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Generalized Normal Derivatives and Their Applications in DDMs with Nonmatching Grids and DG Methods

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Abstract. A class of normal-like derivatives for functions with low regularity defined on Lipschitz domains are introduced and studied. It is shown that the new normal-like derivatives, which are called the *generalized normal derivatives*, preserve the major properties of the existing standard normal derivatives. The generalized normal derivatives are then applied to analyze the convergence of domain decomposition methods (DDMs) with nonmatching grids and discontinuous Galerkin (DG) methods for second-order elliptic problems. The approximate solutions generated by these methods still possess the optimal energy-norm error estimates, even if the exact solutions to the underlying elliptic problems admit very low regularities.

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1. Introduction

It is known that normal derivative of function is well defined on boundary of any Lipschitz domain, provided that the underlying function is smooth enough (see, [13, 20, 21]). But, if the function under consideration has low regularity only, an additional assumption is needed to guarantee the existence of this normal derivative. For smooth domains, this topic was studied in details (see Chapter 2 in [20]). However, the results (e.g.,Theorem 7.3) in [20] can not be extended to the case of Lipschitz domain. The main difficulty rests in the fact that normal vector is discontinuous at the corners of nonsmooth boundary. For Lipschitz domain, only a few results on normal derivative of low regularity function have been obtained by [11] and [13].

On the other hand, the normal derivative indeed plays an important role in numerical analysis of boundary value problems. For example, one has to use normal derivatives on the underlying interface in analysis of convergence of DDMs with nonmatching grids and of DG methods for elliptic boundary value problems of second order (see

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[1, 3, 5, 7, 15, 18, 19, 23, 24]). Most existing error estimates for these methods was derived only under the assumption that the underlying analytic solution has high regularity, since, as we mentioned above, the normal derivative is not well defined without such high regularity. But, this assumption can not be satisfied in some applications. This problem was investigated for DDMs with nonmatching grids in [4], in which an error estimate was derived for a particular case with low regularity by using Hilbert interpolation technique. However, Hilbert interpolation technique is not available for the general case that the analytic solution has different regularities on different subdomains.

In the present paper, we try to extend Theorems 6.5 and 7.3 of Chapter 2 in [20] to second-order elliptic problems on Lipschitz domains in a new manner. To this end, we introduce a class of *generalized normal derivative*, which is defined by a Green-like formula. The generalized normal derivative is well defined under very weak assumptions. Some similar functionals with this generalized normal derivative were mentioned in literatures. However, to our knowledge, this kind of functional has never been studied in details before. It will be shown that the generalized normal derivative preserves the main properties of the usual normal derivative, although it can not be understood as the usual normal derivative. Such generalized normal derivative will be used to derive the optimal energy error estimates for the approximations generated by DDMs with nonmatching grids or by the DG methods for second-order elliptic problems with low regularity solution. An advantage of the new approach is that one can work the case that the loading function of the equation does not belong to L^2 space (compare [4]).

The outline of the remainder of the paper is as follows. In Section 2, we introduce generalized normal derivatives and investigate main properties of the generalized normal derivatives. In Section 3, we apply the generalized normal derivative to analyzing convergence of DDMs with nonmatching grids and DG methods for second-order elliptic problems with low regularity solution. A Hilbert interpolation result on subspace is derived in Appendix.

2. Generalized normal derivatives

This section is devoted to introducing and studying a class of generalized normal derivative.

2.1. Sobolev spaces

In the rest of the paper, we will use various Sobolev spaces repeatedly.

Let $\hat{\Omega} \subset \mathscr{R}^n$ ($n = 2,\ 3$) be a bounded and connected Lipschitz domain with piecewise smooth boundaries. For convenience, a smooth piece of $\partial\hat{\Omega}$ is called a *face* of $\partial\hat{\Omega}$ in the following. In applications, the domain $\hat{\Omega}$ usually represents a convex subdomain of the underlying Lipschitz domain Ω, which includes polygon (in $\mathscr{R}^2$) and polyhedron (in $\mathscr{R}^3$) with planed or curved faces. Denote by $H^\sigma(\hat{\Omega})$ ($\sigma \in [0,2]$) and $H_0^\sigma(\hat{\Omega})$ ($\sigma \in (0,1]$) the usual Sobolev spaces associated with *weak* derivatives (see [2,13,21]). The norm in $H^\sigma(\hat{\Omega})$ is denoted by $\|\cdot\|_{\sigma,\hat{\Omega}}$. Let $L_{loc}(\hat{\Omega})$ denote the space of locally integrable functions on $\hat{\Omega}$, and let $(\cdot,\cdot)_{\hat{\Omega}}$ denote the $L^2(\hat{\Omega})$-inner product. For $\sigma \in [0,1]$, define (refer to [21])

$$H^{-\sigma}(\hat{\Omega}) = \left\{ v \in L_{loc}(\hat{\Omega}) : \sup_{w \in H^{\sigma}(\hat{\Omega})} \frac{|(v,w)_{\hat{\Omega}}|}{\|w\|_{\sigma,\hat{\Omega}}} < +\infty \right\}$$

and

$$\|v\|_{-\sigma,\hat{\Omega}} = \sup_{w \in H^{\sigma}(\hat{\Omega})} \frac{|(v,w)_{\hat{\Omega}}|}{\|w\|_{\sigma,\hat{\Omega}}}, \quad v \in H^{-\sigma}(\hat{\Omega}).$$

It is certain to regard $v_1 = v_2$ in $H^{-\sigma}(\hat{\Omega})$ if $(v_1 - v_2, w)_{\hat{\Omega}} = 0$ for any $w \in H^{\sigma}(\hat{\Omega})$. Then, $\|\cdot\|_{-\sigma,\hat{\Omega}}$ is indeed a norm, and $H^{-\sigma}(\hat{\Omega})$ is a Banach space. It is clear that

$$H^{\sigma_2}(\hat{\Omega}) \subset H^{\sigma_1}(\hat{\Omega}) \subset L^2(\hat{\Omega}) \subset H^{-\sigma_1}(\hat{\Omega}) \subset H^{-\sigma_2}(\hat{\Omega}), \quad 0 \le \sigma_1 < \sigma_2 \le 1.$$

Accordingly, we define

$$H_0^{-\sigma}(\hat{\Omega}) = \left\{ v \in L_{loc}(\hat{\Omega}) : \sup_{w \in H_0^{\sigma}(\hat{\Omega})} \frac{|(v,w)_{\hat{\Omega}}|}{\|w\|_{\sigma,\hat{\Omega}}} < +\infty \right\}, \quad \sigma \in [0,1].$$

Note that both $H^{-\sigma}(\hat{\Omega})$ and $H_0^{-\sigma}(\hat{\Omega})$ are different slightly from the standard dual space. In fact, we have $H^{-\sigma}(\hat{\Omega}) \subset H_0^{-\sigma}(\hat{\Omega}) \subset (H_0^{\sigma}(\hat{\Omega}))'$ (in the isomorphism sense).

Throughout this paper all differential operators are understood as the *weak* differential operators defined by the integration by parts.

Let $a \in L^{\infty}(\hat{\Omega})$ be a function with positive low bound. For convenience, define A as the operator $Av = -div(a\nabla v)$. For $\sigma \in [1,2]$, define the space

$$H_A^{\sigma}(\hat{\Omega}) = \{v : v \in H^{\sigma}(\hat{\Omega}),\ Av \in H^{\sigma-2}(\hat{\Omega})\}.$$

The space is equipped with the norm

$$\|v\|_{H_A^{\sigma}(\hat{\Omega})} = \left(\|v\|_{\sigma,\hat{\Omega}}^2 + \|Av\|_{\sigma-2,\hat{\Omega}}^2 \right)^{\frac{1}{2}}.$$

Then, $H_A^{\sigma}(\hat{\Omega})$ is a Banach space. In fact, we have

Lemma 2.1. *The space $H_A^{\sigma}(\hat{\Omega})$ is a Hilbert space for each $\sigma \in [1,2]$.*

Proof. Let $\Lambda : H^1(\hat{\Omega}) \to L^2(\hat{\Omega})$ be a positive and self-adjoint operator in $L^2(\hat{\Omega})$ scalar product, such that $H^1(\hat{\Omega})$ is just the definition domain of Λ (see pp. 129-137 in [21] for the details). Then, $H^{\theta}(\hat{\Omega})$ is also the definition domain of Λ^{θ} for each $\theta \in [0,1]$. Accordingly, let $\tilde{H}^{-\theta}(\hat{\Omega})$ denote the definition domain of $\Lambda^{-\theta}$ (namely, $\Lambda^{-\theta}(\tilde{H}^{-\theta}(\hat{\Omega})) = L^2(\hat{\Omega})$). Set

$$\langle \cdot,\cdot \rangle_{\theta,\hat{\Omega}} = (\Lambda^{-\theta}\cdot,\ \Lambda^{\theta}\cdot)_{\hat{\Omega}}.$$

Then, $\tilde{H}^{-\theta}(\hat{\Omega})$ can be viewed as the dual space of $H^{\theta}(\hat{\Omega})$ associated with the duality pairing $\langle \cdot,\cdot \rangle_{\theta,\hat{\Omega}}$. It can be verified that, for any $v \in H_A^{\sigma}(\hat{\Omega})$ ($\sigma \in [1,2]$), there exists a $f_v \in \tilde{H}^{\sigma-2}(\hat{\Omega})$ such that

$$(Av, w)_{\hat{\Omega}} = \langle f_v, w \rangle_{2-\sigma,\hat{\Omega}}, \quad \forall w \in H^{2-\sigma}(\hat{\Omega}).$$

Moreover, we have

$$\|Av\|_{\sigma-2,\hat{\Omega}} = \|\Lambda^{\sigma-2} f_v\|_{0,\hat{\Omega}}.$$

Thus, $H^\sigma_A(\hat{\Omega})$ is a Hilbert space with respect to the inner product

$$(v,w)_{H^\sigma(\hat{\Omega})} + (\Lambda^{\sigma-2} f_v, \Lambda^{\sigma-2} f_w)_{\hat{\Omega}}.$$

This completes the proof of the lemma. □

Let G denote $\partial\hat{\Omega}$ itself or a face of $\partial\hat{\Omega}$. For $\sigma \in (0,1)$, the Sobolev space $H^\sigma(G)$ is defined as usual (see [11,13,16]). The norm in $H^\sigma(G)$ is defined by

$$\|\varphi\|_{\sigma,G} = \left(\|\varphi\|^2_{0,G} + \int_G \int_G \frac{(\varphi(p) - \varphi(q))^2}{|p-q|^{n-1+2\sigma}} ds(p) ds(q) \right)^{\frac{1}{2}}, \quad \sigma \in (0,1).$$

For a face $\mathrm{F} \subset \partial\hat{\Omega}$, let $H^{\frac{1}{2}}_{00}(\mathrm{F})$ be the space with the norm

$$\|\varphi\|_{H^{\frac{1}{2}}_{00}(\mathrm{F})} = \left(\|\varphi\|^2_{\frac{1}{2},\mathrm{F}} + \int_{\mathrm{F}} \frac{\varphi^2(p)}{dist(p,\partial\mathrm{F})} ds(p) \right)^{\frac{1}{2}}.$$

For $\sigma \in [0,1)$, let $H^{-\sigma}(G)$ denote the dual space of $H^\sigma(G)$ associated with the *pivot space* $L^2(G)$ (namely, the dual space of $L^2(G)$ is $L^2(G)$ itself), and let $\langle\cdot,\cdot\rangle_{\sigma,G}$ be the duality pairing between $H^{-\sigma}(G)$ and $H^\sigma(G)$. It is known that $\langle\cdot,\cdot\rangle_{\sigma,G}$ can be viewed as an extension of the $L^2(G)$ scalar product in the sense that

$$\langle w,v\rangle_{\sigma,G} = \int_G wv dx \ \text{ for } \ \mathrm{w} \in \mathrm{L}^2(\mathrm{G})$$

and $v \in H^\sigma(G)$ (refer to Section 4.4 of [21]). It is easy to see that

$$\langle\mu,\varphi\rangle_{\sigma_1,G} = \langle\mu,\varphi\rangle_{\sigma_2,G}, \quad \forall\mu \in H^{-\sigma_1}(G),\ \varphi \in H^{\sigma_2}(G)\ (0 \le \sigma_1 < \sigma_2 < 1). \tag{2.1}$$

The norm of $H^{-\sigma}(G)$ is defined as

$$\|\mu\|_{-\sigma,G} = \sup_{\varphi \in H^\sigma(G)} \frac{|\langle\mu,\varphi\rangle_{\sigma,G}|}{\|\varphi\|_{\sigma,G}}, \quad \mu \in H^{-\sigma}(G)\ (\sigma \in [0,1)). \tag{2.2}$$

Accordingly, one can define the dual space $(H^{\frac{1}{2}}_{00}(\mathrm{F}))'$.

For convenience, the symbols $\lesssim$, $\gtrsim$ and $\eqsim$ will be used in the rest of this paper. That $x_1 \lesssim y_1$, $x_2 \gtrsim y_2$ and $x_3 \eqsim y_3$, mean that $x_1 \le C_1 y_1$, $x_2 \ge c_2 y_2$ and $c_3 x_3 \le y_3 \le C_3 x_3$ for some constants C_1, c_2, c_3 and C_3.

2.2. Definitions of generalized normal derivatives

Let us first give two well-known results (refer to [16], see also Theorem 1.4.2.4 and Theorem 1.4.4.6 in [13]), which can be proved in the standard way.

Proposition 2.1. The space $C_0^\infty(\hat\Omega)$ is dense in $H^\sigma(\hat\Omega)$ for $\sigma\in[0,\frac{1}{2}]$. In particular, we have $H_0^\sigma(\hat\Omega)=H^\sigma(\hat\Omega)$ for $\sigma\in(0,\frac{1}{2})$.

Proposition 2.2. Each (weak) first-order partial differential operator on $\hat\Omega$ is a bounded linear mapping from $H^\sigma(\hat\Omega)$ into $H^{\sigma-1}(\hat\Omega)$ (resp. $H_0^{\sigma-1}(\hat\Omega)$) for $\sigma\in(\frac{1}{2},2]$ (resp. $\sigma\in[0,\frac{1}{2})$). In particular, we have $H_A^\sigma(\hat\Omega)=H^\sigma(\hat\Omega)$ for $\sigma\in(\frac{3}{2},2]$, provided that the known function a satisfies a suitable smoothness.

The desired generalized normal derivative is associated with a Green-like formula.

Theorem 2.1. *Assume that $v\in H_A^{1+s}(\hat\Omega)$ with some $s\in[0,\frac{1}{2})$. Then, there exists a $\mu_{\partial\hat\Omega}(v)\in H^{s-\frac{1}{2}}(\partial\hat\Omega)$ such that*

$$\int_{\hat\Omega}a\nabla v\cdot\nabla w dp+\int_{\hat\Omega}Av\cdot w dp=\langle\mu_{\partial\hat\Omega}(v),w\rangle_{\frac{1}{2}-s,\partial\hat\Omega},\quad\forall w\in H^1(\hat\Omega).\tag{2.3}$$

Moreover, we have

$$\|\mu_{\partial\hat\Omega}(v)\|_{s-\frac{1}{2},\partial\hat\Omega}\lesssim\|v\|_{H_A^{1+s}(\hat\Omega)}.\tag{2.4}$$

Proof. For a $\varphi\in H^{\frac{1}{2}-s}(\partial\hat\Omega)$, let $u_\varphi\in H^{1-s}(\hat\Omega)$ be the unique solution satisfying $u_\varphi|_{\partial\hat\Omega}=\varphi$ and $div(a\nabla u_\varphi)=0$. Then, we have by Lemmas 3.7 and 4.2 of [11] (see also [16])

$$\|a\frac{\partial u_\varphi}{\partial\mathbf{n}}\|_{-(\frac{1}{2}+s),\partial\hat\Omega}\lesssim\|\varphi\|_{\frac{1}{2}-s,\partial\hat\Omega}\ \text{ and }\ \|u_\varphi\|_{1-s,\hat\Omega}\lesssim\|\varphi\|_{\frac{1}{2}-s,\partial\hat\Omega}.\tag{2.5}$$

Since $v\in H^{1+s}(\hat\Omega)$ with $s\in[0,\frac{1}{2})$, the trace theorem yields

$$\|v\|_{\frac{1}{2}+s,\partial\hat\Omega}\lesssim\|v\|_{1+s,\hat\Omega}.\tag{2.6}$$

Define

$$F(\varphi)=\langle a\frac{\partial u_\varphi}{\partial\mathbf{n}}|_{\partial\hat\Omega},v\rangle_{\frac{1}{2}+s,\partial\hat\Omega}+\int_{\hat\Omega}Av\cdot u_\varphi dp,\quad\varphi\in H^{\frac{1}{2}-s}(\partial\hat\Omega).$$

It follows by (2.5) and (2.6) that

$$\begin{aligned}|F(\varphi)|&\le\|a\frac{\partial u_\varphi}{\partial\mathbf{n}}\|_{-(\frac{1}{2}+s),\partial\hat\Omega}\cdot\|v\|_{\frac{1}{2}+s,\partial\hat\Omega}+\|Av\|_{s-1,\hat\Omega}\cdot\|u_\varphi\|_{1-s,\hat\Omega}\\&\lesssim\|v\|_{1+s,\hat\Omega}\cdot\|\varphi\|_{\frac{1}{2}-s,\partial\hat\Omega}+\|Av\|_{s-1,\hat\Omega}\cdot\|u_\varphi\|_{1-s,\hat\Omega}\\&\lesssim\|v\|_{H_A^{1+s}(\hat\Omega)}\cdot\|\varphi\|_{\frac{1}{2}-s,\partial\hat\Omega}.\end{aligned}$$

Thus, $F(\varphi)$ is a bounded linear functional on $H^{\frac{1}{2}-s}(\partial\hat{\Omega})$. Since $\langle\cdot,\cdot\rangle_{\frac{1}{2}-s,\partial\hat{\Omega}}$ is the duality pairing between $H^{s-\frac{1}{2}}(\partial\hat{\Omega})$ and $H^{\frac{1}{2}-s}(\partial\hat{\Omega})$, there exists a $\mu_{\partial\hat{\Omega}}(v)\in H^{s-\frac{1}{2}}(\partial\hat{\Omega})$ such that

$$\langle\mu_{\partial\hat{\Omega}}(v),\varphi\rangle_{\frac{1}{2}-s,\partial\hat{\Omega}}=F(\varphi),\quad \forall\varphi\in H^{\frac{1}{2}-s}(\partial\hat{\Omega}). \tag{2.7}$$

Moreover, we have

$$\|\mu_{\partial\hat{\Omega}}(v)\|_{s-\frac{1}{2},\partial\hat{\Omega}}\lesssim\|F\|\lesssim\|v\|_{H_A^{1+s}(\hat{\Omega})}.$$

It suffices to prove the Green-like formula (2.3) for such $\mu_{\partial\hat{\Omega}}(v)$.

Let $w\in H^1(\hat{\Omega})$, and set $\varphi=w|_{\partial\hat{\Omega}}$. Then, $\varphi\in H^{\frac{1}{2}}(\partial\hat{\Omega})$, $u_\varphi\in H^1(\hat{\Omega})$ and $\chi=w-u_\varphi\in H_0^1(\hat{\Omega})$. Using Green's formula (see [11]), yields

$$F(\varphi)=\int_{\hat{\Omega}}a\nabla u_\varphi\cdot\nabla v\,dp+\int_{\hat{\Omega}}Av\cdot u_\varphi\,dp,\quad \varphi=w|_{\partial\hat{\Omega}}\in H^{\frac{1}{2}}(\partial\hat{\Omega}). \tag{2.8}$$

Note that $C_0^\infty(\hat{\Omega})$ is dense in $H_0^1(\hat{\Omega})$, by the definition of weak derivatives, one can verify that (refer to p. 165 in [2])

$$\int_{\hat{\Omega}}a\nabla v\cdot\nabla\chi\,dp+\int_{\hat{\Omega}}Av\cdot\chi\,dp=0,\quad \chi=w-u_\varphi\in H_0^1(\hat{\Omega}). \tag{2.9}$$

Combining (2.8) and (2.9), leads to

$$\int_{\hat{\Omega}}a\nabla v\cdot\nabla w\,dp+\int_{\hat{\Omega}}Av\cdot w\,dp=F(\varphi),\quad \varphi=w|_{\partial\hat{\Omega}},\quad \forall w\in H^1(\hat{\Omega}).$$

This, together with (2.7), gives (2.3). □

Definition 2.1. Let $v\in H_A^{1+s}(\hat{\Omega})$ with $s\in[0,\frac{1}{2})$. The function $\mu_{\partial\hat{\Omega}}(v)$ defined by Theorem 2.1 is called the generalized normal derivative of v on $\partial\hat{\Omega}$.

Remark 2.1. If the known function $a(p)$ is smooth enough, the linear functional $F(\varphi)$ can be defined directly by

$$F(\varphi)=\int_{\hat{\Omega}}a\nabla v\cdot\nabla u_\varphi\,dp+\int_{\hat{\Omega}}Av\cdot u_\varphi\,dp,\quad \forall\varphi\in H^{\frac{1}{2}-s}(\partial\hat{\Omega}).$$

It can be verified that such linear functional $F(\varphi)$ is also bounded on $H^{\frac{1}{2}-s}(\partial\hat{\Omega})$. Then, the Green-like formula (2.3) is valid for any $w\in H^{1-s}(\hat{\Omega})$. In this case, the condition $v\in H_A^{1+s}(\hat{\Omega})$ becomes a necessary and sufficient condition for the existence of the generalized normal derivatives.

Remark 2.2. The space $C^\infty(\hat{\Omega})$ is not dense in $H_A^{1+s}(\hat{\Omega})$ with $s\in[0,\frac{1}{2})$, so the generalized normal derivative $\mu_{\partial\hat{\Omega}}(v)$ is not a continuous extension of $a\frac{\partial v}{\partial\mathbf{n}}|_{\partial\hat{\Omega}}$. This means that $\mu_{\partial\hat{\Omega}}(v)$ can not be understood as the usual normal derivative. In fact, it seems that there is no simple relation between $\mu_{\partial\hat{\Omega}}(v)$ and the normal vector $\mathbf{n}$.

Remark 2.3. It is clear that the assumption $v \in H_A^{1+s}(\hat{\Omega})$ is the weakest condition such that the functional $F(\varphi)$ is well defined for any $\varphi \in H^{\frac{1}{2}-s}(\partial\hat{\Omega})$. Thus, Theorem 2.1 is sharp. For smooth domains, various trace theorems have been built in [20] (see, for example, Theorems 6.5 and 7.3). It was shown that $\frac{\partial v}{\partial \mathbf{n}}|_{\partial\hat{\Omega}}$ is well defined when $v \in D_A^\sigma(\hat{\Omega}) \subset H_A^\sigma(\hat{\Omega})$ (see p. 227 in [20] for the definition of the space $D_A^\sigma(\hat{\Omega})$). But, such Theorem 6.5 and Theorem 7.3 can not be extended to the current case (see p. 54 and p. 57 in [13]).

Remark 2.4. We fails to prove the result of Theorem 2.1 for $s = \frac{1}{2}$. The main difficulty comes from the fact that the trace inequality (2.6) is not valid for $s = \frac{1}{2}$ yet (see [11] and [16]). It seems that a stronger assumption is needed for such case.

Proposition 2.3. Assume that $v \in H^{\frac{3}{2}}(\hat{\Omega})$ and $Av \in H^{-\frac{1}{2}+\varepsilon}(\hat{\Omega})$ with (small) $\varepsilon > 0$. Then, the usual normal derivative $a\frac{\partial v}{\partial \mathbf{n}}|_{\partial\hat{\Omega}}$ is well defined and $a\frac{\partial v}{\partial \mathbf{n}}|_{\partial\hat{\Omega}} \in L^2(\partial\hat{\Omega})$.

Proof. Let v^0 be the unique solution of the problem:

$$div(a\nabla v^0) = Av \ (\text{in } \hat{\Omega}) \ \text{ and } \ v^0|_{\partial\hat{\Omega}} = 0.$$

This means that $v^0 \in H^{\frac{3}{2}+\varepsilon}(\hat{\Omega})$. Thus, the normal derivative $\frac{\partial v^0}{\partial \mathbf{n}}|_{\partial\hat{\Omega}}$ is well defined, and $\frac{\partial v^0}{\partial \mathbf{n}}|_{\mathrm{F}} \in H^\varepsilon(\mathrm{F})$ for each face $\mathrm{F} \subset \partial\hat{\Omega}$. In particular, we have $a\frac{\partial v^0}{\partial \mathbf{n}}|_{\partial\hat{\Omega}} \in L^2(\partial\hat{\Omega})$. Define $v^{har} = v - v^0$. Then, $v^{har} \in H^{\frac{3}{2}}(\hat{\Omega})$, and v^{har} satisfies the homogeneous equation

$$div(a\nabla v^{har}) = 0 \ (\text{in } \hat{\Omega}).$$

By the result in [17], we have $v^{har}|_{\partial\hat{\Omega}} \in H^1(\partial\hat{\Omega})$. We emphasize that this result is not valid for solution of inhomogeneous problem on general Lipschitz domains (see p. 166 in [16]). It follows, by Lemma 3.7 in [11], that $a\frac{\partial v^{har}}{\partial \mathbf{n}}|_{\partial\hat{\Omega}} \in L^2(\partial\hat{\Omega})$. Since $v = v^0 + v^{har}$, one obtains $a\frac{\partial v}{\partial \mathbf{n}}|_{\partial\hat{\Omega}} \in L^2(\partial\hat{\Omega})$. □

2.3. Properties of the generalized normal derivatives

To our knowledge, there is no literature to investigate the function $\mu_{\partial\hat{\Omega}}(v)$ even if the coefficient $a(p)$ is smooth. In this subsection, we study properties of the generalized normal derivatives $\mu_{\partial\hat{\Omega}}(v)$ in details.

Let $\sigma \in [1, \frac{3}{2}]$. For $p \in [1, +\infty)$, define

$$H_A^{\sigma,p}(\hat{\Omega}) = \{v \in H^\sigma(\hat{\Omega}) : Av \in L^p(\hat{\Omega})\},$$

which is a Banach space with respect to the norm

$$\|v\|_{H_A^{\sigma,p}(\hat{\Omega})} = \left(\|v\|_{\sigma,\hat{\Omega}}^2 + \|Av\|_{L^p(\hat{\Omega})}^2\right)^{\frac{1}{2}}.$$

The following result provides simple condition to guarantee the existence of the generalized normal derivative $\mu_{\partial\hat{\Omega}}(v)$.

Theorem 2.2. (simple sufficient condition) *Let* $s \in [0, \frac{1}{2})$. *Assume that* $v \in H_A^{1+s,p}(\hat{\Omega})$ *with* $p > \frac{2n}{n+2(1-s)}$. *Then the generalized normal derivative* $\mu_{\partial\hat{\Omega}}(v)$ *is well defined. Moreover, we have*

$$\|\mu_{\partial\hat{\Omega}}(v)\|_{s-\frac{1}{2},\partial\hat{\Omega}} \lesssim \|v\|_{H_A^{1+s,p}(\hat{\Omega})}, \tag{2.10}$$

and

$$\int_{\hat{\Omega}} Av \cdot w dp + \int_{\hat{\Omega}} a\nabla v \cdot \nabla w dp = \langle \mu_{\partial\hat{\Omega}}(v), w\rangle_{\frac{1}{2}-s,\partial\hat{\Omega}}, \quad \forall w \in H^1(\hat{\Omega}). \tag{2.11}$$

Proof. Set $q = \frac{p}{p-1}$ for $p > \frac{2n}{n+2(1-s)}$, and let $w \in H^{1-s}(\hat{\Omega})$. By the Sobolev embedding theorem we have $w \in L^q(\hat{\Omega})$ and

$$\|w\|_{L^q(\hat{\Omega})} \lesssim \|w\|_{1-s,\hat{\Omega}}.$$

Then (note that $Av \in L^p(\hat{\Omega})$),

$$\left|\int_{\hat{\Omega}} Av \cdot w dp\right| \leq \|Av\|_{L^p(\hat{\Omega})} \cdot \|w\|_{L^q(\hat{\Omega})} \lesssim \|Av\|_{L^p(\hat{\Omega})} \cdot \|w\|_{1-s,\hat{\Omega}}, \quad \forall w \in H^{1-s}(\hat{\Omega}).$$

Hence, $Av \in H^{s-1}(\hat{\Omega})$. Furthermore, we have $v \in H_A^{1+s}(\hat{\Omega})$. It follows by Theorem 2.1 that the generalized normal derivative $\mu_{\partial\hat{\Omega}}(v)$ is well defined. Moreover, $\mu_{\partial\hat{\Omega}}(v)$ satisfies (2.10) and (2.11). □

Remark 2.5. When $v \in H_A^{1+s,2}(\hat{\Omega})$ with $s \in [0, \frac{1}{2})$, the usual normal derivative $\frac{\partial v}{\partial \mathbf{n}}|_{\partial\hat{\Omega}}$ is well defined (see Lemma 4.3 in [11]). If $\hat{\Omega}$ is a polygon in $\mathscr{R}^2$, then the local normal derivative $\frac{\partial v}{\partial \mathbf{n}}|_{\mathrm{F}} \in (H_{00}^{\frac{1}{2}}(\mathrm{F}))'$ is well defined for any $v \in H_A^{1,p}(\hat{\Omega})$ with $p > 1$ and each face $\mathrm{F} \subset \partial\hat{\Omega}$ (see Theorem 1.5.4 in [14]).

It is well known that, when $v \in H^{1+s}(\hat{\Omega})$ with $s \in (\frac{1}{2}, 1]$, the usual normal derivative $\frac{\partial v}{\partial \mathbf{n}}|_{\partial\hat{\Omega}}$ is well defined. Moreover, we have $\frac{\partial v}{\partial \mathbf{n}}|_{\mathrm{F}} \in H^{s-\frac{1}{2}}(\mathrm{F})$ for each face $\mathrm{F} \subset \partial\hat{\Omega}$, which implies that $\frac{\partial v}{\partial \mathbf{n}}|_{\partial\hat{\Omega}} \in L^2(\partial\hat{\Omega})$.

The following result indicates that the generalized normal derivative is consistent with the usual normal derivative.

Theorem 2.3. (consistency) *Let* $v \in H^{1+s}(\hat{\Omega})$ *with some* $s \in (\frac{1}{2}, 1]$. *Then the generalized normal derivative* $\mu_{\partial\hat{\Omega}}(v)$ *is well defined. Moreover, we have* $\mu_{\partial\hat{\Omega}}(v) \in H^{-\varepsilon}(\partial\hat{\Omega})$ *and*

$$\langle \mu_{\partial\hat{\Omega}}(v) - a\frac{\partial v}{\partial \mathbf{n}}|_{\partial\hat{\Omega}},\ \varphi\rangle_{\varepsilon,\partial\hat{\Omega}} = 0, \quad \forall \varphi \in H^{\varepsilon}(\partial\hat{\Omega}) \tag{2.12}$$

for arbitrarily small $\varepsilon > 0$. *Namely,* $\mu_{\partial\hat{\Omega}}(v) = a\frac{\partial v}{\partial \mathbf{n}}|_{\partial\hat{\Omega}}$ *in* $H^{-\varepsilon}(\partial\hat{\Omega})$ *for arbitrarily small* $\varepsilon > 0$.

Proof. Since $s-1>-\frac{1}{2}$, it follows from Proposition 2.2 that $Av\in H^{s-1}(\hat{\Omega})$. Thus, $v\in H_A^{\frac{3}{2}-\varepsilon}(\hat{\Omega})$ for arbitrarily small $\varepsilon>0$. It follows by Theorem 2.1 that the generalized normal derivative $\mu_{\partial\hat{\Omega}}(v)\in H^{-\varepsilon}(\partial\hat{\Omega})$ is well defined. Moreover, we have

$$\int_{\hat{\Omega}} a\nabla v\cdot\nabla w dp+\int_{\hat{\Omega}} Av\cdot w dp=\langle\mu_{\partial\hat{\Omega}}(v),w\rangle_{\varepsilon,\partial\hat{\Omega}},\quad \forall w\in H^1(\hat{\Omega}). \tag{2.13}$$

On the other hand, by the standard Green's formula and the density of $H^2(\hat{\Omega})$ in $H^{1+s}(\hat{\Omega})$ ($s\in(\frac{1}{2},1]$), one can verify that (note that $\frac{\partial v}{\partial\mathbf{n}}|_{\partial\hat{\Omega}}\in L^2(\partial\hat{\Omega})$)

$$\int_{\hat{\Omega}} a\nabla v\cdot\nabla w dp+\int_{\hat{\Omega}} Av\cdot w dp=\langle a\frac{\partial v}{\partial\mathbf{n}}|_{\partial\hat{\Omega}},w\rangle_{\varepsilon,\partial\hat{\Omega}},\quad \forall w\in H^1(\hat{\Omega}). \tag{2.14}$$

Combining (2.13) and (2.14), we can deduce that

$$\langle\mu_{\partial\hat{\Omega}}(v)-a\frac{\partial v}{\partial\mathbf{n}}|_{\partial\hat{\Omega}},\varphi\rangle_{\varepsilon,\partial\hat{\Omega}}=0,\quad \forall\varphi\in H^{\frac{1}{2}}(\partial\hat{\Omega}).$$

Furthermore, one obtains (2.12) by the fact that $H^{\frac{1}{2}}(\partial\hat{\Omega})$ is dense in $H^{\varepsilon}(\partial\hat{\Omega})$. □

Remark 2.6. One can not infer that $\mu_{\partial\hat{\Omega}}(v)=a\frac{\partial v}{\partial\mathbf{n}}|_{\partial\hat{\Omega}}$ from (2.12), since it is unclear whether we have $\mu_{\partial\hat{\Omega}}(v)\in L^2(\partial\hat{\Omega})$ or not.

To describe a local property of $\mu_{\partial\hat{\Omega}}(v)$, we first give an auxiliary result on zero extensions of functions defined in F.

Lemma 2.2. *Let* F *be a face of* $\partial\hat{\Omega}$. *Assume that* $\varphi\in H^{\sigma}(\mathrm{F})$ *for* $\sigma\in[0,\frac{1}{2})$, *or* $\varphi\in H_{00}^{\frac{1}{2}}(\mathrm{F})$. *Then the zero extension* $\tilde{\varphi}\in H^{\sigma}(\partial\hat{\Omega})$ *for any* $\sigma\in[0,\frac{1}{2}]$. *Moreover, we have*

$$\|\tilde{\varphi}\|_{\sigma,\partial\hat{\Omega}}\lesssim\|\varphi\|_{\sigma,\mathrm{F}} \text{ for } \sigma\in[0,\frac{1}{2}) \text{ and } \|\tilde{\varphi}\|_{\frac{1}{2},\partial\hat{\Omega}}\lesssim\|\varphi\|_{H_{00}^{\frac{1}{2}}(\mathrm{F})}. \tag{2.15}$$

Proof. This result can be viewed as an extension of Theorem 11.4 in Chapter 1 of [20] to Lipschitz domains, but it can not be proved as in [20]. If φ is a piecewise linear polynomial associated with some triangulation on F, the inequality (2.15) can be verified as Lemma 4.10 of [25]. For the general case, one has to use the Hilbert interpolation theory. To this end, define the linear mapping $\imath: L^2(\mathrm{F})\to L^2(\partial\hat{\Omega})$ as $\imath\varphi=\tilde{\varphi}$. It is clear that

$$\|\imath\varphi\|_{0,\partial\hat{\Omega}}=\|\varphi\|_{0,\mathrm{F}},\quad \forall\varphi\in L^2(\mathrm{F})$$

and

$$\|\imath\varphi\|_{1,\partial\hat{\Omega}}=\|\varphi\|_{1,\mathrm{F}},\quad \forall\varphi\in H_0^1(\mathrm{F}).$$

Thus, we have by the Hilbert interpolation theory

$$\|\imath\varphi\|_{\theta,\partial\hat{\Omega}}\lesssim\|\varphi\|_{\theta,\mathrm{F}},\quad \forall\varphi\in[H_0^1(\mathrm{F}),L^2(\mathrm{F})]_{\theta}\quad(0<\theta\le 1). \tag{2.16}$$

It is known that $[H_0^1(\mathrm{F}), L^2(\mathrm{F})]_\theta = H_0^\theta(\mathrm{F})$ when $\theta \neq \frac{1}{2}$, but $[H_0^1(\mathrm{F}), L^2(\mathrm{F})]_{\frac{1}{2}} = H_{00}^{\frac{1}{2}}(\mathrm{F})$. By Proposition 2.1, we have $H_0^\theta(\mathrm{F}) = H^\theta(\mathrm{F})$ for $\theta \in (0, \frac{1}{2})$. Set $\theta = \sigma$ in (3.31), one obtains (2.15). □

For $v \in H_A^{1+s}(\hat{\Omega})$ with $s \in [0, \frac{1}{2})$, let $\mu_{\partial\hat{\Omega}}(v) \in H^{s-\frac{1}{2}}(\partial\hat{\Omega})$ be the generalized normal derivative of v on $\partial\hat{\Omega}$. For a face $\mathrm{F} \subset \partial\hat{\Omega}$, define the linear functional $\mu_\mathrm{F}(v)$ as follows:

$$\langle \mu_\mathrm{F}(v), \varphi \rangle_{\frac{1}{2}-s,\mathrm{F}} = \langle \mu_{\partial\hat{\Omega}}(v), \tilde{\varphi} \rangle_{\frac{1}{2}-s,\partial\hat{\Omega}} \text{ for } \varphi \in H^{\frac{1}{2}-s}(\mathrm{F}) \text{ if } s \in (0, \frac{1}{2}),$$

or for $\varphi \in H_{00}^{\frac{1}{2}}(\mathrm{F})$ if $s = 0$. The functional $\mu_\mathrm{F}(v)$, which is well defined by Lemma 2.2, is called the *locally* generalized normal derivative on F.

The following result will simplify derivations of error estimates in the next section.

Theorem 2.4. (local duality) *Assume that $v \in H_A^{1+s}(\hat{\Omega})$ with $s \in [0, \frac{1}{2})$. Then, $\mu_\mathrm{F}(v) \in H^{s-\frac{1}{2}}(\mathrm{F})$ if $s \in (0, \frac{1}{2})$, or $\mu_\mathrm{F}(v) \in (H_{00}^{\frac{1}{2}}(\mathrm{F}))'$ if $s = 0$. Moreover, we have*

$$\|\mu_\mathrm{F}(v)\|_{s-\frac{1}{2},\mathrm{F}} \lesssim \|v\|_{H_A^{1+s}(\hat{\Omega})} \; (s \in (0, \frac{1}{2})) \text{ and } \|\mu_\mathrm{F}(v)\|_{(H_{00}^{\frac{1}{2}}(\mathrm{F}))'} \lesssim \|v\|_{H_A^1(\hat{\Omega})}, \quad \mathrm{F} \subset \partial\hat{\Omega}. \tag{2.17}$$

Proof. It follows by (2.2) that

$$|\langle \mu_\mathrm{F}(v), \varphi \rangle_{\frac{1}{2}-s,\mathrm{F}}| = |\langle \mu_{\partial\hat{\Omega}}(v), \tilde{\varphi} \rangle_{\frac{1}{2}-s,\partial\hat{\Omega}}| \leq \|\mu_{\partial\hat{\Omega}}(v)\|_{s-\frac{1}{2},\partial\hat{\Omega}} \cdot \|\tilde{\varphi}\|_{\frac{1}{2}-s,\partial\hat{\Omega}}.$$

This, together with (2.15), leads to

$$|\langle \mu_\mathrm{F}(v), \varphi \rangle_{\frac{1}{2}-s,\mathrm{F}}| \lesssim \begin{cases} \|\mu_{\partial\hat{\Omega}}(v)\|_{s-\frac{1}{2},\partial\hat{\Omega}} \cdot \|\varphi\|_{\frac{1}{2}-s,\mathrm{F}}, & \text{if } s \in (0, \frac{1}{2}), \\ \|\mu_{\partial\hat{\Omega}}(v)\|_{-\frac{1}{2},\partial\hat{\Omega}} \cdot \|\varphi\|_{H_{00}^{\frac{1}{2}}(\mathrm{F})}. \end{cases}$$

Now, the inequality (2.17) follows from (2.2) together with (2.4). □

Remark 2.7. Theorem 2.4 is consistent with Theorem 1.5.4 in [14]. Note that, for the case with $v \in H_A^1(\hat{\Omega})$, we have only

$$\|\mu_\mathrm{F}(v)\|_{(H_{00}^{\frac{1}{2}}(\mathrm{F}))'} \lesssim \|v\|_{H_A^1(\hat{\Omega})}, \tag{2.18}$$

instead of the ideal result

$$\|\mu_\mathrm{F}(v)\|_{-\frac{1}{2},\mathrm{F}} = \|\mu_\mathrm{F}(v)\|_{(H^{\frac{1}{2}}(\mathrm{F}))'} \lesssim \|v\|_{H_A^1(\hat{\Omega})}. \tag{2.19}$$

Fortunately, the drawback will not affect applications to elliptic boundary value problems, since the underlying solution always belongs to $H^{1+\varepsilon}(\hat{\Omega})$ with some $\varepsilon > 0$.

In most applications, the generalized normal derivative will be usually associated with boundary value problems (see the next section). For the applications, we need to study continuity of the generalized normal derivative.

Let Ω be a bounded and connected Lipschitz domain in $\mathscr{R}^n$ $(n=2,\ 3)$. Consider the model problem

$$\begin{cases} -div(a\nabla u)=f, & \text{in } \Omega, \\ \ \ u=0, & \text{on } \partial\Omega, \end{cases} \tag{2.20}$$

where $f\in H_0^{-1}(\Omega)$, and $a\in L^\infty(\Omega)$ is a function with a positive lower bound.

Let $(\cdot,\cdot)$ denote the $L^2(\Omega)$-inner product. The weak formulation of (2.20) in $H_0^1(\Omega)$ is then given by the following: Find $u\in H_0^1(\Omega)$ such that

$$(a\nabla u,\nabla w)=(f,w),\quad \forall v\in H_0^1(\Omega). \tag{2.21}$$

Let $\hat{\Omega}_1,\hat{\Omega}_2\subset\Omega$ be two connected Lipschitz subdomains with piecewise smooth boundaries. Denote by F the common face between $\hat{\Omega}_1$ and $\hat{\Omega}_2$. Assume that $\hat{u}_k=u|_{\hat{\Omega}_k}\in H^{1+s_k}(\hat{\Omega}_k)$ and $f|_{\hat{\Omega}_k}\in H^{s_k-1}(\hat{\Omega}_k)$ with $s_k\in[0,\frac{1}{2})$ $(k=1,\ 2)$. Let $\mu_{\partial\hat{\Omega}_k}(\hat{u}_k)$ be defined as in Theorem 2.1 (replacing Av with $-f|_{\hat{\Omega}_k}$), and let $\mu_{\mathrm{F}}(\hat{u}_k)$ denote the locally generalized normal derivative of $\hat{u}_k$ on F. Set $s_{12}=\max\{\frac{1}{2}-s_1,\frac{1}{2}-s_2\}$.

Theorem 2.5. (continuity) *Assume that $\hat{u}_k\in H^{1+s_k}(\hat{\Omega}_k)$ and $f|_{\Omega_k}\in H^{s_k-1}(\Omega_k)$ with $s_k\in[0,\frac{1}{2})$ for $k=1,\ 2$. Then, we have*

$$\langle\mu_{\mathrm{F}}(\hat{u}_1),\varphi\rangle_{\frac{1}{2}-s_1,\mathrm{F}}+\langle\mu_{\mathrm{F}}(\hat{u}_2),\varphi\rangle_{\frac{1}{2}-s_2,\mathrm{F}}=0,\quad \forall\varphi\in H^{s_{12}}(\mathrm{F})\ (\text{if } s_{12}\in(0,\frac{1}{2})), \tag{2.22}$$

or

$$\langle\mu_{\mathrm{F}}(\hat{u}_1),\varphi\rangle_{\frac{1}{2}-s_1,\mathrm{F}}+\langle\mu_{\mathrm{F}}(\hat{u}_2),\varphi\rangle_{\frac{1}{2}-s_2,\mathrm{F}}=0,\quad \forall\varphi\in H_{00}^{\frac{1}{2}}(\mathrm{F})\ (\text{if } s_{12}=\frac{1}{2}). \tag{2.23}$$

Similarly, if $\hat{u}_2\in H^{1+s_2}(\hat{\Omega}_2)$ with $s_2\in(\frac{1}{2},1]$, then

$$\langle\mu_{\mathrm{F}}(\hat{u}_1),\varphi\rangle_{\frac{1}{2}-s_1,\mathrm{F}}+\int_{\mathrm{F}}a\frac{\partial\hat{u}_2}{\partial\mathbf{n}}\varphi ds=0,\quad \forall\varphi\in H^{\frac{1}{2}}(\mathrm{F}). \tag{2.24}$$

Proof. One needs only to verify (2.22). It follows by Theorem 2.4 that $\mu_{\mathrm{F}}(\hat{u}_1)+\mu_{\mathrm{F}}(\hat{u}_2)\in H^{-s_{12}}(\mathrm{F})$. Without loss of generality, we assume that $\|\mu_{\mathrm{F}}(\hat{u}_1)+\mu_{\mathrm{F}}(\hat{u}_2)\|_{-s_{12},\mathrm{F}}\neq 0$. Since $0<s_{12}<\frac{1}{2}$, the space $C_0^\infty(\mathrm{F})$ is dense in $H^{s_{12}}(\mathrm{F})$ (refer to Proposition 2.1). For arbitrarily small $\varepsilon>0$, there exists $\varphi_\varepsilon\in C_0^\infty(\mathrm{F})$ such that

$$\|\varphi_\varepsilon-\varphi\|_{s_{12},\mathrm{F}}\leq\frac{\varepsilon}{\|\mu_{\mathrm{F}}(\hat{u}_1)+\mu_{\mathrm{F}}(\hat{u}_2)\|_{-s_{12},\mathrm{F}}}.$$

Hence,

$$\begin{aligned}&|\langle\mu_{\mathrm{F}}(\hat{u}_1)+\mu_{\mathrm{F}}(\hat{u}_2),\varphi-\varphi_\varepsilon\rangle_{s_{12},\mathrm{F}}|\\ \leq&\|\mu_{\mathrm{F}}(\hat{u}_1)+\mu_{\mathrm{F}}(\hat{u}_2)\|_{-s_{12},\mathrm{F}}\cdot\|\varphi_\varepsilon-\varphi\|_{s_{12},\mathrm{F}}\leq\varepsilon.\end{aligned} \tag{2.25}$$

Note that $\varphi_\varepsilon \in C_0^\infty(\mathrm{F})$, there exists an extension $\tilde{\varphi}_\varepsilon$, which satisfies $\tilde{\varphi}_\varepsilon \in H_0^1(\hat{\Omega}_1 \cup \hat{\Omega}_2)$ and $\tilde{\varphi}_\varepsilon|_\mathrm{F} = \varphi_\varepsilon$. Thus (by (2.21)),

$$\int_{\hat{\Omega}_1 \cup \hat{\Omega}_2} a\nabla u \cdot \nabla \tilde{\varphi}_\varepsilon dp - \int_{\hat{\Omega}_1 \cup \hat{\Omega}_2} f \cdot \tilde{\varphi}_\varepsilon dp = 0. \tag{2.26}$$

On the other hand, it follows by (2.3) that

$$\int_{\hat{\Omega}_k} a\nabla \hat{u}_k \cdot \nabla \tilde{\varphi}_\varepsilon dp - \int_{\hat{\Omega}_k} f \cdot \tilde{\varphi}_\varepsilon dp = \langle \mu_{\partial\hat{\Omega}_k}(\hat{u}_k), \varphi_\varepsilon \rangle_{\frac{1}{2}-s_k,\mathrm{F}}, \quad k = 1,\ 2.$$

This, together with (2.26), leads to

$$\langle \mu_{\partial\hat{\Omega}_1}(\hat{u}_1) + \mu_{\partial\hat{\Omega}_2}(\hat{u}_2), \varphi_\varepsilon \rangle_{s_{12},\mathrm{F}} = 0.$$

Combining this with (2.25), gives

$$|\langle \mu_{\partial\hat{\Omega}_1}(\hat{u}_1) + \mu_{\partial\hat{\Omega}_2}(\hat{u}_2), \varphi \rangle_{s_{12},\mathrm{F}}| \le \varepsilon, \quad \forall \varepsilon > 0,$$

which implies (2.22). □

Remark 2.8. Theorems 2.1 and 2.5 indicate that the generalized normal derivative preserves the main properties of the usual normal derivative. Thus, the generalized normal derivative can be applied to numerical analysis for second-order elliptic boundary value problems.

3. Applications of the generalized normal derivatives

In this section, we present some applications of the results introduced in the last section to solving the variational problem (2.21).

One can define a discrete problem of (2.21) in various manners. Among them DDMs with nonmatching grids and DG methods become popular in recent years. However, convergence results of these methods have been obtained only when u is very smooth. For example, $u \in H^2(\Omega)$ is required for DG methods in literature. It is known that this high smoothness can not be satisfied if the domain Ω is a concave polyhedron, or the loading function f does not belong to $L^2(\Omega)$. To analyze convergence of these methods for the case that u has low regularity only, one needs to use the generalized normal derivatives.

3.1. An application to DDMs with nonmatching grids

For simplicity of exposition, let Ω be a bounded polyhedral domain in $\mathscr{R}^3$.

Let the domain Ω be decomposed into $\bar{\Omega} = \bigcup_{k=1}^{\mathrm{N}} \bar{\Omega}_k$, which satisfy $\Omega_i \cap \Omega_j = \emptyset$ when $i \neq j$. We only consider the case of geometrically conforming partitioning of the region into subdomains:

(i) if $\bar{\Omega}_i \cap \bar{\Omega}_j \neq \emptyset$ for some $i \neq j$, then $\partial\Omega_i \cap \partial\Omega_j$ is a common vertex, or a common edge or a common face of Ω_i and Ω_j. In particular, we set $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$ when $\partial\Omega_i \cap \partial\Omega_j$ is just a common face of Ω_i and Ω_j, and define $\Gamma = \cup\Gamma_{ij}$;

(ii) each subdomain Ω_k has the same "size" d in the usual sense (see [25]).

As usual, we assume that each Ω_k is a polyhedron. With each subdomain Ω_k we associate a regular and quasi-uniform triangulation $\mathscr{T}_k$ made of elements that are either hexahedra or tetrahedra. We denote by h_k the mesh size of $\mathscr{T}_k$, i.e., h_k denotes the maximum diameter of any tetrahedra in the mesh $\mathscr{T}_k$. The triangulations in the subdomains are independent of each other and generally do not match at the interfaces between subdomains. Hence, each interface Γ_{ij} is provided with two different ($2D$) meshes $\mathscr{T}_{ij}$ and $\mathscr{T}_{ji}$, which are associated with $\mathscr{T}_i$ and $\mathscr{T}_j$ respectively. Define $V(\Omega_k)$ as the space consisting of continuous piecewise linear functions associated with $\mathscr{T}_k$. If $\partial\Omega_k \cap \partial\Omega \neq \emptyset$, we require that each function in $V(\Omega_k)$ vanishes on $\partial\Omega_k \cap \partial\Omega$. Let $V(\partial\Omega_k)$ denotes the trace space associated with $V(\Omega_k)$.

For each local interface Γ_{ij}, let $W(\Gamma_{ij})$ be a given finite dimensional space on Γ_{ij}. Define

$$V(\Omega) = \left\{ v \in \prod_{k=1}^{\mathrm{N}} V(\Omega_k) : \int_{\Gamma_{ij}} v_i \varphi ds = \int_{\Gamma_{ij}} v_j \varphi ds, \ \forall \varphi \in W(\Gamma_{ij}) \text{ for each } \Gamma_{ij} \subset \Gamma \right\},$$

where $v = (v_1, v_2, \cdots, v_{\mathrm{N}})$. Note that we do not require $V(\Omega) \subset H^1(\Omega)$. In fact, a function $v \in V(\Omega)$ may be discontinuous on the set of all faces and all edges.

The discrete problem of (2.21) is: *Find* $u_h = (u_{h1}, u_{h2}, \cdots, u_{h\mathrm{N}}) \in V(\Omega)$ *such that*

$$\sum_{k=1}^{\mathrm{N}} (a\nabla u_{hk}, \nabla v_k)_{\Omega_k} = \sum_{k=1}^{\mathrm{N}} (f, v_k)_{\Omega_k}, \quad \forall v = (v_1, v_2, \cdots, v_{\mathrm{N}}) \in V(\Omega). \tag{3.1}$$

Hereafter, we assume that $f|_{\Omega_k} \in H^{-1}(\Omega_k)$ for each Ω_k. It is known that the system (3.1) has a unique solution under suitable assumptions.

The space $W(\Gamma_{ij})$, which is called local multiplier space, can be defined such that $W(\Gamma_{ij})$ is associated with the triangulation $\mathscr{T}_{ij}$ or $\mathscr{T}_{ji}$. It is also possible that $W(\Gamma_{ij})$ is independent of both $\mathscr{T}_{ij}$ and $\mathscr{T}_{ji}$ (refer to [19]). When one grid is finer than another grid between $\mathscr{T}_{ij}$ and $\mathscr{T}_{ji}$, the local multiplier space $W(\Gamma_{ij})$ should be associated with the coarser one to reduce the cost of calculation. Without loss of generality, we assume that $h_i \geq h_j$ for each Γ_{ij}, and we choose $\mathscr{T}_{ij}$ to define $W(\Gamma_{ij})$.

For each Γ_{ij}, define

$$V_i^0(\Gamma_{ij}) = V_i(\Gamma_{ij}) \cap H_0^1(\Gamma_{ij}) \ \text{ with } \ V_i(\Gamma_{ij}) = \{v|_{\Gamma_{ij}}, \ v \in V(\Omega_i)\}.$$

For simplicity of exposition, we assume that $dim(W(\Gamma_{ij})) = dim(V_i^0(\Gamma_{ij}))$.

Three conditions for $W(\Gamma_{ij})$ have been introduced in literature (refer to [18,23]):

A$_1$: each local multiplier space $W(\Gamma_{ij})$ possesses the optimal approximation

$$\inf_{w\in W(\Gamma_{ij})} \|w-v\|_{0,\Gamma_{ij}} \lesssim h_i^{\frac{1}{2}} \|v\|_{\frac{1}{2},\Gamma_{ij}}, \quad \forall v \in H^{\frac{1}{2}}(\Gamma_{ij}).$$

A$_2$: for each $\Gamma_{ij} \subset \Gamma$ and any $\varphi \in W(\Gamma_{ij})$, there is a function $\psi \in V_i^0(\Gamma_{ij})$ such that

$$(\psi,\varphi)_{\Gamma_{ij}} \gtrsim \|\psi\|_{0,\Gamma_{ij}} \cdot \|\varphi\|_{0,\Gamma_{ij}}.$$

A$_3$: each $W(\Gamma_{ij})$ contains the constant functions.

These conditions can guarantee the optimal approximate property of u_h. Some examples for $W(\Gamma_{ij})$ have been constructed in [3] (the mortar element multiplier) and [18] (the dual basis multiplier and the finite volume multiplier). We would like to point out that the dual basis multiplier was first introduced in [24] for two-dimensional problems.

For ease of notation, set $u_k = u|_{\Omega_k}$. Define the norm $\|\cdot\|$ by

$$\|v\| = \left(\sum_{k=1}^{\mathrm{N}} |v_k|_{1,\Omega_k}\right)^{\frac{1}{2}}, \quad v = (v_1, v_2, \cdots, v_{\mathrm{N}}) \in \prod_{k=1}^{\mathrm{N}} H^1(\Omega_k).$$

The following result has been proved in [3,23,24] (see [15] for a similar result involving numerical integrations).

Proposition 3.1. Let the conditions **A**$_1$-**A**$_3$ be satisfied. Assume that $u_k \in H^{1+\alpha_k}(\Omega_k)$ with $\alpha_k \in (\frac{1}{2},1]$ $(k=1,\cdots,\mathrm{N})$. Then, the error of the nonconforming approximation u_h can be estimated by

$$\|u-u_h\| \lesssim \left(\sum_{k=1}^{\mathrm{N}} h_k^{2\alpha_k} \|u\|^2_{1+\alpha_k,\Omega_k}\right)^{\frac{1}{2}}. \tag{3.2}$$

Since the efficient $a(p)$ may have jumps across the faces Γ_{ij}, we set $a_k(p) = a(p)$ for $p \in \Omega_k$. Let $\mathbf{n}_k$ denote the outer normal vector on $\partial\Omega_k$. The proof of Proposition 3.1 is based on the Strang's lemma (see [22]) below.

Lemma 3.1. *The accuracy of u_h can be estimated by the following inequality*

$$\|u-u_h\| \leq \inf_{v\in V(\Omega)} \|u-v\| + \sup_{w\in V(\Omega)} \frac{\left|\sum_{k=1}^{\mathrm{N}} \int_{\partial\Omega_k} a_k \frac{\partial u_k}{\partial \mathbf{n}_k} w_k ds\right|}{\|w\|}. \tag{3.3}$$

The normal derivatives $\frac{\partial u_k}{\partial \mathbf{n}_k}|_{\partial\Omega_k}$ are involved in the second term in the right-hand side of (**??**). But, this kind of normal derivative does not well defined when u_k belongs only to $H^{1+\alpha_k}(\Omega_k)$ with $\alpha_k \leq \frac{1}{2}$. This means that the restriction $\alpha_k \in (\frac{1}{2},1]$ $(k=1,\cdots,\mathrm{N})$ is essential in the proof of Proposition 3.1. To our knowledge, the work trying to remove such restriction is [4], in which a different estimate from (3.2) has been derived for the case of $\alpha_1 = \cdots = \alpha_{\rm N}$ by using the Hilbert interpolation technique.

The main result of this section is described as follows.

Theorem 3.1. *Let the conditions* $\mathbf{A}_1$-$\mathbf{A}_3$ *be satisfied. Assume that: (i)* $u_k \in H^{1+\alpha_k}(\Omega_k)$ *with* $\alpha_k \in (0,1]$ *for* $k = 1,\cdots,{\rm N}$*; (ii)* $f|_{\Omega_k} \in H^{\alpha_k-1}(\Omega_k)$ *if* $\alpha_k \in (0,\frac{1}{2}]$*. Then, the error of the nonconforming approximation* u_h *can be estimated by*

$$\|u-u_h\| \lesssim \left(\sum_{k=1}^{\rm N} h_k^{2\alpha_k}\|u_k\|^2_{1+\alpha_k,\Omega_k} + \sum_{\alpha_k\in(0,\frac{1}{2}]} h_k^{2\alpha_k}\|f\|^2_{\alpha_k-1,\Omega_k}\right)^{\frac{1}{2}}. \tag{3.4}$$

The proof of Theorem 3.1 will depend on the results obtained in Section 2 and some other auxiliary results. Define

$$F_k(u_k) = \begin{cases} \|u_k\|^2_{1+\alpha_k,\Omega_k}, & \text{if } \alpha_k \in (\frac{1}{2},1], \\ \|u_k\|^2_{1+\alpha_k,\Omega_k} + \|f\|^2_{L^{p_k}(\Omega_k)}, & \text{if } \alpha_k \in (0,\frac{1}{2}] \end{cases}.$$

By Theorems 2.2 and 3.1, one immediately obtains

Corollary 3.1. Let the conditions $\mathbf{A}_1$- $\mathbf{A}_3$ be satisfied. Assume that: (i) $u_k \in H^{1+\alpha_k}(\Omega_k)$ with $\alpha_k \in (0,1]$ $(k = 1,\cdots,{\rm N})$; (ii) $f|_{\Omega_k} \in L^{p_k}(\Omega_k)$ with $p_k > \frac{6}{3+2(1-\alpha_k)}$ when $\alpha_k \in (0,\frac{1}{2}]$. Then, the error of the nonconforming approximation u_h can be estimated by

$$\|u-u_h\| \lesssim \left(\sum_{k=1}^{\rm N} h_k^{2\alpha_k} F_k(u_k)\right)^{\frac{1}{2}}, \tag{3.5}$$

Before proving Theorem 3.1, we give several useful lemmas. For ease of notation, define the index sets of k

$$\Lambda_+ = \{k:\ \alpha_k \in (1/2,\ 1]\} \ \text{ and } \ \Lambda_- = \{k:\ \alpha_k \in (0,\ 1/2)\}.$$

Let $\mu_{\partial\Omega_k}(u_k)$ denote the generalized normal derivative associated with the subdomain Ω_k.

By Strang's lemma, together with the standard Green's formula and (2.3), one obtains

Lemma 3.2. *Assume that* $\alpha_k \neq \frac{1}{2}$ *for all k. Then, the accuracy of* u_h *can be estimated by the following inequality*

$$\|u-u_h\| \leq \inf_{v\in V(\Omega)} \|u-v\| + \sup_{w\in V(\Omega)} \frac{\left|\sum_{k\in\Lambda_+}\int_{\partial\Omega_k} a_k\frac{\partial u_k}{\partial \mathbf{n}_k}\cdot w_k ds + \sum_{k\in\Lambda_-}\langle \mu_{\partial\Omega_k}(u_k), w_k\rangle_{\frac{1}{2}-\alpha_k,\partial\Omega_k}\right|}{\|w\|}. \tag{3.6}$$

For each Γ_{ij}, let $P_{ij} : L^2(\Gamma_{ij}) \rightarrow W(\Gamma_{ij})$ be the usual L^2 projection. It follows by $\mathbf{A}_1$ that

$$\|P_{ij}v - v\|_{0,\Gamma_{ij}} \lesssim h_i^{\frac{1}{2}} \|v\|_{\frac{1}{2},\Gamma_{ij}}, \quad \forall v \in H^{\frac{1}{2}}(\Gamma_{ij}).$$

Moreover, it is known that

$$\|P_{ij}v\|_{0,\Gamma_{ij}} \leq \|v\|_{0,\Gamma_{ij}}, \quad \forall v \in L^2(\Gamma_{ij}),$$

and

$$\|P_{ij}v\|_{\frac{1}{2},\Gamma_{ij}} \lesssim \|v\|_{\frac{1}{2},\Gamma_{ij}}, \quad \forall v \in H^{\frac{1}{2}}(\Gamma_{ij}).$$

Thus, by Hilbert interpolation theory, one obtains for any $s \in [0, \frac{1}{2}]$

$$\|P_{ij}v - v\|_{0,\Gamma_{ij}} \lesssim h_i^s \|v\|_{s,\Gamma_{ij}}, \quad \forall v \in H^s(\Gamma_{ij}), \tag{3.7}$$

and

$$\|P_{ij}v - v\|_{s,\Gamma_{ij}} \lesssim h_i^{\frac{1}{2}-s} \|v\|_{\frac{1}{2},\Gamma_{ij}}, \quad \forall v \in H^s(\Gamma_{ij}). \tag{3.8}$$

For $v \in L^2(\Gamma_{ij})$, let $\gamma_{\Gamma_{ij}}(v)$ denote the integration average of v on Γ_{ij}. For ease of notation, we define

$$w_{ij}^i = w_i|_{\Gamma_{ij}} - \gamma_{\Gamma_{ij}}(w_i) \ \text{ and } \ w_{ij}^j = w_j|_{\Gamma_{ij}} - \gamma_{\Gamma_{ij}}(w_j) \tag{3.9}$$

for $w = (w_1, w_2, \cdots, w_{\rm N}) \in V(\Omega)$.

The following result is a direct consequence of $\mathbf{A}_3$ (which implies that $1 \in W(\Gamma_{ij})$).

Lemma 3.3. *Assume that $w = (w_1, w_2, \cdots, w_{\rm N}) \in V(\Omega)$. Then, we have for each Γ_{ij}*

$$\gamma_{\Gamma_{ij}}(w_i) = \gamma_{\Gamma_{ij}}(w_j), \tag{3.10}$$

and

$$(w_i - w_j)|_{\Gamma_{ij}} = w_{ij}^i - w_{ij}^j. \tag{3.11}$$

Lemma 3.4. *Assume that $w = (w_1, w_2, \cdots, w_{\rm N}) \in V(\Omega)$. Then,*

$$\|w_{ij}^i - w_{ij}^j\|_{0,\Gamma_{ij}} \lesssim h_i^{\frac{1}{2}} (|w_i|_{\frac{1}{2},\partial\Omega_i} + |w_j|_{\frac{1}{2},\partial\Omega_j}) \tag{3.12}$$

and

$$\|w_{ij}^i - w_{ij}^j\|_{s,\Gamma_{ij}} \lesssim h_i^{\frac{1}{2}-s} (|w_i|_{\frac{1}{2},\partial\Omega_i} + |w_j|_{\frac{1}{2},\partial\Omega_j}), \quad (s \in [0, \frac{1}{2}]). \tag{3.13}$$

Proof. One needs only to prove (3.13). By the definition of $V(\Omega)$, together with (3.10), leads to

$$(w_{ij}^i - w_{ij}^j)|_{\Gamma_{ij}} = (w_{ij}^i - P_{ij}w_{ij}^i) + (P_{ij}w_{ij}^j - w_{ij}^j). \tag{3.14}$$

This, together with (3.8), yields

$$\|w_{ij}^i - w_{ij}^j\|_{s,\Gamma_{ij}} \lesssim h_i^{\frac{1}{2}-s} (\|w_{ij}^i\|_{\frac{1}{2},\Gamma_{ij}} + \|w_{ij}^j\|_{\frac{1}{2},\Gamma_{ij}}).$$

Furthermore, we obtain (3.13) by Friedrich's inequality. □

To deal with the particular case $\alpha_k = \frac{1}{2}$, we give an interpolation result between two complex spaces.

Let $1 \le \sigma'_k \le \sigma''_k \le 2$ ($k = 1, \cdots, \mathrm{N}$). Define (using similar notation to Subsection 2.1)

$$Z'(\Omega) = \left\{ v \in H_0^1(\Omega) : v \in \prod_{k=1}^{\mathrm{N}} H^{\sigma'_k}(\Omega_k),\ Av \in \prod_{k=1}^{\mathrm{N}} H^{\sigma'_k - 2}(\Omega_k) \right\},$$

and

$$Z''(\Omega) = \left\{ v \in H_0^1(\Omega) : v \in \prod_{k=1}^{\mathrm{N}} H^{\sigma''_k}(\Omega_k),\ Av \in \prod_{k=1}^{\mathrm{N}} H^{\sigma''_k - 2}(\Omega_k) \right\}.$$

Hereafter, $v \in \prod_{k=1}^{\mathrm{N}} H^{\sigma'_k}(\Omega_k)$ means $v|_{\Omega_k} \in H^{\sigma'_k}(\Omega_k)$ for each k. The spaces $Z'(\Omega)$ and $Z''(\Omega)$ are Banach spaces when they are equipped with obvious norms.

We assume that the coefficient $a(p)$ is smooth on each Ω_k.

Lemma 3.5. *For a given $\theta \in [0,1]$, set $\sigma_k = \theta \sigma'_k + (1-\theta)\sigma''_k$ for $k = 1, \cdots, \mathrm{N}$. Then,*

$$[Z'(\Omega),\ Z''(\Omega)]_\theta = \left\{ v \in H_0^1(\Omega) : v \in \prod_{k=1}^{\mathrm{N}} H^{\sigma_k}(\Omega_k),\ Av \in \prod_{k=1}^{\mathrm{N}} H^{\sigma_k - 2}(\Omega_k) \right\}. \tag{3.15}$$

This result will be proved in Appendix.

Proof of Theorem 3.1. The proof is divided into three steps.

Step 1. Estimate the first term in the right of (3.3). As in [3, 5, 23, 24] (with an obvious change), one can verify by $\mathbf{A}_2$ that

$$\inf_{v \in V(\Omega)} \|u - v\| \lesssim \left(\sum_{k=1}^{\mathrm{N}} h_k^{2\alpha_k} \|u_k\|^2_{1+\alpha_k,\Omega_k} \right)^{\frac{1}{2}}. \tag{3.16}$$

Step 2. Estimate the second term in the right of (3.3) when none α_k equals $\frac{1}{2}$. For a face $\mathrm{F} \subset \partial\Omega_k$, let $\mu_{\mathrm{F}}(u_k)$ denote the locally generalized normal derivative of u_k on F. By Theorem 2.4, we have $\mu_{\mathrm{F}}(u_k) \in H^{\alpha_k - \frac{1}{2}}(\mathrm{F})$. It is clear that

$$\int_{\partial\Omega_k} a_k \frac{\partial u_k}{\partial \mathbf{n}_k} \cdot w_k ds = \sum_{\mathrm{F} \subset \partial\Omega_k} \int_{\mathrm{F}} a_k \frac{\partial u_k}{\partial \mathbf{n}_k} \cdot w_k ds, \quad k \in \Lambda_+$$

and

$$\langle \mu_{\partial\Omega_k}(u_k), w_k \rangle_{\frac{1}{2}-\alpha_k, \partial\Omega_k} = \sum_{\mathrm{F} \subset \partial\Omega_k} \langle \mu_{\mathrm{F}}(u_k), w_k \rangle_{\frac{1}{2}-\alpha_k, \mathrm{F}}, \quad k \in \Lambda_-.$$

Thus,

$$\begin{aligned}
&\sum_{k\in\Lambda_+}\int_{\partial\Omega_k} a_k\frac{\partial u_k}{\partial \mathbf{n}_k}\cdot w_k ds+\sum_{k\in\Lambda_-}\langle\mu_{\partial\Omega_k}(u),w_k\rangle_{\frac{1}{2}-\alpha_k,\partial\Omega_k}\\
=&\sum_{\Gamma_{ij}\subset\Gamma}\Bigg\{\sum_{i\in\Lambda_+}\sum_{j\in\Lambda_+}\Big[\int_{\Gamma_{ij}} a_i\frac{\partial u_i}{\partial \mathbf{n}_i}\cdot w_i ds+\int_{\Gamma_{ij}} a_j\frac{\partial u_j}{\partial \mathbf{n}_j}\cdot w_j ds\Big]\\
&+\sum_{i\in\Lambda_+}\sum_{j\in\Lambda_-}\Big[\int_{\Gamma_{ij}} a_i\frac{\partial u_i}{\partial \mathbf{n}_i}\cdot w_i ds+\langle\mu_{\Gamma_{ij}}(u_j),w_j\rangle_{\frac{1}{2}-\alpha_j,\Gamma_{ij}}\Big]\\
&+\sum_{i\in\Lambda_-}\sum_{j\in\Lambda_-}\Big[\langle\mu_{\Gamma_{ij}}(u_i),w_i\rangle_{\frac{1}{2}-\alpha_i,\Gamma_{ij}}+\langle\mu_{\Gamma_{ij}}(u_j),w_j\rangle_{\frac{1}{2}-\alpha_j,\Gamma_{ij}}\Big]\Bigg\}.
\end{aligned}$$

This, together with Theorem 2.5, leads to

$$\begin{aligned}
&\sum_{k\in\Lambda_+}\int_{\partial\Omega_k} a_k\frac{\partial u_k}{\partial \mathbf{n}_k}\cdot w_k ds+\sum_{k\in\Lambda_-}\langle\mu_{\partial\Omega_k}(u),w_k\rangle_{\frac{1}{2}-\alpha_k,\partial\Omega_k}\\
=&\sum_{\Gamma_{ij}}\Bigg\{\sum_{i\in\Lambda_+}\int_{\Gamma_{ij}} a_i\frac{\partial u_i}{\partial \mathbf{n}_i}\cdot(w_i-w_j)ds+\sum_{i\in\Lambda_-}\sum_{j\in\Lambda_-}\langle\mu_{\Gamma_{ij}}(u_i),w_i-w_j\rangle_{\frac{1}{2}-\alpha_i,\Gamma_{ij}}\Bigg\}. \qquad (3.17)
\end{aligned}$$

By Lemmas 3.3 and 3.4, one obtains for $\alpha_i\in(\frac{1}{2},1]$

$$\begin{aligned}
\Big|\int_{\Gamma_{ij}} a_i\frac{\partial u_i}{\partial \mathbf{n}_i}\cdot(w_i-w_j)ds\Big| &= \Big|\int_{\Gamma_{ij}}(a_i\frac{\partial u_i}{\partial \mathbf{n}_i}-P_{ij}(a_i\frac{\partial u_i}{\partial \mathbf{n}_i}|_{\Gamma_{ij}}))\cdot(w^i_{ij}-w^j_{ij})ds\Big|\\
&\le \|a_i\frac{\partial u_i}{\partial \mathbf{n}_i}-P_{ij}(a_i\frac{\partial u_i}{\partial \mathbf{n}_i}|_{\Gamma_{ij}})\|_{0,\Gamma_{ij}}\cdot\|w^i_{ij}-w^j_{ij}\|_{0,\Gamma_{ij}}\\
&\le h_i^{\alpha_i}\|a_i\frac{\partial u_i}{\partial \mathbf{n}_i}\|_{\alpha_i-\frac{1}{2},\Gamma_{ij}}\cdot(|w_i|_{\frac{1}{2},\partial\Omega_i}+|w_j|_{\frac{1}{2},\partial\Omega_j}), \qquad (3.18)
\end{aligned}$$

and for $\alpha_i\in(0,\frac{1}{2})$

$$\begin{aligned}
\big|\langle\mu_{\Gamma_{ij}}(u_i),w_i-w_j\rangle_{\frac{1}{2}-\alpha_i,\Gamma_{ij}}\big| &= \big|\langle\mu_{\Gamma_{ij}}(u_i),w^i_{ij}-w^j_{ij}\rangle_{\frac{1}{2}-\alpha_i,\Gamma_{ij}}\big|\\
&\le \|\mu_{\Gamma_{ij}}(u_i)\|_{\alpha_i-\frac{1}{2},\Gamma_{ij}}\cdot\|w^i_{ij}-w^j_{ij}\|_{\frac{1}{2}-\alpha_i,\Gamma_{ij}}\\
&\le h_i^{\alpha_i}\|\mu_{\Gamma_{ij}}(u_i)\|_{\alpha_i-\frac{1}{2},\Gamma_{ij}}\cdot\big(|w_i|_{\frac{1}{2},\partial\Omega_i}+|w_j|_{\frac{1}{2},\partial\Omega_j}\big). \qquad (3.19)
\end{aligned}$$

Substituting (3.18) and (3.19) into (3.17), and using the trace theorem, the inequality (2.17) and Cauchy-Schwarz inequality, leads to

$$\Big|\sum_{k\in\Lambda_+}\int_{\partial\Omega_k} a_k\frac{\partial u_k}{\partial \mathbf{n}_k}\cdot w_k ds+\sum_{k\in\Lambda_-}\langle \mu_{\partial\Omega_k}(u),w_k\rangle_{\frac{1}{2}-\alpha_k,\partial\Omega_k}\Big|$$

$$\lesssim \|w\|\cdot\Big(\sum_{k=1}^{\mathrm{N}} h_k^{2\alpha_k}\|u_k\|^2_{1+\alpha_k,\Omega_k}+\sum_{k\in\Lambda_-}h_k^{2\alpha_k}\|f\|^2_{\alpha_k-1,\Omega_k}\Big)^{\frac{1}{2}}\quad(\alpha_k\neq\frac{1}{2}).\tag{3.20}$$

Now, combining Lemma 3.2 with (3.16) and (3.20), one deduces that

$$\|u-u_h\|\lesssim\Big(\sum_{k=1}^{\mathrm{N}} h_k^{2\alpha_k}\|u_k\|^2_{1+\alpha_k,\Omega_k}+\sum_{k\in\Lambda_-}h_k^{2\alpha_k}\|f\|^2_{\alpha_k-1,\Omega_k}\Big)^{\frac{1}{2}}\quad(\alpha_k\neq\frac{1}{2}).\tag{3.21}$$

Step 3. Derive (3.4) for the case with some $\alpha_k=\frac{1}{2}$. It is clear that a slightly weaker result can be obtained by Proposition 2.3 for such case. To get the current result, we have to use Lemma 3.5. Without loss of generality, we assume that $\alpha_k=\frac{1}{2}$ when $k=1,\cdots,\mathrm{N}_0$, and $\alpha_k\neq\frac{1}{2}$ for $k=\mathrm{N}_0+1,\cdots,\mathrm{N}$. Then, there exist $\alpha'_k\in(0,\frac{1}{2})$ and $\alpha''_k\in(\frac{1}{2},1)$ ($k=1,\cdots,\mathrm{N}_0$) such that $\frac{1}{2}=\frac{\alpha'_k+\alpha''_k}{2}$ ($k=1,\cdots,\mathrm{N}_0$). For convergence, set $\alpha'_k=\alpha''_k=\alpha_k$ for $k=\mathrm{N}_0+1,\cdots,\mathrm{N}$. Define

$$Y'(\Omega)=\Big\{v\in H_0^1(\Omega):v\in\prod_{k=1}^{\mathrm{N}}H^{1+\alpha'_k}(\Omega_k),\ Av\in\prod_{k=1}^{\mathrm{N}}H^{\alpha'_k-1}(\Omega_k)\Big\}$$

and

$$Y''(\Omega)=\Big\{v\in H_0^1(\Omega):v\in\prod_{k=1}^{\mathrm{N}}H^{1+\alpha''_k}(\Omega_k),\ Av\in\prod_{k=1}^{\mathrm{N}}H^{\alpha''_k-1}(\Omega_k)\Big\}.$$

Then, $T:u\to u-u_h$ is a linear mapping from $Y'(\Omega)+Y''(\Omega)$ into $X(\Omega)$ defined by

$$X(\Omega)=\Big\{v\in L^2(\Omega):v|_{\partial\Omega}=0,\ v\in\prod_{k=1}^{\mathrm{N}}H^1(\Omega_k)\Big\}.$$

It follows by (3.21) that

$$\|Tu\|\lesssim\Big[\sum_{k=1}^{\mathrm{N}}h_k^{2\alpha'_k}(\|u\|^2_{1+\alpha'_k,\Omega_k}+\|Au\|^2_{\alpha'_k-1,\Omega_k})\Big]^{\frac{1}{2}},\quad\forall u\in Y'(\Omega)\tag{3.22}$$

and

$$\|Tu\|\lesssim\Big[\sum_{k=1}^{\mathrm{N}}h_k^{2\alpha''_k}(\|u\|^2_{1+\alpha''_k,\Omega_k}+\|Av\|^2_{\alpha''_k-1,\Omega_k})\Big]^{\frac{1}{2}},\quad\forall u\in Y''(\Omega).\tag{3.23}$$

Thus, by Hilbert interpolation theory, gives

$$\|Tu\| \lesssim \left[\sum_{k=1}^{N} h_k^{2\alpha_k}(\|u\|^2_{1+\alpha_k,\Omega_k} + \|Au\|^2_{\alpha_k-1,\Omega_k})\right]^{\frac{1}{2}}, \quad \forall u \in [Y'(\Omega),\ Y''(\Omega)]_{\frac{1}{2}}. \tag{3.24}$$

On the other hand, it follows by Lemma 3.5 that

$$[Y'(\Omega), Y''(\Omega)]_{\frac{1}{2}} = \left\{ v \in H_0^1(\Omega) : v \in \prod_{k=1}^{N} H^{1+\alpha_k}(\Omega_k),\ Av \in \prod_{k=1}^{N} H^{\alpha_k-1}(\Omega_k) \right\}.$$

Furthermore, using (3.24) we deduce to (3.4) for the case with some $\alpha_k = \frac{1}{2}$.

Finally, we obtain (3.4) for $\alpha_k \in (0,1]$ $(k = 1, \cdots, N)$. □

Remark 3.1. Step 3 above is similar to the proof of Theorem 2.1 in [4] (for the case with $\alpha_1 = \cdots = \alpha_N$). Such interpolation technique can be also used to derive Theorem 3.1 directly if the parameters $\alpha_1, \cdots, \alpha_N$ have some particular relation. In fact, using (3.1), together with Cauchy-Schwarz inequality and the generalized Poincáre inequality (see [5]), yields

$$\|u_h\| \leq \left(\sum_{k=1}^{N} \|f\|^2_{-1,\Omega_k}\right)^{\frac{1}{2}}.$$

Thus,

$$\|u_h - u\| \lesssim \left[\sum_{k=1}^{N} (\|u_k\|^2_{1,\Omega_k} + \|f\|^2_{-1,\Omega_k})|\right]^{\frac{1}{2}} = \left(\sum_{k=1}^{N} \|u_k\|^2_{H_A^1(\Omega_k)}\right)^{\frac{1}{2}}, \quad u_k \in H_A^1(\Omega_k). \tag{3.25}$$

Define a set of the parameters $\{\alpha_k\}_{k=1}^N$ as

$$\mathscr{C} = \left\{ \{\alpha_k\}_{k=1}^N \subset (0, \frac{1}{2}] :\ \exists\ \alpha_k^0 \in (\frac{1}{2},\ 1] \text{ s.t. } \frac{\alpha_1}{\alpha_1^0} = \cdots = \frac{\alpha_N}{\alpha_N^0} = \theta \in (0,1) \right\}.$$

When $(\alpha_1, \cdots, \alpha_N) \in \mathscr{C}$, the estimate (3.4) can be derived by Hilbert interpolation result, together with (3.2) and (3.25). However, if $(\alpha_1, \cdots, \alpha_N) \notin \mathscr{C}$, then the estimate (3.4) can not be obtained directly by such simple method. This means that the proof of Theorem 2.1 in [4] is not complete.

3.2. An application to DG methods

There are many variations of DG methods (refer to [1]). As an application example, we consider only the DG method described in [7], and use the same notation as that in [7].

Let us first describe the DG method. Without loss of generality, we assume that Ω is a bounded polygonal domain in $\mathscr{R}^2$.

Let $\{\mathcal{T}_h\}_h$ be a regular and quasi-uniform family of triangulations of Ω; we shall indicate by E the triangles of $\mathcal{T}_h$, and set $h = \max_{E\in\mathcal{T}_h} diam(E)$. We denote by $\mathcal{E}_h$ the set of all internal edges of $\mathcal{T}_h$. Define

$$V_h = \{v_h \in L^2(\Omega) : v_h|_E \text{ is linear}, \quad \forall\, E \in \mathcal{T}_h\}$$

and

$$\mathbf{W}_h = \{\tau_h \in (L^2(\Omega))^2 : \text{ each complement of } \tau_h|_E \text{ is linear}, \quad \forall\, E \in \mathcal{T}_h\}.$$

To define discontinuous approximations $u_h \in V_h$ of (2.20), let us define the *average* of a function vector $\theta_h \in \mathbf{W}_h$ and the *jump* of a function $v_h \in V_h$ as follows

- $\theta_h^0 = \theta_h + \theta_h^{ext}/\,2$ on internal edges $e \in \mathcal{E}_h$;
- $[[v_h]] = v_h^+\mathbf{n}^+ + v_h^-\mathbf{n}^-$ on internal edges.

Hereafter, θ_h^{ext} denotes the value of θ_h on the element facing the element under consideration across the edge e; the notation $(\cdot)^+$ and $(\cdot)^-$ indicates the value of the generic quantity $(\cdot)$ on the two elements sharing the same edge e.

For the space

$$\tilde{V} = \prod_{E\in\mathcal{T}_h} H^1(E),$$

define the affine operators $\mathbf{R}$, $\mathbf{r}_e : \tilde{V} \to \mathbf{W}_h$ by

$$\int_\Omega \mathbf{R}(w)\cdot\tau_h dp = -\sum_{e\in\mathcal{E}_h}\int_e [[w]]\cdot\tau_h^0 ds, \quad \forall \tau_h \in \mathbf{W}_h$$

and

$$\int_\Omega \mathbf{r}_e(w)\cdot\tau_h dp = -\int_e [[w]]\cdot\tau_h^0 ds, \quad e \in \mathcal{E}_h\ \forall \tau_h \in \mathbf{W}_h.$$

Now, the bilinear form $a_h(\cdot,\cdot)$ can be defined as

$$\begin{aligned} a_h(u_h, v_h) = &\sum_{E\in\mathcal{T}_h}\int_E a[\nabla u_h\cdot\nabla v_h + \nabla u_h\cdot\mathbf{R}(v_h) + \mathbf{R}(u_h)\cdot\nabla v_h]dp \\ &+ c_0\sum_{e\in\mathcal{E}_h}\int_\Omega a\mathbf{r}_e(u_h)\cdot\mathbf{r}_e(v_h)dp. \end{aligned} \tag{3.26}$$

Hereafter, c_0 is a suitable positive number. The discrete variational problem for (3.1) becomes: Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = \sum_{E\in\mathcal{T}_h}(f, v_h)_E, \quad \forall v_h \in V_h. \tag{3.27}$$

It is certain to assume that $\sum_{E\in\mathcal{T}_h}(f, v_h)_E$ is well defined for each $v_h \in V_h$.

Define the mesh-dependent norm

$$|||v||| = \Big(\sum_{E\in\mathscr{T}_h} |v|^2_{1,E} + \sum_{e\in\mathscr{E}_h} \|\mathbf{r}_e(v)\|^2_{0,\Omega} \Big)^{\frac{1}{2}}, \quad \forall v \in \tilde{V}.$$

All existing error estimates for DG methods were obtained under the assumption $u \in H^2(\Omega)$, since the derivation of the error estimates involves the normal derivative $\frac{\partial u}{\partial \mathbf{n}}|_e$ for each edge $e \in \mathscr{E}_h$ (see [7]). Of course, the existing results can be extended directly to the case when $u \in H^\sigma(\Omega)$ with $\sigma \in (\frac{3}{2}, 2]$.

By the generalized normal derivative introduced in the last section, one can prove the following optimal error estimate for the case with low regularity.

Theorem 3.2. *Let $u \in H^s(\Omega)$ for some $s > 1$, and let u_h be defined by* (3.27). *Assume that, for each $E \in \mathscr{T}_h$, $u|_E \in H^{1+\alpha_E}(E)$ with $\alpha_E \in (0,1]$, and $f|_E \in H^{\alpha_E-1}(E)$ if $\alpha_E \in (0,\frac{1}{2}]$. Then, we have*

$$|||u - u_h||| \lesssim \Big(\sum_{E\in\mathscr{T}_h} h^{2\alpha_E} \|u\|^2_{1+\alpha_E,E} \Big)^{\frac{1}{2}}. \tag{3.28}$$

Proof. The bilinear form $a_h(\cdot,\cdot)$ can be written as (see (30) in [7])

$$\begin{aligned} a_h(u_h, v_h) = &\sum_{E\in\mathscr{T}_h} \int_E a\nabla u_h \cdot \nabla v_h dp - \int_e [[u_h]] \cdot (a\nabla v_h)^0 ds \\ &- \int_e [[v_h]] \cdot (a\nabla u_h)^0 ds \,+\, c_0 \sum_{e\in\mathscr{E}_h} \int_\Omega a\mathbf{r}_e(u_h) \cdot \mathbf{r}_e(v_h)\, dp. \end{aligned} \tag{3.29}$$

In the above formula, u_h can not be replaced by u, since ∇u has no sense on $e \in \mathscr{E}_h \cap \partial E$ for $\alpha_E \in (0,\frac{1}{2}]$. Thus, the estimate (3.28) can not be proved as in Theorem 1 of [7]. One needs only to consider the case when none $\alpha_E = \frac{1}{2}$. If some $\alpha_E = \frac{1}{2}$, the result can be derived from Hilbert interpolation technique as in Theorem 3.1.

It can be verified that (refer to the proof of Lemma 1 in [7])

$$[[w]] \cdot (a\nabla v_h)^0 = \frac{1}{2}(w^+ - w^-)\Big[a^+ \frac{\partial v_h^+}{\partial \mathbf{n}^+} - a^- \frac{\partial v_h^-}{\partial \mathbf{n}^-}\Big] \text{ on each edge } e, \quad \forall w \in H^{\frac{1}{2}}(e).$$

For convenience, set $\alpha_0 = \max_E(\frac{1}{2} - \alpha_E)$. Since $(w^+ - w^-)|_e \in H^{\frac{1}{2}}(e)$ and $v_h|_E \in H^2(E)$ for each $E \in \mathscr{T}_h$, we have by Theorems 2.3 and 2.4 (note (2.1))

$$\int_e [[w]] \cdot (a\nabla v_h)^0 ds = \frac{1}{2} \langle \mu_e(v_h^+) - \mu_e(v_h^-), w^+ - w^- \rangle_{\alpha_0,e} \text{ on each edge } e.$$

Substituting this relation into (3.29), yields

$$\begin{aligned} a_h(u_h, v_h) = &\sum_{E\in\mathscr{T}_h} \int_E a\nabla u_h \cdot \nabla v_h dp - \frac{1}{2} \sum_{e\in\mathscr{E}_h} \langle \mu_e(v_h^+) - \mu_e(v_h^-), u_h^+ - u_h^- \rangle_{\alpha_0,e} \\ &- \frac{1}{2} \sum_{e\in\mathscr{E}_h} \langle \mu_e(u_h^+) - \mu_e(u_h^-), v_h^+ - v_h^- \rangle_{\alpha_0,e} + c_0 \sum_{e\in\mathscr{E}_h} \int_\Omega a\mathbf{r}_e(u_h) \cdot \mathbf{r}_e(v_h)\, dp. \end{aligned} \tag{3.30}$$

Let $\tilde{V}_h$ be the usual conforming finite element space

$$\tilde{V}_h = \{v_h \in H_0^1(\Omega) : v_h|_E \text{ is linear for each } E \in \mathscr{T}_h\},$$

and let $\tilde{u}_h$ be the interpolation of u. By the assumptions, the interpolation $\tilde{u}_h$ is well defined, and belongs to $\tilde{V}_h$. It is known that

$$|||u - \tilde{u}_h|||^2 = \sum_{E \in \mathscr{T}_h} \|u - \tilde{u}_h\|_{1,E}^2 \lesssim \sum_{E \in \mathscr{T}_h} h^{2\alpha_E} \|u\|_{1+\alpha_E,E}^2. \tag{3.31}$$

From Theorem 2.4, one can replace u_h and v_h in (3.30) by $u - u_h$ and $\tilde{u}_h - u_h$ respectively, and define $a_h(u - u_h, \tilde{u}_h - u_h)$ accordingly. As in [7], we have

$$|||\tilde{u}_h - u_h|||^2 \lesssim |||\tilde{u}_h - u||| \cdot |||\tilde{u}_h - u_h||| + a_h(u - u_h, \tilde{u}_h - u_h).$$

If one can verify that

$$a_h(u - u_h, \tilde{u}_h - u_h) = 0, \tag{3.32}$$

then

$$|||\tilde{u}_h - u_h||| \lesssim |||\tilde{u}_h - u|||. \tag{3.33}$$

Note that $\mathbf{r}_e(u) = 0$ for each $e \in \mathscr{E}_h$, it follows by (3.30) that

$$\begin{aligned} a_h(u, \tilde{u}_h - u_h) = &\sum_{E \in \mathscr{T}_h} \int_E a \nabla u \cdot \nabla(\tilde{u}_h - u_h) dp \\ &- \frac{1}{2} \sum_{e \in \mathscr{E}_h} \langle \mu_e(u^+) - \mu_e(u^-), (\tilde{u}_h - u_h)^+ - (\tilde{u}_h - u_h)^- \rangle_{\alpha_0,e}. \end{aligned}$$

This, together with Theorem 2.5, leads to

$$\begin{aligned} &a_h(u, \tilde{u}_h - u_h) \\ &= \sum_{E \in \mathscr{T}_h} \int_E a \nabla u \cdot \nabla(\tilde{u}_h - u_h) dp - \sum_{e \in \mathscr{E}_h} \langle \mu_e(u), (\tilde{u}_h - u_h)^+ - (\tilde{u}_h - u_h)^- \rangle_{\alpha_0,e} \\ &= \sum_{E \in \mathscr{T}_h} \left[\int_E a \nabla u \cdot \nabla(\tilde{u}_h - u_h) dp - \langle \mu_{\partial E}(u), \tilde{u}_h - u_h \rangle_{\alpha_E, \partial E} \right]. \end{aligned} \tag{3.34}$$

Here, we have used the relation (2.1). Combining (3.34) with (3.27), gives

$$\begin{aligned} &a_h(u - u_h, \tilde{u}_h - u_h) = a_h(u, \tilde{u}_h - u_h) - a_h(u_h, \tilde{u}_h - u_h) \\ &= \sum_{E \in \mathscr{T}_h} \left[\int_E a \nabla u \cdot \nabla(\tilde{u}_h - u_h) dp - \langle \mu_{\partial E}(u), \tilde{u}_h - u_h \rangle_{\alpha_E, \partial E} \right] - \sum_{E \in \mathscr{T}_h} (f, \tilde{u}_h - u_h)_E \\ &= \sum_{E \in \mathscr{T}_h} \left[\int_E a \nabla u \cdot \nabla(\tilde{u}_h - u_h) dp - (f, \tilde{u}_h - u_h)_E \right] - \langle \mu_{\partial E}(u), \tilde{u}_h - u_h \rangle_{\alpha_E, \partial E}, \end{aligned}$$

which, together with (2.3), yields to (3.32). Then, (3.28) is a direct consequence of (3.31) and (3.33). $\square$

Remark 3.2. If $u \in H^1(\Omega)$ only, one needs to replace the usual interpolation $\tilde{u}_h$ in (3.31) by Clement interpolation [8].

Corollary 3.2. Let u_h be defined by (3.27). Assume that $u \in H^{1+\alpha}(\Omega)$ with $\alpha \in (0,1]$ and $f \in L^p(\Omega)$ with $p > \frac{2}{2-\alpha}$. Then we have

$$|||u - u_h||| \lesssim h^\alpha \|u\|_{1+\alpha,\Omega}. \tag{3.35}$$

Remark 3.3. Note that the norm $\|f\|_{\alpha_E-1,E}$ (resp. $\|f\|_{L^p(\Omega)}$) does not appear in (3.28) (resp. (3.35)). In fact, the generalized normal derivative is only a “bridge" in the proof of Theorem 3.2.

Appendix

In this appendix, we prove Lemma 3.5. To this end, we need to use a Hilbert interpolation result in subspace (see [20]).

Let Φ and Ψ be two locally convex topological vector spaces, and let X and Y be two Banach spaces which satisfy $X \subset \Phi$ and $Y \subset \Phi$. Assume that $\partial \in \mathscr{L}(\Phi, \Psi)$, which denotes the set consisting of all continuous and linear operators from Φ into Ψ. For two Banach spaces $\mathscr{X}$ and $\mathscr{Y}$, define

$$X_{\partial,\mathscr{X}} = \{v : v \in X,\ \partial v \in \mathscr{X}\} \ \text{ and } \ Y_{\partial,\mathscr{Y}} = \{v : v \in Y,\ \partial v \in \mathscr{Y}\}.$$

Assume that there exist two Banach spaces $\tilde{\mathscr{X}}$ and $\tilde{\mathscr{Y}}$ and an operator $\mathscr{S}$, such that

(i) $\mathscr{X} \subset \tilde{\mathscr{X}} \subset \Psi$ and $\mathscr{Y} \subset \tilde{\mathscr{Y}} \subset \Psi$;
(ii) $\partial \in \mathscr{L}(X, \tilde{\mathscr{X}}) \cap \mathscr{L}(Y, \tilde{\mathscr{Y}})$;
(iii) $\mathscr{S} \in \mathscr{L}(\tilde{\mathscr{X}}, X) \cap \mathscr{L}(\tilde{\mathscr{Y}}, Y)$ and $\mathscr{S}\partial x = x$ for any $x \in \tilde{\mathscr{X}} + \tilde{\mathscr{Y}}$.

The following result can be viewed as a particular case of Theorem 14.3 in Chapter 1 of [20](with the mapping $r = 0$).

Lemma A.1 *Let the conditions (i), (ii) and (iii) above be satisfied. Then, the following relation holds for any* $\theta \in [0,1]$

$$[X_{\partial,\mathscr{X}}, Y_{\partial,\mathscr{Y}}]_\theta = ([X,Y]_\theta)_{\partial,[\mathscr{X},\mathscr{Y}]_\theta}. \tag{A.1}$$

In the following we prove Lemma 3.5 by this result.

Proof of Lemma 3.5. Define

$$X = H_0^1(\Omega) \cap \prod_{k=1}^{N} H^{\sigma'_k}(\Omega_k),\ \mathscr{X} = H_0^{-1}(\Omega) \cap \prod_{k=1}^{N} H^{\sigma'_k - 2}(\Omega_k),$$

and

$$Y = H_0^1(\Omega) \cap \prod_{k=1}^{N} H^{\sigma''_k}(\Omega_k),\ \mathscr{Y} = H_0^{-1}(\Omega) \cap \prod_{k=1}^{N} H^{\sigma''_k - 2}(\Omega_k).$$

Moreover, define $\partial = A$. As in Section 12.5 of Chapter 1 in [20], one can verify that

$$[H^{\sigma'_k-2}(\Omega_k), H^{\sigma''_k-2}(\Omega_k)]_\theta = H^{\sigma_k-2}(\Omega_k), \quad k=1,\cdots,\mathrm{N}.$$

Thus, we have by the interpolation of product spaces (see p. 178 in [20])

$$\left[\prod_{k=1}^{\mathrm{N}} H^{\sigma'_k}(\Omega_k),\ \prod_{k=1}^{\mathrm{N}} H^{\sigma''_k}(\Omega_k)\right]_\theta = \prod_{k=1}^{\mathrm{N}} H^{\sigma_k}(\Omega_k),$$

and

$$\left[\prod_{k=1}^{\mathrm{N}} H^{\sigma'_k-2}(\Omega_k),\ \prod_{k=1}^{\mathrm{N}} H^{\sigma''_k-2}(\Omega_k)\right]_\theta = \prod_{k=1}^{\mathrm{N}} H^{\sigma_k-2}(\Omega_k).$$

Furthermore, one can verify, by Lemma 2.1 and the technique developed by the works of Bernardi, Dauge and Maday (refer to [6]), that

$$[X,\ Y]_\theta = H_0^1(\Omega)\cap\prod_{k=1}^{\mathrm{N}} H^{\sigma_k}(\Omega_k),\ \ [\mathscr{X},\ \mathscr{Y}]_\theta = H_0^{-1}(\Omega)\cap\prod_{k=1}^{\mathrm{N}} H^{\sigma_k-2}(\Omega_k).$$

In the following, we derive the equality (3.15) by Lemma A.1 (note that $A(H_0^1(\Omega)) \subset H_0^{-1}(\Omega)$). To this end, we need to define suitable $\tilde{\mathscr{X}}$, $\tilde{\mathscr{Y}}$, Φ, Ψ and $\mathscr{S}$, so that the conditions (i), (ii) and (iii) in Lemma A.1 can be satisfied.

Define

$$\Phi = H_0^1(\Omega),\ \Psi = H_0^{-1}(\Omega),\ \tilde{\mathscr{X}} = \Psi\cap\prod_{k=1}^{\mathrm{N}} H_0^{\sigma'_k-2}(\Omega_k)\ \text{ and }\ \tilde{\mathscr{Y}} = \Psi\cap\prod_{k=1}^{\mathrm{N}} H_0^{\sigma''_k-2}(\Omega_k).$$

Besides, set $A_0 = A|_{H_0}$ with $H_0 = \prod_{k=1}^{\mathrm{N}} H_0^1(\Omega_k)$, and define $\mathscr{S} = (A_0)^{-1}$. For such definitions, the conditions (i) and (ii) in Lemma A.1 are satisfied by Proposition 2.2. It suffices to consider the condition (iii). Since Ω_k is a convex polyhedron, we have

$$\mathscr{S}\in\mathscr{L}\Big(\tilde{\mathscr{X}},\ \prod_{k=1}^{\mathrm{N}}(H^{\sigma'_k}(\Omega_k)\cap H_0^1(\Omega_k)\Big)\bigcap\mathscr{L}\Big(\tilde{\mathscr{Y}},\ \prod_{k=1}^{\mathrm{N}}(H^{\sigma''_k}(\Omega_k)\cap H_0^1(\Omega_k))\Big).$$

Thus,

$$\mathscr{S}\in\mathscr{L}(\tilde{\mathscr{X}},\ X)\cap\mathscr{L}(\tilde{\mathscr{Y}},\ Y).$$

Moreover, for any $z\in\tilde{\mathscr{X}}+\tilde{\mathscr{Y}}$ we have $\mathscr{S}z\in\prod_{k=1}^{\mathrm{N}} H_0^1(\Omega_k)$, so

$$\partial\mathscr{S}z = A_0(A_0)^{-1}z = z,\quad \forall z\in\tilde{\mathscr{X}}+\tilde{\mathscr{Y}}.$$

All this shows that the condition (iii) in Lemma A.1 is also satisfied. □

Remark A.1 From the above proof, we know that the following interpolation result also holds

$$[H_A^{\sigma_1}(\Omega_k),\ H_A^{\sigma_2}(\Omega_k)]_\theta = H_A^{\theta\sigma_1+(1-\theta)\sigma_2}(\Omega_k), \quad 1 \le \sigma_1 \le \sigma_2 \le 2;\ \theta \in [0,1].$$

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