alpha}(\|\chi_\omega\bar\varphi(t+\alpha)\|-\|\chi_\omega\bar\varphi(t)\|)^2dt \\ &\leq \int_0^{T-\alpha}\|\chi_\omega\bar\varphi(t+\alpha)-\chi_\omega\bar\varphi(t)\|^2dt \\ &\leq \int_0^{T-\alpha}\|\bar\varphi(t+\alpha)-\bar\varphi(t)\|^2dt\end{aligned}$$

$$\leq \quad M\alpha^2.$$

By making use of Lemma 2.2 again, we obtain that $f \in H^1(0,T)$. Thus, it holds that $k \in H^1(0,T)$. Moreover, we can easily derive that for almost every $t \in [0,T)$,

$$\begin{aligned} |f'(t)| &= \lim_{\alpha \to 0^+} \frac{|f(t+\alpha)-f(t)|}{\alpha} \\ &\leq \lim_{\alpha \to 0^+} \frac{\|\chi_\omega \bar{\varphi}(t+\alpha) - \chi_\omega \bar{\varphi}(t)\|}{\alpha} \\ &\leq \lim_{\alpha \to 0^+} \left\| \frac{\bar{\varphi}(t+\alpha)-\bar{\varphi}(t)}{\alpha} \right\| \\ &= \|\partial_t \bar{\varphi}(t)\|. \end{aligned}$$

Hence, we have proved (2.13).

On the other hand, it follows from (2.7) that $\bar{u} \in H^1(0,T;L^2(\Omega))$ and

$$\partial_t \bar{u}(t) = \frac{\chi_\omega \partial_t \bar{\varphi}(t)(1+k(t)) - k'(t)\chi_\omega \bar{\varphi}(t)}{(1+k(t))^2} \qquad \text{for a.e. } t \in [0,T].$$

Thus, we obtain from (2.13) that

$$\begin{aligned} \int_0^T \|\partial_t \bar{u}(t)\|^2 dt &\leq 2\int_0^T \|\chi_\omega \partial_t \bar{\varphi}(t)\|^2 dt + 2\int_0^T |k'(t)|^2 \|\chi_\omega \bar{\varphi}(t)\|^2 dt \\ &\leq 2\int_0^T \|\partial_t \bar{\varphi}(t)\|^2 dt + 2\|\bar{\varphi}\|^2_{C([0,T],L^2(\Omega))} \int_0^T |k'(t)|^2 dt \\ &\leq C(1+\|\bar{\varphi}\|^2_{C([0,T],L^2(\Omega))}) \int_0^T \|\partial_t \bar{\varphi}(t)\|^2 dt, \end{aligned}$$

from which, it follows that

$$\|\bar{u}\|_{H^1(0,T;L^2(\Omega))} \leq C(1+\|\bar{\varphi}\|_{C([0,T],L^2(\Omega))})\|\bar{\varphi}\|_{H^1(0,T;L^2(\Omega))}.$$

This completes the proof. □

3. Semi-discrete finite element approximation of the problem (P)

In this section, we shall set up a semi-discrete finite element approximation problem (P_h) for the problem (P), and then discuss the similar subjects for the problem (P_h) as those for the problem (P) in section 2. We recall that the Salter condition (A) is assumed to be true in the whole paper.

First of all we introduce certain notations and assumptions, which will be used in what follows. Associated with a parameter h with $h > 0$, we take a family of triangulations $\{\mathcal{T}^h\}$ in $\bar{\Omega}$. For every element $S \in \mathcal{T}^h$, we write $\rho(S)$ and $\sigma(S)$ for the diameter of the set S and the diameter of the greatest ball included in S, respectively. Let $h = \max_{S\in\mathcal{T}^h} \rho(S)$. From now on, we assume that *the domain Ω is convex and the following regularity properties on the triangulations hold:*

(i) *There exists two positive constants ρ and σ such that*

$$\frac{\rho(S)}{\sigma(S)} \leq \sigma, \qquad \frac{h}{\rho(S)} \leq \rho$$

for every element $S \in \mathcal{T}^h$ and all $h > 0$.

(ii) *Let $\bar{\Omega}_h = \cup_{S\in\mathcal{T}^h} S$ be the polygonal approximation of $\bar{\Omega}$. Write Ω_h and $\partial\Omega_h$ for the interior and boundary of the set $\bar{\Omega}_h$, respectively. Then all such vertices of $\mathcal{T}^h$ that are on the boundary $\partial\Omega_h$ stay on the boundary $\partial\Omega$.*

Since the domain Ω is convex, by the inequality (5.2.19) in [25], we have

$$\text{measure}(\Omega \setminus \Omega_h) \le Ch^2. \tag{3.1}$$

Here and in what follows, C stands for several positive constants independent of h, which may be different in the different contexts.

Associated with every triangulation $\mathcal{T}^h$, we define a finite dimensional space as follows:

$$V^h = \{v_h \in C(\bar{\Omega}) \ ; \ v_h|_S \in P_1(S) \text{ for every } S \in \mathcal{T}^h, \text{ and } v|_{\bar{\Omega}\setminus\Omega_h} = 0\},$$

where $P_1(S)$ denotes the space of all polynomials defined on S and of degree less than or equal to one. It is clear that $V^h \subset H_0^1(\Omega)$. Moreover, under the assumptions (i) and (ii), the following inverse inequality holds (see [11]):

$$\|v_h\|_1 \le Ch^{-1}\|v_h\|, \quad \forall\, v_h \in V^h,$$

which will be used later.

Let P_h be the L^2-projection from $L^2(\Omega)$ to V^h, defined by

$$(P_h v, v_h) = (v, v_h), \quad \forall\, v \in L^2(\Omega), v_h \in V^h.$$

Then, by (3.1) and by making use of the similar argument as that used in the proof of (3.5.22) in [24], we see that for $m = 0, 1$ and for any $v \in H^{m+1}(\Omega) \cap H_0^1(\Omega)$,

$$\|v - P_h v\| + h\|v - P_h v\|_1 \le Ch^{m+1}\|v\|_{m+1}. \tag{3.2}$$

Define a bilinear form $a(\cdot,\cdot)$ over $H_0^1(\Omega) \times H_0^1(\Omega)$ by setting

$$a(f,g) = \int_\Omega (\nabla f, \nabla g)_{\mathbb{R}^d}\, dx, \qquad \forall\, f, g \in H_0^1(\Omega),$$

where $(\cdot,\cdot)_{R^d}$ stands for the usual inner product of R^d. Consider the following equation:

$$\begin{cases} (\partial_t y_h(t), v_h) + a(y_h(t), v_h) = (\chi_\omega u, v_h), \quad \forall\, v_h \in V^h, \text{ for a.e. } t \in [0,T], \\ y_h(0) = P_h y_0. \end{cases} \tag{3.3}$$

One can easily verify the following result.

Lemma 3.1. *Let $y_0 \in H_0^1(\Omega)$. Then for any $u \in L^2(Q)$, the equation (3.3) has a unique solution y_h in the space $H^1(0,T;V_h)$ with the following estimate:*

$$\sup_{t\in[0,T]} \|y_h(t)\|_1^2 + \|\partial_t y_h\|_{L^2(Q)}^2 \le C(\|y_0\|_1^2 + \|u\|_{L^2(Q)}^2).$$

Moreover, we have the following error estimates for the solutions of equation (1.1) and the equation (3.3).

Lemma 3.2. *Let y and y_h be the solutions of the equation (1.1) and the equation (3.3), respectively. Then $(y - y_h)$ enjoys the following properties:*

$$\begin{aligned} &\|y - y_h\|_{L^2(Q)} + h(\|y - y_h\|_{C([0,T],L^2(\Omega))} + \|y - y_h\|_{L^2(0,T;H^1(\Omega))}) \\ \le\ & Ch^2(\|y_0\|_1 + \|u\|_{L^2(Q)}), \end{aligned}$$

provided that $y_0 \in H_0^1(\Omega)$ and $u \in L^2(Q)$; and

$$\|y(t) - y_h(t)\| \le Ch^2(\frac{1}{t}\|y_0\| + \|u\|_{H^1(0,T;L^2(\Omega))}) \ \textit{for all} \ t \in (0,T]$$

provided that $y_0 \in L^2(\Omega)$ *and* $u \in H^1(0,T;L^2(\Omega))$.

Proof. The first estimate is a direct consequence of Theorem 3.2 and Theorem 3.5 in [10] while the second one can be derived easily from Theorem 2.10 in [14]. Thus, the proof is complete. $\square$

Next, we set

$$U^h = \{w \in L^2(\Omega); w|_S \text{ is a constant function for each } S \in \mathcal{T}^h, w|_{\bar{\Omega}\setminus\Omega_h} = 0\}$$

and

$$\mathcal{K}^h = \{v \in L^2(0,T;U^h); \|v(t)\| \leq 1 \text{ a.e. } t \in [0,T]\}.$$

Then we define a semi-discrete finite element approximation for the problem (P) as follows:

$$(P_h) \qquad \min\left\{\frac{1}{2}\int_0^T\int_\Omega (y_h - y_d)^2 + \frac{1}{2}\int_0^T\int_\Omega u_h^2 dxdt\right\}$$

over all such controls $u_h \in \mathcal{K}^h$ that the corresponding solution y_h to the equation (3.3) has the property: $y_h(T) \in K$.

It is clear that the problem (P_h) is an optimal control problem governed by a system of linear ordinary differential equations. In the following, we will derive the first order optimality conditions for the problem (P_h). For this purpose, we define a projection operator Π_h from $L^2(\Omega)$ to U^h by setting

$$(\Pi_h v, v_h) = (v, v_h), \quad \forall\, v \in L^2(\Omega), v_h \in U^h.$$

It follows at once that for any $v \in L^2(\Omega)$ and any $S \in \mathcal{T}^h$

$$\Pi_h v|_S = \frac{\int_S v dx}{|S|}, \qquad \Pi_h v|_{\bar{\Omega}\setminus\Omega_h} = 0 \text{ and } \|\Pi_h v\| \leq \|v\|.$$

Moreover, according to the well known Poincaré inequality [15] and by making use of the density of the subspace $H_0^1(\Omega)$ in the space $L^2(\Omega)$, one can easily check that for any $v \in L^2(\Omega)$,

$$\|v - \Pi_h v\| \longrightarrow 0 \quad \text{as } h \longrightarrow 0. \tag{3.4}$$

Now, we first deal with the Slater property for the problem (P_h).

Lemma 3.3. *There exists a positive number* h_0 *having the following property: For any* h *with* $0 < h \leq h_0$, *there is an element* $u_{0h} \in \mathcal{K}^h$, *such that* $y_h(u_{0h})(T) \in intK$, *where* $y_h(u_{0h})(\cdot)$ *denotes the solution of the equation (3.3) with* $u_h = u_{0h}$. *Moreover, such an element* u_{0h} *can be taken as* $\Pi_h u_0$, *where* u_0 *is an element satisfying the Slater condition* (A).

Proof. By the Slater condition (A), there is a control $u_0 \in \mathcal{K}$ such that $y(u_0)(T) \in intK$, where $y(u_0)(\cdot)$ is the solution of the equation (1.1) with $u = u_0$. Thus, we can find a number γ with $0 \leq \gamma < 1$ such that $\|y(u_0)(T)\| \leq \gamma$. Now, we define a function u_{0h} by $u_{0h}(t) = \Pi_h u_0(t)$ for each $t \in [0,T]$. It can be verified easily that

$$\|u_{0h}(t)\|^2 = \|\Pi_h u_0(t)\|^2 \leq \|u_0(t)\|^2 \leq 1, \qquad \text{for a.e. } t \in [0,T],$$

which shows $u_{0h} \in \mathcal{K}^h$ for any $h > 0$. Moreover, it follows from (3.4) that as $h \longrightarrow 0$,

$$\|u_{0h}(t) - u_0(t)\| \longrightarrow 0, \qquad \text{for a.e. } t \in [0,T].$$

By making use of Lebesgue's dominated convergence theorem, we obtain

$$\|u_{0h} - u_0\|_{L^2(Q)} \longrightarrow 0 \qquad \text{as } h \longrightarrow 0. \tag{3.5}$$

By (3.5) and according to Lemma 2.1 and Lemma 3.2, we can find a positive number h_0 such that the following properties hold: For any number h with $0 < h \leq h_0$,

$$\|y(u_0)(T) - y(u_{0h})(T)\| \leq C\|u_0 - u_{0h}\|_{L^2(Q)} \leq \frac{1-\gamma}{4},$$

and

$$\|y(u_{0h})(T) - y_h(u_{0h})(T)\| \leq Ch \leq \frac{1-\gamma}{4}.$$

This implies that for any number h with $0 < h \leq h_0$,

$$\begin{aligned} & \|y(u_0)(T) - y_h(u_{0h})(T)\| \\ \leq \quad & \|y(u_0)(T) - y(u_{0h})(T)\| + \|y(u_{0h})(T) - y_h(u_{0h})(T)\| \leq \frac{1-\gamma}{2}. \end{aligned}$$

Thus, we obtain that for each h with $0 < h \leq h_0$,

$$\|y_h(u_{0h})(T)\| \leq \|y_h(u_{0h})(T) - y(u_0)(T)\| + \|y(u_0)(T)\| \leq \frac{1+\gamma}{2} < 1. \tag{3.6}$$

This completes the proof. □

Theorem 3.1. *There exists a positive number h_0 such that for any number h with $0 < h < h_0$, the optimal control problem (P_h) has a unique solution. Moreover, a function $\bar{u}_h$ is an optimal control for the problem (P_h) if and only if there exist functions $\bar{\mu}_h \in V^h, \bar{y}_h, \bar{\varphi}_h \in H^1(0,T;V_h)$ such that for all $z \in K$, $v_h \in V^h$, $u_h \in \mathcal{K}^h$ and for almost every $t \in [0,T]$,*

$$\bar{y}_h(T) \in K, \quad (\bar{\mu}_h, z - \bar{y}_h(T)) \leq 0, \tag{3.7}$$

$$\begin{cases} (\partial_t \bar{y}_h(t), v_h) + a(\bar{y}_h(t), v_h) = (\chi_\omega \bar{u}_h(t), v_h), \\ \bar{y}_h(0) = P_h y_0, \end{cases} \tag{3.8}$$

$$\begin{cases} (\partial_t \bar{\varphi}_h(t), v_h) - a(\bar{\varphi}_h(t), v_h) = (\bar{y}_h(t) - y_d(t), v_h), \\ \bar{\varphi}_h(T) = -\bar{\mu}_h, \end{cases} \tag{3.9}$$

$$\bar{u} \in \mathcal{K}^h, \quad \int_0^T \int_\Omega (\bar{u}_h - \chi_\omega \bar{\varphi}_h)(u_h - \bar{u}_h)\,dxdt \geq 0. \tag{3.10}$$

Proof. By Lemma 3.3, there is a positive number h_0 such that the semi-discrete version of Slater condition for (P_h) holds for each h with $0 < h \leq h_0$. Then, by making use of the very similar argument as that used in the proof of Theorem 2.1, we can get the desired results. This completes the proof. □

Next, we shall make an explicit expression of the optimal control for the problem (P_h). To this end, we express each triangulation $\mathcal{T}_h$ by $\mathcal{T}_h = \{S_1, S_2, \cdots, S_l\}$. Let

$$K^h = K \cap U^h.$$

By the similar argument as that used in the proof of Proposition 2.1, it can be verified easily that the property (3.10) is equivalent to

$$\int_\Omega (\bar{u}_h(t) - \chi_\omega \bar{\varphi}_h(t))(v_h - \bar{u}_h(t))\,dx \geq 0, \qquad \text{for a.e. } t \in [0,T], \forall\, v_h \in K^h.$$

Namely, for all $v_h \in K^h$ and for a.e. $t \in [0,T]$, it holds that

$$\sum_{i=1}^{l} |S_i| \frac{\int_{S_i} (\bar{u}_h(t) - \chi_\omega \bar{\varphi}_h(t))\,dx}{|S_i|} (v_h|_{S_i} - \bar{u}_h(t)|_{S_i}) \geq 0. \tag{3.11}$$

Recall $\Omega \in \mathbb{R}^d$ with $d \leq 3$. In the case where $d = 2$, by the assumption (i) on the triangulations, we have

$$\frac{1}{4}\frac{\pi h^2}{(\sigma\rho)^2} \leq |S_i| \leq \frac{1}{4}\pi h^2, \qquad \forall\, i = 1, 2, \cdots, l.$$

The similar result holds for the case that either $d = 1$ or $d = 3$. Thus, we can define a new inner product in $\mathbb{R}^l$ by setting

$$(x, y)_h = \sum_{i=1}^{l} |S_i| x_i y_i, \quad \forall\, x, y \in \mathbb{R}^l,$$

which induces a norm as follows:

$$|x|_h = (\sum_{k=1}^{l} |S_i| x_i^2)^{\frac{1}{2}}, \quad \forall\, x \in \mathbb{R}^l.$$

Moreover, if we view a function v_h in the set $K^h \subset U^h$, where U^h is a $l-$dimensional space, as an element of $\mathbb{R}^l$, namely, $v_h = (v_h|_{S_1}, v_h|_{S_2}, \cdots, v_h|_{S_l})$, then

$$\|v_h\| = \left(\int_\Omega |v_h(x)|^2 dx\right)^{\frac{1}{2}} = \left(\sum_{i=1}^{l} |S_i| (v_h|_{S_i})^2\right)^{\frac{1}{2}} = |v_h|_h.$$

Let

$$\bar{d}_i(t) = \frac{\int_{S_i} (\bar{u}_h(t) - \chi_\omega \bar{\varphi}_h(t)) dx}{|S_i|}$$

for $i = 1, 2, \cdots, l$ and write $\bar{d}_h(t) = (\bar{d}_1(t), \bar{d}_2(t), \cdots, \bar{d}_l(t)) \in \mathbb{R}^l$. Then, one can check directly that inequality (3.11) is equivalent to

$$(\bar{d}_h(t), v_h - \bar{u}_h(t))_h \geq 0, \qquad \text{for a.e. } t \in [0, T],\ \forall\, v_h \text{ with } |v_h|_h \leq 1. \tag{3.12}$$

Now, we are ready to give the following explicit expression of the optimal control for the problem (P_h).

Proposition 3.1. *Let $\bar{u}_h$ be the optimal control for the problem (P_h). Then there exists a non-negative function $k_h(t)$ such that for almost every $t \in [0, T]$,*

$$\bar{u}_h(t) = \frac{\Pi_h \chi_\omega \bar{\varphi}_h(t)}{1 + k_h(t)}, \tag{3.13}$$

where

$$k_h(t) = \begin{cases} 0, & \text{for a.e. } t \in [0, T] \text{ if } \|\Pi_h \chi_\omega \bar{\varphi}_h(t)\| < 1, \\ \|\Pi_h \chi_\omega \bar{\varphi}_h(t)\| - 1, & \text{for a.e. } t \in [0, T] \text{ if } \|\Pi_h \chi_\omega \bar{\varphi}_h(t)\| \geq 1. \end{cases} \tag{3.14}$$

Proof. Since $\bar{u}_h$ is the optimal control for the problem (P_h), we can get (3.12) from (3.10). By (3.12), we have $-\bar{d}_h(t) = k_h(t)\bar{u}_h(t)$ for a.e. $t \in [0, T]$. Namely, for all $i = 1, 2, \cdots, l$ and for a.e. $t \in [0, T]$, it holds that

$$-\frac{\int_{S_i} (\bar{u}_h(t) - \chi_\omega \bar{\varphi}_h(t)) dx}{|S_i|} = k_h(t) \bar{u}_h(t)|_{S_i}, \tag{3.15}$$

where

$$k_h(t) \begin{cases} = 0, & \text{for a.e. } t \in [0, T] \text{ if } |\bar{u}_h(t)|_h < 1, \\ \geq 0, & \text{for a.e. } t \in [0, T] \text{ if } |\bar{u}_h(t)|_h = 1, \end{cases}$$

Since $\bar{u}_h(t)|_S$ is a constant function for any $S \in \mathcal{T}^h$, (3.13) follows at once from (3.15). Then, by the similar argument as that used in the proof of Proposition 2.1, we get the desired result. This completes the proof. $\square$

According to Proposition 3.1, and by making use of the same argument as that used in the proof of Proposition 2.2, we can have the following regularity result on the optimal control for the problem (P_h).

Proposition 3.2. *Let $\bar{u}_h$ be the optimal control for the problem (P_h). Then it holds that $\bar{u}_h \in H^1(0,T;L^2(\Omega))$. Moreover, we have the following estimate:*

$$\|\bar{u}_h\|_{H^1(0,T;L^2(\Omega))} \leq C(1 + \|\bar{\varphi}_h\|_{C([0,T],L^2(\Omega))})\|\bar{\varphi}_h\|_{H^1(0,T;L^2(\Omega))}.$$

4. Error estimate between the solutions of (P) and (P_h)

In this section, we shall establish an error estimate between the optimal controls of the problem (P) and the problem (P_h). We make an additional assumption on ω and $\mathcal{T}^h$:

(iii) *The subset ω is a polygon. Moreover, for any triangulation $\mathcal{T}^h$, there exist a subset $\bar{\mathcal{T}}^h \subset \mathcal{T}^h$ such that $\omega = \cup_{S \in \bar{\mathcal{T}}^h} S$.*

We would like emphasize that from the point of view of control theory, this assumption is acceptable. By the above assumption, we see that for any $S \in \mathcal{T}^h$, either $S \subset \bar{\omega}$ or $S \subset \bar{\Omega}/\omega$. Moreover, the operator χ_ω and the operator Π_h are commutative.

Lemma 4.1. *Let $\bar{\mu}_h \in V^h$, together with $\bar{u}_h \in \mathcal{K}^h$, $\bar{y}_h \in H^1(0,T,V^h)$ and $\bar{\varphi}_h \in H^1(0,T,V^h)$ satisfy (3.7)-(3.10). Then it holds that*

$$\|\bar{\mu}_h\|_1 \leq C$$

for any number h with $0 < h \leq h_0$, where h_0 is the positive number given in Theorem 3.1.

Proof. We first prove that the family $\{\bar{\mu}_h\}_{0<h\leq h_0}$ is bounded in $L^2(\Omega)$. Let u_{0h} be the control given in Lemma 3.3. Since $u_{0h}, \bar{u}_h \in \mathcal{K}^h \subset \mathcal{K}$, it holds that

$$\|u_{0h}\|_{L^2(Q)} \leq C \quad \text{and} \quad \|\bar{u}_h\|_{L^2(Q)} \leq C. \tag{4.1}$$

Then, by Lemma 3.1, we have

$$\|y_h(u_{0h})\| \leq C \quad \text{and} \quad \|\bar{y}_h\| \leq C. \tag{4.2}$$

On the other hand, by (3.6), there exists a number γ with $0 \leq \gamma < 0$ such that for any number h with $0 < h \leq h_0$, $\|y_h(u_{0h})(T)\| \leq \frac{1+\gamma}{2}$. Thus, we can find a positive number ρ, which is independent of h and satisfies $0 < \rho \leq \frac{1-\gamma}{2}$, such that

$$\|y_h(u_{0h})(T) + \rho w\| \leq \|y_h(u_{0h})(T)\| + \rho\|w\| \leq 1$$

for any element $w \in L^2(\Omega)$ with $\|w\| \leq 1$ and for any number h with $0 < h \leq h_0$. This, combined with (3.7), shows that for any element $w \in L^2(\Omega)$ with $\|w\| \leq 1$ and for any number h with $0 < h \leq h_0$, it holds that

$$(\bar{\mu}_h, y_h(u_{0h})(T) + \rho w - \bar{y}_h(T)) \leq 0. \tag{4.3}$$

Now, by the above inequality (4.3), and by (3.8)-(3.10), we get that for each h with $0 < h \leq h_0$,

$$\begin{aligned} \rho\|\bar{\mu}_h\| &\leq -(\bar{\mu}_h, y_h(u_{0h})(T) - \bar{y}_h(T)) \\ &= \int_0^T \int_\Omega \partial_t \bar{\varphi}_h (y_h(u_{0h}) - \bar{y}_h) dxdt + \int_0^T \int_\Omega \bar{\varphi}_h \partial_t (y_h(u_{0h}) - \bar{y}_h) dxdt \end{aligned}$$

$$
\begin{aligned}
&= \int_0^T\int_\Omega \nabla\bar{\varphi}_h\nabla(y_h(u_{0h})-\bar{y}_h)dxdt+\int_0^T\int_\Omega(\bar{y}_h-y_d)(y_h(u_{0h})-\bar{y}_h)dxdt\\
&\qquad +\int_0^T\int_\Omega \bar{\varphi}_h\partial_t(y_h(u_{0h})-\bar{y}_h)dxdt\\
&= \int_0^T\int_\Omega \chi_\omega\bar{\varphi}_h(u_{0h}-\bar{u}_h)dxdt+\int_0^T\int_\Omega(\bar{y}_h-y_d)(y_h(u_{0h})-\bar{y}_h)dxdt\\
&= -\int_0^T\int_\Omega(\bar{u}_h-\chi_\omega\bar{\varphi}_h)(u_{0h}-\bar{u}_h)dxdt+\int_0^T\int_\Omega \bar{u}_h(u_{0h}-\bar{u}_h)dxdt\\
&\qquad +\int_0^T\int_\Omega(\bar{y}_h-y_d)(y_h(u_{0h})-\bar{y}_h)dxdt\\
&\le \int_0^T\int_\Omega \bar{u}_h(u_{0h}-\bar{u}_h)dxdt+\int_0^T\int_\Omega(\bar{y}_h-y_d)(y_h(u_{0h})-\bar{y}_h)dxdt,
\end{aligned}
$$

from which, it follows, by taking into account (4.1) and (4.2), that

$$\|\bar{\mu}_h\|\le C,\quad \text{for all } h \ \text{ with } 0<h\le h_0 \tag{4.4}$$

Next, we will show that the family $\{\bar{\mu}_h\}_{0<h\le h_0}$ is bounded in $H_0^1(\Omega)$. By (3.7) and by the similar argument as that used in the proof of (2.5) and (2.6), we can get

$$\bar{\mu}_h = k_h\bar{y}_h(T),$$

where

$$k_h=\begin{cases} 0 & \text{if } \|\bar{y}_h(T)\|<1,\\ \|\bar{\mu}_h\| & \text{if } \|\bar{y}_h(T)\|=1.\end{cases}$$

Thus, it follows from (4.4) that

$$k_h\le\|\bar{\mu}_h\|\le C.$$

Finally, by making use of Lemma 3.1, we get

$$\|\bar{\mu}_h\|_1=k_h\|\bar{y}_h(T)\|_1\le C.$$

This completes the proof. □

Let $y(\bar{u}_h)$ be the solution of (1.1) with $u=\bar{u}_h$ and $\varphi(\bar{u}_h)$ be the solution of the following equation

$$\begin{cases} \partial_t\varphi(\bar{u}_h)+\triangle\varphi(\bar{u}_h)=y(\bar{u}_h)-y_d & \text{in } \Omega\times(0,T),\\ \varphi(\bar{u}_h)=0 & \text{on } \partial\Omega\times(0,T),\\ \varphi(\bar{u}_h)(T)=-\bar{\mu}_h & \text{in } \Omega.\end{cases}$$

Now, we can give the estimate for the error between the solutions of the problem (P) and the problem (P_h).

Theorem 4.1. *Suppose that $\bar{u}$ and $\bar{u}_h$ are the solutions of the problem (P) and the problem (P_h), respectively. Then, there exists a positive number h_0 such that for all numbers h with $0<h\le h_0$,*

$$\|\bar{u}-\bar{u}_h\|_{L^2(Q)}\le Ch.$$

Proof. Let h_0 be given by Theorem 3.1. Then, it follow from (2.4) and (3.10) that

$$\int_0^T(\bar{u}(t),\bar{u}(t)-\bar{u}_h(t))dt\le\int_0^T(\chi_\omega\bar{\varphi}(t),\bar{u}(t)-\bar{u}_h(t))dt \tag{4.5}$$

and

$$\int_0^T (\bar{u}_h(t) - \chi_\omega \bar{\varphi}_h(t), \Pi_h \bar{u}(t) - \bar{u}_h(t))dt \geq 0. \tag{4.6}$$

Here, we recall that Π_h is the projection operator form $L^2(\Omega)$ to U_h given in section 3. Thus by (4.5), (4.6) and (3.10), we obtain that

$$\begin{aligned}
&\|\bar{u} - \bar{u}_h\|^2_{L^2(Q)} \\
= \;& \int_0^T (\bar{u}(t), \bar{u}(t) - \bar{u}_h(t))dt - \int_0^T (\bar{u}_h(t), \bar{u}(t) - \bar{u}_h(t))dt \\
\leq \;& \int_0^T (\chi_\omega \bar{\varphi}(t), \bar{u}(t) - \bar{u}_h(t))dt - \int_0^T (\bar{u}_h(t), \bar{u}(t) - \bar{u}_h(t))dt \\
\leq \;& \int_0^T (\chi_\omega \bar{\varphi}(t), \bar{u}(t) - \bar{u}_h(t))dt - \int_0^T (\bar{u}_h(t), \bar{u}(t) - \bar{u}_h(t))dt \\
& + \int_0^T (\bar{u}_h(t) - \chi_\omega \bar{\varphi}_h(t), \Pi_h \bar{u}(t) - \bar{u}_h(t))dt \\
\leq \;& \int_0^T (\chi_\omega \bar{\varphi}(t), \bar{u}(t) - \bar{u}_h(t))dt - \int_0^T (\bar{u}_h(t), \bar{u}(t) - \bar{u}_h(t))dt \\
& + \int_0^T (\bar{u}_h(t) - \chi_\omega \bar{\varphi}_h(t), \Pi_h \bar{u}(t) - \bar{u}(t))dt \\
& + \int_0^T (\bar{u}_h(t) - \chi_\omega \bar{\varphi}_h(t), \bar{u}(t) - \bar{u}_h(t))dt \\
= \;& \int_0^T (\chi_\omega (\bar{\varphi}(t) - \bar{\varphi}_h(t)), \bar{u}(t) - \bar{u}_h(t))dt \\
& + \int_0^T (\bar{u}_h(t) - \chi_\omega \bar{\varphi}_h(t), \Pi_h \bar{u}(t) - \bar{u}(t))dt \\
= \;& \int_0^T (\chi_\omega (\bar{\varphi}(t) - \varphi(\bar{u}_h)(t)), \bar{u}(t) - \bar{u}_h(t))dt \\
& + \int_0^T (\chi_\omega (\varphi(\bar{u}_h)(t) - \bar{\varphi}_h(t)), \bar{u}(t) - \bar{u}_h(t))dt \\
& + \int_0^T (\bar{u}_h(t) - \chi_\omega \bar{\varphi}_h(t), \Pi_h \bar{u}(t) - \bar{u}(t))dt \\
\equiv \;& I_1 + I_2 + I_3.
\end{aligned} \tag{4.7}$$

Now, we shall estimate terms I_i for $i = 1, 2, 3$ one by one. Since the family $\{\bar{\mu}_h\}_{0<h\leq h_0}$ is bounded in $H_0^1(\Omega)$ by Lemma 4.1, and $-\bar{\mu}_h$ is the initial data for the adjoint equation (3.9) for the problem (P_h), we can have the same estimate for the solution $\bar{\varphi}_h$ as that in Lemma 3.1, namely,

$$\sup_{t\in[0,T]} \|\bar{\varphi}_h(t)\|_1^2 + \|\partial_t \bar{\varphi}_h\|^2_{L^2(Q)} \leq C(\|\bar{\mu}_h\|_1^2 + \|\bar{y}_h - y_d\|^2_{L^2(Q)}) \leq C. \tag{4.8}$$

Thus it follows from Proposition 3.2 that

$$\|\bar{u}_h\|_{H^1(0,T;L^2(\Omega))} \leq C,$$

which, together with (2.1), (3.7) and the second estimate of Lemma 3.2, yields

$$\begin{aligned}
& I_1 \\
= \;& \int_0^T (\chi_\omega (\bar{\varphi}(t) - \varphi(\bar{u}_h)(t)), \bar{u}(t) - \bar{u}_h(t))dt
\end{aligned} \tag{4.9}$$

$$
\begin{aligned}
&= \int_0^T (\bar\varphi(t)-\varphi(\bar u_h)(t), \partial_t(\bar y(t)-y(\bar u_h)(t)))dt \\
&\quad + \int_0^T a(\bar\varphi(t)-\varphi(\bar u_h)(t), \bar y(t)-y(\bar u_h)(t))dt \\
&= -(\bar\mu-\bar\mu_h, \bar y(T)-y(\bar u_h)(T)) - \int_0^T (\partial_t(\bar\varphi(t)-\varphi(\bar u_h)(t)), \bar y(t)-y(\bar u_h)(t))dt \\
&\quad + \int_0^T a(\bar\varphi(t)-\varphi(\bar u_h)(t), \bar y(t)-y(\bar u_h)(t)) \\
&= -(\bar\mu-\bar\mu_h, \bar y(T)-y(\bar u_h)(T)) - \int_0^T \|\bar y(t)-y(\bar u_h)(t)\|^2 dt \\
&\le -(\bar\mu-\bar\mu_h, \bar y(T)-\bar y_h(T)) - (\bar\mu-\bar\mu_h, \bar y_h(T)-y(\bar u_h)(T)) \\
&\le -(\bar\mu-\bar\mu_h, \bar y_h(T)-y(\bar u_h)(T)) \ \ \textit{(by (2.1) and (3.7))} \\
&\le (\|\bar\mu\|+\|\bar\mu_h\|)\|\bar y_h(T)-y(\bar u_h)(T)\| \\
&\le C(\|\bar\mu\|+\|\bar\mu_h\|)h^2(\frac{1}{T}\|y_0\|+\|\bar u_h\|_{H^1(0,T;L^2(\Omega))}) \\
&\le Ch^2.
\end{aligned}
$$

Similar to the first estimate of Lemma 3.2, we can easily get

$$\|\varphi(\bar u_h)-\bar\varphi_h\|_{L^2(Q)} \le Ch^2.$$

Thus it follows that

$$
\begin{aligned}
I_2 &= \int_0^T (\chi_\omega(\varphi(\bar u_h)(t)-\bar\varphi_h(t)), \bar u(t)-\bar u_h(t))dt \\
&\le C\|\varphi(\bar u_h)-\bar\varphi_h\|^2_{L^2(Q)} + \frac12\|\bar u-\bar u_h\|^2_{L^2(Q)} \\
&\le Ch^2 + \frac12\|\bar u-\bar u_h\|^2_{L^2(Q)}.
\end{aligned} \tag{4.10}
$$

In order to estimate the term I_3, we first observe that for any $t\in[0,T]$, it holds that

$$\Pi_h\bar u(t)|_S = \begin{cases} \frac{1}{|S|}\int_S \bar u dx, & S\in\bar{\mathcal{T}}^h, \\ 0, & \text{otherwise.}\end{cases},$$

Here $\bar{\mathcal{T}}^h$ is given in assumption (iii). Moreover, it follows from Proposition 2.1 that $\bar u(t)\in H^1(S)$ and $\|\bar u(t)\|_{H^1(S)} \le \|\bar\varphi\|_{H^1(S)}$ for any $S\in\bar{\mathcal{T}}^h$. By the well known Poincaré inequality [15], it follows that, for any $S\in\bar{\mathcal{T}}^h$,

$$\|\bar u(t)-\Pi_h\bar u(t)\|^2_{L^2(S)} \le Ch^2\|\bar u(t)\|^2_{H^1(S)} \le Ch^2\|\bar\varphi(t)\|^2_{H^1(S)},$$

where the positive constant C is independent of h, t and S. By the assumption (iii), we have

$$
\begin{aligned}
\int_0^T \|\bar u(t)-\Pi_h u(t)\|^2 dt &= \int_0^T\int_\omega |\bar u(t,x)-\Pi_h u(t,x)|^2 dxdt \\
&= \int_0^T \sum_{S\subset\omega}\|\bar u(t)-\Pi_h u(t)\|^2_{L^2(S)}dt \le Ch^2\|\bar\varphi\|^2_{L^2(0,T;H^1(\Omega))}.
\end{aligned} \tag{4.11}
$$

Similarly, we can obtain

$$\int_0^T \|\bar\varphi_h(t)-\Pi_h\bar\varphi_h(t)\|^2 dt \le Ch^2\|\bar\varphi_h\|^2_{L^2(0,T;H^1(\Omega))}. \tag{4.12}$$

We recall that $\bar{u}_h \in \mathcal{K}^h$. Thus, $\bar{u}_h(t) \in U^h$ for almost all $t \in [0, T]$. Since the operator χ_ω and the operator Π_h are commutative, it follows from (4.8), (4.11) and (4.12) that

$$
\begin{aligned}
I_3 &= \int_0^T \int_\Omega (\bar{u}_h(t) - \chi_\omega \bar{\varphi}_h(t))(\Pi_h \bar{u}(t) - \bar{u}(t)) dx dt \\
&= \int_0^T (\chi_\omega \bar{\varphi}_h(t), \bar{u}(t) - \Pi_h \bar{u}(t)) dt \\
&= \int_0^T (\chi_\omega(\bar{\varphi}_h(t) - \Pi_h \bar{\varphi}_h(t)), \bar{u}(t) - \Pi_h \bar{u}(t)) dt \\
&\leq \|\bar{\varphi}_h - \Pi_h \bar{\varphi}_h\|_{L^2(Q)} \|\bar{u} - \Pi_h \bar{u}\|_{L^2(Q)} \\
&\leq Ch^2 \|\bar{\varphi}_h\|_{L^2(0,T;H^1(\Omega))} \|\bar{\varphi}\|_{L^2(0,T;H^1(\Omega))} \\
&\leq Ch^2.
\end{aligned} \tag{4.13}
$$

Finally, the desired estimate follows immediately from (4.7), (4.9),(4.10) and (4.13). This completes the proof. □

5. Fully discrete approximation of the problem (P)

In this section, we shall first set up a fully discrete approximation problem $(P_{h\tau})$ for the semi-discrete problem (P_h) by making use of the backward Euler method. Then we derive the first order optimality conditions for the solution of the problem $(P_{h\tau})$. For this purpose, we partition the time interval $[0, T]$ into N subintervals with a uniform time step τ by the following nodal points:

$$0 = t_0 < t_1 < \cdots < t_N = T,$$

where $t_i = i\tau$ for $i = 0, 1, \cdots, N$ and $\tau = T/N$. For any function $f \in L^2(0, T; L^2(\Omega))$, we write the average of function f on $[t_{i-1}, t_i]$ for $\tilde{f}^i$, namely,

$$\tilde{f}^i(\cdot) = \frac{1}{\tau} \int_{t_{i-1}}^{t_i} f(\cdot, t) dt \quad \text{for } i = 1, 2, \cdots, N.$$

For any function $f \in C([0, T], L^2(\Omega))$, we write $f^i(\cdot) = f(\cdot, t_i)$ for $i = 0, 1, \cdots, N$. Recall the definition of the space V^h given in the section 3, namely,

$$V^h = \{v_h \in C(\bar{\Omega}) \ ; \ v_h|_S \in P_1(S) \text{ for every } S \in \mathcal{T}^h, \text{ and } v|_{\bar{\Omega} \setminus \Omega_h} = 0\}.$$

Let $U_{h\tau} = (U^1, U^2, \cdots, U^N)$ be a given function or a control in the space $(L^2(\Omega))^N$. Write $Y_{h\tau} = (Y_h^1, Y_h^2, \cdots, Y_h^N)$ for a function in the space $(V^h)^N$. Denote by $\partial_\tau Y^i$ the difference quotient $\frac{Y^i - Y^{i-1}}{\tau}$ for $i = 1, 2, \cdots, N$. Consider the following equation

$$
\begin{cases}
(\partial_\tau Y_h^i, v_h) + a(Y_h^i, v_h) = (\chi_\omega U^i, v_h), & \forall\, v_h \in V^h, \quad 1 \leq i \leq N. \\
Y_h^0 = P_h y_0.
\end{cases} \tag{5.1}
$$

We shall first give a stability estimate for the equation (5.1) as follows. By taking $v_h = \tau \partial_\tau Y_h^i$ in (5.1), we get

$$\tau \|\partial_\tau Y_h^i\|^2 + \frac{1}{2}\|Y_h^i\|_1^2 - \frac{1}{2}\|Y_h^{i-1}\|_1^2 \leq C\tau \|U^i\|^2 + \frac{1}{2}\tau \|\partial_\tau Y_h^i\|^2, \quad 1 \leq i \leq N.$$

Here and in what follows, C stands for a positive constant independent of h and τ, which may be different in the different contexts. Summing the above equations over i from 1 to m with $1 \leq m \leq N$, we get

$$\|Y_h^m\|_1^2 + \tau \sum_{i=1}^m \|\partial_\tau Y_h^i\|^2 \leq C(\tau \sum_{i=1}^m \|U^i\|^2 + \|Y_h^0\|_1^2),$$

which shows

$$\max_{1\le i\le N}\|Y_h^i\|_1^2+\tau\sum_{i=1}^N\|\partial_\tau Y_h^i\|^2\le C(\tau\sum_{i=1}^N\|U^i\|^2+\|y_0\|_1^2).$$

Thus, we have already proved the following result.

Lemma 5.1. *Let $y_0\in H_0^1(\Omega)$ and $U_{h\tau}=(U^1,U^2,\cdots,U^N)\in(L^2(\Omega))^N$. Then, the equation (5.1) has a unique solution $Y_{h\tau}=(Y^1,Y^2,\cdots,Y^N)\in(V^h)^N$. Moreover, the following estimate holds:*

$$\max_{1\le i\le N}\|Y_h^i\|_1^2+\tau\sum_{i=1}^N\|\partial_\tau Y_h^i\|^2\le C(\|y_0\|_1^2+\tau\sum_{i=1}^N\|U^i\|^2).$$

The next result concerns an estimate between the solutions of the equation (3.3) and the equation (5.1). We would like to mention that in order to get the estimate, the higher regularity for both control and the initial data, namely, the $H^1(0,T;L^2(\Omega))-$regularity for the control u in the equation (3.3) and the $H^2(\Omega)\cap H_0^1(\Omega)-$regularity for the initial data y_0, are required. This is a difference from Lemma 3.2.

Lemma 5.2. *Let $u\in H^1(0,T;L^2(\Omega))$, $U_{h\tau}\in(L^2(\Omega))^N$ and $y_0\in H^2(\Omega)\cap H_0^1(\Omega)$. Assume that y_h and $Y_{h\tau}$ are the solutions of the equations (3.3) and (5.1), respectively. Then it holds that*

$$\begin{aligned}&\max_{1\le i\le N}\|y_h^i-Y_h^i\|^2+\tau\sum_{i=1}^N\|y_h^i-Y_h^i\|_1^2\\ \le\ & C\left(\tau\sum_{i=1}^N\|\tilde u^i-U^i\|^2+\tau^2(\|u\|^2_{H^1(0,T;L^2(\Omega))}+\|y_0\|_2^2)\right),\end{aligned}\tag{5.2}$$

where $y_h^i(\cdot)=y_h(\cdot,t_i)$ and $\tilde u^i$ is the average of u on $[t_{i-1},t_i]$ for $i=1,2,\cdots,N$.

Proof. We first give an estimate for $\partial_t y_h$. Since $u\in H^1(0,T;L^2(\Omega))$, the term on the right hand of the equation (3.3) is continuous. Thus, we can make use of the standard ODE theories to get $y_h\in C^1([0,T],V^h)$. Hence, the equation (3.3) holds for every $t\in[0,T]$. By making use of (3.1) and by taking $t=0$ and $v_h=\partial_t y_h(0)$ in the equation (3.3), we get that

$$\begin{aligned}\|\partial_t y_h(0)\|^2 &= (\chi_\omega u(0),\partial_t y_h(0))-a(y_0,\partial_t y_h(0))+a(y_0-y_h(0),\partial_t y_h(0))\\ &\le \|u(0)\|\|\partial_t y_h(0)\|+\|\triangle y_0\|\|\partial_t y_h(0)\|+C\|y_0-P_hy_0\|_1\|\partial_t y_h(0)\|_1\\ &\le \|u(0)\|\|\partial_t y_h(0)\|+\|\triangle y_0\|\|\partial_t y_h(0)\|+Ch\|y_0\|_2\frac{1}{h}\|\partial_t y_h(0)\|,\end{aligned}$$

which implies

$$\|\partial_t y_h(0)\|\le C(\|u(0)\|+\|y_0\|_2).\tag{5.3}$$

By differentiating the equation (3.3) with respect to t, and then, by taking $v_h=\partial_t y_h(t)$, we obtain that

$$\frac12\frac{d}{dt}\|\partial_t y_h\|^2+a(\partial_t y_h,\partial_t y_h)=(\chi_\omega\partial_t u,\partial_t y_h),$$

which, together with (5.3), yields

$$\sup_{t\in[0,T]}\|\partial_t y_h(t)\|^2+\int_0^T\|\partial_t y_h(t)\|_1^2dt\le C(\|u\|^2_{H^1(0,T;L^2(\Omega))}+\|y_0\|_2^2).\tag{5.4}$$

Now, by integrating (3.3) from t_{i-1} to t_i, we get that

$$(\partial_\tau y_h^i, v_h) + a(\tilde{y}_h^i, v_h) = (\chi_\omega \tilde{u}^i, v_h), \qquad \forall\, v_h \in V^h, 1 \le i \le N,$$

where $\partial_\tau y_h^i = [y_h(t_i) - y_h(t_{i-1})]/\tau$. Write $e_h^i = y_h^i - Y_h^i$. By subtracting the equation (5.1) from above equation, we see

$$(\partial_\tau e_h^i, v_h) + a(e_h^i, v_h) = (\chi_\omega(\tilde{u}^i - U^i), v_h) - a(\tilde{y}_h^i - y_h^i, v_h), \quad \forall\, v_h \in V^h, 1 \le i \le N.$$

Then, taking $v_h = \tau e_h^i$ in the above equation yields

$$\frac{1}{2}\|e_h^i\|^2 - \frac{1}{2}\|e_h^{i-1}\|^2 + \tau\|e_h^i\|_1^2 \le C\tau\|\tilde{u}^i - U^i\|^2 + \frac{\tau}{4}\|e_h^i\|_1^2 + C\tau\|\tilde{y}_h^i - y_h^i\|_1^2 + \frac{\tau}{4}\|e_h^i\|_1^2.$$

Summing the above equations over i from 1 to m with $1 \le m \le N$, we get that

$$\|e_h^m\|^2 + \tau\sum_{i=1}^m \|e_h^i\|_1^2 \le C(\tau\sum_{i=1}^m \|\tilde{u}^i - U^i)\|^2 + \tau\sum_{i=1}^m \|\tilde{y}_h^i - y_h^i\|_1^2),$$

from which, it follows that

$$\|e_h^m\|^2 + \tau\sum_{i=1}^N \|e_h^i\|_1^2 \le C(\tau\sum_{i=1}^N \|\tilde{u}^i - U^i)\|^2 + \tau\sum_{i=1}^N \|\tilde{y}_h^i - y_h^i\|_1^2). \tag{5.5}$$

On the other hand, by (5.4), we see that

$$\begin{aligned}
\tau\sum_{i=1}^N \|\tilde{y}_h^i - y_h^i\|_1^2 &= \tau\sum_{i=1}^N \|\frac{1}{\tau}\int_{t_{i-1}}^{t_i} y_h(t)dt - y_h(t_i)\|_1^2 \\
&= \tau\sum_{i=1}^N \|\frac{1}{\tau}\int_{t_{i-1}}^{t_i} (y_h(t) - y_h(t_i))dt\|_1^2 \\
&\le \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \|y_h(t) - y_h(t_i)\|_1^2 dt \\
&\le \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \|\int_t^{t_i} \partial_s y_h(s)ds\|_1^2 dt \\
&\le \tau^2\sum_{i=1}^N \int_{t_{i-1}}^{t_i} \|\partial_t y_h(t)\|_1^2 dt \\
&= \tau^2\int_0^T \|\partial_t y_h(t)\|_1^2 dt \\
&\le C\tau^2(\|u\|_{H^1(0,T;L^2(\Omega))}^2 + \|y_0\|_2^2).
\end{aligned}$$

The estimate (5.5), together with the above inequality, gives us the desired estimate. This completes the proof. □

Corollary 5.1. *Assume $u \in L^2(Q)$ and $y_0 \in H_0^1(\Omega)$. Let y_h be the solution of the equation (3.3) and Y_h be the solution of the equation (5.1), where $U^i = \tilde{u}^i$ for $1 = 1, 2, \cdots, N$. Then we have*

$$\max_{1\le i\le N} \|y_h^i - Y_h^i\| \longrightarrow 0$$

uniformly with respect to h as $\tau \longrightarrow 0$.

Proof. Since the space $H^1(0,T;L^2(\Omega))$ and the space $H^2(\Omega)\cap H_0^1(\Omega)$ are dense subspaces of $L^2(Q)$ and $H_0^1(\Omega)$, respectively, it holds that, for any $\varepsilon>0$, there exist functions $u_\delta\in H^1(0,T;L^2(\Omega))$ and $y_{0\delta}\in H^2(\Omega)\cap H_0^1(\Omega)$ such that

$$\|u-u_\delta\|_{L^2(Q)}\le\varepsilon,\quad\text{and}\ \|y_0-y_{0\delta}\|_1\le\varepsilon.$$

Let $y_{h\delta}\in H^1(0,T;V^h)$ be the solution of the following equation:

$$\begin{aligned}&(\partial_t y_{h\delta}(t),v_h)+a(y_{h\delta}(t),v_h)=(\chi_\omega u_\delta,v_h),\quad\forall\, v_h\in V^h,\ \ t\in[0,T],\\&y_{h\delta}(0)=P_hy_{0\delta},\end{aligned}$$

and let $(Y_{h\delta}^1,Y_{h\delta}^2,\cdots,Y_{h\delta}^N)\subset(V^h)^N$ be the solution of the equation:

$$\begin{aligned}&(\partial_\tau Y_{h\delta}^i,v_h)+a(Y_{h\delta}^i,v_h)=(\chi_\omega\tilde u_\delta^i,v_h),\qquad\forall\, v\in V^h,\quad 1\le i\le N,\\&Y_{h\delta}^0=P_hy_{0\delta}.\end{aligned}$$

Then according to Lemma 5.2, we can find a positive number $\bar\tau=\bar\tau(\varepsilon)$ such that for any number τ with $0<\tau<\bar\tau$,

$$\max_{1\le i\le N}\|y_{h\delta}^i-Y_{h\delta}^i\|\le C\tau(\|u_\delta\|_{H^1(0,T;L^2(\Omega))}+\|y_{0\delta}\|_2)\le\varepsilon. \tag{5.6}$$

Moreover, by making use of Lemma 3.1 and Lemma 5.1, we derive that

$$\sup_{t\in[0,T]}\|y_h(t)-y_{h\delta}(t)\|\le C(\|y_0-y_{0\delta}\|_1+\|u-u_\delta\|_{L^2(Q)})\le C\varepsilon \tag{5.7}$$

and

$$\begin{aligned}&\max_{1\le i\le N}\|Y_h^i-Y_{h\delta}^i\|\\
\le\ & C\left(\|y_0-y_{0\delta}\|_1+(\tau\sum_{i=1}^N\|\tilde u^i-\tilde u_\delta^i\|^2)^{\frac12}\right)\\
=\ & C\left(\|y_0-y_{0\delta}\|_1+(\tau\sum_{i=1}^N\|\frac1\tau\int_{t_{i-1}}^{t_i}(u-u_\delta)dt\|^2)^{\frac12}\right)\\
\le\ & C\left(\|y_0-y_{0\delta}\|_1+(\sum_{i=1}^N\int_{t_{i-1}}^{t_i}\|u-u_\delta\|^2dt)^{\frac12}\right)\\
=\ & C\left(\|y_0-y_{0\delta}\|_1+\|u-u_\delta\|_{L^2(Q)}\right)\\
\le\ & C\varepsilon.\end{aligned} \tag{5.8}$$

Now, it follows at once from (5.6), (5.7) and (5.8) that

$$\|y_h^i-Y_h^i\|\le\|y_h(t_i)-y_{h\delta}(t_i)\|+\|y_{h\delta}^i-Y_{h\delta}^i\|+\|Y_{h\delta}^i-Y_h^i\|\le C\varepsilon.$$

This completes the proof. $\square$

Now, we are on the position to set up a fully discrete approximation for the problem (P). For this purpose, we shall, from now on, assume $y_d\in H^1(0,T;L^2(\Omega))$. Write

$$\mathcal K^{h\tau}=\{V_{h\tau}=(V^1,V^2,\cdots,V^N)\in(U^h)^N;\|V^i\|\le 1,i=1,2,\cdots,N\}.$$

Then, the fully discrete approximation problem reads

$$(P_{\tau h})\qquad \min\left\{\frac\tau2\sum_{i=1}^N\left(\|Y_h^i-y_d^i\|^2+\|U_h^i\|^2\right)\right\},$$

subject to

$$U_{h\tau}=(U_h^1,U_h^2,\cdots,U_h^N)\in\mathcal K^{h\tau}\qquad\text{and}\qquad Y_h^N\in K,$$

where $Y_{h\tau} = (Y_h^1, Y_h^2, \cdots, Y_h^N) \in (V^h)^N$ is the solution of the equation (5.1). We recall $y_d \in H^1(0,T; L^2(\Omega) \subset C([0,T]; L^2(\Omega))$. Thus, $y_d^i(\cdot) = y_d(\cdot, t_i)$ is well-defined.

The next result concerns the Slater condition for the problem $(P_{h\tau})$.

Lemma 5.3. *Let $y_0 \in H_0^1(\Omega)$. Then there exist two positive numbers $\bar{h}$ and $\bar{\tau}$ having the following property: For each pair (h,τ) with $0 < h \leq \bar{h}, 0 < \tau \leq \bar{\tau}$, there exists a control $U_{0h\tau} = (U_{0h}^1, U_{0h}^2, \cdots, U_{0h}^N) \in \mathcal{K}^{h\tau}$ such that the corresponding solution $Y_{h\tau}(U_{0h\tau}) = (Y_h^1(U_{0h\tau}), Y_h^2(U_{0h\tau}), \cdots, Y_h^N(U_{0h\tau}))$ of the equation (5.1), where $U^i = U_{0h}^i, i = 1, 2, \cdots, N$, has such a property that $Y_h^N(U_{0h\tau}) \in intK$.*

Proof. By the condition (A) and according to Lemma 3.3, we can take elements u_0 and u_{0h} from the sets $\mathcal{K}$ and $\mathcal{K}^h$, respectively, such that the solution $y(u_0)$ of the equation (1.1) with $u = u_0$ and the solution $y_h(u_{0h})$ of the equation (3.3) with $u_h = u_{0h}$ take values in the set $intK$ at time T, namely, $y(u_0)(T) \in intK$ and $y_h(u_{0h})(T) \in intK$. Moreover, by (3.5) and (3.6), for any given $\varepsilon > 0$, we can find a positive number h_1 with $h_1 < h_0$ such that for any h with $0 < h \leq h_1$,

$$\|u_0 - u_{0h}\|_{L^2(Q)} \leq \varepsilon \tag{5.9}$$

and

$$\|y_h(u_{0h})(T)\| \leq \frac{1+\gamma}{2} < 1, \quad \text{for a certain number } \gamma \text{ with } 0 \leq \gamma < 1. \tag{5.10}$$

Write

$$U_{0h}^i = \tilde{u}_{0h}^i = \frac{1}{\tau}\int_{t_{i-1}}^{t_i} u_{0h} dt, \ \ i = 1, 2, \cdots, N, \ \ 0 < h \leq h_1$$

and

$$U_{0h\tau} = (U_{0h}^1, U_{0h}^2, \cdots, U_{0h}^N).$$

Since $u_{0h} \in \mathcal{K}^h$, it holds that for $i = 1, 2, \cdots, N$, $\|U_{0h}^i\| \leq 1$. Thus, we have $U_{0h\tau} = (U_{0h}^1, U_{0h}^2, \cdots, U_{0h}^N) \in \mathcal{K}^{h\tau}$.

Let $Y_h^N(U_{0h\tau})$ and $Y_h^N(U_{0h_1\tau})$ be the solutions of the equation (5.1) corresponding to $U_{0h\tau}$ and $U_{0h_1\tau}$, respectively. Then, by (5.9) and according to Lemma 5.1, we get, for any h with $0 < h \leq h_1$ and for any τ with $\tau > 0$,

$$\begin{aligned}
& \|Y_h^N(U_{0h\tau}) - Y_h^N(U_{0h_1\tau})\| \\
\leq\ & C\left(\tau\sum_{i=1}^N \|U_{0h}^i - U_{0h_1}^i\|^2\right)^{\frac{1}{2}} \\
=\ & C\left(\tau\sum_{i=1}^N \|\frac{1}{\tau}\int_{t_{i-1}}^{t_i}(u_{0h} - u_{0h_1})dt\|^2\right)^{\frac{1}{2}} \\
\leq\ & C\left(\sum_{i=1}^N \int_{t_{i-1}}^{t_i} \|u_{0h} - u_{0h_1}\|^2 dt\right)^{\frac{1}{2}} \\
=\ & C\|u_{0h} - u_{0h_1}\|_{L^2(Q)} \\
\leq\ & C(\|u_{0h} - u_0\|_{L^2(Q)} + \|u_0 - u_{0h_1}\|_{L^2(Q)}) \\
\leq\ & 2C\varepsilon.
\end{aligned} \tag{5.11}$$

Now, by making use of Corollary 5.1, we can select a positive number τ_0 such that for any τ with $0 < \tau \leq \tau_0$ and for any h with $0 < h \leq h_1$,

$$\|Y_h^N(U_{0h_1\tau}) - y_h(u_{0h_1})(T)\| \leq \varepsilon. \tag{5.12}$$

Moreover, according to Lemma 3.1, we get, for any number h with $0 < h \leq h_1$,

$$\|y_h(u_{0h_1})(T) - y_h(u_{0h})(T)\| \tag{5.13}$$

$$
\begin{aligned}
&\le C\|u_{0h_1} - u_{0h}\|_{L^2(Q)}\\
&\le C(\|u_{0h_1} - u_0\|_{L^2(Q)} + \|u_0 - u_{0h}\|_{L^2(Q)})\\
&\le 2C\varepsilon.
\end{aligned}
$$

Hence, it follows at once from (5.11), (5.12) and (5.13) that

$$
\begin{aligned}
&\|Y_h^N(U_{0h\tau}) - y_h(u_{0h})(T)\|\\
\le\ &\|Y_h^N(U_{0h\tau}) - Y_h^N(U_{0h_1\tau})\| + \|Y_h^N(U_{0h_1\tau}) - y_h(u_{0h_1})(T)\|\\
&+\|y_h(u_{0h_1})(T) - y_h(u_{0h})(T)\| \longrightarrow 0 \qquad \text{as } h,\tau \longrightarrow 0.
\end{aligned}
$$

Thus there exist positive constants $\bar{h}$ and $\bar{\tau}$ such that for any h with $0 < h \le \bar{h}$ and for any τ with $0 < \tau \le \bar{\tau}$,

$$\|Y_h^N(U_{0h\tau}) - y_h(u_{0h})(T)\| \le \frac{1-\gamma}{4},$$

where γ is exactly the number given in the inequality (5.10). The later, combined with (5.10), gives

$$
\begin{aligned}
\|Y_h^N(U_{0h\tau})\| &\le \|Y_h^N(U_{0h\tau}) - y_h(u_{0h})(T)\| + \|y_h(u_{0h})(T)\|\\
&\le \frac{1-\gamma}{4} + \frac{1+\gamma}{2} = \frac{3+\gamma}{4} < 1.
\end{aligned}
\tag{5.14}
$$

This completes the proof. □

According to Lemma 5.3 and by making use of the same arguments as those used in the proofs of Theorem 2.1 and Proposition 2.1, we can get the following first order optimality conditions for the problem $(P_{h\tau})$.

Theorem 5.1. *Let $y_0 \in H_0^1(\Omega)$ and $y_d \in H^1(0,T;L^2(\Omega))$. Then there exists two positive numbers $\bar{h}$ and $\bar{\tau}$ such that for all numbers h and τ with $0 < h \le \bar{h}$ and $0 < \tau \le \bar{\tau}$, the problem $(P_{h\tau})$ has a unique solution. Moreover, $\bar{U}_{h\tau} = (\bar{U}_h^1, \bar{U}_h^2, \cdots, \bar{U}_h^N)$ is the solution of the problem $(P_{h\tau})$ if and only if there exist $\bar{\mu}_{h\tau} \in V^h$, $\bar{Y}_{h\tau} = (\bar{Y}_h^1, \bar{Y}_h^2, \cdots, \bar{Y}_h^N) \in (V^h)^N$ and $\bar{\Phi}_{h\tau} = (\bar{\Phi}_h^1, \bar{\Phi}_h^2, \cdots, \bar{\Phi}_h^N) \in (V^h)^N$ such that*

$$\bar{Y}_h^N \in K, \quad (\bar{\mu}_{h\tau}, z - \bar{Y}_h^N) \le 0, \quad \forall\, z \in K, \tag{5.15}$$

$$
\left\{
\begin{array}{ll}
(\partial_\tau \bar{Y}_h^i, v_h) + a(\bar{Y}_h^i, v_h) = (\chi_\omega \bar{U}_h^i, v_h), & \forall\, v_h \in V^h, \quad 1 \le i \le N,\\
\bar{Y}_h^0 = P_h y_0. &
\end{array}
\right.
\tag{5.16}
$$

$$
\left\{
\begin{array}{ll}
(\partial_\tau \bar{\Phi}_h^i, v_h) - a(\bar{\Phi}_h^{i-1}, v_h) = (\bar{Y}_h^i - y_d^i, v_h), & \forall\, v_h \in V^h, \quad 1 \le i \le N,\\
\bar{\Phi}_h^N = -\bar{\mu}_{h\tau}. &
\end{array}
\right.
\tag{5.17}
$$

(5.18)

$$\bar{U}_{h\tau} \in \mathcal{K}^{h\tau},\ \sum_{i=1}^{N} (\bar{U}_h^i - \chi_\omega \bar{\Phi}_h^{i-1}, U_h^i - \bar{U}_h^i) \ge 0, \quad \forall\, U_{h\tau} = (U_h^1, U_h^2, \cdots, U_h^N) \in \mathcal{K}^{h\tau}.$$

Furthermore, the optimal control $\bar{U}_{h\tau}$ for the problem $(P_{h\tau})$ has the following explicit expression:

$$\bar{U}_h^i = \frac{\Pi_h \chi_\omega \bar{\Phi}_h^{i-1}}{1 + k_h^i}, \qquad i = 1, 2, \cdots, N, \tag{5.19}$$

where

$$
k_h^i = \left\{
\begin{array}{ll}
0 & \text{if } \|\Pi_h \chi_\omega \bar{\Phi}_h^{i-1}\| < 1,\\
\|\Pi_h \chi_\omega \bar{\Phi}_h^{i-1}\| - 1 & \text{if } \|\Pi_h \chi_\omega \bar{\Phi}_h^{i-1}\| \ge 1.
\end{array}
\right.
\tag{5.20}
$$

Proposition 5.1. *Let $\bar{U}_{h\tau} = (\bar{U}_h^1, \bar{U}_h^2, \cdots, \bar{U}_h^N)$ be the solution of the problem $(P_{h\tau})$. Then it holds that*

$$\sum_{i=2}^{N} \|\bar{U}_h^i - \bar{U}_h^{i-1}\|^2 \le C(1 + \max_{1\le i\le N} \|\bar{\Phi}_h^{i-1}\|) \sum_{i=2}^{N} \|\bar{\Phi}_h^{i-1} - \bar{\Phi}_h^{i-2}\|^2. \tag{5.21}$$

Proof. By (5.19) and (5.20), we obtain that

$$\begin{aligned}
&\sum_{i=2}^{N} \|\bar{U}_h^i - \bar{U}_h^{i-1}\|^2 \\
= \ & \sum_{i=2}^{N} \|\frac{\Pi_h\chi_\omega\bar{\Phi}_h^{i-1}}{1+k_h^i} - \frac{\Pi_h\chi_\omega\bar{\Phi}_h^{i-2}}{1+k_h^{i-1}}\|^2 \\
= \ & \sum_{i=2}^{N} \|\frac{\Pi_h\chi_\omega\bar{\Phi}_h^{i-1}(1+k_h^{i-1}) - \Pi_h\chi_\omega\bar{\Phi}_h^{i-2}(1+k_h^i)}{(1+k_h^i)(1+k_h^{i-1})}\|^2 \\
\le \ & C\sum_{i=2}^{N} \|\Pi_h\chi_\omega(\bar{\Phi}_h^{i-1} - \bar{\Phi}_h^{i-2})\|^2 + C\sum_{i=2}^{N} \|\Pi_h\chi_\omega\bar{\Phi}_h^{i-1}\|^2 |k_h^{i-1} - k_h^i|^2 \\
& \qquad + C\sum_{i=2}^{N} \|\Pi_h\chi_\omega(\bar{\Phi}_h^{i-1} - \bar{\Phi}_h^{i-2})\|^2 \\
\le \ & C(1 + \max_{1\le i\le N} \|\bar{\Phi}_h^{i-1}\|) \sum_{i=2}^{N} \|\bar{\Phi}_h^{i-1} - \bar{\Phi}_h^{i-2}\|^2,
\end{aligned}$$

which completes this proof. □

Remark 5.1. *Obviously, we can rewrite (5.21) as*

$$\tau\sum_{i=2}^{N} \|\partial_\tau\bar{U}_h^i\|^2 \le C(1 + \max_{1\le i\le N} \|\bar{\Phi}_h^{i-1}\|)\tau\sum_{i=2}^{N} \|\partial_\tau\bar{\Phi}_h^{i-1}\|^2,$$

which is the fully discrete version of Proposition 2.2.

6. Error estimate between the solutions of (P_h) and $(P_{h\tau})$

Lemma 6.1. *Suppose that all assumptions in Theorem 5.1 hold. Let $\bar{U}_{h\tau} = (\bar{U}_h^1, \bar{U}_h^2, \cdots, \bar{U}_h^N) \in \mathcal{K}^{h\tau}$ be the solution of the problem $(P_{h\tau})$. Then there exists a positive constant C independent of h, τ such that*

$$\|\bar{\mu}_{h\tau}\|_1 \le C$$

for any h, τ with $0 < h \le \bar{h}, 0 < \tau \le \bar{\tau}$, where m $\bar{h}$ and $\bar{\tau}$ are the numbers given in Theorem 5.1.

Proof. We first prove that the family $\{\bar{\mu}_{h\tau}\}_{0<h\le\bar{h},0<\tau\le\bar{\tau}}$ is bounded in $L^2(\Omega)$. Let $U_{0h\tau} = (U_{0h}^1, \cdots, U_{0h}^N) \in \mathcal{K}^{h\tau}$ be given in Lemma 5.3. Write $Y_{h\tau}(U_{0h\tau}) = (Y_h^1(U_{0h\tau}), \cdots, Y_h^N(U_{0h\tau}))$ for the solution of the equation (5.1) with $U_{h\tau} = U_{0h\tau}$. As a matter of convenience, we write Y_h^i for $Y_h^i(U_{0h\tau})$ where $i = 1, 2, \cdots, N$. By the equation (5.14), we see that for all $h \in (0, \bar{h})$ and $\tau \in (0, \bar{\tau})$,

$$\|Y_h^N\| = \|Y_h^N(U_{0h\tau})\| \le \frac{3+\gamma}{4} < 1 \tag{6.1}$$

for a certain constant γ with $0 \leq \gamma < 1$. Since $U_{0h\tau}$ and $\bar{U}_{h\tau}$ are in the set of $\mathcal{K}^{h\tau}$, it follows that

$$\|U_{0h}^i\| \leq 1 \quad \text{and} \quad \|\bar{U}_h^i\| \leq 1, \qquad 1 \leq i \leq N. \tag{6.2}$$

Then, by Lemma 5.1, we obtain that

$$\max_{1\leq i\leq N} \|Y_h^i\| \leq C \quad \text{and} \quad \max_{1\leq i\leq N} \|\bar{Y}_h^i\| \leq C. \tag{6.3}$$

Because of (6.1), we can find a positive number $\rho > 0$, which is independent of h, τ and satisfies $0 < \rho \leq \frac{1-\gamma}{4}$, such that

$$\|Y_h^N + \rho w\| \leq \|Y_h^N\| + \rho\|w\| \leq \frac{3+\gamma}{4} + \frac{1-\gamma}{4} \leq 1 \quad \text{for all } \omega \text{ with } \|w\| \leq 1.$$

Thus it follows from (5.15) that for any element $w \in L^2(\Omega)$ with $\|w\| \leq 1$ and for all numbers h, τ with $0 < h \leq \bar{h}, 0 < \tau \leq \bar{\tau}$,

$$(\bar{\mu}_{h\tau}, Y_h^N + \rho w - \bar{Y}_h^N) \leq 0.$$

By applying the following discrete integration by parts formula

$$\sum_{i=1}^N (a_i - a_{i-1})b_i = a_N b_N - a_0 b_0 - \sum_{i=1}^N a_{i-1}(b_i - b_{i-1}), \tag{6.4}$$

where $a_i = \bar{\Phi}_h^i, b_i = Y_h^i - \bar{Y}_h^i$, and by (5.16)-(5.18), we derive that

$$\begin{aligned}
\rho\|\bar{\mu}_{h\tau}\| &\leq -(\bar{\mu}_{h\tau}, Y_h^N - \bar{Y}_h^N) \\
&= \tau\sum_{i=1}^N(\partial_\tau\bar{\Phi}_h^i, Y_h^i - \bar{Y}_h^i) + \tau\sum_{i=1}^N(\bar{\Phi}_h^{i-1}, \partial_\tau(Y_h^i - \bar{Y}_h^i)) \\
&= \tau\sum_{i=1}^N a(\bar{\Phi}_h^{i-1}, Y_h^i - \bar{Y}_h^i) + \tau\sum_{i=1}^N(\bar{Y}_h^i - y_d^i, Y_h^i - \bar{Y}_h^i) \\
&\qquad +\tau\sum_{i=1}^N(\bar{\Phi}_h^{i-1}, \partial_\tau(Y_h^i - \bar{Y}_h^i)) \\
&= \tau\sum_{i=1}^N(\bar{\Phi}_h^{i-1}, \chi_\omega(U_{0h}^i - \bar{U}_h^i)) + \tau\sum_{i=1}^N(\bar{Y}_h^i - y_d^i, Y_h^i - \bar{Y}_h^i) \\
&= -\tau\sum_{i=1}^N(\bar{U}_h^i - \chi_\omega\bar{\Phi}_h^{i-1}, U_{0h}^i - \bar{U}_h^i) + \tau\sum_{i=1}^N(\bar{U}_h^i, U_{0h}^i - \bar{U}_h^i) \\
&\qquad +\tau\sum_{i=1}^N(\bar{Y}_h^i - y_d^i, Y_h^i - \bar{Y}_h^i) \\
&\leq \tau\sum_{i=1}^N(\bar{U}_h^i, U_{0h}^i - \bar{U}_h^i) + \tau\sum_{i=1}^N(\bar{Y}_h^i - y_d^i, Y_h^i - \bar{Y}_h^i).
\end{aligned}$$

This together with (6.2) and (6.3) yields

$$\|\bar{\mu}_{h\tau}\| \leq C \text{ for all } h, \tau \text{ with } 0 < h \leq \bar{h}, 0 < \tau \leq \bar{\tau}. \tag{6.5}$$

Next, we will show that the family $\{\bar{\mu}_{h\tau}\}_{0<h\leq\bar{h}, 0<\tau\leq\bar{\tau}}$ is bounded in $H_0^1(\Omega)$. By (5.15) and by making use of the same arguments as those used in the proof of (2.5) and (2.6), we can easily derive that

$$\bar{\mu}_{h\tau} = k_{h\tau}\bar{Y}_h^N,$$

where

$$k_{h\tau} = \begin{cases} 0 & \text{if } \|\bar{Y}_h^N\| < 1, \\ \|\bar{\mu}_{h\tau}\| & \text{if } \|\bar{Y}_h^N\| = 1. \end{cases}$$

Thus it follows from (6.5) that

$$k_{h\tau} \le \|\bar{\mu}_{h\tau}\| \le C.$$

According to Lemma 5.1, we obtain that

$$\|\bar{\mu}_{h\tau}\|_1 = k_{h\tau}\|\bar{Y}_h^N\|_1 \le C.$$

This completes the proof. □

Lemma 6.2. *Assume $y_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Let $\bar{h}$ and $\bar{\tau}$ be the positive numbers given in Theorem 5.1 and $\bar{U}_{h\tau} = (\bar{U}_h^1, \bar{U}_h^2, \cdots, \bar{U}_h^N) \in \mathcal{K}^{h\tau}$ be the solution of the problem $(P_{h\tau})$. Then it holds that*

$$\tau \sum_{i=1}^N \|\bar{Y}_h^i - \bar{Y}_h^{i-1}\|_1^2 \le C\tau^2$$

for all h, τ with $0 < h \le \bar{h}, 0 < \tau \le \bar{\tau}$.

Proof. Write $E_h^i = \bar{Y}_h^i - \bar{Y}_h^{i-1}$. Subtracting two consecutive equations in (5.16) gives

$$(\partial_\tau E_h^i, v_h) + a(E_h^i, v_h) = (\chi_\omega(\bar{U}^i - \bar{U}^{i-1}), v_h), \qquad \forall\, v_h \in V^h, \quad 2 \le i \le N.$$

By taking $v_h = \tau E_h^i$ in the above equation, we obtain that

$$\frac{1}{2}\|E_h^i\|^2 - \frac{1}{2}\|E_h^{i-1}\|^2 + \tau C_a \|E_h^i\|_1^2 \le C\tau\|\bar{U}^i - \bar{U}^{i-1}\|^2 + \frac{C_a\tau}{2}\|E_h^i\|_1^2.$$

Summing the above equations over i from $i = 2$ to N yields

$$\frac{1}{2}\|E_h^N\|^2 - \frac{1}{2}\|E_h^1\|^2 + \tau\frac{C_a}{2}\sum_{i=2}^N \|E_h^i\|_1^2 \le C\tau\sum_{i=2}^N \|\bar{U}^i - \bar{U}^{i-1}\|^2,$$

from which, it follows that

$$\tau\sum_{i=2}^N \|E_h^i\|_1^2 \le C\tau\sum_{i=2}^N \|\bar{U}^i - \bar{U}^{i-1}\|^2 + C\|E_h^1\|^2.$$

Thus we have

$$\tau\sum_{i=1}^N \|E_h^i\|_1^2 \le C\tau\sum_{i=2}^N \|\bar{U}^i - \bar{U}^{i-1}\|^2 + C\|E_h^1\|^2 + \tau\|E_h^1\|_1^2. \tag{6.6}$$

By the similar argument as that used in the proof of Lemma 5.1 and by making use of the boundedness of the family $\{\bar{\mu}_{h\tau}\}_{0<h\le\bar{h}, 0<\tau\le\bar{\tau}}$ in $H_0^1(\Omega)$, which is provided by Lemma 6.1, we get

$$\max_{1\le i\le N}\|\Phi_h^{i-1}\|_1 + \tau\sum_{i=1}^N \|\partial_\tau \Phi_h^i\|^2 \le C(\|\bar{\mu}_{h\tau}\|_1^2 + \tau\sum_{i=1}^N \|\bar{Y}_h^i - y_d^i\|^2) \le C. \tag{6.7}$$

Then, according to Proposition 5.1, it follows from (6.7) that

$$\begin{aligned} &\tau\sum_{i=2}^N \|\bar{U}^i - \bar{U}^{i-1}\|^2 \\ \le\ & C(1 + \max_{1\le i\le N}\|\bar{\Phi}^{i-1}\|)\tau\sum_{i=2}^N \|\bar{\Phi}^{i-1} - \bar{\Phi}^{i-2}\|^2 \le C\tau^2. \end{aligned} \tag{6.8}$$

Write $y_h(\cdot) = y_h(\bar{U}_h^1)(\cdot)$ for the solution of the following equation:

$$\begin{cases} (\partial_t y_h(t), v_h) + a(y_h(t), v_h) = (\chi_\omega \bar{U}_h^1, v_h), \quad \forall\, v_h \in V^h, \text{ for a.e. } t \in [0,\tau], \\ y_h(0) = P_h y_0. \end{cases}$$

Because of $\bar{U}_h^1 \in H^1(0,\tau; L^2(\Omega))$, we can use the very similar arguments as those used in the proofs of the inequality (5.2) and the inequality (5.4) to get the following estimates:

$$\|y_h^1 - \bar{Y}_h^1\|^2 + \tau\|y_h^1 - Y_h^1\|_1^2 \le C\tau^2(\|\bar{U}_h^1\|^2 + \|y_0\|_2^2) \tag{6.9}$$

and

$$\sup_{t\in[0,\tau]} \|\partial_t y_h(t)\|^2 + \int_0^\tau \|\partial_t y_h(t)\|_1^2 dt \le C(\|U_h^1\|^2 + \|y_0\|_2^2). \tag{6.10}$$

Thus, it follows from (6.9) and (6.10) that

$$\begin{aligned} \tau\|E_h^i\|_1^2 &\le \tau\|\bar{Y}_h^1 - \bar{Y}_h^0\|_1^2 \\ &\le 2\tau\|\bar{Y}_h^1 - y_h^1\|_1^2 + 2\tau\|y_h(\tau) - y_h(0)\|_1^2 \\ &\le C\tau^2 + C\tau\|\int_0^\tau \partial_t y_h(t)dt\|_1^2 \\ &\le C\tau^2 + C\tau^2 \int_0^\tau \|\partial_t y_h(t)\|_1^2 dt \\ &\le C\tau^2 \end{aligned} \tag{6.11}$$

and

$$\begin{aligned} \|E_h^1\|^2 &= \|\bar{Y}_h^1 - \bar{Y}_h^0\|^2 \\ &\le 2\|\bar{Y}_h^1 - y_h^1\|^2 + 2\|y_h(\tau) - y_h(0)\|^2 \\ &\le C\tau^2 + C\|\int_0^\tau \partial_t y_h(t)dt\|^2 \\ &\le C\tau^2 + C\tau \int_0^\tau \|\partial_t y_h(t)\|^2 dt \\ &\le C\tau^2. \end{aligned} \tag{6.12}$$

Finally, by (6.6), (6.8), (6.11) and (6.12), we can get the desired estimate. This completes the proof of this lemma. □

Now we define functions $\bar{y}_{h\tau} \in H^1(0,T;V^h)$ and $\bar{u}_{h\tau} \in L^2(0,T;U^h)$ by setting

$$\bar{y}_{h\tau}|_{(t_{i-1},t_i]} = \bar{Y}_h^{i-1} + (\frac{t - t_{i-1}}{\tau})(\bar{Y}_h^i - \bar{Y}_h^{i-1}), \ \ i = 1, 2, \cdots, N \tag{6.13}$$

and

$$\bar{u}_{h\tau}|_{(t_{i-1},t_i]} = \bar{U}_h^i, \ \ i = 1, 2, \cdots, N \tag{6.14}$$

respectively. For each function $f \in C([0,T]; L^2(\Omega))$, we define function $\hat{f}$ by

$$\hat{f}(t)|_{(t_{i-1},t_i]} = f(t_i) \quad \text{for } \ i = 1, 2, \cdots, N.$$

Then, it is clear that the function $\bar{y}_{h\tau}$ solves the following equation:

$$\begin{cases} (\partial_t \bar{y}_{h\tau}(t), v_h) + a(\hat{\bar{y}}_{h\tau}(t), v_h) = (\bar{u}_{h\tau}, v_h), \quad \forall\, v_h \in V^h, \text{ for a.e. } t, \\ \bar{y}_{h\tau}(0) = P_h y_0, \end{cases} \tag{6.15}$$

Moreover, we have the following error estimate between the solutions of the semi-discrete problem and the fully discrete problem.

Theorem 6.1. *Assume* $y_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $y_d \in H^1(0,T;L^2(\Omega))$. *Let* $\bar{u}_h$ *and* $\bar{U}_{h\tau} = (\bar{U}_h^1, \bar{U}_h^2, \cdots, \bar{U}_h^N)$ *be the solutions of the problem* (P_h) *and the problem* $(P_{h\tau})$, *respectively. Then there exist two positive numbers* $\bar{h}$ *and* $\bar{\tau}$ *such that for all* h, τ *with* $0 < h \le \bar{h}, 0 < \tau \le \bar{\tau}$,

$$\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \|\bar{u}_h - \bar{U}_h^i\|^2 dt \le C\tau.$$

Proof. Let $\bar{h}$ and $\bar{\tau}$ be the positive numbers given in Theorem 5.1. It follows from (3.10) and (5.18) that

$$\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} (\bar{u}_h(t) - \chi_\omega \bar{\varphi}_h(t), \bar{U}_h^i - \bar{u}_h(t)) dt \ge 0 \tag{6.16}$$

and

$$\tau \sum_{i=1}^{N} (\bar{U}_h^i - \chi_\omega \bar{\Phi}_h^{i-1}, U_h^i - \bar{U}_h^i) = \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} (\bar{U}_h^i - \chi_\omega \bar{\Phi}_h^{i-1}, U_h^i - \bar{U}_h^i) dt \ge 0, \tag{6.17}$$

where

$$U_h^i = \tilde{\bar{u}}_h^i = \frac{1}{\tau} \int_{t_{i-1}}^{t_i} \bar{u}_h dt, \qquad 1 \le i \le N.$$

By making use of the following equation

$$\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} (\bar{U}_h^i - \chi_\omega \bar{\Phi}_h^{i-1}, \tilde{\bar{u}}_h^i - \bar{u}_h(t)) dt = 0,$$

we get from (6.16) and (6.17) that

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \|\bar{u}_h - \bar{U}_h^i\|^2 dt \\
= &\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \|\bar{u}_h(t) - \bar{U}_h^i\|^2 dt \\
= &\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} (\bar{u}_h(t), \bar{u}_h(t) - \bar{U}_h^i) dt - \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} (\bar{U}_h^i, \bar{u}_h(t) - \bar{U}_h^i) dt \\
\le &\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} (\chi_\omega \bar{\varphi}_h(t), \bar{u}_h(t) - \bar{U}_h^i) dt - \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} (\bar{U}_h^i, \bar{u}_h(t) - \bar{U}_h^i) dt \\
\le &\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} (\chi_\omega \bar{\varphi}_h(t), \bar{u}_h(t) - \bar{U}_h^i) dt - \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} (\bar{U}_h^i, \bar{u}_h(t) - \bar{U}_h^i) dt \\
&+ \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} (\bar{U}_h^i - \chi_\omega \bar{\Phi}_h^{i-1}, \tilde{\bar{u}}_h^i - \bar{U}_h^i) dt \\
= &\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} (\chi_\omega \bar{\varphi}_h(t), \bar{u}_h(t) - \bar{U}_h^i) dt - \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} (\bar{U}_h^i, \bar{u}_h(t) - \bar{U}_h^i) dt \\
&+ \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} (\bar{U}_h^i - \chi_\omega \bar{\Phi}_h^{i-1}, \tilde{\bar{u}}_h^i - \bar{u}_h(t)) dt
\end{aligned} \tag{6.18}$$

$$
\begin{aligned}
&+\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}(\bar U_h^i-\chi_\omega\bar\Phi_h^{i-1},\bar u_h(t)-\bar U_h^i)dt\\
=\;&\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}(\chi_\omega(\bar\varphi_h(t)-\bar\Phi_h^{i-1}),\bar u_h(t)-\bar U_h^i)dt\\
&+\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}(\bar U_h^i-\chi_\omega\bar\Phi_h^{i-1},\tilde{\bar u}_h^i-\bar u_h(t))dt\\
=\;&\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}(\chi_\omega(\bar\varphi_h(t)-\bar\Phi_h^{i-1}),\bar u_h(t)-\bar U_h^i)dt\\
=\;&\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}(\chi_\omega\bar\varphi_h(t),\bar u_h(t))dt+\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}(\chi_\omega\bar\Phi_h^{i-1},\bar U_h^i)dt\\
&-\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}(\chi_\omega\bar\varphi_h(t),\bar U_h^i)dt-\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}(\chi_\omega\bar\Phi_h^{i-1},\bar u_h(t))dt\\
=\;&J_1+J_2+J_3+J_4.
\end{aligned}
$$

Now we will rewrite the terms $J_i, 1\le i\le 4$, into the forms which can be estimated easier, one by one. First, it follows from (3.8) and (3.9) that

$$
\begin{aligned}
(6.19)\quad J_1 =\;& \sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}(\chi_\omega\bar\varphi_h(t),\bar u_h(t))dt\\
=\;&\int_0^T(\chi_\omega\bar\varphi_h(t),\bar u_h(t))dt\\
=\;&\int_0^T(\partial_t\bar y_h(t),\bar\varphi_h(t))dt+\int_0^T a(\bar y_h(t),\bar\varphi_h(t))dt\\
=\;&(\bar y_h(T),\bar\varphi_h(T))-(\bar y_h(0),\bar\varphi_h(0))\\
&-\int_0^T(\bar y_h(t),\partial_t\bar\varphi_h(t))dt+\int_0^T a(\bar y_h(t),\bar\varphi_h(t))dt\\
=\;&-(\bar y_h(T),\bar\mu_h)-(P_hy_0,\bar\varphi_h(0))-\int_0^T(\bar y_h(t),\bar y_h(t)-y_d(t))dt.
\end{aligned}
$$

Secondly, by (5.16), (5.17) and (6.4), we get that

$$
\begin{aligned}
(6.20)\quad J_2 =\;& \tau\sum_{i=1}^{N}(\chi_\omega\bar\Phi_h^{i-1},\bar U_h^i)\\
=\;&\tau\sum_{i=1}^{N}(\partial_\tau\bar Y_h^i,\bar\Phi_h^{i-1})+\tau\sum_{i=1}^{N}a(\bar Y_h^i,\bar\Phi_h^{i-1})\\
=\;&(\bar Y_h^N,\bar\Phi_h^N)-(\bar Y_h^0,\bar\Phi_h^0)-\tau\sum_{i=1}^{N}(\bar Y_h^i,\partial_\tau\bar\Phi_h^i)+\tau\sum_{i=1}^{N}a(\bar Y_h^i,\bar\Phi_h^{i-1})\\
=\;&-(\bar Y_h^N,\bar\mu_{h\tau})-(P_hy_0,\bar\Phi_h^0)-\tau\sum_{i=1}^{N}(\bar Y_h^i,\bar Y_h^i-y_d^i).
\end{aligned}
$$

Thirdly, let $\bar{y}_{h\tau}$ be given by (6.13), which satisfies the equation (6.15). Let $\bar{u}_{h\tau}$ be defined by (6.14). Then by making use of (3.9) and (6.15), we obtain that

$$
\begin{aligned}
(6.21)\quad J_3 &= -\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}(\chi_\omega\bar{\varphi}_h(t),\bar{U}_h^i)dt\\
&= -\int_0^T(\chi_\omega\bar{\varphi}_h(t),\bar{u}_{h\tau})dt\\
&= -\int_0^T(\partial_t\bar{y}_{h\tau}(t),\bar{\varphi}_h(t))dt-\int_0^T a(\hat{\bar{y}}_{h\tau}(t),\bar{\varphi}_h(t))dt\\
&= -\int_0^T(\partial_t\bar{y}_{h\tau}(t),\bar{\varphi}_h(t))dt-\int_0^T a(\bar{y}_{h\tau}(t),\bar{\varphi}_h(t))dt\\
&\quad +\int_0^T a(\bar{y}_{h\tau}(t)-\hat{\bar{y}}_{h\tau}(t),\bar{\varphi}_h(t))dt\\
&= -(\bar{y}_{h\tau}(T),\bar{\varphi}_h(T))+(\bar{y}_{h\tau}(0),\bar{\varphi}_h(0))+\int_0^T(\bar{y}_{h\tau}(t),\partial_t\bar{\varphi}_h(t))dt\\
&\quad -\int_0^T a(\bar{y}_{h\tau}(t),\bar{\varphi}_h(t))dt+\int_0^T a(\bar{y}_{h\tau}(t)-\hat{\bar{y}}_{h\tau}(t),\bar{\varphi}_h(t))dt\\
&= (\bar{Y}_h^N,\bar{\mu}_h)+(P_hy_0,\bar{\varphi}_h(0))+\int_0^T(\bar{y}_{h\tau}(t),\bar{y}_h(t)-y_d(t))dt\\
&\qquad +\int_0^T a(\bar{y}_{h\tau}(t)-\hat{\bar{y}}_{h\tau}(t),\bar{\varphi}_h(t))dt.
\end{aligned}
$$

Finally, we deal with the term J_4. To this end, we integrate the equation (3.8) from t_{i-1} to t_i to get

$$
(6.22)\qquad (\partial_\tau\bar{y}_h^i,v_h)+a(\tilde{\bar{y}}_h^i,v_h)=(\chi_\omega\tilde{\bar{u}}^i,v_h),\qquad \forall\, v_h\in V^h, 1\le i\le N.
$$

Then by (5.17), (6.4) and (6.22), we see that

$$
\begin{aligned}
(6.23)\quad J_4 &= -\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}(\chi_\omega\bar{\Phi}_h^{i-1},\bar{u}_h(t))dt\\
&= -\tau\sum_{i=1}^{N}(\chi_\omega\bar{\Phi}_h^{i-1},\tilde{\bar{u}}_h^i)\\
&= -\tau\sum_{i=1}^{N}(\partial_\tau\bar{y}_h^i,\bar{\Phi}_h^{i-1})-\tau\sum_{i=1}^{N}a(\tilde{\bar{y}}_h^i,\bar{\Phi}_h^{i-1})\\
&= -(\bar{y}_h^N,\bar{\Phi}_h^N)+(\bar{y}_h^0,\bar{\Phi}_h^0)+\tau\sum_{i=1}^{N}(\bar{y}_h^i,\partial_\tau\bar{\Phi}_h^i)-\tau\sum_{i=1}^{N}a(\tilde{\bar{y}}_h^i,\bar{\Phi}_h^{i-1})\\
&= (\bar{y}_h(T),\bar{\mu}_{h\tau})+(P_hy_0,\bar{\Phi}_h^0)\\
&\quad +\tau\sum_{i=1}^{N}(\bar{y}_h^i,\bar{Y}_h^i-y_d^i)+\tau\sum_{i=1}^{N}a(\bar{y}_h^i-\tilde{\bar{y}}_h^i,\bar{\Phi}_h^{i-1}).
\end{aligned}
$$

Since it follows form (3.7) and (5.15) that

$$
(\bar{Y}_h^N-\bar{y}_h(T),\bar{\mu}_h)\le 0,
$$

$$
(\bar{y}_h(T)-\bar{Y}_h^N,\bar{\mu}_{h\tau})\le 0,
$$

we get from (6.18)-(6.21) and (6.23) that

$$
\begin{aligned}
(6.24)\qquad &\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}\|\bar{u}_h-\bar{U}_h^i\|^2dt\\
\leq\ & -\int_0^T(\bar{y}_h(t),\bar{y}_h(t)-y_d(t))dt-\tau\sum_{i=1}^{N}(\bar{Y}_h^i,\bar{Y}_h^i-y_d^i)\\
&+\int_0^T(\bar{y}_{h\tau}(t),\bar{y}_h(t)-y_d(t))dt+\tau\sum_{i=1}^{N}(\bar{y}_h^i,\bar{Y}_h^i-y_d^i)\\
&+\int_0^T a(\bar{y}_{h\tau}(t)-\hat{\bar{y}}_{h\tau}(t),\bar{\varphi}_h(t))dt+a(\bar{y}_h^i-\tilde{\bar{y}}_h^i,\bar{\Phi}_h^{i-1})\\
=\ & -\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}\|\bar{y}_h(t)-\bar{Y}_h^i\|^2dt+\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}(\bar{y}_{h\tau}(t)-\bar{Y}_h^i,\bar{y}_h(t))dt\\
&+\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}(\bar{y}_h^i-\bar{y}_h(t),\bar{Y}_h^i)dt+\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}(\bar{y}_h(t)-\bar{y}_h^i,y_d(t))dt\\
&+\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}(\bar{y}_h^i,y_d(t)-y_d^i)dt+\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}(\bar{Y}_h^i-\bar{y}_{h\tau}(t),y_d^i)dt\\
&+\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}(\bar{y}_{h\tau}(t),y_d^i-y_d(t))dt+\int_0^T a(\bar{y}_{h\tau}(t)-\hat{\bar{y}}_{h\tau}(t),\bar{\varphi}_h(t))dt\\
&+\tau\sum_{i=1}^{N}a(\bar{y}_h^i-\tilde{\bar{y}}_h^i,\bar{\Phi}_h^{i-1})\\
=\ & Q_1+Q_2+\cdots+Q_9.
\end{aligned}
$$

Now we will estimate the terms $Q_i, 1\leq i\leq 9$. We make the estimates for the terms Q_1, Q_8 and Q_9 one by one, the estimates for the rest terms follow by the similar arguments. It is clear that

$$
(6.25)\qquad Q_1=-\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}\|\bar{y}_h(t)-\bar{Y}_h^i\|^2dt\leq 0.
$$

By (4.8) and according to Lemma 6.2, we see that

$$
\begin{aligned}
(6.26)\qquad Q_8\ =\ & \int_0^T a(\bar{y}_{h\tau}(t)-\hat{\bar{y}}_{h\tau}(t),\bar{\varphi}_h(t))dt\\
\leq\ & C\sup_{t\in[0,T]}\|\bar{\varphi}_h(t)\|_1\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}\|\bar{y}_{h\tau}^i-\bar{Y}_h^i\|_1dt\\
\leq\ & C\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}\frac{t_i-t}{\tau}dt\|\bar{Y}_h^{i-1}-\bar{Y}_h^i\|_1\\
\leq\ & C\tau\sum_{i=1}^{N}\|\bar{Y}_h^i-\bar{Y}_h^{i-1}\|_1\\
\leq\ & C\left(\tau\sum_{i=1}^{N}\|\bar{Y}_h^i-\bar{Y}_h^{i-1}\|_1^2\right)^{\frac{1}{2}}\\
\leq\ & C\tau.
\end{aligned}
$$

According to Proposition 3.2 and by the inequality (4.8), we obtain that

$$\|\bar{u}_h\|_{H^1(0,T;L^2(\Omega))} \leq C,$$

which, together with (5.4) and (6.7), yields the following estimate:

$$\begin{aligned}
Q_9 &= \tau \sum_{i=1}^{N} a(\bar{y}_h^i - \tilde{\bar{y}}_h^i, \bar{\Phi}_h^{i-1}) \\
&\leq C \max_{1\leq i\leq N} \|\bar{\Phi}_h^{i-1}\|_1 \tau \sum_{i=1}^{N} \|\bar{y}_h^i - \tilde{\bar{y}}_h^i\|_1 \\
&\leq C \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \|\bar{y}_h(t_i) - \bar{y}_h(t)\|_1 dt \\
&\leq C \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \| \int_t^{t_i} \partial_s \bar{y}_h(s) ds\|_1 dt \\
&\leq C\tau \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \|\partial_s \bar{y}_h(s)\|_1 ds \\
&\leq C\tau (\int_0^T \|\partial_s \bar{y}_h(s)\|_1^2 ds)^{\frac{1}{2}} \\
&\leq C\tau (\|\bar{u}_h\|_{H^1(0,T;L^2(\Omega))} + \|y_0\|_2) \\
&\leq C\tau.
\end{aligned} \tag{6.27}$$

Similarly, we can show that

$$Q_2 + Q_3 + \cdots + Q_7 \leq C\tau. \tag{6.28}$$

By (6.24)-(6.28), we complete the proof of this theorem. □

As a direct consequence of Theorem 4.1 and Theorem 6.1, we have the main result of the paper as follows.

Theorem 6.2. *Assume* $y_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $y_d \in H^1(0,T;L^2(\Omega))$. *Let* $\bar{u} \in \mathcal{K}$ *and* $\bar{U}_{h\tau} = (\bar{U}_h^1, \bar{U}_h^2, \cdots, \bar{U}_h^N) \in \mathcal{K}^{h\tau}$ *be the solutions of the problem* (P) *and the problem* $(P_{h\tau})$, *respectively. Then there exist two positive numbers* $\bar{h}$ *and* $\bar{\tau}$*such that for all* h, τ *with* $0 < h \leq \bar{h}, 0 < \tau \leq \bar{\tau}$,

$$\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \|\bar{u} - \bar{U}_h^i\|^2 dt \leq C(h^2 + \tau).$$

7. An application to the exactly null controllability of the heat equation

Consider the internally controlled heat equation (1.1). It is well known ([19, 29]) that for each initial data $y_0 \in L^2(\Omega)$, there exists a control $\widetilde{u}(\cdot) \in L^\infty(0,T;L^2(\Omega))$ with the estimate

$$\|\widetilde{u}\|_{L^\infty(0,T;L^2(\Omega))} \leq L\|y_0\|$$

such that the corresponding solution $\widetilde{y}(\cdot)$ to the equation (1.1) with $u = \widetilde{u}$ reaches zero in the state space at the time T, namely, $\widetilde{y}(T) = 0$. Here, L stands for a positive constant depending only on the domain Ω, the subdomain ω, the ending time T and the operator $-\Delta$, which can be estimated in many cases. Such a property is called the exactly null controllability for the heat equation, which has been extensively investigated.

It is significant to give a numerical approach for such a control $\widetilde{u}$. For this purpose, we fix the initial data y_0 in the space $H^2(\Omega)\cap H_0^1(\Omega)$. Without loss of generality, we can assume that $y_0 \neq 0$. Write

$$\mathcal{K}(L)=\{u(\cdot)\in L^2(0,T;L^2(\Omega))\ ;\ \|u(t)\|\leq L\|y_0\|\ \text{ a.e. } t\in[0,T]\}.$$

Now, we set up the following optimal control problem:

$$(\mathcal{P}) \qquad \min\left\{\frac{1}{2}\int_0^T\int_\Omega y^2dxdt+\frac{1}{2}\int_0^T\int_\Omega u^2dxdt\right\}$$

over all pairs (y,u) satisfing the equation (1.1) and the constraints $u\in\mathcal{K}(L)$ and $y(T)=0$.

Since there is a control $\widetilde{u}$ in the set $\mathcal{K}_L$ such that $\widetilde{y}(T)=0$, one can easily prove, by making use of the very similar argument as that used in the proof of Theorem 2.1, that the problem $(\mathcal{P})$ has a unique optimal pair $(\bar{y},\bar{u})$. In general, the qualified Pontryagin maximum principle dose not hold for the problem $(\mathcal{P})$ since the ending point state constraint set is a single point and controls enter the system internally. Thus, we can not use the previous results to approach the control $\bar{u}$ numerically. In what follows, we shall approximate the problem $(\mathcal{P})$ by another optimal control problem, whose solution can be numerically approximated. We consider, for each natural number m, the following optimal control problem:

$$(\mathcal{P}^m) \qquad \min\left\{\frac{1}{2}\int_0^T\int_\Omega y^2dxdt+\frac{1}{2}\int_0^T\int_\Omega u^2dxdt\right\}$$

over all pairs (y,u) solving the equation (1.1) and satisfying

$$u\in\mathcal{K}(L) \quad \text{and} \quad y(T)\in K_m,$$

where

$$K_m=\{w\in L^2(\Omega);\|w\|\leq\frac{1}{m}\}.$$

By Theorem 2.1, the optimal control problem $(\mathcal{P}^m)$ has a unique optimal pair (y^m,u^m).

Theorem 7.1. *Let (y^m,u^m) and $(\bar{y},\bar{u})$ be the optimal pairs for the problem $(\mathcal{P}^m)$ and the problem $(\mathcal{P})$, respectively. Then we have*

$$u^m\to\bar{u}\ \textit{ strongly in } L^2(0,T;L^2(\Omega))\ \textit{ as } m\to\infty.$$

Proof. Take arbitrarily a subsequence of the sequence of pairs $\{(y^m,u^m)\}_{m=1}^\infty$, denoted in the same way. Since the sequence $\{u^m\}_{m=1}^\infty$ of controls is bounded in the space $L^\infty(0,T;L^2(\Omega))$, there exists a subsequence of $\{u^m\}_{m=1}^\infty$, still denoted in the same way, such that

$$u^m\rightharpoonup\widetilde{u}\ \text{ weakly star in } L^\infty(0,T;L^2(\Omega)) \text{ as } m\to\infty, \tag{7.1}$$

from which, it follows that

$$u^m\rightharpoonup\widetilde{u}\ \text{ weakly in } L^2(0,T;L^2(\Omega)) \text{ as } m\to\infty \tag{7.2}$$

and

$$\|\widetilde{u}\|_{L^\infty(0,T;L^2(\Omega))}\leq\liminf_{m\to\infty}\|u^m\|_{L^\infty(0,T;L^2(\Omega))}.$$

Since $u^m\in\mathcal{K}(L)$ for each natural number m, the later implies

$$\widetilde{u}\in\mathcal{K}(L). \tag{7.3}$$

Write $\widetilde{y}$ for the solution of the equation (1.1) with $u = \widetilde{u}$. Because the pair (y^m, u^m) solves the equation (1.1), we get from (7.2) that there is a subsequence of the sequence $\{y^m\}_{m=1}^{\infty}$, denoted in the same way, such that

$$y^m \to \widetilde{y} \text{ strongly in } C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)) \text{ as } m \to \infty, \tag{7.4}$$

from which, it follows, in particular, that for $m \to \infty$,

$$y^m(T) \to \widetilde{y}(T) \text{ strongly in } L^2(\Omega).$$

Since $y^m(T) \in K_m$ for each natural number m, the later gives

$$\widetilde{y}(T) = 0. \tag{7.5}$$

Set $J(y,u) = \frac{1}{2}\int_Q (y^2 + u^2) dxdt$ for $(y,u) \in L^2(Q) \times L^2(Q)$. Then, by (7.2) and (7.4), we have

$$J(\widetilde{y}, \widetilde{u}) \leq \liminf_{m \to \infty} J(y^m, u^m). \tag{7.6}$$

Because the pair $(\widetilde{y}, \widetilde{u})$ solves the equation (1.1) and properties (7.3) and (7.5) hold, we get, by the optimality of the pair $(\bar{y}, \bar{u})$ for the problem $(\mathcal{P})$,

$$J(\widetilde{y}, \widetilde{u}) \geq J(\bar{y}, \bar{u}). \tag{7.7}$$

On the other hand, since $\bar{y}(T) = 0$, we have $\bar{y}(T) \in K_m$ for each natural number m. Then, by making use of the optimality of the pair (y^m, u^m) to the problem $(\mathcal{P}^m)$, we see that for each natural number m,

$$J(y^m, u^m) \leq J(\bar{y}, \bar{u}). \tag{7.8}$$

Now, it follows at once from (7.6)-(7.8) that

$$J(\widetilde{y}, \widetilde{u}) \leq \liminf_{m \to \infty} J(y^m, u^m) \leq \limsup_{m \to \infty} J(y^m, u^m) \leq J(\bar{y}, \bar{u}) \leq J(\widetilde{y}, \widetilde{u}),$$

from which, it follows that

$$\lim_{m \to \infty} J(y^m, u^m) = J(\widetilde{y}, \widetilde{u}) = J(\bar{y}, \bar{u}). \tag{7.9}$$

Thus, the pair $(\widetilde{y}, \widetilde{u})$ is optimal for the problem $(\mathcal{P})$. However, the problem $(\mathcal{P})$ has a unique optimal pair. Hence, we must have

$$\widetilde{y} = \bar{y} \text{ and } \widetilde{u} = \bar{u}. \tag{7.10}$$

By (7.4), (7.9) and (7.10), we get that

$$\|u^m\|_{L^2(Q)} \to \|\bar{u}\|_{L^2(Q)} \text{ as } m \to \infty,$$

which, together with (7.2) and (7.10), gives that

$$u^m \to \bar{u} \text{ strongly in } L^2(0,T; L^2(\Omega)) \text{ as } m \to \infty.$$

Thus, we have proved that for any subsequence $\{u^{m,k}\}_{k=1}^{\infty}$ of the sequence $\{u^m\}_{m=1}^{\infty}$, there is a subsequence of $\{u^{m,k}\}_{k=1}^{\infty}$ converging to $\bar{u}$ strongly in $L^2(0,T; L^2(\Omega))$, from which, it follows that the sequence $\{u^m\}_{m=1}^{\infty}$ converges to the control $\bar{u}$ strongly in $L^2(0,T; L^2(\Omega))$. This completes the proof.

□

Remark 7.1. *By Theorem 7.1, we see that if the optimal control u^m for the problem $(\mathcal{P}^m)$ can be numerically approximated for each m, then the optimal control $\bar{u}$ for the problem $(\mathcal{P})$ is numerically approached. Moreover, the control $\bar{u}$ not only makes the corresponding solution to the equation (1.1) with $u = \bar{u}$ reaches zero in the state space at the time T but also is optimal in the sense that the cost functional $J(\cdot,\cdot)$ is minimized. On the other hand, by Theorem 6.2, we observe that each aforementioned control u^m can really be approximated numerically. Thus, we have provided a way to approximate the control $\bar{u}$ numerically.*

Now, we shall state our last result in the paper. Consider the following discrete optimal control problem:

$$(\mathcal{P}_{h\tau}^m) \quad \min\left\{\frac{\tau}{2}\sum_{i=1}^{N}\left(\|Y_h^i\|^2+\|U_h^i\|^2\right)\right\},$$

subject to

$$U_{h\tau}=(U_h^1,U_h^2,\cdots,U_h^N)\in\mathcal{K}(L)_{h\tau}$$

and

$$Y_h^N\in K_m,$$

where $Y_{h\tau}=(Y_h^1,Y_h^2,\cdots,Y_h^N)\in(V^h)^N$ is the solution of the following equation:

$$\left\{\begin{array}{l}(\partial_\tau Y_h^i,v_h)+a(Y_h^i,v_h)=(\chi_\omega U^i,v_h),\ \forall v_h\in V^h,\ 1\le i\le N,\\ Y_h^0=P_hy_0,\end{array}\right. . \tag{7.11}$$

Theorem 7.2. *For each natural number m and for any sufficient small numbers h and τ, the problem $(\mathcal{P}_{h\tau}^m)$ has a unique optimal control $\bar{U}_{h\tau}^m=(\bar{U}_h^{m1},\bar{U}_h^{m2},\cdots,\bar{U}_h^{mN})\in\mathcal{K}(L)_{h\tau}$. Moreover, for each natural number m, there exits a positive constant $C(m)$ such that*

$$\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}\|\bar{u}-\bar{U}_h^{mi}\|^2dt\le\|\bar{u}-\bar{u}^m\|_{L^2(Q)}^2+C(m)(h^2+\tau).$$

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