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An Explicit-Implicit Predictor-Corrector Domain Decomposition Method for Time Dependent Multi-Dimensional Convection Diffusion Equations

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Abstract. The numerical solution of large scale multi-dimensional convection diffusion equations often requires efficient parallel algorithms. In this work, we consider the extension of a recently proposed non-overlapping domain decomposition method for two dimensional time dependent convection diffusion equations with variable coefficients. By combining predictor-corrector technique, modified upwind differences with explicit-implicit coupling, the method under consideration provides intrinsic parallelism while maintaining good stability and accuracy. Moreover, for multi-dimensional problems, the method can be readily implemented on a multi-processor system and does not have the limitation on the choice of subdomains required by some other similar predictor-corrector or stabilized schemes. These properties of the method are demonstrated in this work through both rigorous mathematical analysis and numerical experiments.

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1. Introduction

Large scale time dependent multi-dimensional convection-diffusion partial differential equations are often used to model many important physical problems. Numerical solutions of these equations are computationally demanding due to the needs to achieve high accuracy and overcome numerical instabilities and stiffness [3, 4, 8]. Solutions on multi-processor computer systems are sometimes the only viable approach for conducting realistic simulation in practice. There are thus considerable interests in developing efficient

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parallel schemes for convection-diffusion problems which share nice stability and accuracy properties.

To help parallelize large scale simulations, domain decomposition can be one powerful tool which has been successfully applied not only to solve many time-independent equations but also to simulate transient phenomena and evolution equations. For time-dependent parabolic problems, there are naturally built decomposition schemes to utilize the time history. For example, the explicit-implicit domain decomposition (EIDD) methods [1,2,7] have been studied for many years. In the EIDD algorithm proposed in [1], the values at inter-boundaries may be calculated by explicit schemes with coarser spatial grid size, while those in subdomains are obtained by implicit computation with the finer grid. Such EIDD methods are globally non-iterative, non-overlapping and are computationally and communicationally efficient for each simulation time step. Due to high parallelism and good stability, the studies on EIDD type algorithms have attracted the interests of many researchers. In [28], by adding a stabilization step to EIDD, some stabilized explicit-implicit domain decomposition (SEIDD) methods for the numerical solution of parabolic equations are proposed. The SEIDD methods retain the time-stepwise efficiency in computation and communication of the EIDD methods while maintaining numerical stability. However, flexibility in domain partitioning has to be sacrificed to some extent due to the non-crossover assumption of interior boundaries. In addition, there is no mathematical proof of the improved stability associated with the SEIDD methods so far, though there have been convincing numerical experiments conducted for a wide range of multidimensional parabolic problems. As a further improvement, in [13], a new class of corrected explicit-implicit domain decomposition (CEIDD) methods is presented. Based on non-crossover and crossover types of zigzag interfaces, the resulting CEIDD-ZI algorithms are shown to be convergent in the discrete H^1 semi-norm and L^2 norm. While the CEIDD-ZI scheme allows crossover interior boundaries, the assumption on zigzag interfaces adds complication to the practical implementation and limits the flexibility of domain partitioning. Later on, some new corrected explicit-implicit algorithms have been considered in [11,12,17,18] which, instead of predicting the values at inter-boundary with explicit schemes like that in [13,28], employ some linear combination of the values on previous two time levels as the predicted values at interior boundary. Some theoretical analysis has also been given in [12].

Recently, we have extended the ideas of [11,12,17,18] to time dependent convection-diffusion equations [27]. In the past, upwind schemes have been widely used in the numerical simulations of such equations. A standard upwind scheme can often avoid numerical oscillations, but it can only get the first-order accuracy. Among many possible improvements, modified upwind difference methods have been used, for example, in the context of explicit-implicit schemes, some works can be found in [19,20] where methods with relaxed CFL stability conditions were considered. In our recent work, combining the new corrected explicit-implicit domain decomposition method proposed in [11] with the modified upwind differences proposed in [9], a new Explicit-Implicit Predictor-Corrector Modified-Upwind (EIPCMU) scheme has been developed. Yet the true advantage of such an algorithm lies in its multidimensional implementation, especially in terms of allowing very flexible domain decompositions. We therefore present in this paper the extension or the analog of such an algorithm for multidimensional convection-diffusion equations with variable coefficients. The algorithm has intrinsic parallelism, nice stability and accuracy features, just like some other stabilized or corrected explicit-implicit schemes with the addition of modified-upwind differencing for convection-diffusion problems. Moreover, the new algorithm is simple and can be readily implemented on multiprocessor computers with flexible domain partitioning. The latter feature is a significant improvement over other existing modifications to the standard explicit-implicit schemes. This is also a very important aspect in practical applications which motivated our current study. For the completeness in theory, we also provide rigorous analysis on the stability and sharp error estimates for the new algorithm in a special but generic two dimensional setting. Numerical experiments illustrating the accuracy, efficiency and parallelism are also given.

The rest of this paper is organized as follows. In Section 2, the algorithms and detailed presentations are given. The stability and error analysis of the modified upwind scheme proposed are given in Sections 3 and 4, respectively, using energy methods. In Section 5, we present some numerical experiments which confirm the theoretical results obtained. In the last section, we conclude the paper and present some future work needed to be further studied.

2. Algorithm presentation

We now present the model equation and the domain decomposition algorithm.

2.1. The model equation and notation

For simplicity, as an illustration, here we only consider the two dimensional case. Let u(**x**,t) be the solution of the following convection-diffusion problem

$$\begin{cases} \frac{\partial u}{\partial t} = Lu + f(\mathbf{x},t), & \mathbf{x}\in\Omega,\ t\in(0,T], \\ u(\mathbf{x},0) = u_0(\mathbf{x}), & \mathbf{x}\in\Omega, \\ u(\mathbf{x},t) = 0, & \mathbf{x}\in\partial\Omega,\ t\in(0,T], \end{cases} \tag{2.1}$$

where

$$Lu = \nabla\cdot((\mathbf{A}(\mathbf{x})\nabla u) + \mathbf{b}(\mathbf{x})\cdot\nabla u - c(\mathbf{x})u,$$

and $\partial\Omega$ is the boundary of an open domain $\Omega\in\mathscr{R}^2$. The diffusion coefficients matrix $\mathbf{A}(\mathbf{x})$ is a diagonal matrix with diagonal entries $\{a^{(1)}(\mathbf{x}), a^{(2)}(\mathbf{x})\}$. And the coefficients $\mathbf{b}(\mathbf{x}) = (b^{(1)}(\mathbf{x}), b^{(2)}(\mathbf{x}))^T$ represent components of the convective velocity. Assume that $0 < a_0 \le a^{(i)}(\mathbf{x}) \le a_1$ and $b^{(i)}(\mathbf{x})$ are all continuously differentiable on $\bar{\Omega}$ for $i = 1,2$, with $c(\mathbf{x})$ and $f(\mathbf{x},t)$ all being uniformly bounded.

Again, without loss of generality, we take $\Omega = (0,1)\times(0,1)$ which is discretized uniformly by Ω_h with grid points $\mathbf{x}_{i,j} = (x_i, y_j) = (ih, jh)$ and a spatial grid size $h = 1/J$ for some integer J. Note that the algorithm and its analysis for general domains and high dimensional cases can be constructed and analyzed by complete analogy. In addition, one can also work with the case of different mesh sizes in different axis directions in a similar fashion.

We now decompose Ω into p subdomains. In general, p is related to the problem size and the number of processors in the computer platform. And the subdomains may be of different sizes. Again for illustration, let us decompose Ω into only four subdomains (that is, $p = 4$ see Fig. 1, left picture): $\Omega_1 = (0, x_k) \times (0, y_l)$; $\Omega_2 = (x_k, 1) \times (0, y_l)$; $\Omega_3 = (0, x_k) \times (y_l, 1)$; $\Omega_4 = (x_k, 1) \times (y_l, 1), (2 < k, l < J - 2)$. We note that this type of configuration allows cross-point in the interfaces, though the method works as well for decompositions without cross-points like that shown in the Fig. 1 (right picture).

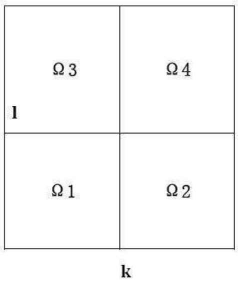

Figure 1: A decomposition used in the analysis with four subdomains having a cross-point (left) and an example of other possible decompositions with p subdomains (right).

Given a final time T of interests, let the time step be $\tau = T/N$ for some integer N, and $\{t^n = n\tau\}$ be the discrete time level. We introduce the parameter $\lambda = \tau/h^2$. In the remainder of the presentation, C means some generic positive constant, which varies independently of h and τ (and thus J and N).

For any grid function $\phi(x, y, t)$, let $\phi^n_{i,j} = \phi(x_i, y_j, t^n)$ and we introduce the following notations

$$\delta_x \phi^n_{i+\frac{1}{2},j} = \frac{1}{h}(\phi^n_{i+1,j} - \phi^n_{i,j}), \qquad \delta^2_{a,x}\phi^n_{i,j} = \frac{1}{h}\Big(a^{(1)}_{i+\frac{1}{2},j}\delta_x\phi^n_{i+\frac{1}{2},j} - a^{(1)}_{i-\frac{1}{2},j}\delta_x\phi^n_{i-\frac{1}{2},j}\Big),$$
$$\delta_y \phi^n_{i,j+\frac{1}{2}} = \frac{1}{h}(\phi^n_{i,j+1} - \phi^n_{i,j}), \qquad \delta^2_{a,y}\phi^n_{i,j} = \frac{1}{h}\Big(a^{(2)}_{i,j+\frac{1}{2}}\delta_x\phi^n_{i,j+\frac{1}{2}} - a^{(2)}_{i,j-\frac{1}{2}}\delta_x\phi^n_{i,j-\frac{1}{2}}\Big),$$
$$\Delta_\tau \phi^n_{i,j} = \frac{1}{\tau}(\phi^{n+1}_{i,j} - \phi^n_{i,j}), \qquad \Delta^2_\tau\phi^n_{i,j} = \frac{1}{\tau}\Big(\Delta_\tau\phi^{n+1}_{i,j} - \Delta_\tau\phi^n_{i,j}\Big).$$

We note that for any (i, j), $a^{(1)}_{i+\frac{1}{2},j}$ and $\phi_{i+\frac{1}{2},j}$ denote respectively the values of grid functions $a^{(1)}$ and ϕ at $(x_{i+\frac{1}{2}}, y_j)$ with $x_{i+\frac{1}{2}} = (x_i + x_{i+1})/2$. A similar convention is applied to the function $a^{(2)}$ and the index j as well. A couple of discrete norms are defined by

$$\|\phi^n_h\|^2_0 = \sum_{i=0}^{J-1}\sum_{j=0}^{J-1}|\phi^n_{i,j}|^2h^2, \qquad \|\phi^n_h\|^2_\infty = \max_{0\le i,j\le J}|\phi^n_{i,j}|^2.$$

To apply the modified upwind difference, we let

$$\tilde{a}^{(m)}_{i,j} = \left(1 + \frac{|b^{(m)}_{i,j}|}{2a^{(m)}_{i,j}} h\right)^{-1}, \quad \tilde{b}^{(m)}_{i,j} = b^{(m)}_{i,j} \frac{a^{(m)}_{i-\frac{1}{2},j}}{a^{(m)}_{i,j}}, \quad \hat{b}^{(m)}_{i,j} = b^{(m)}_{i,j} \frac{a^{(m)}_{i+\frac{1}{2},j}}{a^{(m)}_{i,j}}, \quad (m=1,2). \tag{2.2}$$

Then, we define L_h as the following discrete approximation of the continuous operator L in the subdomains:

$$L_h U^{n+1}_{i,j} = \tilde{a}^{(1)}_{i,j} \delta^2_{a,x} U^{n+1}_{i,j} + \tilde{a}^{(2)}_{i,j} \delta^2_{a,y} U^{n+1}_{i,j} - \delta_{b^{(1)},x} U^{n+1}_{i,j} - \delta_{b^{(2)},y} U^{n+1}_{i,j} - c_{i,j} U^{n+1}_{i,j}.$$

Here,

$$\begin{aligned}
\delta_{b^{(1)},x} U^{n+1}_{i,j} &= \tilde{b}^{(1)}_{i,j} \sigma(b^{(1)}_{i,j}) \delta_x U^{n+1}_{i-\frac{1}{2},j} + (1-\sigma(b^{(1)}_{i,j})) \hat{b}^{(1)}_{i,j} \delta_x U^{n+1}_{i+\frac{1}{2},j}, \\
\delta_{b^{(2)},y} U^{n+1}_{i,j} &= \tilde{b}^{(2)}_{i,j} \sigma(b^{(2)}_{i,j}) \delta_y U^{n+1}_{i,j-\frac{1}{2}} + (1-\sigma(b^{(2)}_{i,j})) \hat{b}^{(2)}_{i,j} \delta_y U^{n+1}_{i,j+\frac{1}{2}},
\end{aligned}$$

with the sign function $\sigma = \sigma(x)$ given by $\sigma(x) = 1$ for $x \geq 0$ while $\sigma(x) = 0$ for $x < 0$. Such choices of using either forward or backward first order differences in space are consistent with the upwind finite difference schemes commonly used for convection terms while the choices of the coefficients $\tilde{a}^{(m)}_{i,j}$, $\tilde{b}^{(m)}_{i,j}$ and $\hat{b}^{(m)}_{i,j}$ are consistent with the modified upwind differencing [9, 27]. While the standard upwind difference provides a stable approximation to convection-dominated problems, it is general of only first order accuracy. The modified upwind differencing, on the other hand, takes a suitable linear combination of the upwind difference for the convection term and the standard second order difference for the diffusion term so that one can ensure the unconditional stability while maintaining the second order accuracy in the truncation error [8, 10]. The coefficients of L_h are then determined accordingly.

To make the notation simpler, without loss of generality, we assume that

$$b^{(i)}(\mathbf{x}) > 0, \quad \forall\, \mathbf{x} \in \bar{\Omega}, \ i = 1, 2.$$

In such a case, the operator L_h can be rewritten as

$$L_h U^{n+1}_{i,j} = \tilde{a}^{(1)}_{i,j} \delta^2_{a,x} U^{n+1}_{i,j} + \tilde{a}^{(2)}_{i,j} \delta^2_{a,y} U^{n+1}_{i,j} - \tilde{b}^{(1)}_{i,j} \delta_x U^{n+1}_{i-\frac{1}{2},j} - \tilde{b}^{(2)}_{i,j} \delta_y U^{n+1}_{i,j-\frac{1}{2}} - c_{i,j} U^{n+1}_{i,j}.$$

2.2. The EIPCMU2D algorithm

The numerical algorithm to be studied in this paper is referred as the EIPCMU2D algorithm. Its main steps are given as follows.

To initialize the procedure, we first set the values at boundary points and the initial time level:

$$U^0_{i,j} = u^0_{i,j}, \quad (0 \leq i,j \leq J); \qquad U^n_{i,j} = 0, \quad (i,j = 0, J; n > 0). \tag{2.3}$$

Since the discrete scheme to be presented involves a three-level time discretization, we need another initialization to get the solution at the first time level. Here we adopt the following fully implicit scheme

$$\Delta_\tau U_{i,j}^1 - L_h U_{i,j}^1 = f_{i,j}^1, \quad i,j=1,2,\cdots,J-1, \tag{2.4}$$

where the operator L_h is defined as in the above.

For the main discretization step of the EIPCMU2D algorithm, our objective is to compute the solution at the $n+1$ level for any $n \geq 1$. In reference to the domain decomposition of the form in Fig. 1 (left), the solution is computed via the following steps:

1). Predict the values at interior boundary with the values on previous two time levels:

$$\tilde{U}_{k,j}^{n+1} = 2U_{k,j}^n - U_{k,j}^{n-1}; \qquad \tilde{U}_{i,l}^{n+1} = 2U_{i,l}^n - U_{i,l}^{n-1}; \quad (i,j=1,2,\cdots,J-1). \tag{2.5}$$

2). Calculate the values at points inside subdomains $\{\Omega_m\}_{m=1}^4$ with fully implicit schemes. Firstly, by replacing $U_{i,j}^{n+1}$ with $\tilde{U}_{i,j}^{n+1}$ in the fully implicit schemes at the points with $i=k$ or $j=l$, we have the following schemes at the points near the cross-over point,

$$\left\{\begin{aligned}
&\Delta_\tau U_{k-1,j}^{n+1} = L_h U_{k-1,j}^{n+1} - \lambda\tau \tilde{a}_{k-1,j}^{(1)} a_{k-\frac{1}{2},j}^{(1)} \Delta_\tau^2 U_{k,j}^n + f_{k-1,j}^{n+1},\ j \neq l, l\pm 1;\\
&\Delta_\tau U_{k+1,j}^{n+1} = L_h U_{k+1,j}^{n+1} - \lambda\tau (\tilde{a}_{k+1,j}^{(1)} a_{k+\frac{1}{2},j}^{(1)} + h\tilde{b}_{k+1,j}^{(1)}) \Delta_\tau^2 U_{k,j}^n\\
&\qquad\qquad + f_{k+1,j}^{n+1},\ j \neq l, l\pm 1;\\
&\Delta_\tau U_{i,l-1}^{n+1} = L_h U_{i,l-1}^{n+1} - \lambda\tau \tilde{a}_{i,l-1}^{(2)} a_{i,l-\frac{1}{2}}^{(2)} \Delta_\tau^2 U_{i,l}^n + f_{i,l-1}^{n+1},\ i \neq k, k\pm 1;\\
&\Delta_\tau U_{i,l+1}^{n+1} = L_h U_{i,l+1}^{n+1} - \lambda\tau (\tilde{a}_{i,l+1}^{(2)} a_{i,l+\frac{1}{2}}^{(2)} + h\tilde{b}_{i,l+1}^{(2)}) \Delta_\tau^2 U_{i,l}^n\\
&\qquad\qquad + f_{i,l+1}^{n+1},\ i \neq k, k\pm 1;\\
&\Delta_\tau U_{k-1,l-1}^{n+1} = L_h U_{k-1,l-1}^{n+1} - \lambda\tau \tilde{a}_{k-1,l-1}^{(1)} a_{k-\frac{1}{2},l-1}^{(1)} \Delta_\tau^2 U_{k,l-1}^n\\
&\qquad\qquad - \lambda\tau \tilde{a}_{k-1,l-1}^{(2)} a_{k-1,l-\frac{1}{2}}^{(2)} \Delta_\tau^2 U_{k-1,l}^n + f_{k-1,l-1}^{n+1},\\
&\Delta_\tau U_{k+1,l-1}^{n+1} = L_h U_{k+1,l-1}^{n+1} - \lambda\tau \tilde{a}_{k+1,l-1}^{(2)} a_{k+1,l-\frac{1}{2}}^{(2)} \Delta_\tau^2 U_{k,l-1}^n\\
&\qquad\qquad - \lambda\tau (\tilde{a}_{k+1,l-1}^{(1)} a_{k+\frac{1}{2},l-1}^{(1)} + h\tilde{b}_{k+1,l-1}^{(1)}) \Delta_\tau^2 U_{k+1,l}^n + f_{k+1,l-1}^{n+1},\\
&\Delta_\tau U_{k-1,l+1}^{n+1} = L_h U_{k-1,l+1}^{n+1} + f_{k-1,l+1}^{n+1} - \lambda\tau \tilde{a}_{k-1,l+1}^{(1)} a_{k-\frac{1}{2},l+1}^{(1)} \Delta_\tau^2 U_{k,l+1}^n\\
&\qquad\qquad - \lambda\tau (\tilde{a}_{k-1,l+1}^{(2)} a_{k-1,l+\frac{1}{2}}^{(2)} + h\tilde{b}_{k-1,l+1}^{(2)}) \Delta_\tau^2 U_{k-1,l}^n,\\
&\Delta_\tau U_{k+1,l+1}^{n+1} = L_h U_{k+1,l+1}^{n+1} - \lambda\tau (\tilde{a}_{k+1,l+1}^{(1)} a_{k+\frac{1}{2},l+1}^{(1)} + h\tilde{b}_{k+1,l+1}^{(1)}) \Delta_\tau^2 U_{k,l+1}^n\\
&\qquad\qquad - \lambda\tau (\tilde{a}_{k+1,l+1}^{(2)} a_{k+1,l+\frac{1}{2}}^{(2)} + h\tilde{b}_{k+1,l+1}^{(2)}) \Delta_\tau^2 U_{k+1,l}^n + f_{k+1,l+1}^{n+1}.
\end{aligned}\right. \tag{2.6}$$

Then the rest of $\{U_{i,j}^{n+1}\}$ inside the subdomains $\{\Omega_m\}_{m=1}^4$ can be computed by the following fully implicit schemes in parallel,

$$\Delta_\tau U_{i,j}^{n+1} = L_h U_{i,j}^{n+1} + f_{i,j}^{n+1}, \qquad i \neq k-1, k, k+1,\ \ j \neq l-1, l, l+1. \tag{2.7}$$

3). Correct the values at interior boundary in between the subdomains:

$$\begin{cases} \Delta_\tau U_{k,j}^{n+1} = L_h U_{k,j}^{n+1} + f_{k,j}^{n+1}, & j \neq l-1, l, l+1; \\ \Delta_\tau U_{i,l}^{n+1} = L_h U_{i,l}^{n+1} + f_{i,l}^{n+1}, & i \neq k-1, k, k+1; \end{cases} \tag{2.8}$$

$$\begin{cases} \Delta_\tau U_{k,l-1}^{n+1} = L_h U_{k,l-1}^{n+1} + f_{k,l-1}^{n+1} - \lambda\tau \tilde{a}_{k,l-1}^{(2)} a_{k,l-\frac{1}{2}}^{(2)} \Delta_\tau^2 U_{k,l}^n\,, \\ \Delta_\tau U_{k,l+1}^{n+1} = L_h U_{k,l+1}^{n+1} + f_{k,l+1}^{n+1} - \lambda\tau (\tilde{a}_{k,l+1}^{(2)} a_{k,l+\frac{1}{2}}^{(2)} + h\tilde{b}_{k,l+1}^{(2)}) \Delta_\tau^2 U_{k,l}^n\,, \\ \Delta_\tau U_{k-1,l}^{n+1} = L_h U_{k-1,l}^{n+1} + f_{k-1,l}^{n+1} - \lambda\tau \tilde{a}_{k-1,l}^{(1)} a_{k-\frac{1}{2},l}^{(1)} \Delta_\tau^2 U_{k,l}^n\,, \\ \Delta_\tau U_{k+1,l}^{n+1} = L_h U_{k+1,l}^{n+1} + f_{k+1,l}^{n+1} - \lambda\tau (\tilde{a}_{k+1,l}^{(1)} a_{k-\frac{1}{2},l}^{(1)} - \tilde{b}_{k+1,l}^{(1)} h) \Delta_\tau^2 U_{k,l}^n\,, \end{cases} \tag{2.9}$$

$$\Delta_\tau U_{k,l}^{n+1} = L_h U_{k,l}^{n+1} + f_{k,l}^{n+1}. \tag{2.10}$$

To clarify further, after computing $U_{i,j}^{n+1}$ inside each subdomain by Eqs. (2.6)-(2.7), the values on interior boundary can be computed by (2.8)-(2.9). As the values at the time step $n+1$ inside the subdomains have been already computed, only some tridiagonal linear systems need to be solved in order to obtain the values on interior boundary. Finally, the solution $U_{k,l}$ at the cross point gets corrected, in an *explicit* manner, by the fully implicit scheme (2.10) based on the already computed solutions at the other interface and interior points.

For easy reference and to distinguish from the name of its one-dimensional version studied in [27], as indicated in the start of this subsection, we name the above algorithm as the EIPCMU2D algorithm in short. In comparison with other parallel explicit-implicit difference schemes [9, 13, 28], the EIPCMU2D scheme is very simple in structure and allows flexible domain partitioning. It can thus be easily implemented on massively parallel computer systems. Furthermore, the new algorithm is unconditionally stable and has second order accuracy with respect to the discretization in spatial variables, as shown in the later sections.

3. Stability analysis by energy method

In this section, we perform the stability analysis for the EIPCMU2D schemes (2.3)-(2.10) by energy method.

3.1. Some technical lemmas

First, we give a couple of preliminary lemmas which will be used in the later proof of the stability theorem.

Lemma 3.1. *Let* $w_{i,j}^{n+1} = \triangle_\tau U_{i,j}^{n+1}$ $(0 \le i, j \le J)$, *we have*

$$\delta_x U_{i-\frac{1}{2},j}^{n+1} w_{i,j}^{n+1} h^2 = \frac{h^3}{2\tau}\Big(|\delta_x U_{i-\frac{1}{2},j}^{n+1}|^2 - |\delta_x U_{i-\frac{1}{2},j}^{n}|^2\Big) + \frac{h\tau}{2}|w_{i,j}^{n+1} - w_{i-1,j}^{n+1}|^2 + h^2(\delta_x U_{i-\frac{1}{2},j}^{n+1}) w_{i-1,j}^{n+1}, \tag{3.1}$$

$$\delta_y U_{i,j-\frac{1}{2}}^{n+1} w_{i,j}^{n+1} h^2 = \frac{h^3}{2\tau}\Big(|\delta_y U_{i,j-\frac{1}{2}}^{n+1}|^2 - |\delta_y U_{i,j-\frac{1}{2}}^{n}|^2\Big) + \frac{h\tau}{2}|w_{i,j}^{n+1} - w_{i,j-1}^{n+1}|^2 + h^2(\delta_y U_{i,j-\frac{1}{2}}^{n+1}) w_{i,j-1}^{n+1}. \tag{3.2}$$

Proof. On one hand, by inserting the term $U_{i-1,j}^{n+1}$,

$$\delta_x U_{i-\frac{1}{2},j}^{n+1} w_{i,j}^{n+1} h^2 = \frac{h}{\tau}(U_{i,j}^{n+1} - U_{i-1,j}^{n+1})(U_{i,j}^{n+1} - U_{i-1,j}^{n+1} + U_{i-1,j}^{n+1} - U_{i,j}^{n}), \tag{3.3}$$

and on the other hand, by inserting the term $U_{i-1,j}^{n}$,

$$\delta_x U_{i-\frac{1}{2},j}^{n+1} w_{i,j}^{n+1} h^2 = \frac{h}{\tau}(U_{i,j}^{n+1} - U_{i-1,j}^{n+1})(U_{i,j}^{n+1} - U_{i-1,j}^{n} + U_{i-1,j}^{n} - U_{i,j}^{n}). \tag{3.4}$$

Adding (3.3) to (3.4), we have

$$\begin{aligned}&2\delta_x U_{i-\frac{1}{2},j}^{n+1} w_{i,j}^{n+1} h^2\\ =&\frac{h^3}{\tau}\Big(|\delta_x U_{i-\frac{1}{2},j}^{n+1}|^2 - \delta_x U_{i-\frac{1}{2},j}^{n+1}\delta_x U_{i-\frac{1}{2},j}^{n}\Big) + h^2(\delta_x U_{i-\frac{1}{2},j}^{n+1})(w_{i,j}^{n+1} + w_{i-1,j}^{n+1}).\end{aligned}$$

Then, we can get that

$$\begin{aligned}\delta_x U_{i-\frac{1}{2},j}^{n+1} w_{i,j}^{n+1} h^2 =& \frac{h^3}{\tau}(|\delta_x U_{i-\frac{1}{2},j}^{n+1}|^2 - \delta_x U_{i-\frac{1}{2},j}^{n+1}\delta_x U_{i-\frac{1}{2},j}^{n}) + h^2(\delta_x U_{i-\frac{1}{2},j}^{n+1}) w_{i-1,j}^{n+1}\\ =& \frac{h^3}{\tau}\Big(|\delta_x U_{i-\frac{1}{2},j}^{n+1}|^2 - \frac{1}{2}|\delta_x U_{i-\frac{1}{2},j}^{n+1}|^2 - \frac{1}{2}|\delta_x U_{i-\frac{1}{2},j}^{n}|^2\\ &+ \frac{1}{2}|\delta_x U_{i-\frac{1}{2},j}^{n+1} - \delta_x U_{i-\frac{1}{2},j}^{n}|^2\Big) + h^2(\delta_x U_{i-\frac{1}{2},j}^{n+1}) w_{i-1,j}^{n+1}.\end{aligned} \tag{3.5}$$

Notice that the following equivalence relation holds,

$$\begin{aligned}&\delta_x U_{i+\frac{1}{2},j}^{n+1} - \delta_x U_{i+\frac{1}{2},j}^{n}\\ =&\frac{1}{h}\Big((U_{i+1,j}^{n+1} - U_{i,j}^{n+1}) - (U_{i+1,j}^{n} - U_{i,j}^{n})\Big) = \frac{\tau}{h}(w_{i+1,j}^{n+1} - w_{i,j}^{n+1}).\end{aligned} \tag{3.6}$$

By (3.5) and (3.6), we have (3.1). Similar derivation gives (3.2). ☐

Define the following energy norm,

$$\|U_h^{n+1}\|_1^2 = \sum_{i=0}^{J-1}\sum_{j=0}^{J-1}\Big[\tilde{a}_{i,j}^{(1)} a_{i+\frac{1}{2},j}^{(1)}|\delta_x U_{i+\frac{1}{2},j}^{n+1}|^2 + \tilde{a}_{i,j}^{(2)} a_{i,j+\frac{1}{2}}^{(2)}|\delta_y U_{i,j+\frac{1}{2}}^{n+1}|^2$$
$$+|\tilde{b}_{i,j}^{(1)}||\delta_x U_{i-\frac{1}{2},j}^{n+1}|^2 h + |\tilde{b}_{i,j}^{(2)}||\delta_y U_{i,j-\frac{1}{2}}^{n+1}|^2 h + |c_{i,j}||U_{i,j}^{n+1}|^2\Big] h^2. \tag{3.7}$$

Then, we have

Lemma 3.2. *Let $U_{i,j}^{n+1}$ be the solution of the EIPCMU2D scheme* (2.3)-(2.10). *Then,*

$$\frac{\|U_h^{n+1}\|_1^2 - \|U_h^n\|_1^2}{2\tau} = \sum_{i=1}^{J-1}\sum_{j=1}^{J-1}\Big[-(1+\frac{\tau}{2}c_{i,j})|w_{i,j}^{n+1}|^2 h^2 + f_{i,j}^{n+1} w_{i,j}^{n+1} h^2\Big] + I_x + I_y, \tag{3.8}$$

where

$$\begin{aligned}
I_x = &-\sum_{j=1}^{J-1}\sum_{i=0}^{J-1} a_{i+\frac{1}{2},j}^{(1)}\Big[\frac{\tau}{2}\tilde{a}_{i+1,j}^{(1)}|w_{i+1,j}^{n+1} - w_{i,j}^{n+1}|^2 + \delta_x U_{i+\frac{1}{2},j}^{n+1}\Big(\frac{\partial \tilde{a}^{(1)}}{\partial x}(\xi_i, y_j)\Big) w_{i,j}^{n+1} h^2\Big]\\
&-\frac{h\tau}{2}\sum_{i=1, i\neq k+1}^{J-1}\sum_{j=1}^{J-1}\tilde{b}_{i,j}^{(1)}|w_{i,j}^{n+1} - w_{i-1,j}^{n+1}|^2 - h^2\sum_{i=1}^{J-1}\sum_{j=1}^{J-1}\tilde{b}_{i,j}^{(1)}(\delta_x U_{i-\frac{1}{2},j}^{n+1}) w_{i-1,j}^{n+1}\\
&+\sum_{j=1}^{J-1}[\lambda(\tilde{a}_{k-1,j}^{(1)} a_{k-\frac{1}{2},j}^{(1)} w_{k-1,j}^{n+1} + \tilde{a}_{k+1,j}^{(1)} a_{k+\frac{1}{2},j}^{(1)} w_{k+1,j}^{n+1})(w_{k,j}^n - w_{k,j}^{n+1})h^2]\\
&-\frac{h\tau}{2}\sum_{j=1}^{J-1}\tilde{b}_{k+1,j}^{(1)}\Big[|w_{k+1,j}^{n+1} - w_{k,j}^n|^2 + (|w_{k,j}^{n+1}|^2 - |w_{k,j}^n|^2)\Big],\\
I_y = &-\sum_{i=1}^{J-1}\sum_{j=0}^{J-1} a_{i,j+\frac{1}{2}}^{(2)}\Big[\frac{\tau}{2}\tilde{a}_{i,j+1}^{(2)}|w_{i,j+1}^{n+1} - w_{i,j}^{n+1}|^2 + \delta_y U_{i,j+\frac{1}{2}}^{n+1}\Big(\frac{\partial \tilde{a}^{(2)}}{\partial y}(x_i, \eta_j)\Big) w_{i,j}^{n+1} h^2\Big]\\
&-\frac{h\tau}{2}\sum_{i=1}^{J-1}\sum_{j=1, j\neq l+1}^{J-1}\tilde{b}_{i,j}^{(2)}|w_{i,j}^{n+1} - w_{i,j-1}^{n+1}|^2 - h^2\sum_{i=1}^{J-1}\sum_{j=1}^{J-1}\tilde{b}_{i,j}^{(2)}(\delta_y U_{i,j-\frac{1}{2}}^{n+1}) w_{i,j-1}^{n+1}\\
&+\sum_{i=1}^{J-1}\Big[\lambda(\tilde{a}_{i,l-1}^{(2)} a_{i,l-\frac{1}{2}}^{(2)} w_{i,l-1}^{n+1} + \tilde{a}_{i,l+1}^{(2)} a_{i,l+\frac{1}{2}}^{(2)} w_{i,l+1}^{n+1})(w_{i,l}^n - w_{i,l}^{n+1})h^2\Big]\\
&-\frac{h\tau}{2}\sum_{i=1}^{J-1}\tilde{b}_{i,l+1}^{(2)}\Big[|w_{i,l+1}^{n+1} - w_{i,l}^n|^2 + (|w_{i,l}^{n+1}|^2 - |w_{i,l}^n|^2)\Big],
\end{aligned}$$

where ξ_i, η_j are some points in the intervals (x_i, x_{i+1}) and (y_j, y_{j+1}), respectively.

Proof. For $0 \le i, j \le J$, multiplying $w_{i,j}^{n+1}h^2 = h^2\triangle_\tau U_{i,j}^{n+1}$ to both sides of equations in

(2.3)-(2.10) and adding the results together, we have

$$\begin{aligned}\sum_{i,j=1}^{J-1}|w_{i,j}^{n+1}|^2h^2=&\sum_{i,j=1}^{J-1}\Big(\tilde{a}_{i,j}^{(1)}\delta_{a,x}^2U_{i,j}^{n+1}+\tilde{a}_{i,j}^{(2)}\delta_{a,y}^2U_{i,j}^{n+1}-\tilde{b}_{i,j}^{(1)}\delta U_{i-\frac{1}{2},j}^{n+1}-\tilde{b}_{i,j}^{(2)}\delta U_{i,j-\frac{1}{2}}^{n+1}\\&\qquad-c_{i,j}U_{i,j}^{n+1}+f_{i,j}^{n+1}\Big)w_{i,j}^{n+1}h^2\\&+\sum_{j=1}^{J-1}\Big(\lambda(\tilde{a}_{k-1,j}^{(1)}a_{k-\frac{1}{2},j}^{(1)}w_{k-1,j}^{n+1}+\tilde{a}_{k+1,j}^{(1)}a_{k+\frac{1}{2},j}^{(1)}w_{k+1,j}^{n+1})(w_{k,j}^{n}-w_{k,j}^{n+1})h^2\Big)\\&+\sum_{j=1}^{J-1}\Big(\lambda(h\tilde{b}_{k+1,j}^{(1)}w_{k+1,j}^{n+1})(w_{k,j}^{n}-w_{k,j}^{n+1})h^2\Big)\\&+\sum_{i=1}^{J-1}\Big(\lambda(\tilde{a}_{i,l-1}^{(2)}a_{i,l-\frac{1}{2}}^{(2)}w_{i,l-1}^{n+1}+\tilde{a}_{i,l+1}^{(2)}a_{i,l+\frac{1}{2}}^{(2)}w_{i,l+1}^{n+1})(w_{i,l}^{n}-w_{i,l}^{n+1})h^2\Big)\\&+\sum_{i=1}^{J-1}\Big(\lambda(h\tilde{b}_{i,l+1}^{(2)}w_{i,l+1}^{n+1})(w_{i,l}^{n}-w_{i,l}^{n+1})h^2\Big).\end{aligned}\tag{3.9}$$

We analyze each term on the right hand of (3.9). First, by the discrete Green formula, the boundary condition

$$w_{0,j}^{n+1}=w_{J,j}^{n+1}=w_{i,0}^{n+1}=w_{i,J}^{n+1}=0,$$

and the equivalence relation (3.6), we have

$$\begin{aligned}&\sum_{i=1}^{J-1}\sum_{j=1}^{J-1}\tilde{a}_{i,j}^{(1)}\delta_{a,x}^2U_{i,j}^{n+1}w_{i,j}^{n+1}h^2\\=&\sum_{i=1}^{J-1}\sum_{j=1}^{J-1}\tilde{a}_{i,j}^{(1)}\Big(a_{i+\frac{1}{2},j}^{(1)}\delta_xU_{i+\frac{1}{2},j}^{n}-a_{i-\frac{1}{2},j}^{(1)}\delta_xU_{i-\frac{1}{2},j}^{n}\Big)w_{i,j}^{n+1}h\\=&\sum_{j=1}^{J-1}\sum_{i=0}^{J-1}-a_{i+\frac{1}{2},j}^{(1)}\delta_xU_{i+\frac{1}{2},j}^{n+1}(\tilde{a}_{i+1,j}^{(1)}w_{i+1,j}^{n+1}-\tilde{a}_{i,j}^{(1)}w_{i,j}^{n+1})h\\=&\sum_{j=1}^{J-1}\sum_{i=0}^{J-1}-a_{i+\frac{1}{2},j}^{(1)}\delta_xU_{i+\frac{1}{2},j}^{n+1}\Big[\tilde{a}_{i+1,j}^{(1)}(w_{i+1,j}^{n+1}-w_{i,j}^{n+1})h+(\tilde{a}_{i+1,j}^{(1)}-\tilde{a}_{i,j}^{(1)})(w_{i,j}^{n+1})h\Big]\\=&-\sum_{j=1}^{J-1}\sum_{i=0}^{J-1}\Big[\frac{h^2}{2\tau}\tilde{a}_{i+1,j}^{(1)}a_{i+\frac{1}{2},j}^{(1)}\Big(|\delta_xU_{i+\frac{1}{2},j}^{n+1}|^2-|\delta_xU_{i+\frac{1}{2},j}^{n}|^2\Big)\\&+\frac{\tau}{2}\tilde{a}_{i+1,j}^{(1)}a_{i+\frac{1}{2},j}^{(1)}|w_{i+1,j}^{n+1}-w_{i,j}^{n+1}|^2+a_{i+\frac{1}{2},j}^{(1)}\delta_xU_{i+\frac{1}{2},j}^{n+1}\frac{\partial\tilde{a}^{(1)}}{\partial x}(\xi_i,y_j)w_{i,j}^{n+1}h^2\Big],\end{aligned}\tag{3.10}$$

where ξ_i is some point in the interval (x_i,x_{i+1}) due to the mean value theorem. Similar calculation gives

$$
\begin{aligned}
&\sum_{i=1}^{J-1}\sum_{j=1}^{J-1}\tilde{a}_{i,j}^{(2)}\delta_{a,y}^2 U_{i,j}^{n+1}w_{i,j}^{n+1}h^2\\
=&-\sum_{i=1}^{J-1}\sum_{j=0}^{J-1}\Bigg[\frac{h^2}{2\tau}\tilde{a}_{i,j+1}^{(2)}a_{i,j+\frac{1}{2}}^{(2)}(|\delta_y U_{i,j+\frac{1}{2}}^{n+1}|^2-|\delta_y U_{i,j+\frac{1}{2}}^{n}|^2)\\
&+\frac{\tau}{2}\tilde{a}_{i,j+1}^{(2)}a_{i,j+\frac{1}{2}}^{(2)}|w_{i,j+1}^{n+1}-w_{i,j}^{n+1}|^2+a_{i,j+\frac{1}{2}}^{(2)}\delta_y U_{i,j+\frac{1}{2}}^{n+1}\frac{\partial\tilde{a}^{(2)}}{\partial y}(x_i,\eta_j)w_{i,j}^{n+1}h^2\Bigg],
\end{aligned}
\tag{3.11}
$$

where η_j is some point in the interval (y_j, y_{j+1}) by the mean value theorem.

Now, simple calculation gives

$$
\begin{aligned}
&\sum_{i=1}^{J-1}\sum_{j=1}^{J-1}c_{i,j}U_{i,j}^{n+1}w_{i,j}^{n+1}h^2=\frac{h^2}{\tau}\sum_{i=1}^{J-1}\sum_{j=1}^{J-1}c_{i,j}U_{i,j}^{n+1}(U_{i,j}^{n+1}-U_{i,j}^{n})\\
=&\frac{h^2}{2\tau}\sum_{i=1}^{J-1}\sum_{j=1}^{J-1}c_{i,j}\left(|U_{i,j}^{n+1}|^2-|U_{i,j}^{n}|^2+|U_{i,j}^{n+1}-U_{i,j}^{n}|^2\right)\\
=&\frac{h^2}{2\tau}\sum_{i=1}^{J-1}\sum_{j=1}^{J-1}c_{i,j}\left(|U_{i,j}^{n+1}|^2-|U_{i,j}^{n}|^2\right)+\frac{\tau h^2}{2}\sum_{i=1}^{J-1}\sum_{j=1}^{J-1}c_{i,j}|w_{i,j}^{n+1}|^2.
\end{aligned}
\tag{3.12}
$$

Finally, it is easy to see that

$$2(w_{k,j}^{n}-w_{k,j}^{n+1})w_{k+1,j}^{n+1}=|w_{k+1,j}^{n+1}-w_{k,j}^{n+1}|^2-|w_{k+1,j}^{n+1}-w_{k,j}^{n}|^2-|w_{k,j}^{n+1}|^2+|w_{k,j}^{n}|^2, \tag{3.13}$$

$$2(w_{i,l}^{n}-w_{i,l}^{n+1})w_{i,l+1}^{n+1}=|w_{i,l+1}^{n+1}-w_{i,l}^{n+1}|^2-|w_{i,l+1}^{n+1}-w_{i,l}^{n}|^2-|w_{i,l}^{n+1}|^2+|w_{i,l}^{n}|^2. \tag{3.14}$$

By (3.1)-(3.2) and (3.9)-(3.14), we get the conclusion in Lemma 3.2 □

Lemma 3.3. *Let $U_{i,j}^1$ be obtained by the fully implicit scheme* (2.4). *Then, for τ small enough, there exists a positive constant C such that,* $\forall\ \tau > 0$,

$$\|U_h^1\|_1^2 \le C\left(\|U_h^0\|_1^2+\tau\|f_h^1\|_0^2\right). \tag{3.15}$$

Note that the energy norm $\|U_h^n\|_1$ is defined by (3.7). Here and hereafter, the restriction on the time step τ being small enough is only dependent on the coefficients of the PDEs, but not on the spatial mesh size. The above result is for the standard fully-implicit scheme, and the proof in the two dimensional setting is nearly identical to that in the one dimensional case which can be found in textbooks and also [27]. We thus omit the detailed proof.

3.2. The stability theorem

Now we give the stability theorem for the EIPCMU2D scheme (2.3)-(2.10).

Theorem 3.1. *Assume that $U_{i,j}^n$ is the solution of the EIPCMU2D scheme* (2.3)-(2.10)*. Then when τ is small enough, there exists a positive constant C such that,*

$$\|U_h^n\|_1^2 \le C\left(\|U_h^0\|_1^2 + \tau^2\|L_hU_h^0\|_\infty^2 + \sum_{l=0}^{n-1}\tau\|f^{l+1}\|_0^2\right). \tag{3.16}$$

Proof. Let $w_{i,j}^n = \Delta_\tau U_{i,j}^n$. We now estimate the right hand of (3.8). First we consider the following terms in the expression of I_x:

$$\begin{aligned}
I_1 =& -\frac{\tau}{2}\sum_{j=1}^{J-1}\sum_{i=0}^{J-1}\tilde{a}_{i+1,j}^{(1)}a_{i+\frac{1}{2},j}^{(1)}\left(|w_{i+1,j}^{n+1}-w_{i,j}^{n+1}|^2\right)\\
&+\sum_{j=1}^{J-1}\left(\sum_{|p-k|=1}\tilde{a}_{p,j}^{(1)}a_{\frac{k+p}{2},j}^{(1)}w_{p,j}^{n+1}\right)(w_{k,j}^n-w_{k,j}^{n+1})\lambda h^2\\
=& -\frac{\tau}{2}\sum_{j=1}^{J-1}\sum_{\substack{i=0\\ i\neq k,k-1}}^{J-1}\tilde{a}_{i+1,j}^{(1)}a_{i+\frac{1}{2},j}^{(1)}|w_{i+1,j}^{n+1}-w_{i,j}^{n+1}|^2\\
&-\frac{\tau}{2}\sum_{j=1}^{J-1}\Big(\tilde{a}_{k+1,j}^{(1)}a_{k+\frac{1}{2},j}^{(1)}|w_{k+1,j}^{n+1}-w_{k,j}^{n+1}|^2+\tilde{a}_{k,j}^{(1)}a_{k-\frac{1}{2},j}^{(1)}|w_{k,j}^{n+1}-w_{k-1,j}^{n+1}|^2\\
&-2\tilde{a}_{k-1,j}^{(1)}a_{k-\frac{1}{2},j}^{(1)}w_{k-1,j}^{n+1}(w_{k,j}^n-w_{k,j}^{n+1})-2\tilde{a}_{k+1,j}^{(1)}a_{k+\frac{1}{2},j}^{(1)}w_{k+1,j}^{n+1}(w_{k,j}^n-w_{k,j}^{n+1})\Big).
\end{aligned}$$

We use the mean value theorem on the term $\tilde{a}_{k,j}^{(1)}$ to get

$$\begin{aligned}
I_1 =& -\frac{\tau}{2}\sum_{j=1}^{J-1}\Bigg[\sum_{\substack{i=0\\ i\neq k,k-1}}^{J-1}\tilde{a}_{i+1,j}^{(1)}a_{i+\frac{1}{2},j}^{(1)}|w_{i+1,j}^{n+1}-w_{i,j}^{n+1}|^2+\tilde{a}_{k+1,j}^{(1)}a_{k+\frac{1}{2},j}^{(1)}|w_{k+1,j}^{n+1}-w_{k,j}^{n+1}|^2\\
&+\Big(\tilde{a}_{k-1,j}^{(1)}+h\frac{\partial\tilde{a}^{(1)}}{\partial x}(\xi_{k-1},y_j)\Big)a_{k-\frac{1}{2},j}^{(1)}|w_{k,j}^{n+1}-w_{k-1,j}^{n+1}|^2\\
&-2\tilde{a}_{k-1,j}^{(1)}a_{k-\frac{1}{2},j}^{(1)}w_{k-1,j}^{n+1}(w_{k,j}^n-w_{k,j}^{n+1})-2\tilde{a}_{k+1,j}^{(1)}a_{k+\frac{1}{2},j}^{(1)}w_{k+1,j}^{n+1}(w_{k,j}^n-w_{k,j}^{n+1})\Bigg]\\
=& -\frac{\tau}{2}\sum_{j=1}^{J-1}\Bigg(\sum_{\substack{i=0\\ i\neq k,k-1}}^{J-1}\tilde{a}_{i+1,j}^{(1)}a_{i+\frac{1}{2},j}^{(1)}|w_{i+1,j}^{n+1}-w_{i,j}^{n+1}|^2\\
&+h\frac{\partial\tilde{a}^{(1)}}{\partial x}(\xi_{k-1},y_j)a_{k-\frac{1}{2},j}^{(1)}|w_{k,j}^{n+1}-w_{k-1,j}^{n+1}|^2\Bigg)\\
&-\frac{\tau}{2}\sum_{j=1}^{J-1}\sum_{|p-k|=1}\tilde{a}_{p,j}^{(1)}a_{\frac{p+k}{2},j}^{(1)}\Big(|w_{p,j}^{n+1}-w_{k,j}^n|^2+|w_{k,j}^{n+1}|^2-|w_{k,j}^n|^2\Big),
\end{aligned}\tag{3.17}$$

where ξ_{k-1} is some point in the interval (x_{k-1}, x_k). Similarly, for the corresponding terms in I_y of Eq. (3.8), we have

$$\begin{aligned}
I_2 =& -\frac{\tau}{2}\sum_{i=1}^{J-1}\sum_{j=0}^{J-1}\tilde{a}^{(2)}_{i,j+1}a^{(2)}_{i,j+\frac{1}{2}}|w^{n+1}_{i,j+1}-w^{n+1}_{i,j}|^2 \\
&+\sum_{i=1}^{J-1}\Big(\sum_{|p-l|=1}\tilde{a}^{(2)}_{i,p}a^{(2)}_{i,\frac{p+l}{2}}w^{n+1}_{i,p}\Big)(w^n_{i,l}-w^{n+1}_{i,l})\lambda h^2 \\
=& -\frac{\tau}{2}\sum_{i=1}^{J-1}\Big(\sum_{\substack{j=0\\ j\neq l,l-1}}^{J-1}\tilde{a}^{(2)}_{i,j+1}a^{(2)}_{i,j+\frac{1}{2}}|w^{n+1}_{i,j+1}-w^{n+1}_{i,j}|^2 \\
&+h\frac{\partial\tilde{a}^{(2)}}{\partial y}(x_i,\eta_{l-1})a^{(2)}_{i,l-\frac{1}{2}}|w^{n+1}_{i,l}-w^{n+1}_{i,l-1}|^2\Big) \\
&-\frac{\tau}{2}\sum_{j=1}^{J-1}\sum_{|p-l|=1}\tilde{a}^{(2)}_{i,p}a^{(2)}_{i,\frac{l+p}{2}}(|w^{n+1}_{i,p}-w^n_{i,l}|^2+|w^{n+1}_{i,l}|^2-|w^n_{i,l}|^2).
\end{aligned}\tag{3.18}$$

Note that $a^{(m)}(x,y)$ and $b^{(m)}(x,y)$ $(m=1,2)$ are all continuously differentiable, $\frac{\partial\tilde{a}^{(1)}}{\partial x}$, $\frac{\partial\tilde{a}^{(2)}}{\partial y}$ and $\tilde{a}^{(1)}$, $\tilde{a}^{(2)}$ are all uniformly bounded. By the definition (2.2) of $\tilde{a}^{(m)}$ $(m=1,2)$, we have

$$\frac{\partial\tilde{a}^{(m)}}{\partial x}(\xi_{k-1},y_j)=-h(\tilde{a}^{(m)})^2(\xi_{k-1},y_j)\frac{\partial}{\partial x}\left(\frac{b^{(m)}}{2a^{(m)}}\right)(\xi_{k-1},y_j),$$

which lead to,

$$\begin{aligned}
&-\frac{\tau}{2}\sum_{j=1}^{J-1}h\frac{\partial\tilde{a}^{(1)}}{\partial x}(\xi_{k-1},y_j)a^{(1)}_{k-\frac{1}{2},j}|w^{n+1}_{k,j}-w^{n+1}_{k-1,j}|^2 \\
\le& C\tau\sum_{j=1}^{J-1}\Big(|w^{n+1}_{k-1,j}|^2+|w^{n+1}_{k,j}|^2\Big)h^2
\end{aligned}\tag{3.19}$$

for some generic positive constant C independent of τ and h. Similarly, we have

$$\begin{aligned}
&-\frac{\tau}{2}\sum_{i=1}^{J-1}h\frac{\partial\tilde{a}^{(2)}}{\partial y}(x_i,\eta_{l-1})a^{(2)}_{i,l-\frac{1}{2}}|w^{n+1}_{i,l}-w^{n+1}_{i,l-1}|^2 \\
\le& C\tau\sum_{i=1}^{J-1}\Big(|w^{n+1}_{i,l-1}|^2+|w^{n+1}_{i,l}|^2\Big)h^2.
\end{aligned}\tag{3.20}$$

By the positivity assumptions on the coefficients **A**(x,y) and **b**(x,y), using Young's inequality together with (3.8) and (3.17)-(3.18), we have

$$
\begin{aligned}
&\frac{1}{2\tau}\left(\|U_h^{n+1}\|_1^2-\|U_h^n\|_1^2\right)\\
\le&-\sum_{i=1}^{J-1}\sum_{j=1}^{J-1}\left(1+\frac{\tau}{2}c_{i,j}\right)|w_{i,j}^{n+1}|^2h^2+\sum_{i=1}^{J-1}\sum_{j=1}^{J-1}f_{i,j}^{n+1}w_{i,j}^{n+1}h^2\\
&-\sum_{j=1}^{J-1}\sum_{i=0}^{J-1}a^{(1)}_{i+\frac12,j}\delta_xU^{n+1}_{i+\frac12,j}\Big(\frac{\partial\tilde a^{(1)}}{\partial x}(\xi_i,y_j)\Big)w_{i,j}^{n+1}h^2-h^2\sum_{i=1}^{J-1}\sum_{j=1}^{J-1}\tilde b^{(1)}_{i,j}(\delta_xU^{n+1}_{i-\frac12,j})w^{n+1}_{i-1,j}\\
&-\sum_{i=1}^{J-1}\sum_{j=0}^{J-1}a^{(2)}_{i,j+\frac12}\delta_yU^{n+1}_{i,j+\frac12}\Big(\frac{\partial\tilde a^{(2)}}{\partial y}(x_i,\eta_j)\Big)w_{i,j}^{n+1}h^2-h^2\sum_{i=1}^{J-1}\sum_{j=1}^{J-1}\tilde b^{(2)}_{i,j}(\delta_yU^{n+1}_{i,j-\frac12})w^{n+1}_{i,j-1}\\
&-\frac{\tau}{2}\Bigg\{\sum_{j=1}^{J-1}\Big(h\tilde b^{(1)}_{k+1,j}+\tilde a^{(1)}_{k+1,j}a^{(1)}_{k+\frac12,j}+\tilde a^{(1)}_{k-1,j}a^{(1)}_{k-\frac12,j}\Big)\Big(|w^{n+1}_{k,j}|^2-|w^n_{k,j}|^2\Big)\\
&+\sum_{i=1}^{J-1}\Big(h\tilde b^{(2)}_{i,l+1}+\tilde a^{(2)}_{i,l+1}a^{(2)}_{i,l+\frac12}+\tilde a^{(2)}_{i,l-1}a^{(2)}_{i,l-\frac12}\Big)|w^{n+1}_{i,l}|^2-|w^n_{i,l}|^2\\
&+\sum_{j=1}^{J-1}\Big(\frac{\partial\tilde a^{(1)}}{\partial x}(\xi_{k-1},y_j)\Big)ha^{(1)}_{k-\frac12,j}|w^{n+1}_{k,j}-w^{n+1}_{k-1,j}|^2\\
&+\sum_{i=1}^{J-1}\Big(\frac{\partial\tilde a^{(2)}}{\partial y}(x_i,\eta_{l-1})\Big)ha^{(2)}_{i,l-\frac12}(|w^{n+1}_{i,l}-w^{n+1}_{i,l-1}|^2)\Bigg\}\\
\le&-\Big(\frac14-C\tau\Big)\|w_h^n\|_0+\sum_{i=1}^{J-1}\sum_{j=1}^{J-1}\Big(\frac12|f^{n+1}_{i,j}|^2h^2+C(|\delta_xU^{n+1}_{i+\frac12,j}|^2+|\delta_yU^{n+1}_{i,j+\frac12}|^2)h^2\Big)\\
&-\frac{\tau}{2}\sum_{j=1}^{J-1}\Big(h\tilde b^{(1)}_{k+1,j}+\tilde a^{(1)}_{k+1,j}a^{(1)}_{k+\frac12,j}+\tilde a^{(1)}_{k-1,j}a^{(1)}_{k-\frac12,j}\Big)\Big(|w^{n+1}_{k,j}|^2-|w^n_{k,j}|^2\Big)\\
&-\frac{\tau}{2}\sum_{i=1}^{J-1}\Big(h\tilde b^{(2)}_{i,l+1}+\tilde a^{(2)}_{i,l+1}a^{(2)}_{i,l+\frac12}+\tilde a^{(2)}_{i,l-1}a^{(2)}_{i,l-\frac12}\Big)\Big(|w^{n+1}_{i,l}|^2-|w^n_{i,l}|^2\Big),
\end{aligned}
\tag{3.21}
$$

for some generic constant $C > 0$, dependent only on the coefficients but independent of the mesh size and time step. Taking τ to be small enough, such that

$$\frac14-C\tau\ge 0\,.$$

Thus, we get from (3.21) the following recurrent inequality

$$
\begin{aligned}
&\|U_h^{n+1}\|_1^2+\tau^2\sum_{j=1}^{J-1}\Big(h\tilde{b}^{(1)}_{k+1,j}+\tilde{a}^{(1)}_{k+1,j}a^{(1)}_{k+\frac{1}{2},j}+\tilde{a}^{(1)}_{k-1,j}a^{(1)}_{k-\frac{1}{2},j}\Big)|w^{n+1}_{k,j}|^2\\
&\quad+\tau^2\sum_{i=1}^{J-1}\Big(h\tilde{b}^{(2)}_{i,l+1}+\tilde{a}^{(2)}_{i,l+1}a^{(2)}_{i,l+\frac{1}{2}}+\tilde{a}^{(2)}_{i,l-1}a^{(2)}_{i,l-\frac{1}{2}}\Big)|w^{n+1}_{i,l}|^2\\
&\le\|U_h^{n}\|_1^2+C\tau\|U_h^{n+1}\|_1^2+\tau^2\sum_{j=1}^{J-1}\Big(h\tilde{b}^{(1)}_{k+1,j}+\tilde{a}^{(1)}_{k+1,j}a^{(1)}_{k+\frac{1}{2},j}+\tilde{a}^{(1)}_{k-1,j}a^{(1)}_{k-\frac{1}{2},j}\Big)|w^{n}_{k,j}|^2\\
&\quad+\tau^2\sum_{i=1}^{J-1}\Big(h\tilde{b}^{(2)}_{i,l+1}+\tilde{a}^{(2)}_{i,l+1}a^{(2)}_{i,l+\frac{1}{2}}+\tilde{a}^{(2)}_{i,l-1}a^{(2)}_{i,l-\frac{1}{2}}\Big)|w^{n}_{i,l}|^2+\tau\|f^{n+1}_{i,j}\|_0^2.
\end{aligned}
$$

Summing up with respect to n, we get that

$$
\begin{aligned}
&\|U_h^{n+1}\|_1^2+\tau^2\sum_{j=1}^{J-1}\Big(\tilde{a}^{(1)}_{k+1,j}a^{(1)}_{k+\frac{1}{2},j}+\tilde{a}^{(1)}_{k-1,j}a^{(1)}_{k-\frac{1}{2},j}+h\tilde{b}^{(1)}_{k+1,j}\Big)|w^{n+1}_{k,j}|^2\\
&\quad+\tau^2\sum_{i=1}^{J-1}\Big(\tilde{a}^{(2)}_{i,l+1}a^{(2)}_{i,l+\frac{1}{2}}+\tilde{a}^{(1)}_{i,l-1}a^{(2)}_{i,l-\frac{1}{2}}+h\tilde{b}^{(2)}_{i,l+1}\Big)|w^{n+1}_{i,l}|^2\\
&\le\|U_h^{1}\|_1^2+\tau^2\sum_{j=1}^{J-1}\Big(\tilde{a}^{(1)}_{k+1,j}a^{(1)}_{k+\frac{1}{2},j}+\tilde{a}^{(1)}_{k-1,j}a^{(1)}_{k-\frac{1}{2},j}+h\tilde{b}^{(1)}_{k+1,j}\Big)|w^{1}_{k,j}|^2\\
&\quad+\tau^2\sum_{i=1}^{J-1}\Big(\tilde{a}^{(2)}_{i,l+1}a^{(2)}_{i,l+\frac{1}{2}}+\tilde{a}^{(1)}_{i,l-1}a^{(2)}_{i,l-\frac{1}{2}}+h\tilde{b}^{(2)}_{i,l+1}\Big)|w^{1}_{i,l}|^2\\
&\quad+\sum_{l=1}^{n}2\tau C\|U_h^{l+1}\|_1^2+\sum_{l=1}^{n}\tau\|f^{l+1}\|_0^2.
\end{aligned}
\tag{3.22}
$$

By the discrete Gronwall inequality, we have

$$
\begin{aligned}
\|U_h^{n+1}\|_1^2\le C\Bigg(&\|U_h^{1}\|_1^2+\tau^2\sum_{j=1}^{J-1}\Big(\tilde{a}^{(1)}_{k+1,j}a^{(1)}_{k+\frac{1}{2},j}+\tilde{a}^{(1)}_{k-1,j}a^{(1)}_{k-\frac{1}{2},j}+h\tilde{b}^{(1)}_{k+1,j}\Big)|w^{1}_{k,j}|^2\\
&+\tau^2\sum_{i=1}^{J-1}\Big(\tilde{a}^{(2)}_{i,l+1}a^{(2)}_{i,l+\frac{1}{2}}+\tilde{a}^{(1)}_{i,l-1}a^{(2)}_{i,l-\frac{1}{2}}+h\tilde{b}^{(2)}_{i,l+1}\Big)|w^{1}_{i,l}|^2+\sum_{l=1}^{n}\tau\|f^{l+1}\|_0^2\Bigg).
\end{aligned}
\tag{3.23}
$$

Since the discrete operator L_h is linear, we have

$$
w^1_{i,j}=\tau L_h\frac{U^1_{i,j}-U^0_{i,j}}{\tau}+L_hU^0_{i,j}=\tau L_hw^1_{i,j}+L_hU^0_{i,j}.
$$

Because the first level is fully implicit scheme, by the discrete maximum principle, it is easy to show that

$$
\|w^1_h\|_\infty\le\|L_hU^0_h\|_\infty. \tag{3.24}
$$

By (3.15) and (3.23)-(3.24), we have Theorem 3.1. □

Remark 3.1. The above results can be established by complete analogy for more general cases where the coefficients may be time dependent as long as they are differentiable with respect to time.

Remark 3.2. Note that the restrictions on the time step τ and mesh size h being small are only related to the coefficients of the PDEs and not on each other nor the solutions, thus they should not be viewed as stability conditions. In this sense, the EIPCMU2D scheme remains unconditionally stable.

4. Error estimate

In this section, we deduce the error estimate for the EIPCMU2D scheme (2.3)-(2.10) via the energy method.

4.1. Local truncation error analysis

We first give one lemma on the local truncation error.

Lemma 4.1. *Assume that $G_{i,j}^{n+1}$ is the local truncation error of the approximation $U_{i,j}^{n+1}$ generated by the numerical scheme* (2.3)-(2.10) *at point (x_i, y_j, t^{n+1}). Then, for τ and h small, we have*

$$|G_{i,j}^{n+1}| \le C(\tau + h^2), \qquad |i-k| \neq 1 \quad and \quad |j-l| \neq 1, \tag{4.1}$$

$$|G_{i,j}^{n+1}| \le C(\tau + h^2 + \frac{\tau^2}{h^2}), \qquad |i-k| = 1 \quad or \quad |j-l| = 1, \tag{4.2}$$

$$|G_{i,j}^{n+1} - G_{i,j}^{n}| \le C(\tau + h^2 + \frac{\tau^2}{h^2})\tau, \qquad |i-k| = 1 \quad or \quad |j-l| = 1, \tag{4.3}$$

where C is a positive constant which is independent of τ and h.

Proof. The calculation of the truncation error for $G_{i,j}^{n+1}$ with $|i-k| \neq 1$ and $|j-l| \neq 1$, is standard. So, we firstly focus on the case $i = k-1$ and $j \neq l, l \pm 1$. By (2.6), we can know that the only difference of this case with the standard case is the appearance of an extra term

$$-\lambda\tau\tilde{a}_{k-1,j}^{(1)} a_{k-\frac{1}{2},j}^{(1)} \Delta_\tau^2 U_{k,j}^n.$$

By Taylor expansion, the truncation error on this term is

$$-\tilde{a}_{k-1,j} a_{k-\frac{1}{2},j}^{(1)} \lambda\tau \left(\frac{\partial^2 u}{\partial t^2}(x_k, y_j, t^n) + \mathscr{O}(\tau) \right)$$

which is on the order of $\mathscr{O}(\lambda\tau) = \mathscr{O}(\tau^2/h^2)$. By analogy, we have the same results of other cases, so we have (4.2). Similar calculation gives (4.3). □

4.2. Discretization errors

The truncation error terms given in the above are similar to those associated to the well-known DuFort-Frankel scheme for diffusion equations which typically has an error order $\mathcal{O}(\tau + h^2 + \tau^2/h^2)$. The term τ^2/h^2 is largely due to the use of predictor-corrector or the replacement of time extrapolated values. It would severely limit the time step size of the DuFort-Frankel scheme even though unconditionally stability is assured. Yet, due to the limited use in the EIPCMU2D scheme of the predictor-corrector steps at the interface boundary points only, as in the one-dimensional case, the contribution from such a term to the overall scheme is multiplied by a factor of h which thus has much less impact on the accuracy of EIPCMU2D scheme.

Theorem 4.1. *Assume that $u = u(x,y,t)$ is the solution of the PDE (2.1) and $\{U^n_{i,j}\}$ is the solution of the EIPCMU2D scheme (2.3)-(2.10) respectively. Let $e^n_{i,j} = u^n_{i,j} - U^n_{i,j}$. Then, when τ is small enough, there exists a positive constant C independent of τ and h, such that*

$$\|e^n_h\|_1 \le C\Big(\tau + h^2 + \frac{\tau^2}{h}\Big), \tag{4.4}$$

where the energy norm $\|e^n_h\|_1$ is defined by (3.7).

Proof. Let $e^n_{i,j} = u^n_{i,j} - U^n_{i,j}$. Then, by (2.6)-(2.10), it is easy to see that $\{e^{n+1}_{i,j}\}$ satisfies the following error equations,

$$\Delta_\tau e^{n+1}_{k-1,j} = L_h e^{n+1}_{k-1,j} + G^{n+1}_{k-1,j} - \lambda \tilde{a}^{(1)}_{k-1,j} a^{(1)}_{k-\frac{1}{2},j} \Delta^2_\tau e^n_{k,j}, \quad j \ne l, l\pm 1; \tag{4.5}$$

$$\begin{aligned}\Delta_\tau e^{n+1}_{k+1,j} =& L_h e^{n+1}_{k+1,j} + G^{n+1}_{k+1,j}\\ &- \lambda\Big(\tilde{a}^{(1)}_{k+1,j} a^{(1)}_{k+\frac{1}{2},j} + h\tilde{b}^{(1)}_{k+1,j}\Big)\Delta^2_\tau e^n_{k,j}, \qquad j \ne l, l\pm 1;\end{aligned} \tag{4.6}$$

$$\Delta_\tau e^{n+1}_{i,l-1} = L_h e^{n+1}_{i,l-1} + G^{n+1}_{i,l-1} - \lambda \tilde{a}^{(2)}_{i,l-1} a^{(2)}_{i,l-\frac{1}{2}} \Delta^2_\tau e^n_{i,l}, \qquad i \ne k, k\pm 1; \tag{4.7}$$

$$\begin{aligned}\Delta_\tau e^{n+1}_{i,l+1} =& L_h e^{n+1}_{i,l+1} + G^{n+1}_{i,l+1}\\ &- \lambda\Big(\tilde{a}^{(2)}_{i,l+1} a^{(2)}_{i,l+\frac{1}{2}} + h\tilde{b}^{(2)}_{i,l+1}\Big)\Delta^2_\tau e^n_{i,l}, \qquad i \ne k, k\pm 1;\end{aligned} \tag{4.8}$$

$$\begin{aligned}\Delta_\tau e^{n+1}_{k-1,l-1} =& L_h e^{n+1}_{k-1,l-1} + G^{n+1}_{k-1,l-1} - \lambda\tau \tilde{a}^{(1)}_{k-1,l-1} a^{(1)}_{k-\frac{1}{2},l-1} \Delta^2_\tau e^n_{k,l-1}\\ &- \lambda\tau \tilde{a}^{(2)}_{k-1,l-1} a^{(2)}_{k-1,l-\frac{1}{2}} \Delta^2_\tau e^n_{k-1,l},\end{aligned} \tag{4.9}$$

$$\begin{aligned}\Delta_\tau e^{n+1}_{k+1,l-1} =& L_h e^{n+1}_{k+1,l-1} + G^{n+1}_{k+1,l-1} - \lambda\tau \tilde{a}^{(2)}_{k+1,l-1} a^{(2)}_{k+1,l-\frac{1}{2}} \Delta^2_\tau e^n_{k,l-1}\\ &- \lambda\tau\Big(\tilde{a}^{(1)}_{k+1,l-1} a^{(1)}_{k+\frac{1}{2},l-1} + h\tilde{b}^{(1)}_{k+1,l-1}\Big)\Delta^2_\tau e^n_{k+1,l},\end{aligned} \tag{4.10}$$

$$\begin{aligned}\Delta_\tau e^{n+1}_{k-1,l+1} =& L_h e^{n+1}_{k-1,l+1} + G^{n+1}_{k-1,l+1} - \lambda\tau \tilde{a}^{(1)}_{k-1,l+1} a^{(1)}_{k-\frac{1}{2},l+1} \Delta^2_\tau e^n_{k,l+1}\\ &- \lambda\tau\Big(\tilde{a}^{(2)}_{k-1,l+1} a^{(2)}_{k-1,l+\frac{1}{2}} + h\tilde{b}^{(2)}_{k-1,l+1}\Big)\Delta^2_\tau e^n_{k-1,l},\end{aligned} \tag{4.11}$$

$$\begin{aligned}\Delta_\tau e^{n+1}_{k+1,l+1} =& L_h e^{n+1}_{k+1,l+1} + G^{n+1}_{k+1,l+1} - \lambda\tau\left(\tilde{a}^{(1)}_{k+1,l+1} a^{(1)}_{k+\frac{1}{2},l+1} + h\tilde{b}^{(1)}_{k+1,l+1}\right)\Delta^2_\tau e^n_{k,l+1}\\ &- \lambda\tau\left(\tilde{a}^{(2)}_{k+1,l+1} a^{(2)}_{k+1,l+\frac{1}{2}} + h\tilde{b}^{(2)}_{k+1,l+1}\right)\Delta^2_\tau e^n_{k+1,l},\end{aligned} \tag{4.12}$$

$$\Delta_\tau e^{n+1}_{i,j} = L_h e^{n+1}_{i,j} + G^{n+1}_{i,j}, \qquad i \neq k-1, k, k+1, \;\; j \neq l-1, l, l+1; \tag{4.13}$$

$$\Delta_\tau e^{n+1}_{k,j} = L_h e^{n+1}_{k,j} + G^{n+1}_{k,j}, \qquad j \neq l-1, l, l+1; \tag{4.14}$$

$$\Delta_\tau e^{n+1}_{i,l} = L_h e^{n+1}_{i,l} + G^{n+1}_{i,l}, \qquad i \neq k-1, k, k+1; \tag{4.15}$$

$$\Delta_\tau e^{n+1}_{k,l-1} = L_h e^{n+1}_{k,l-1} + G^{n+1}_{k,l-1} - \lambda\tau\tilde{a}^{(2)}_{k,l-1} a^{(2)}_{k,l-\frac{1}{2}}\Delta^2_\tau e^n_{k,l}, \tag{4.16}$$

$$\Delta_\tau e^{n+1}_{k,l+1} = L_h e^{n+1}_{k,l+1} + G^{n+1}_{k,l+1} - \lambda\tau\left(\tilde{a}^{(2)}_{k,l+1} a^{(2)}_{k,l+\frac{1}{2}} + h\tilde{b}^{(2)}_{k,l+1}\right)\Delta^2_\tau e^n_{k,l}, \tag{4.17}$$

$$\Delta_\tau e^{n+1}_{k-1,l} = L_h e^{n+1}_{k-1,l} + G^{n+1}_{k-1,l} - \lambda\tau\tilde{a}^{(1)}_{k-1,l} a^{(1)}_{k-\frac{1}{2},l}\Delta^2_\tau e^n_{k,l}, \tag{4.18}$$

$$\Delta_\tau e^{n+1}_{k+1,l} = L_h e^{n+1}_{k+1,l} + G^{n+1}_{k+1,l} - \lambda\tau\left(\tilde{a}^{(1)}_{k+1,l} a^{(1)}_{k-\frac{1}{2},l} - \tilde{b}^{(1)}_{k+1,l} h\right)\Delta^2_\tau e^n_{k,l}, \tag{4.19}$$

and

$$\begin{cases} e^0_{i,j} = 0, \quad i, j = 0, 1, \cdots, J; \\ e^{n+1}_{i,0} = e^{n+1}_{i,J} = e^{n+1}_{0,j} = e^{n+1}_{J,j} = 0, \quad i, j = 0, 1, \cdots, J, \;\; n \geq 0, \end{cases} \tag{4.20}$$

where $G^{n+1}_{i,j}$ is the local truncation error at point (x_i, y_j, t^{n+1}).

Let $w^{n+1}_{i,j} = \Delta_\tau e^{n+1}_{i,j}$. Similar to the derivation of inequality (3.23), by multiplying $w^{n+1}_{i,j} h^2$ $(1 \leq j \leq J-1)$ to both sides of (4.5)-(4.19) and summing the results up, for h small enough, we get

$$\begin{aligned}\|e^{n+1}_h\|^2_1 \leq C \Bigg(& \|e^1_h\|^2_1 + \tau^2 \sum_{j=1}^{J-1} \left(\tilde{a}^{(1)}_{k+1,j} a^{(1)}_{k+\frac{1}{2},j} + \tilde{a}^{(1)}_{k-1,j} a^{(1)}_{k-\frac{1}{2},j} + h\tilde{b}^{(1)}_{k+1,j}\right)|w^1_{k,j}|^2 \\ & + \tau^2 \sum_{i=1}^{J-1} \left(\tilde{a}^{(2)}_{i,l+1} a^{(2)}_{i,l+\frac{1}{2}} + \tilde{a}^{(1)}_{i,l-1} a^{(2)}_{i,l-\frac{1}{2}} + h\tilde{b}^{(2)}_{i,l+1}\right)|w^1_{i,l+1}|^2 + \sum_{m=1}^{n} \sum_{|i-k|\neq 1}^{J} \sum_{|j-l|\neq 1}^{J} \tau h^2 |G^m_{i,j}|^2 \\ & + \sum_{m=1}^{n} \sum_{|i-k|=1} \sum_{j=1}^{J} 2\tau h^2 G^m_{i,j} w^m_{i,j} - \sum_{m=1}^{n} \sum_{i=1}^{J} \sum_{|j-l|=1} \tau h^2 G^m_{i,j} w^m_{i,j} \Bigg),\end{aligned} \tag{4.21}$$

where C is a positive constant independent of τ and h. By the discrete Green formula and the boundary condition (4.20), we have

$$\sum_{j=1}^{J} \sum_{m=0}^{n} 2\tau h^2 G^{m+1}_{k-1,j} w^{m+1}_{k-1,j} = -2h^2 \sum_{j=1}^{J} \left(\sum_{m=1}^{n} (G^{m+1}_{k-1,j} - G^m_{k-1,j}) e^{m+1}_{k-1,j} + G^{m+1}_{k-1,j} e^{m+1}_{k-1,j} \right).$$

Notice that, by a discrete trace theorem, there exists a generic constant C independent of τ and h, such that

$$\sum_{j=1}^{J} |e^{n+1}_{k-1,j}|^2 h \leq C \|e^{n+1}_h\|^2_1.$$

Then, by (4.3), the above two inequalities, and the Young inequality, we have

$$\begin{aligned}
&\sum_{j=1}^{J}\sum_{m=0}^{n} 2\tau G_{k-1,j}^{m+1} h^2 w_{k-1,j}^{m+1} \\
\le &\sum_{j=1}^{J}\Big(\sum_{m=0}^{n}\tau h|e_{k-1,j}^{m+1}|^2+\sum_{m=0}^{n}\tau\Big((\tau+h^2+\frac{\tau^2}{h^2})h\Big)^2 h+\frac{2}{\epsilon}(G_{k-1,j}^{m+1}h)^2h+\frac{\epsilon}{2}|e_{k-1,j}^{n+1}|^2h\Big) \\
\le &C\sum_{m=0}^{n}\tau\|e_h^{m+1}\|_1^2+C\Big((\tau+h^2+\frac{\tau^2}{h^2})h\Big)^2+C\epsilon\|e_h^{n+1}\|_1^2,
\end{aligned} \tag{4.22}$$

where ϵ is any positive real number due to the Young inequality. Similarly, we may deal with the terms

$$\sum_{j=1}^{J}\sum_{m=0}^{n}2\tau G_{k+1,j}^{m+1}h^2w_{k+1,j}^{m+1},\quad \sum_{i=1}^{J}\sum_{m=0}^{n}2\tau G_{i,l-1}^{m+1}h^2w_{i,l-1}^{m+1},\quad \sum_{i=1}^{J}\sum_{m=0}^{n}2\tau G_{i,l+1}^{m+1}h^2w_{i,l+1}^{m+1}$$

respectively. Then, by (4.21) and (4.22), we have

$$\begin{aligned}
\|e_h^{n+1}\|_1^2 \le &C\|e_h^1\|_1^2+C\Big(\frac{\tau^2}{h}+\tau^2\Big)\|w_h^1\|_\infty^2+C(\tau+h^2)^2 \\
&+C\sum_{m=0}^{n}\tau\|e_h^{m+1}\|_1^2+C\Big((\tau+h^2+\frac{\tau^2}{h^2})h\Big)^2+C\epsilon\|e_h^{n+1}\|_1^2.
\end{aligned} \tag{4.23}$$

Taking ϵ to be small enough such that $C\epsilon < 1$, then, we have

$$\begin{aligned}
\|e_h^{n+1}\|_1^2 \le &C\|e_h^1\|_1^2+C\Big(\frac{\tau^2}{h}+\tau^2\Big)\|w_h^1\|_\infty^2+C(\tau+h^2)^2 \\
&+C\sum_{l=0}^{n}\tau\|e_h^{l+1}\|_1^2+C\Big((\tau+h^2+\frac{\tau^2}{h^2})h\Big)^2.
\end{aligned} \tag{4.24}$$

By the discrete Gronwall inequality and (4.24), we have for τ small enough that,

$$\|e_h^{n+1}\|_1^2 \le C\|e_h^1\|_1^2+C\Big(\frac{\tau^2}{h}+\tau^2\Big)\|w_h^1\|_\infty^2+C(\tau+h^2)^2+C\Big(\frac{\tau^2}{h}\Big)^2. \tag{4.25}$$

Since the first level is computed by full-implicit scheme, from (2.4), we have

$$w_{i,j}^1-L_he_{i,j}^1=G_{i,j}^1. \tag{4.26}$$

Note that $e_j^0=0$ and $e_{i,j}^1=\tau w_{i,j}^1$, we have

$$w_{i,j}^1=\tau L_hw_{i,j}^1+G_{i,j}^1. \tag{4.27}$$

After multiplying $w^1_{i,j}h^2$ to the above equation and summing up, similarly to the derivation of (3.8), we get

$$
\begin{aligned}
\frac{\tau\|w^1_h\|_1^2}{2} =& -\sum_{i=1}^{J-1}\sum_{j=1}^{J-1}\Big(1+\frac{\tau}{2}c_{i,j}\Big)|w^1_{i,j}|^2h^2+\sum_{i=1}^{J-1}\sum_{j=1}^{J-1}G^1_{i,j}|w^1_{i,j}|h^2\\
&-\sum_{j=1}^{J-1}\sum_{i=0}^{J-1}a^{(1)}_{i+\frac{1}{2},j}\left[\frac{\tau}{2}\tilde{a}^{(1)}_{i+1,j}|w^1_{i+1,j}-w^1_{i,j}|^2+\tau\delta w^1_{i+\frac{1}{2},j}\frac{\partial\tilde{a}^{(1)}(x,y)}{\partial x}(\xi_i)w^1_{i,j}h^2\right]\\
&-\sum_{i=1}^{J-1}\sum_{j=0}^{J-1}a^{(2)}_{i,j+\frac{1}{2}}\left[\frac{\tau}{2}\tilde{a}^{(2)}_{i,j+1}|w^1_{i,j+1}-w^1_{i,j}|^2+\tau\delta w^1_{i,j+\frac{1}{2}}\frac{\partial\tilde{a}^{(2)}(x,y)}{\partial y}(\eta_j)w^1_{i,j}h^2\right]\\
&-\frac{h\tau}{2}\sum_{i=1}^{J-1}\sum_{j=1}^{J-1}\tilde{b}^{(1)}_{i,j}|w^1_{i,j}-w^1_{i-1,j}|^2-\tau h^2\sum_{i=1}^{J-1}\sum_{j=1}^{J-1}\tilde{b}^{(1)}_{i,j}\delta w^1_{i-\frac{1}{2},j}w^1_{i-1,j}\\
&-\frac{h\tau}{2}\sum_{i=1}^{J-1}\sum_{j=1}^{J-1}\tilde{b}^{(2)}_{i,j}|w^1_{i,j}-w^1_{i,j-1}|^2-\tau h^2\sum_{i=1}^{J-1}\sum_{j=1}^{J-1}\tilde{b}^{(2)}_{i,j}\delta w^1_{i,j-\frac{1}{2}}w^1_{i,j-1}.
\end{aligned}
$$

It follows from the above equation that

$$\tau\|w^1_h\|_1^2\le C\tau^2\|w^1_h\|_1^2+C\sum_{i=1}^{J-1}\sum_{j=1}^{J-1}|G^1_{i,j}|^2h^2$$

for some constant $C>0$. So for τ small enough (depending only on the coefficients), we have

$$\|e^1_h\|_1^2\le C\tau\sum_{i=1}^{J-1}\sum_{j=1}^{J-1}|G^1_{i,j}|^2h^2\le C\tau(\tau+h^2)^2. \tag{4.28}$$

Moreover, by (4.27) and the discrete extreme value principle, it is easy to show that

$$\|w^1_h\|_\infty\le\|G^1_h\|_\infty\le C(\tau+h^2). \tag{4.29}$$

By (4.25) and (4.28)-(4.29), we have

$$\|e^{n+1}_h\|_1^2\le C\Big(\tau+h^2+\frac{\tau^2}{h}\Big)^2, \tag{4.30}$$

which is the desired estimate (4.4). □

5. Numerical examples

In this section, we present some numerical examples which confirm the theoretical results in the above sections and demonstrate that the EIPCMU2D scheme enjoys good stability, accuracy and efficiency.

5.1. A model equation

For the test problem, we consider the transport of a rotating Gaussian pulse in a two-dimensional square domain, which has been widely used for convection diffusion problems to test for numerical artifacts of different schemes such as the numerical instability spurious dispersion, undershoot or overshoot (see, e.g., [5,6,14,15]). The related equations are as following,

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla(\mathbf{A}\cdot\nabla u - \mathbf{b}u), & (x,y)\in\Omega,\quad t\in(0,T], \\ u(x,y,0) = u_0(x,y), & (x,y)\in\Omega, \\ u(x,y,t) = g(x,y,t), & (x,y)\in\partial\Omega,\quad t\in(0,T], \end{cases} \tag{5.1}$$

where the velocity field is given by $b_1 = -4y$, $b_2 = 4x$, and the diffusion tensor is taken as $\mathbf{A} = D\mathbf{I}$ with D being a positive constant. The source $f = 0$ and the analytical solution for this problem is given by

$$u(x,y,t) = \frac{2\sigma^2}{2\sigma^2 + 4Dt}\exp\Big(-\frac{(x^\star - x_c)^2 + (y^\star - y_c)^2}{2\sigma^2 + 4Dt}\Big),$$

where

$$\begin{cases} x^\star = (\cos 4t)x + (\sin 4t)y, \\ y^\star = -(\sin 4t)x + (\cos 4t)y, \end{cases}$$

(x_c, y_c) and σ are the center and standard deviation, respectively. Here we take $\Omega = [-0.5, 0.5]\times[-0.5,-0.5], T = \pi/2, D = 0.005, (x_c, y_c) = (-0.25, 0), \sigma = 0.0447$. The initial value $u_0(x,y)$, and boundary condition $g(x,y,t)$ are decided by the above exact solution.

The experiments are carried out on the LSSC II at the Lab for Scientific and Engineering Computation, Chinese Academy of Sciences. The system was the first terascale cluster built in China for research and educational use, see [22] for system specifications and benchmark performances.

5.2. Summary of numerical results

To examine the accuracy of the spatial discretization of the EIPCMU2D scheme, we first take a small enough time step $\tau = 1.0$E-6. Four processors are used to calculate the numerical solutions on meshes of different sizes. At this moment, we adopt the domain decomposition in Fig. 3. For illustration, both the initial analytic solution for $T = 0$ and the analytic solution for $T = \pi/2$ on meshes with a 128×128 spatial mesh are given in Fig. 2. We note that the pulse is initially situated at $(-0.25, 0)$, and by the time $T = \pi/2$, the center of the pulse gets moved round and arrives at $(-0.25, 0)$, but the peak of the pulse decreases due to the diffusion.

In Table 1, the errors between the numerical solution and the exact solution in the energy norm are given for different spatial meshes. By comparing errors on different spatial meshes, the convergence order r is also computed. It is easy to see that for small enough time step, the convergence order of the proposed parallel schemes tends to two

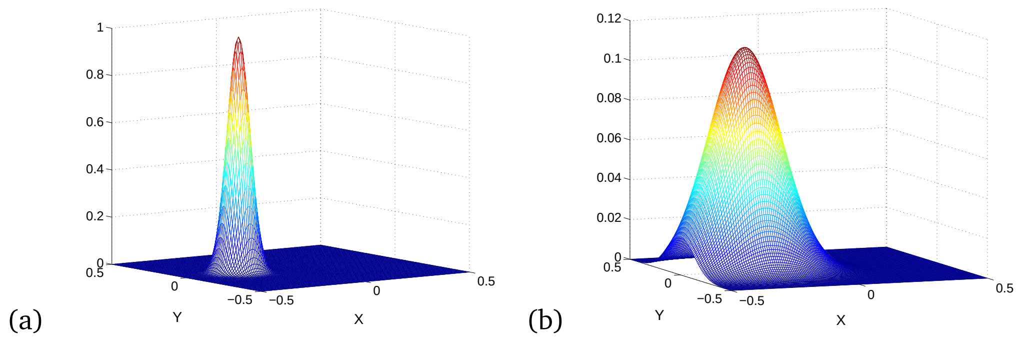

Figure 2: (a) The initial conditions at $T=0$; and (b) the exact solution at $T=\pi/2$.

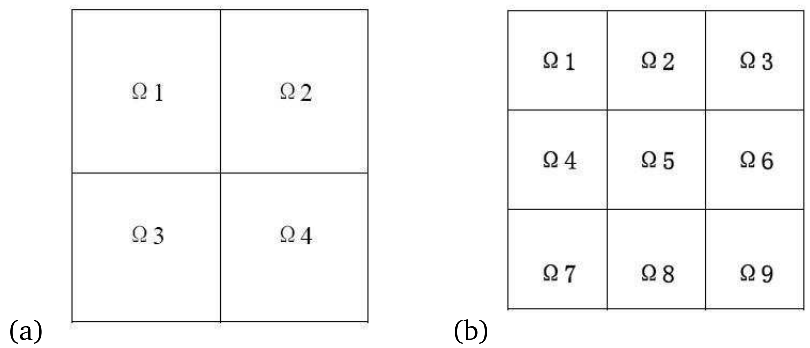

Figure 3: The domain decomposition with (a) four subdomains, and (b) nine subdomains.

as the mesh sizes increases, which is consistent with the theoretical results obtained in Section 4.

In Table 2, we illustrate the numerical results when both the spatial mesh sizes and the time step are varied. Here again, we take $T = \pi/2$ and four processors are employed. It can be seen, from Table 2, that the error in energy norm begins increasing when spatial mesh size becomes sufficiently small with respect to the time step, which is a reflection that the term τ^2/h starts to affect the error when h is small enough. This observation is again consistent with the theoretical results in Theorem 4.1.

The data in Table 2 also illustrate the unconditional stability of our EIPCMU2D scheme. It is evident from the table that on a given spatial mesh, even with significant increase in τ, the numerical solutions still show good convergence. These numerical results verify the theoretical results on the stability given in Section 3.

In addition, in Table 3, the high parallel efficiency of the EIPCMU2D scheme is presented. In these runs, the spatial mesh is taken as 240×240 together with $T=\pi/2$ and τ=1.5625E-5. The CPUs in the table denote the number of the processors, T_{all} is the total computation time, S_p is the relative speedup, E_{ff} is the parallel efficiency. The domain decompositions with CPUs=4 and CPUs=9 are illustrated in Fig. 3. The domain decompositions with CPUs=16 and CPUs=25 are obtained by analogy. The solution for a single CPU

Table 1: The errors of the EIPCMU2D scheme for meshes with different mesh sizes.

mesh	32×32	64×64	128×128	256×256	512×512
$\|u-U\|_1$	1.966E-2	9.371E-3	3.568E-3	1.126E-3	3.175E-4
r	-	1.07	1.39	1.66	1.83

Table 2: The errors of the EIPCMU2D scheme for different meshes and time steps.

mesh$\backslash\tau$	6.25E-5	2.5E-4	1.0E-3	4.0E-3	1.6E-2
120×120	3.991E-3	4.146E-3	4.729E-3	6.598E-3	8.101E-3
240×240	1.321E-3	1.503E-3	2.188E-3	4.279E-3	4.279E-3
480×480	4.209E-4	6.107E-4	1.318E-3	3.359E-3	6.661E-3

Table 3: The parallelism of the EIPCMU2D scheme with 240×240 mesh and τ=1.5625E-5.

CPUs	1	4	9	16	25
$\|u-U\|_1$	1.275E-3	1.275E-3	1.275E-3	1.275E-3	1.275E-3
T_{all}(sec.)	6596.46	1701.29	690.32	334.49	180.92
S_p	1.00	3.88	9.56	19.72	36.46
E_{ff}(%)	100	97	106	123	146

corresponds to the fully implicit scheme. From Table 3, we can see that the EIPCMU2D scheme studied here has a super-linear speedup and enjoys high efficiency.

6. Conclusion

In this paper, the multi-dimensional extension of an explicit-implicit predictor-corrector modified upwind (EIPCMU2D) difference scheme with intrinsic parallelism for time dependent convection diffusion equations is studied. The unconditional stability and second-order (in space) convergence are established by the energy method. Extensions to higher dimensional cases and nonuniform grids can be made in the same spirit. The advantage of the EIPCMU2D algorithm over other similar type of schemes can be best illustrated in the two and higher dimensional settings due to the simplicity in the implementation and the flexibility in the subdomain partitions. Our analysis and computational results demonstrate the good performance of the EIPCMU2D algorithm and its potential. The current study is limited to model equations with continuous coefficients, further investigation on the performance of the EIPCMU2D type methods to linear convection-diffusion equations involving discontinuous coefficients can be of significant interests. Applications to concrete problems in real world applications and complex nonlinear time dependent systems will be studied further in the future.

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