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OPTIMAL $\mathcal{H}_2$ MODEL REDUCTION FOR LARGE SCALE MIMO SYSTEMS VIA TANGENTIAL INTERPOLATION

YUESHENG XU AND TAISHAN ZENG

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Abstract. We consider the optimal $\mathcal{H}_2$ model reduction for large scale multi-input multi-output systems via tangential interpolation. Specifically, we prove that for general multi-input multi-output systems, the tangential interpolation-based optimality conditions and the gramian-based optimality conditions are equivalent. Based on the tangential interpolation, a fast algorithm is proposed for the optimal $\mathcal{H}_2$ model reduction. Numerical examples are presented to demonstrate the approximation accuracy and computational efficiency of the proposed algorithm.

Key Words. Optimal $\mathcal{H}_2$ model reduction, tangential interpolation, multi-input multi-output system.

1. Introduction

Model reduction is to approximate a high-order dynamic system by a low-order one. It is a fundamental tool in reducing the computational complexity of control and numerical simulation of large scale dynamical systems. It has been widely used in many applications, such as the design of very large scale integration chips, the simulation and control of microelectromechanical system devices and weather predictions. For an overview of model reduction, we refer to [1]. See also [5, 23] for more approximation problems related to control.

A commonly used method for model reduction constructs the reduced order system via tangential interpolation [11]. Interpolation-based model reduction methods produce reduced-order systems whose transfer function interpolates the transfer function of the full-order system at selected interpolation points. This class of methods is suitable for the reduction of large scale dynamical systems. However, selection of optimal interpolation points remains a challenging issue.

In this paper, we study selection of interpolation points and tangential interpolation directions that can produce the optimal $\mathcal{H}_2$ reduced-order models. The optimal $\mathcal{H}_2$ model reduction problem has been studied extensively, (see, for instance [9, 10, 12, 14, 18, 21, 22] and the references cited therein). Most researchers used first-order optimality conditions in constructing numerical algorithms. Because of the popularity of interpolation-based model reduction methods, many authors consider the problem of characterizing the optimality conditions via interpolation for the optimal $\mathcal{H}_2$ model reduction. The optimality conditions for single-input single-output (SISO) systems via interpolation were given in [12, 15]. For the multi-input

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multi-output (MIMO) systems with simple poles, the stationary conditions of the cost function were characterized via interpolation in [9]. It was shown in [8] that the stationary points of the cost function for MIMO systems (without the assumption that they have only first-order poles), can be characterized via tangential interpolation. In the literature, there is another kind of first-order optimality conditions for the optimal $\mathcal{H}_2$ model reduction problem: the gramian-based conditions. The Wilson conditions [20] and Hyland-Bernstein conditions [14] are gramian-based conditions, which use gramians of the systems.

It is important to study the connections between the two different kinds of first-order optimality conditions for better understanding existing algorithms and constructing new algorithms. In [12], the Wilson conditions and the Hyland-Bernstein conditions were proved to be equivalent. Also in [12], the interpolation-based necessary conditions for SISO systems with simple poles was proved to be equivalent to the Wilson conditions and the Hyland-Bernstein conditions. For the discrete MIMO dynamical system with simple poles, the equivalence between gramian-based conditions and interpolation-based conditions was established in [6].

Interpolation-based algorithms for the optimal $\mathcal{H}_2$ model reduction were proposed in [2, 3, 4, 12]. Algorithms were proposed in [3, 12] for SISO systems and in [2, 4] for MIMO systems. Since all these algorithms were based on the assumption that the target systems have simple poles, the ill-conditioned behavior can be expected when the target systems has multiple poles or with nearly multiple poles (cf. [8]).

In this paper, for general MIMO systems which are allowed to have multiple poles, we prove that the tangential interpolation-based optimality conditions are equivalent to the gramian-based optimality conditions. The proof of the equivalence between first-order optimality conditions for SISO systems with simple poles in [12] is based on the fact that the set of all proper rational functions with specified simple poles constitute a subspace of $\mathcal{H}_2$. However, this is not the case when the transfer functions have multiple poles. Furthermore, the transfer function of MIMO systems is a rational matrix function, which will also increase the difficulties. In this paper, we accomplish the proof by finding the relationship between the gradients of the cost function and utilizing the partial expansion of the transfer function via Jordan decomposition. Moreover, the equivalence of the first-order optimality conditions leads to the proposed tangential interpolation-based minimization algorithm. The proposed algorithm is based on solving two Sylvester equations, and then constructing the reduced order system by projection. Since the proposed algorithm does not assume that the target system has simple poles, it remains robust when the target system has multiple poles. Unlike many of existing $\mathcal{H}_2$ model reduction methods, the proposed algorithm is numerical efficient. As a result, it is suitable for very large scale dynamical systems.

The paper is organized in six sections. In Section 2, we introduce the $\mathcal{H}_2$ optimal model reduction problem and the first-order optimality conditions. In Section 3, we prove the equivalence between the two first-order optimality conditions. We propose in Section 4 a numerical algorithm based on tangential interpolation. In Section 5, numerical examples are presented to demonstrate the efficiency of the algorithm. Finally, we draw our conclusions.

2. Optimal $\mathcal{H}_2$ model reduction and first-order optimality conditions

In this section, we first describe the optimal $\mathcal{H}_2$ model reduction problem, and then introduce two first-order optimality conditions: gramian-based conditions and tangential interpolation-based conditions.

Given matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$ and $C \in \mathbb{R}^{q\times n}$, the linear time invariant system can be described by

$$\begin{cases} \frac{dx}{dt} = Ax + Bu, \\ y = Cx, \end{cases} \tag{2.1}$$

where $t \geq 0$ is the time variable, $u \in \mathbb{R}^p$ is the *input*, $y \in \mathbb{R}^q$ is the *output* and $x \in \mathbb{R}^n$ is the *state* of the system. Here, n is the *system order*, p and q are the number of system inputs and outputs, respectively. The triple (A, B, C) is called the *system realization*. The *transfer function* of the system is defined by $H(s) := C(sI-A)^{-1}B$, $s \in \mathbb{C}$. The transfer function H is called *stable* if the eigenvalues of A have strictly negative real parts.

If the system order N is too big, it is not computationally efficient to solve various control problems. We seek a reduced order system

$$\begin{cases} \frac{d\widehat{x}}{dt} = \widehat{A}\widehat{x} + \widehat{B}u, \\ \widehat{y} = \widehat{C}\widehat{x}, \end{cases} \tag{2.2}$$

to approximate the full order system, where $\widehat{A} \in \mathbb{R}^{m\times m}, \widehat{B} \in \mathbb{R}^{m\times p}, \widehat{C} \in \mathbb{R}^{q\times m}$. The transfer function of the reduced order system is $\widehat{H}(s) := \widehat{C}(sI - \widehat{A})^{-1}\widehat{B}$. The dimension m satisfies $m \ll n$.

The error of the two transfer functions is defined by

$$E(s) := H(s) - \widehat{H}(s), \quad s \in \mathbb{C}. \tag{2.3}$$

For a stable transfer function H, the square of the $\mathcal{H}_2$ norm is defined as the trace of a matrix integral (cf. [24])

$$\|H\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{trace}\{H(i\omega)^* H(i\omega)\} d\omega,$$

where $i = \sqrt{-1}$ is the imaginary unit and $H(i\omega)^*$ is the conjugate transpose of the matrix $H(i\omega)$. The cost function is defined by

$$\mathcal{J}(\widehat{A}, \widehat{B}, \widehat{C}) := \|E\|_2^2 = \|C(sI - A)^{-1}B - \widehat{C}(sI - \widehat{A})^{-1}\widehat{B}\|_2^2. \tag{2.4}$$

Given a stable system with system realization (A, B, C), the *optimal $\mathcal{H}_2$ model reduction* is to find a small scale system with system realization $(\widehat{A}, \widehat{B}, \widehat{C})$ which minimizes the $\mathcal{H}_2$ norm of the error system

$$\min_{\widehat{A}\in\mathbb{R}^{m\times m}, \widehat{B}\in\mathbb{R}^{m\times p}, \widehat{C}\in\mathbb{R}^{q\times m}} \mathcal{J}(\widehat{A}, \widehat{B}, \widehat{C}). \tag{2.5}$$

To solve this problem, it is beneficial to have an explicit formula for $\mathcal{J}(\widehat{A}, \widehat{B}, \widehat{C})$. Noting that the function E defined in (2.3) can be rewritten as

$$E(s) = C_e(sI - A_e)^{-1}B_e, \quad s \in \mathbb{C},$$

where

$$A_e := \begin{bmatrix} A & \\ & \widehat{A} \end{bmatrix}, B_e := \begin{bmatrix} B \\ \widehat{B} \end{bmatrix}, C_e = \begin{bmatrix} C & -\widehat{C} \end{bmatrix},$$

(A_e, B_e, C_e) is the system realization of the error system, whose transfer function is E. The controllability gramian P_e and observability gramian Q_e of the error system can be obtained by solving the following Lyapunov equations,

$$A_e P_e + P_e A_e^T + B_e B_e^T = 0, \tag{2.6}$$

$$A_e^T Q_e + Q_e A_e + C_e^T C_e = 0. \tag{2.7}$$

Partition the controllability gramian P_e and observability gramian Q_e, respectively, into

$$P_e := \begin{bmatrix} P & X \\ X^T & \widehat{P} \end{bmatrix}, \quad Q_e := \begin{bmatrix} Q & Y \\ Y^T & \widehat{Q} \end{bmatrix}.$$

The Lyapunov equations (2.6) and (2.7) can be written as

$$PA^T + AP + BB^T = 0, \tag{2.8}$$

$$A^TQ + QA + C^TC = 0, \tag{2.9}$$

$$\widehat{P}\widehat{A}^T + \widehat{A}\widehat{P} + \widehat{B}\widehat{B}^T = 0, \tag{2.10}$$

$$\widehat{A}^T\widehat{Q} + \widehat{Q}\widehat{A} + \widehat{C}^T\widehat{C} = 0, \tag{2.11}$$

$$A^TY + Y\widehat{A} - C^T\widehat{C} = 0, \tag{2.12}$$

$$X^TA^T + \widehat{A}X^T + \widehat{B}B^T = 0. \tag{2.13}$$

It is well-known (cf. [24]) that the cost function $\mathcal{J}$ can be expressed in terms of the observability gramian Q_e,

$$\mathcal{J}(\widehat{A}, \widehat{B}, \widehat{C}) = \operatorname{trace}\left(B_e^TQ_eB_e\right) = \operatorname{trace}\left(B^TQB + 2B^TY\widehat{B} + \widehat{B}^T\widehat{Q}\widehat{B}\right), \tag{2.14}$$

or equivalently in terms of controllability gramian P_e,

$$\mathcal{J}(\widehat{A}, \widehat{B}, \widehat{C}) = \operatorname{trace}\left(C_eP_eC_e^T\right) = \operatorname{trace}\left(CPC^T - 2CX\widehat{C}^T + \widehat{C}\widehat{P}\widehat{C}^T\right). \tag{2.15}$$

We describe the first-order optimality conditions in terms of the gradients of $\mathcal{J}$ via $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$. The gradient of a real-valued function $f(Z)$ of a real matrix variable $Z \in \mathbb{R}^{n_1\times n_2}$ is the real matrix $\nabla_Z f \in \mathbb{R}^{n_1\times n_2}$ defined by

$$[\nabla_Z f]_{i,j} = \frac{\partial}{\partial Z_{i,j}} f(Z), \quad i = 1, 2, \ldots, n_1, \quad j = 1, 2, \ldots, n_2.$$

Starting from the characterizations (2.6)–(2.15), it is proved in [9, 20] that the gradients $\nabla_{\widehat{A}}\mathcal{J}$, $\nabla_{\widehat{B}}\mathcal{J}$ and $\nabla_{\widehat{C}}\mathcal{J}$ of the cost function $\mathcal{J}$, are given, respectively, by

$$\nabla_{\widehat{A}}\mathcal{J} = 2(\widehat{Q}\widehat{P} + Y^TX), \quad \nabla_{\widehat{B}}\mathcal{J} = 2(\widehat{Q}\widehat{B} + Y^TB), \quad \nabla_{\widehat{C}}\mathcal{J} = 2(\widehat{C}\widehat{P} - CX). \tag{2.16}$$

The first-order optimality conditions for the optimal $\mathcal{H}_2$ model reduction are then given by

$$\nabla_{\widehat{A}}\mathcal{J} = 0, \quad \nabla_{\widehat{B}}\mathcal{J} = 0 \quad \text{and} \quad \nabla_{\widehat{C}}\mathcal{J} = 0, \tag{2.17}$$

that is,

$$\widehat{Q}\widehat{P} + Y^TX = 0, \quad \widehat{Q}\widehat{B} + Y^TB = 0 \quad \text{and} \quad \widehat{C}\widehat{P} - CX = 0.$$

The above conditions are called *the Wilson conditions* [20]. The Wilson conditions are gramian-based conditions since they are related to gramians $\widehat{P}$, $\widehat{Q}$, X and Y. The Hyland-Bernstein conditions [14] are another gramian-based first-order optimality conditions, which were shown to be equivalent to the Wilson conditions (cf. [12]).

It was showed in [8] that the stationary point of the cost function $\mathcal{J}$ can also be characterized by the tangential interpolation conditions for MIMO systems. In the remaining part of this section, we will give a brief introduction of the tangential interpolation-based optimality conditions proposed in [8].

We introduce the partial expansion of the transfer function via the Jordan canonical form. For each i, let S_i and T_i^* be the matrices whose columns span the (complex) left and right eigenspaces of the real matrix $\widehat{A}$ corresponding to the eigenvalue $\widehat{\lambda}_i$, respectively. The Jordan block $\widehat{A}_i$ associated with the eigenvalue $\widehat{\lambda}_i$ is a $k_i \times k_i$

matrix whose entries are equal to $\widehat{\lambda}_i$ on the diagonal, equal to -1 on the superdiagonal, and equal to 0 elsewhere. Then we have

$$\widehat{A}S_i = S_i\widehat{A}_i, \quad \widehat{C}_i := \widehat{C}S_i, \quad T_i^*\widehat{A} = \widehat{A}_iT_i^*, \quad \widehat{B}_i^* := T_i^*\widehat{B}, \quad T_i^*S_i = I_{k_i}, \tag{2.18}$$

where $\widehat{A}_i \in \mathbb{C}^{k_i\times k_i}$, $\widehat{B}_i^* \in \mathbb{C}^{k_i\times p}$, $\widehat{C}_i \in \mathbb{C}^{q\times k_i}$. Accordingly, the transfer function $\widehat{H}$ has the following representation

$$\widehat{H}(s) = \sum_{i=1}^{\ell} \widehat{H}_i(s), \quad \widehat{H}_i(s) := \widehat{C}_i(sI - \widehat{A}_i)^{-1}\widehat{B}_i^*. \tag{2.19}$$

When there are more than one Jordan blocks associated with the same eigenvalue $\widehat{\lambda}_i$, we associate matrices S_i, T_i with each individual Jordan block A_i. For more discussions about the partial expansion (2.19), see [8].

Next, we describe the characterization of the stationary points via tangential interpolation. To this end, we define two vector functions

$$\begin{aligned}\psi_{\widehat{\lambda}_i}(s) &:= \begin{bmatrix}(s+\widehat{\lambda}_i)^{k_i-1} & \dots & (s+\widehat{\lambda}_i) & 1\end{bmatrix},\\ \phi_{\widehat{\lambda}_i}(s) &:= \begin{bmatrix}1 & (s+\widehat{\lambda}_i) & \dots & (s+\widehat{\lambda}_i)^{k_i-1}\end{bmatrix}^T, \qquad i = 1, 2, \dots, \ell.\end{aligned}$$

If the Taylor expansion of a rational matrix function R about the point λ can be written as

$$R(s) = \sum_{i=k}^{\infty} R_i(s-\lambda)^i,$$

then we write $R(s) = \mathcal{O}(s-\lambda)^k$, where $k \in \mathbb{Z}$ and R_i are constant matrices. Exploiting the Jordan canonical form, we obtain that

$$-\frac{1}{2}(\nabla_{\widehat{B}}\mathcal{J})^T S_i\phi_{\widehat{\lambda}_i}(s) = [H^T(s) - \widehat{H}^T(s)]\widehat{C}_i\phi_{\widehat{\lambda}_i}(s) + \mathcal{O}(s+\widehat{\lambda}_i)^{k_i}, \tag{2.20}$$

$$-\frac{1}{2}\psi_{\widehat{\lambda}_i}(s)T_i^*(\nabla_{\widehat{C}}\mathcal{J})^T = \psi_{\widehat{\lambda}_i}(s)\widehat{B}_i^*[H^T(s) - \widehat{H}^T(s)] + \mathcal{O}(s+\widehat{\lambda}_i)^{k_i}. \tag{2.21}$$

Furthermore, if the cost function $\mathcal{J}$ satisfies the Wilson conditions, $\nabla_{\widehat{A}}\mathcal{J} = 0$, $\nabla_{\widehat{B}}\mathcal{J} = 0$ and $\nabla_{\widehat{C}}\mathcal{J} = 0$, then the following tangential interpolation-based optimality conditions (cf. [8]) are satisfied for $i = 1, 2, \dots, \ell$:

$$[H^T(s) - \widehat{H}^T(s)]\widehat{c}_i(s) = \mathcal{O}(s+\widehat{\lambda}_i)^{k_i}, \tag{2.22}$$

$$\widehat{b}_i(s)^*[H^T(s) - \widehat{H}^T(s)] = \mathcal{O}(s+\widehat{\lambda}_i)^{k_i}, \tag{2.23}$$

$$\widehat{b}_i(s)^*[H^T(s) - \widehat{H}^T(s)]\widehat{c}_i(s) = \mathcal{O}(s+\widehat{\lambda}_i)^{2k_i}, \tag{2.24}$$

where $\widehat{b}_i^*(s) := \psi_{\widehat{\lambda}_i}(s)\widehat{B}_i^*$ and $\widehat{c}_i(s) := \widehat{C}_i\phi_{\widehat{\lambda}_i}(s)$.

The equivalence between the tangential interpolation-based optimality conditions and the Wilson conditions will be proved in the next section.

3. Equivalence between the first-order optimality conditions

In this section, we will show that the tangential interpolation-based optimality conditions are equivalent to the gramian-based conditions.

The following theorem shows that if the the transfer functions H and $\widehat{H}$ satisfy the optimal tangential interpolation conditions (2.22) and (2.23), then the cost function $\mathcal{J}$ satisfies $\nabla_{\widehat{B}}\mathcal{J} = 0$ and $\nabla_{\widehat{C}}\mathcal{J} = 0$.

Theorem 3.1. *Suppose that $-\widehat{\lambda}_i$, $i = 1, 2, \ldots, \ell$, are not a pole of H. If for all $\widehat{\lambda}_i$, $i = 1, \ldots, \ell$, the transfer functions H and $\widehat{H}$ satisfy the conditions* (2.22) *and* (2.23)*, then the cost function $\mathcal{J}$ satisfies*

$$\nabla_{\widehat{B}}\mathcal{J} = 0 \quad \textit{and} \quad \nabla_{\widehat{C}}\mathcal{J} = 0. \tag{3.1}$$

Proof. First, we show that $\nabla_{\widehat{B}}\mathcal{J} = 0$. From equations (2.20) and (2.22), we have that

$$-\frac{1}{2}(\nabla_{\widehat{B}}\mathcal{J})^T S_i \phi_{\widehat{\lambda}_i}(s) = \mathcal{O}(s + \widehat{\lambda}_i)^{k_i}, \quad i = 1, 2, \ldots, \ell. \tag{3.2}$$

Differentiating both sides of equation (3.2), we obtain that

$$(\nabla_{\widehat{B}}\mathcal{J})^T S_i \phi^{(\alpha)}_{\widehat{\lambda}_i}(-\widehat{\lambda}_i) = 0, \qquad i = 1, 2, \ldots, \ell, \quad \alpha = 0, 1, \ldots, k_i - 1.$$

¿From the definition of $\phi_{\widehat{\lambda}_i}$, it can be shown that $\phi^{(\alpha)}_{\widehat{\lambda}_i}(-\widehat{\lambda}_i)$ is nonzero only at its $(\alpha+1)$-th component, that is,

$$\phi^{(\alpha)}_{\widehat{\lambda}_i}(-\widehat{\lambda}_i) = [0, \ldots, \alpha!, \ldots, 0]^T, \qquad i = 1, 2, \ldots, \ell, \quad \alpha = 0, 1, \ldots, k_i - 1.$$

It is easy to show that

$$(\nabla_{\widehat{B}}\mathcal{J})^T S_i = 0, \quad i = 1, 2, \ldots, \ell.$$

Denoting $S := [S_1, S_2, \ldots, S_\ell]$, we conclude that $(\nabla_{\widehat{B}}\mathcal{J})^T S = 0$. From the definition of S_i, we know that S_i are linear independent. This implies that S is a nonsingular matrix. Then we conclude that the first equation of (3.1) holds. Likewise, we can show the second equation of (3.1). □

The following lemma shows the relationship among $\nabla_{\widehat{A}}\mathcal{J}$, $\nabla_{\widehat{B}}\mathcal{J}$ and $\nabla_{\widehat{C}}\mathcal{J}$. This is important for derivation of the equivalence of the first-order optimality conditions.

Lemma 3.2. *For all $i, j = 1, 2, \ldots, \ell$, there holds*

$$-\widehat{A}_i T_i^* (\nabla_{\widehat{A}}\mathcal{J})^T S_j + T_i^* (\nabla_{\widehat{A}}\mathcal{J})^T S_j \widehat{A}_j = T_i^* \widehat{B} (\nabla_{\widehat{B}}\mathcal{J})^T S_j - T_i^* (\nabla_{\widehat{C}}\mathcal{J})^T \widehat{C} S_j. \tag{3.3}$$

Proof. The proof of this lemma is based on exploiting the Jordan canonical form and expressing for $\nabla_{\widehat{A}}\mathcal{J}$, $\nabla_{\widehat{B}}\mathcal{J}$ and $\nabla_{\widehat{B}}\mathcal{J}$.

Recalling that $\nabla_{\widehat{B}}\mathcal{J} = 2(\widehat{Q}\widehat{B} + Y^T B)$, we find that

$$\frac{1}{2}\widehat{B}(\nabla_{\widehat{B}}\mathcal{J})^T = \widehat{B}\widehat{B}^T\widehat{Q}^T + \widehat{B}B^T Y.$$

Combining this formula with (2.12) and (2.13) leads to

$$\frac{1}{2}\widehat{B}(\nabla_{\widehat{B}}\mathcal{J})^T = -\widehat{P}\widehat{A}^T\widehat{Q}^T - \widehat{A}\widehat{P}\widehat{Q}^T - X^T A^T Y - \widehat{A}X^T Y. \tag{3.4}$$

Likewise, we obtain that

$$\frac{1}{2}(\nabla_{\widehat{C}}\mathcal{J})^T\widehat{C} = -\widehat{P}\widehat{A}^T\widehat{Q}^T - \widehat{P}\widehat{Q}^T\widehat{A} - X^T A^T Y - X^T Y\widehat{A}. \tag{3.5}$$

Subtracting (3.4) from (3.5) yields that

$$\frac{1}{2}\widehat{B}(\nabla_{\widehat{B}}\mathcal{J})^T - \frac{1}{2}(\nabla_{\widehat{C}}\mathcal{J})^T\widehat{C} = -\widehat{A}\widehat{P}\widehat{Q}^T - \widehat{A}X^T Y + \widehat{P}\widehat{Q}\widehat{A} + X^T Y\widehat{A}. \tag{3.6}$$

From the first and third formulas of (2.18), we have that

$$\begin{aligned}
&\frac{1}{2}T_i^*\widehat{B}(\nabla_{\widehat{B}}\mathcal{J})^T S_j - \frac{1}{2}T_i^*(\nabla_{\widehat{C}}\mathcal{J})^T\widehat{C}S_j \\
&= -T_i^* A\widehat{P}\widehat{Q}S_j - T_i^*\widehat{A}X^TYS_j + T_i^*\widehat{P}\widehat{Q}\widehat{A}S_j + T_i^*X^TY\widehat{A}S_j \\
&= -\widehat{A}_iT_i^*\widehat{P}\widehat{Q}S_j - \widehat{A}_iT_i^*X^TYS_j + T_i^*\widehat{P}\widehat{Q}S_j\widehat{A}_j + T_i^*X^TYS_j\widehat{A}_j \\
&= -\frac{1}{2}\widehat{A}_iT_i^*(\nabla_{\widehat{A}}\mathcal{J})^TS_j + \frac{1}{2}T_i^*(\nabla_{\widehat{A}}\mathcal{J})^TS_j\widehat{A}_j.
\end{aligned}$$

The last step uses the fact that $\nabla_{\widehat{A}}\mathcal{J} = 2(\widehat{Q}\widehat{P} + Y^TX)$. Finally, we conclude (3.3) for all $i, j = 1, 2, \ldots, \ell$. □

The next lemma useful for deriving of the main theorem was established in [8].

Lemma 3.3. *Let F be a $k \times k$ matrix whose entries are equal to $\widehat{\lambda}_i$ on the diagonal, equal to -1 on the superdiagonal, and equal to 0 elsewhere. If $-\lambda$ is not an eigenvalue of A, then the following statements hold:*

(1) *The solution of the matrix equation*

$$A^TY + YF - C^TL = 0,$$

with $L := \begin{bmatrix} l_0 & l_1 & \ldots & l_{k-1}\end{bmatrix}$, is given by

$$Y = \begin{bmatrix}(A^T+\lambda I)^{-1}C^T, & \ldots, & (A^T+\lambda I)^{-k}C^T\end{bmatrix}\begin{bmatrix} l_0 & l_1 & \ldots & l_{k-1} \\ & l_0 & \ddots & \vdots \\ & & \ddots & l_1 \\ & & & l_0 \end{bmatrix},$$

(2) *The solution of the matrix equation*

$$X^*A^T + FX^* - R^*B^T = 0$$

with $R := \begin{bmatrix} r_{k-1} & r_{k-2} & \ldots & r_0\end{bmatrix}$, is given by

$$X^* = \begin{bmatrix} r_0^* & r_1^* & \ldots & r_{k-1}^* \\ & r_0^* & \ddots & \vdots \\ & & \ddots & r_1^* \\ & & & r_0^* \end{bmatrix}\begin{bmatrix} B^T(A^T+\lambda I)^{-k} \\ \vdots \\ B^T(A^T+\lambda I)^{-2} \\ B^T(A^T+\lambda I)^{-1}\end{bmatrix}.$$

The following is a main result of the paper. It points out that the equivalence of the two optimality conditions.

Theorem 3.4. *The Wilson conditions* (2.17) *are equivalent to the tangential interpolation-based optimality conditions* (2.22)–(2.24).

Proof. Since [8] has proved that the Wilson conditions imply the tangential interpolation-based optimality conditions, it remains to show the reverse. By Theorem 3.1, it suffices to prove the first equation of (2.17). To this end, we show that

$$T^*(\nabla_{\widehat{A}}\mathcal{J})^TS = 0, \tag{3.7}$$

where $T = [T_1, T_2, \ldots, T_\ell]$ and $S = [S_1, S_2, \ldots, S_\ell]$.

For notation convenience, for $i, j = 1, 2, \ldots, \ell$, we let $J_{ij} := T_i^*(\nabla_{\widehat{A}}\mathcal{J})^TS_j$. We first show that $J_{ii} = 0$, $i = 1, 2, \ldots, \ell$. Conditions (2.22)–(2.24) can be expressed in terms of the Taylor expansion of the error function. Let

$$E(s) := \sum_{j=0}^{\infty} E_i(s + \widehat{\lambda}_i)^j$$

be the Taylor expansion of the functions E about the point $-\widehat{\lambda}_i$. For $k \geq 1$, using the coefficient of the Taylor expansion of the error function, we define the following Toeplitz matrix

$$\widetilde{\mathbf{E}}_k^T := \begin{bmatrix} E_k^T & \dots & E_{2k-1}^T \\ \vdots & \ddots & \vdots \\ E_1^T & \dots & E_k^T \end{bmatrix}.$$

Recalling that E_{f+g-1} is the coefficient of the Taylor Expansion of error function E, we have that

$$E_{f+g-1}^T = \widehat{B}^T(\widehat{A}^T + \widehat{\lambda}_i I)^{-f}(\widehat{A}^T + \widehat{\lambda}_i I)^{-g}\widehat{C}^T - B^T(A^T + \widehat{\lambda}_i I)^{-f}(A^T + \widehat{\lambda}_i I)^{-g}C^T.$$

Define the matrices

$$\Psi_{k_i,\widehat{\lambda}_i} := \begin{bmatrix} B^T(A^T + \widehat{\lambda}_i I)^{-k_i} \\ B^T(A^T + \widehat{\lambda}_i I)^{-k_i+1} \\ \vdots \\ B^T(A^T + \widehat{\lambda}_i I)^{-1} \end{bmatrix}, \quad \widehat{\Psi}_{k_i,\widehat{\lambda}_i} := \begin{bmatrix} \widehat{B}^T(\widehat{A}^T + \widehat{\lambda}_i I)^{-k_i} \\ \widehat{B}^T(\widehat{A}^T + \widehat{\lambda}_i I)^{-k_i+1} \\ \vdots \\ \widehat{B}^T(\widehat{A}^T + \widehat{\lambda}_i I)^{-1} \end{bmatrix},$$

$$\Phi_{k_i,\widehat{\lambda}_i} := \begin{bmatrix} (A^T + \widehat{\lambda}_i I)^{-1}C^T, & \dots, & (A^T + \widehat{\lambda}_i I)^{-k_i}C^T \end{bmatrix},$$

$$\widehat{\Phi}_{k_i,\widehat{\lambda}_i} := \begin{bmatrix} (\widehat{A}^T + \widehat{\lambda}_i I)^{-1}\widehat{C}^T, & \dots, & (\widehat{A}^T + \widehat{\lambda}_i I)^{-k_i}\widehat{C}^T \end{bmatrix}.$$

Then we have the following identity

$$\widetilde{\mathbf{E}}_{k_i}^T = \widehat{\Psi}_{k_i,\widehat{\lambda}_i}\widehat{\Phi}_{k_i,\widehat{\lambda}_i} - \Psi_{k_i,\widehat{\lambda}_i}\Phi_{k_i,\widehat{\lambda}_i}. \tag{3.8}$$

On the other hand, let

$$\widehat{c}_i(s) := \sum_{j=0}^{k_i} l_j(s + \widehat{\lambda}_i)^j \quad \text{and} \quad \widehat{b}_i^*(s) := \sum_{j=0}^{k_i} r_j^*(s + \widehat{\lambda}_i)^j$$

be the Taylor expansions of the functions $\widehat{c}_i$ and $\widehat{b}_i^*$ about the point $-\widehat{\lambda}_i$. For $k \geq 1$, we define the block upper triangular matrices

$$\mathbf{E}_k^T := \begin{bmatrix} E_0^T & \dots & E_{k-1}^T \\ & \ddots & \vdots \\ & & E_0^T \end{bmatrix}, \quad \mathbf{L}_k := \begin{bmatrix} l_0 & \dots & l_{k-1} \\ & \ddots & \vdots \\ & & l_0 \end{bmatrix}, \quad \mathbf{R}_k^* := \begin{bmatrix} r_0^* & \dots & r_{k-1}^* \\ & \ddots & \vdots \\ & & r_0^* \end{bmatrix}.$$

Then conditions (2.22), (2.23), (2.24) are respectively equivalent to

$$\mathbf{E}_{k_i}^T\mathbf{L}_{k_i} = 0, \tag{3.9}$$

$$\mathbf{R}_{k_i}^*\mathbf{E}_{k_i}^T = 0, \tag{3.10}$$

$$\mathbf{R}_{2k_i}^*\mathbf{E}_{2k_i}^T\mathbf{L}_{2k_i} = 0. \tag{3.11}$$

Define $Y_i := YS_i$, $\widehat{Q}_i := -\widehat{Q}S_i$, $X_i^* := -T_i^*X^T$, $\widehat{P}_i^* := -T_i^*\widehat{P}$. Using equation (2.16), we find that $J_{ii} = \widehat{P}_i^*\widehat{Q}_i - X_i^*Y_i$. It is easy to see from Lemma 3.3 that

$$J_{ii} = \mathbf{R}_{k_i}^*\widehat{\Psi}_{k_i,\widehat{\lambda}_i}\widehat{\Phi}_{k_i,\widehat{\lambda}_i}\mathbf{L}_{k_i} - \mathbf{R}_{k_i}^*\Psi_{k_i,\widehat{\lambda}_i}\Phi_{k_i,\widehat{\lambda}_i}\mathbf{L}_{k_i}.$$

Combining this formula with (3.8) leads to

$$J_{ii} = \mathbf{R}_{k_i}^*\widetilde{\mathbf{E}}_{k_i}^T\mathbf{L}_{k_i}. \tag{3.12}$$

Since the identities (3.9) and (3.10) hold, we observe that

$$\mathbf{R}_{2k_i}^*\mathbf{E}_{2k_i}^T\mathbf{L}_{2k_i} = \begin{bmatrix} 0 & \mathbf{R}_{k_i}^*\widetilde{\mathbf{E}}_{k_i}^T\mathbf{L}_{k_i} \\ 0 & 0 \end{bmatrix}.$$

From the identity (3.12), we have that

$$\mathbf{R}^*_{2k_i}\mathbf{E}^T_{2k_i}\mathbf{L}_{2k_i} = \begin{bmatrix} 0 & J_{ii} \\ 0 & 0 \end{bmatrix}. \tag{3.13}$$

Using the identity (3.11) and (3.13), we conclue that $J_{ii} = 0$, $i = 1, 2, \ldots, \ell$. Similarly, we can prove that $J_{ij} = 0$ if $\widehat{\lambda}_i = \widehat{\lambda}_j$, for some $i \neq j$.

Next, we show that for the case $\widehat{\lambda}_i \neq \widehat{\lambda}_j$, we also have that $J_{ij} = 0$, where $i \neq j$. By Lemma 3.2, for any $i, j = 1, 2, \ldots, \ell$, the following identity holds

$$-\widehat{A}_i J_{ij} + J_{ij}\widehat{A}_j = T_i^* \widehat{B}(\nabla_{\widehat{B}}\mathcal{J})^T S_j - T_i^*(\nabla_{\widehat{C}}\mathcal{J})^T \widehat{C} S_j. \tag{3.14}$$

Using the second and third formulas of (2.17), we observe that

$$-\widehat{A}_i J_{ij} + J_{ij}\widehat{A}_j = 0. \tag{3.15}$$

Since $\widehat{\lambda}_i \neq \widehat{\lambda}_j$, and by Lemma 3.3, we see that $J_{ij} = 0$. This ensures that (3.7) holds. Since T and S are nonsingular, we have that $\nabla_{\widehat{A}}\mathcal{J} = 0$. □

Because of the equivalence of the gramian-based Wilson conditions and the Hyland-Bernstein conditions (cf. [12]), according to the last theorem, interpolation-based conditions and the gramian-based conditions are equivalent for general multi-input multi-output (MIMO) systems which may have multiple poles.

In the special case of first order poles, we have the following result.

Corollary 3.5. *Suppose that for each $i = 1, 2, \ldots, \ell$, the size k_i of Jordan block $\widehat{A}_i$ is equal to 1. Then the gradients of the cost function $\mathcal{J}$ satisfy* (2.17) *if and only if for all $\widehat{\lambda}_i$, $i = 1, 2, \ldots, n$, the transfer functions H and $\widehat{H}$ satisfy the following tangential interpolation-based optimality conditions:*

$$\begin{aligned} [H^T(-\widehat{\lambda}_i) - \widehat{H}^T(-\widehat{\lambda}_i)]\widehat{c}_i &= 0, \\ \widehat{b}_i^*[H^T(-\widehat{\lambda}_i) - \widehat{H}^T(-\widehat{\lambda}_i)] &= 0, \\ \widehat{b}_i^* \frac{d}{ds}\left[H^T(s) - \widehat{H}^T(s)\right]\Big|_{s=-\widehat{\lambda}_i} \widehat{c}_i &= 0. \end{aligned}$$

The special case consider in Corollary 3.5 with $p = q = 1$ and the eigenvalues of $\widehat{A}$ being distinct corresponds to the equivalence proved in [12].

The equivalence of the tangential interpolation-based and the gramian-based first-order optimality conditions leads to an iterated tangential interpolation algorithm, which we will discuss in the next section.

4. Iterated tangential interpolation

In this section, we describe an effective numerical algorithm that generates a reduced-order model satisfying the tangential interpolation-based optimality conditions (2.22) – (2.24). This algorithm is based on solving two Sylvester equations, and then constructing the reduced order system by projection.

We now describe the mathematical idea that leads to the algorithm. We recall that in the optimal tangential interpolation conditions (2.22)–(2.24), the transfer function $\widehat{H}$ interpolates H at the mirror images of the poles of $\widehat{H}$ with the tangential interpolation directions determined by the columns of $\widehat{C}$ and $\widehat{B}^T$ (cf. [8]). Constructing the optimal tangential interpolation $\widehat{H}$ requires knowing the poles of $\widehat{H}$ and its tangential interpolation directions which also depend on $\widehat{H}$. This is a nonlinear problem. As a result, it requires an iterative scheme to solve it. In the iterative scheme that we describe below, each iterative step requires solving an interpolation problem. This interpolation problem can be described in terms of

the following Sylvester equations. For given matrices $\Sigma \in \mathbb{R}^{m\times m}$, $L \in \mathbb{R}^{q\times m}$ and $R \in \mathbb{R}^{m\times p}$, let $U \in \mathbb{R}^{n\times m}$ and $V \in \mathbb{R}^{n\times m}$ be the solutions of the equations

$$\begin{aligned} AV + V\Sigma^T + BR^T &= 0, \\ A^TU + U\Sigma + C^TL &= 0. \end{aligned}$$

It is shown in [11] that construction of a reduced-order system by

$$(A_r, B_r, C_r) = ((U^TV)^{-1}U^TAV, (U^TV)^{-1}U^TB, CV),$$

assuming that U^TV is non-singular, is equivalent to a tangential interpolation of the transfer functions H and H_r at $-\sigma_i$, where σ_i are the eigenvalues of Σ, with the left and right tangential interpolation directions determined by the columns of L and R, respectively. The left and right interpolation condition pairs Σ, L and Σ, R uniquely determine the reduced-order system (A_r, B_r, C_r). This observation will be used to construct the tangential interpolation at each iterative step.

We construct a iterative interpolation algorithm to solve the optimal $\mathcal{H}_2$ problem. Suppose that for $k = 0, 1, \ldots$, the system realization (A_k, B_k, C_k) of the reduced-order system at the k-th step has been obtained. Next, we describe the construction of the reduced-order system $(A_{k+1}, B_{k+1}, C_{k+1})$. Compute X_k and Y_k by solving the Sylvester equations

$$AX_k + X_kA_k^T + BB_k^T = 0, \tag{4.1}$$

$$A^TY_k + Y_kA_k - C^TC_k = 0. \tag{4.2}$$

Construct the projection matrices W_{k+1} and V_{k+1} by setting $W_{k+1} = Y_k(X_k^TY_k)^{-1}$ and $V_{k+1} = X_k$. Construct the reduced order system via projection $A_{k+1} = W_{k+1}^TAV_{k+1}$, $B_{k+1} = W_{k+1}^TB$, $C_{k+1} = CV_{k+1}$. Specifically, we present the two-sided iteration algorithm (TSIA) as follows.

Input: A stable linear time invariant system with system realization (A, B, C), the size of the reduced order system m, iteration step N.
Output: Reduced model realization $(\widehat{A}, \widehat{B}, \widehat{C})$.

1 Choose matrices $W_0, V_0 \in \mathbb{R}^{n\times m}$ such that $W_0^TV_0 = I$;
2 Compute $A_0 = W_0^TAV_0$, $B_0 = W_0^TB$ and $C_0 = CV_0$, set $k = 0$;
3 **while** $k \le N - 1$ **do**
4 Compute X_k and Y_k by solving the Sylvester equations (4.1) and (4.2);
5 Compute $W_{k+1} = Y_k(X_k^TY_k)^{-1}$, $V_{k+1} = X_k$;
6 Compute $A_{k+1} = W_{k+1}^TAV_{k+1}$, $B_{k+1} = W_{k+1}^TB$ and $C_{k+1} = CV_{k+1}$;
7 Set $k = k + 1$;
8 **end**
9 Set $\widehat{A} = A_N, \widehat{B} = B_N, \widehat{C} = C_N$.

Algorithm 1: Two-sided iteration algorithm (**TSIA**)

The above algorithm solves a tangential interpolation problem at each iterative step. Since X_k and Y_k are the solutions of Sylvester equations (4.1) and (4.2), the left and right interpolation condition pairs A_k, C_k and A_k, B_k uniquely determine the projected system $(A_{k+1}, B_{k+1}, C_{k+1}) = (W_{k+1}^TAV_{k+1}, W_{k+1}^TB, CV_{k+1})$. The interpolation points are $-\lambda_i(A_k)$ which are the mirror images of eigenvalues of A_k. The columns of C_k and B_k determine the left and right tangential interpolation directions, respectively. Upon convergence, the tangential interpolation-based optimality conditions (2.22)–(2.24) will be satisfied.

We apply the above algorithm to many different large scale systems. Despite its simplicity, the proposed algorithm works very effectively in all our numerical

experiments. It converges within a few steps and produces stable reduced-order systems.

In Algorithm 1, at each iterative step, we need to solve two Sylvester equations (4.1) and (4.2) to compute X_k and Y_k. Note that these equations are special type of Sylvester equations

$$AZ + Z\widehat{A} + M = 0, \tag{4.3}$$

where A is a $n \times n$ sparse matrix, $\widehat{A} \in \mathbb{R}^{m\times m}$, $M \in \mathbb{R}^{n\times m}$ and $m \ll n$. This special kind Sylvester equation (4.3) can be solved efficiently by the method proposed in [19]. The main idea of the algorithm is to first transform the small matrix $\widehat{A}$ into a upper triangular matrix by computing the Schur decomposition, and then obtain the solution by sequentially solving m sparse linear equations of the type

$$(A - \eta I)x = b, \quad \eta \notin \sigma(A), b \in \mathbb{R}^n, \tag{4.4}$$

where $\sigma(A)$ denotes the set of all eigenvalues of A. For more details of the implementation, we refer to [19].

As we can see, the proposed algorithm TSIA does not require to solve any Lyapunov equations. Note that many of the existing $\mathcal{H}_2$ model reduction methods require to solve a series of large scale Lyapunov equations, which is a severe computational challenge.

The following theorem shows that if the above algorithm TSIA converges, then the transfer function of the reduced-order system satisfies the tangential interpolation-based optimality conditions.

Theorem 4.1. *Suppose that the algorithm TSIA converges,*

$$(\widehat{A}, \widehat{B}, \widehat{C}) = \lim_{k\to\infty} (A_k, B_k, C_k).$$

Let $\widehat{P}$, $\widehat{Q}$, X and Y be the solutions of the equations (2.10)–(2.13)*, respectively. If the transfer function $\widehat{H}$ is stable and X^TY is nonsingular, then the transfer function $\widehat{H}$ satisfies the tangential interpolation-based optimality conditions* (2.22)–(2.24).

Proof. We first show that gradients of $\mathcal{J}$ with respect to $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ satisfy the Wilson conditions. Then the proof is done by utilizing the equivalence between the Wilson conditions and the tangential interpolation-based optimality conditions.

¿From equation (2.13), we find that

$$X^TA^TY(X^TY)^{-1} + \widehat{A} + \widehat{B}B^TY(X^TY)^{-1} = 0.$$

Since $\widehat{A} = (Y^TX)^{-1}Y^TAX$ and $\widehat{B} = (Y^TX)^{-1}Y^TB$, we have that

$$\widehat{A}^T + \widehat{A} + \widehat{B}\widehat{B}^T = 0.$$

Combining this identity with the fact the Lyapunov equation (2.10) has one unique solution (cf. [24]), we conclude that $\widehat{P} = I$. Since $\widehat{A}$ and $\widehat{C}$ satisfy equation (2.12) and $\widehat{C} = CX$, it is easy to see that

$$\widehat{A}^T(X^TY) + (X^TY)\widehat{A} - \widehat{C}^T\widehat{C} = 0.$$

Combining this formula with equation (2.11) yields that

$$\widehat{A}^T(X^TY + \widehat{Q}) + (X^TY + \widehat{Q})\widehat{A} = 0.$$

Since the solution of the above equation has one unique solution and $\widehat{Q}$ is symmetry, we find that $Y^TX + \widehat{Q} = 0$. This identity with the fact that $\widehat{P} = I$ ensures that the Wilson conditions are satisfied. □

Next, we turn to analyzing the computational cost of the proposed algorithm TSIA. The computational cost will be measured by the number of floating point multiplications. Let $\mathcal{N}(A)$ denote the number of nonzero elements of A. Let N denote the maximum number of iterations for the algorithm TSIA.

Theorem 4.2. *Let (A, B, C) be the system realization of the full order system. If there exists a solver which requires $O(rn+\mathcal{N}(A))$ multiplications to solve the sparse linear equation (4.4), where r is a fixed integer number, then the computational complexity of the algorithm TSIA is $O(N(nmr + m\mathcal{N}(A)))$.*

Proof. For $k = 0, 1, \ldots$, at the k-th step of iterations, the main computational cost of the algorithm TSIA comes from solving the Sylvester equation (4.1) and (4.2). Most of the computational cost of solving (4.1) and (4.2) lies in solving the sparse linear equations of type (4.4). Since solving the sparse linear equation (4.4) requires $O(rn + \mathcal{N}(A))$ multiplications by assumption, the computational cost for solving (4.1) and (4.2) will be $O(nmr+m\mathcal{N}(A))$. Therefore, for each iteration the algorithm TSIA requires $O(nmr + m\mathcal{N}(A))$ number of floating point multiplications. Since the maximum number of iterations of the algorithm TSIA is N, we obtain the desired overall computational complexity for the algorithm. □

We remark that the GMRES(r) algorithm proposed in [17] requires $O(rn+\mathcal{N}(A))$ floating point multiplications to solve the large sparse linear equation (4.4).

In most of practical cases, the order m of the reduced-order system is much smaller than the order n of the full-order system, $m \ll n$. If the original full-order system is sparse, the computational cost of the algorithm TSIA grows *linearly* as the order of the full order system for fixed m.

5. Numerical examples

We present in this section numerical experiments to confirm the approximation accuracy and computational efficiency of the algorithm introduced in Section 4. All numerical programs are run by using Matlab 7.7.0 (R2008b), on a PC with a 3 GHz processor and 4 GBytes of RAM.

5.1. CD player model. This is a model of a portable CD player (cf. [7]). It has 120 states, 2 inputs and 2 outputs, that is $n = 120$, $p = 2$ and $q = 2$. As illustrated in [7], the Hankel singular values of this model do not decay rapidly. Hence this model is relatively hard to reduce.

We compare the proposed algorithm TSIA with the balanced truncation. The balanced truncation method (BT) in this paper uses the matlab library function *balancmr*. Balanced truncation is well-known to give good approximations with respect to the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm (cf. [12]). We reduce the original system to the order m, which varies from 2 to 40. For each m, we compare the relative $\mathcal{H}_2$ error norm of the two algorithms. The relative $\mathcal{H}_2$ error is computed by using $\frac{\|H-\widehat{H}\|_2}{\|H\|_2}$. For the proposed algorithm TSIA, the initial projection matrix W_0 and V_0 are constructed by the tangential interpolation algorithm proposed in [11]. We try two different selections of initial interpolation points. The first one is a random selection of m interpolation points with real parts in the interval $[10^{-1}, 10^3]$ and the imaginary parts in the intervals $[-5\times10^3, 5\times10^3]$. To make this selection, it shows in [7] that most of the mirror images of eigenvalues lies in the selected region. The second one is a selection of the mirror images of the m eigenvalues corresponding to largest residuals computed by the algorithm proposed in paper [16]. The relative $\mathcal{H}_2$ error for each m are shown in Figure 1. It shows that both initial point selection

strategies work quite well. The large residual selection strategy gives better results than the random selection and outperforms the balanced truncation for almost all m. It is worth noting that the random selection strategy also gives satisfactory results.

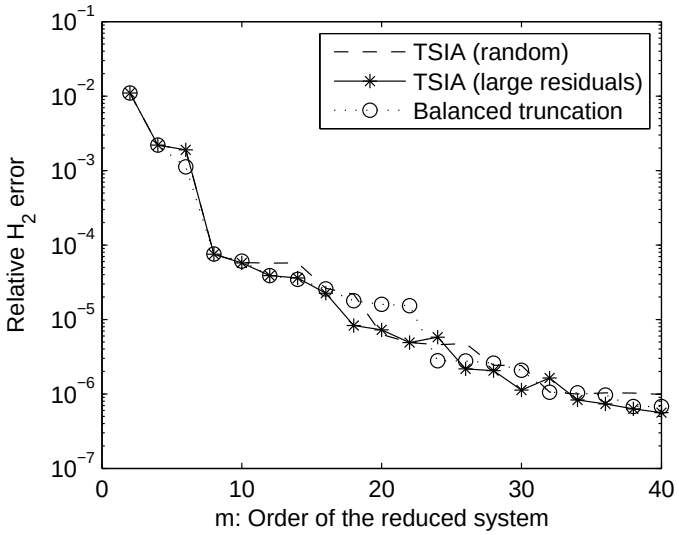

FIGURE 1. Relative $\mathcal{H}_2$ error vs. order m

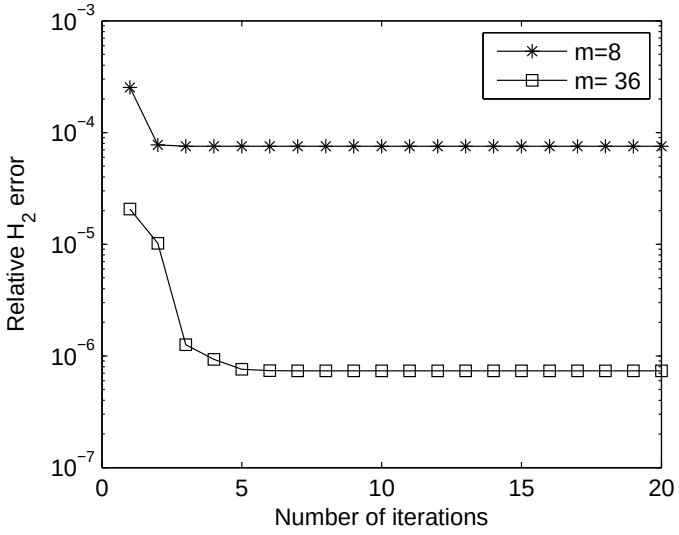

FIGURE 2. Relative $\mathcal{H}_2$ error vs. the number of iterations

We investigate this example further by reporting the convergence curve for $m = 8$ and $m = 36$ in Figure 2. At each step of the iteration, we compute the relative $\mathcal{H}_2$ error and plot this error vs the number of iterations. We can see from Figure 2 that the relative $\mathcal{H}_2$ error decreases rapidly in both cases. The algorithm TSIA converges globally after just a few steps.

TABLE 1. Heat transfer problem . Comparison.

	Relative error	CPU time (s)	(n, m)
BT	6.08×10^{-3}	49.6	(900, 3)
TSIA	4.03×10^{-3}	0.95	
	Relative error	CPU time (s)	(n, m)
BT	1.22×10^{-2}	251.3	(1600, 3)
TSIA	5.88×10^{-3}	1.6	
	Relative error	CPU time (s)	(n, m)
BT	1.17×10^{-2}	2895.8	(3600, 3)
TSIA	7.10×10^{-3}	3.7	
	Relative error (estimate)	CPU time (s)	(n, m)
BT			(25600, 3)
TSIA	3.51×10^{-4}	37.6	

5.2. A semi-discretized heat transfer problem. This model is from discretizing the heat transfer equation (cf. [13, 21])

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{5.1}$$

in the domain $\Omega = (0, 1)^2$, with $u = u(t, x, y)$, $(x, y) \in \Omega$, $t \in [0, \infty)$, (cf. [13]). The differential equation (5.1) is discretized by finite differences using a uniform grid with $d \times d$ grid points. The resulting stiffness matrix $A \in \mathbb{R}^{d^2 \times d^2}$ is a sparse matrix with bandwidth d. The system order is $n = d^2$. Let $b_1 \in \mathbb{R}^n$ be a vector with each entry equal to 1 and $b_2 \in \mathbb{R}^n$ be a random vector. Let $B = [b_1, b_2]$ and $C = B^T$. The resulting system with system realization (A, B, C) has 2 inputs and 2 outputs, that is, $p = 2$, $q = 2$.

We compare the proposed algorithm TSIA with the balanced truncation algorithm (BT). The numerical results are presented in Table 1. The CPU time is the total computing time for each algorithm measured in seconds. In the last column, n is the order of the full-order system and m is the order of the reduced-order system.

It can be seen from Table 1 that the proposed algorithm TSIA spends much less computing time than the balanced truncation, and obtains better results. Moreover, the computational time for the proposed algorithm TSIA grows linearly, while that for the balanced truncation grows cubically. In the case $n = 25600$, the balanced truncation method fails to compute the reduced system because of the large computational costs.

6. Conclusion

In this paper, we consider the optimal $\mathcal{H}_2$ model reduction for large scale MIMO via tangential interpolation. We prove that the tangential interpolation-based optimality conditions are equivalent to the gramian-based optimality conditions for MIMO systems with multiple poles. Furthermore, based on the tangential interpolation, we propose a numerical algorithm which is numerically effective and suitable for large scale MIMO systems. Numerical examples are presented to demonstrate the efficiency of the proposed algorithm and its outperformance over the existing algorithms.

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Department of Mathematics, Syracuse University, Syracuse, NY 13244, USA, and School of Mathematics and Computational Sciences, Sun Yat-sen University, Guangzhou, China, 510275.

E-mail: yxu06@syr.edu

School of Mathematics and Computational Sciences, Sun Yat-sen University, Guangzhou, China, 510275.

E-mail: zengtsh@gmail.com