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Spectral Analysis for HSS Preconditioners

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Abstract. In this paper, we are interested in HSS preconditioners for saddle point linear systems with a nonzero $(2,2)$-th block. We study an approximation of the spectra of HSS preconditioned matrices and use these results to illustrate and explain the spectra obtained from numerical examples, where the previous spectral analysis of HSS preconditioned matrices does not cover.

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1. Introduction

Saddle point linear problems arise in many different practical applications [4]. In this paper we study iterative methods for solving saddle point problems $\mathscr{A}'x = b$ where

$$\mathscr{A}' = \begin{bmatrix} W & K \\ K^T & -\mu I \end{bmatrix}, \tag{1.1}$$

$W \in \mathbb{R}^{m\times m}$ is symmetric positive semi-definite and $K \in \mathbb{R}^{m\times n}$ (so $I \in \mathbb{R}^{n\times n}$). We confine our discussion that $m \geq n$ (or even $m \gg n$) and K is assumed to have full (column) rank. Examples include discrete Stokes equations [6], and weighted Toeplitz least squares problems [5] where applications arise from image reconstruction [1, 10] and nonlinear image restoration [9]. In imaging applications, μ refers to the regularization parameter.

In the literature, there are many solution methods for solving saddle point linear systems; see [4] for references. In this paper, we are interested in HSS preconditioning techniques [3]. The idea is first to transform $\mathscr{A}'$ in (1.1) to a nonsymmetric matrix

$$\mathscr{A} = \begin{bmatrix} W & K \\ -K^T & \mu I \end{bmatrix}. \tag{1.2}$$

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Then we split $\mathscr{A}$ into $\mathscr{H}+\mathscr{S}$ where

$$\mathscr{H}=\begin{bmatrix} W & 0 \\ 0 & \mu I \end{bmatrix} \quad \text{and} \quad \mathscr{S}=\begin{bmatrix} 0 & K \\ -K^T & 0 \end{bmatrix}. \tag{1.3}$$

Here $\mathscr{H}$ is the symmetric part of $\mathscr{A}$ and $\mathscr{S}$ is the skew-symmetric part of $\mathscr{A}$. The HSS preconditioner is in the form

$$\mathscr{P}=\frac{1}{2\alpha}(\mathscr{H}+\alpha I)(\mathscr{S}+\alpha I), \tag{1.4}$$

where $\alpha>0$ is referred to as the preconditioning parameter for the HSS preconditioner. The idea of HSS preconditioner is motivated from the Hermitian and skew-Hermitian splitting (HSS) method [2]. It was shown in [3] that the eigenvalues of $I-\mathscr{P}^{-1}\mathscr{A}$ lie in $\overline{\mathbf{D}(1)}=\{z\in\mathbb{C}\mid |z|\leq 1\}$ and in particular $\mathbf{D}(1)=\{z\in\mathbb{C}\mid |z|<1\}$ if W is positive definite.

In [11] a more thorough spectral analysis for the case of $\mu=0$ is presented where a more refined inclusion region is given, and sufficient conditions with respect to α for a real or a clustered spectrum are provided.

When $\mu\neq 0$, an analysis under the special case when $\alpha=\mu$ can be found in [5]. However, when $\alpha\neq\mu$ it remains a rather difficult task to make any conclusion on the spectrum.

The main aim of this paper is to explain the spectra of HSS preconditioned matrices when $\mu\neq 0$. The argument involved is to approximate the transformed preconditioned matrices for which the eigenvalues of the approximations can be determined and analyzed. When α is smaller than the eigenvalues of W our approximations are quite accurate. Our analysis can be used to study the spectra of HSS preconditioned matrices when $\mu=0$.

The outline of this paper is as follows. In Section 2, we give some preparatory work. In Section 3, we study an approximation of spectra of HSS preconditioned matrices. In Section 4, numerical examples are presented to illustrate the approximation scheme. In Section 5, we study the spectra when $\mu=0$. Finally, we give some concluding remarks in Section 6.

2. Decomposition of matrices

For simplification, throughout this paper we shall denote by $\mathbf{R}(r)$, $\mathbf{D}(r)$ and $\overline{\mathbf{D}(r)}$ the sets $\{z\in\mathbb{C}\mid |z|=r\}$, $\{z\in\mathbb{C}\mid |z|<r\}$ and $\{z\in\mathbb{C}\mid |z|\leq r\}$ respectively. We shall also denote by I_k the $k\times k$ identity matrix, whenever ambiguities may arise from the size of the identity matrix.

We first note that the spectrum of the matrix $(\alpha I-\mathscr{H})(\alpha I+\mathscr{H})^{-1}(\alpha I-\mathscr{S})(\alpha I+\mathscr{S})^{-1}$ is the same as the spectrum of $I-\mathscr{P}^{-1}\mathscr{A}$.

Lemma 2.1. ([11]) *Consider the HSS preconditioner $\mathscr{P}$ to a linear system with coefficient matrix $\mathscr{A}=\mathscr{H}+\mathscr{S}$, where $\mathscr{H}$ and $\mathscr{S}$ are respectively the Hermitian part and skew-Hermitian part of $\mathscr{A}$. Then the matrices $I-\mathscr{P}^{-1}\mathscr{A}$ and $(\alpha I-\mathscr{H})(\alpha I+\mathscr{H})^{-1}(\alpha I-\mathscr{S})(\alpha I+\mathscr{S})^{-1}$ have the same spectrum.*

As $(\alpha I - \mathscr{S})(\alpha I + \mathscr{S})^{-1}$ is the well-known Cayley transform of the skew-symmetric $\mathscr{S}$, it is orthogonal and its spectrum is a subset of $\mathbf{R}(1)$. Next we see that if the singular values of K are known, the precise locations of the eigenvalues of $(\alpha I - \mathscr{S})(\alpha I + \mathscr{S})^{-1}$ can be determined. Let

$$K = V\Lambda U^T$$

be a singular value decomposition of K. Here $V \in \mathbb{R}^{m\times m}$, $U \in \mathbb{R}^{n\times n}$ are orthogonal and $\Lambda = [\Sigma \quad 0]^T \in \mathbb{R}^{m\times n}$ with $m \geq n$ and $\Sigma \in \mathbb{R}^{n\times n}$ is a diagonal matrix (i.e., $\Sigma^T = \Sigma$) with non-negative diagonal entries containing the singular values of K. When K is of full column rank, Σ is positive diagonal. We note that

$$\alpha I + \mathscr{S} = \begin{bmatrix} \alpha I & K \\ -K^T & \alpha I \end{bmatrix} = \begin{bmatrix} \alpha I & V\Lambda U^T \\ -U\Lambda^T V^T & \alpha I \end{bmatrix} = \mathscr{V} \begin{bmatrix} \alpha I & 0 & \Sigma \\ 0 & \alpha I & 0 \\ -\Sigma & 0 & \alpha I \end{bmatrix} \mathscr{V}^T,$$

where

$$\mathscr{V} = \begin{bmatrix} V & 0 \\ 0 & U \end{bmatrix}.$$

Using the following identity

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \frac{1}{\alpha}\Sigma & 0 & I \end{bmatrix} \begin{bmatrix} \alpha I & 0 & \Sigma \\ 0 & \alpha I & 0 \\ -\Sigma & 0 & \alpha I \end{bmatrix} = \begin{bmatrix} \alpha I & 0 & \Sigma \\ 0 & \alpha I & 0 \\ 0 & 0 & D \end{bmatrix}$$

with $D = \frac{1}{\alpha}\Sigma^2 + \alpha I$, we have

$$\begin{bmatrix} \alpha I & 0 & \Sigma \\ 0 & \alpha I & 0 \\ -\Sigma & 0 & \alpha I \end{bmatrix}^{-1} = \begin{bmatrix} D^{-1} & 0 & -\frac{1}{\alpha}\Sigma D^{-1} \\ 0 & \frac{1}{\alpha}I & 0 \\ \frac{1}{\alpha}\Sigma D^{-1} & 0 & D^{-1} \end{bmatrix}.$$

It follows that

$$(\alpha I + \mathscr{S})^{-1} = \mathscr{V} \begin{bmatrix} D^{-1} & 0 & -\frac{1}{\alpha}\Sigma D^{-1} \\ 0 & \frac{1}{\alpha}I & 0 \\ \frac{1}{\alpha}\Sigma D^{-1} & 0 & D^{-1} \end{bmatrix} \mathscr{V}^T.$$

Similarly, we obtain

$$\alpha I - \mathscr{S} = \begin{bmatrix} \alpha I & -K \\ K^T & \alpha I \end{bmatrix} = \begin{bmatrix} \alpha I & -V\Lambda U^T \\ U\Lambda^T V^T & \alpha I \end{bmatrix} = \mathscr{V} \begin{bmatrix} \alpha I & 0 & -\Sigma \\ 0 & \alpha I & 0 \\ \Sigma & 0 & \alpha I \end{bmatrix} \mathscr{V}^T,$$

and therefore we have

$$(\alpha I - \mathscr{S})(\alpha I + \mathscr{S})^{-1} \quad = \quad \mathscr{V} \begin{bmatrix} 2\alpha D^{-1} - I & 0 & -2\Sigma D^{-1} \\ 0 & I & 0 \\ 2\Sigma D^{-1} & 0 & 2\alpha D^{-1} - I \end{bmatrix} \mathscr{V}^T. \tag{2.1}$$

The following lemma is straightforward and the proof can be omitted.

Lemma 2.2. *Let*

$$\mathscr{T} = \begin{bmatrix} D_1 & 0 & -D_2 \\ 0 & I & 0 \\ D_2 & 0 & D_1 \end{bmatrix},$$

where D_1 and D_2 are $n \times n$ diagonal matrices, I is an identity matrix of size $m-n$. Then the eigenvalues of $\mathscr{T}$ are given by

(i) 1 with multiplicity $m-n$; and

(ii) the roots of the quadratic equations

$$\lambda^2 - 2[D_1]_{i,i}\lambda + ([D_1]_{i,i}^2 + [D_2]_{i,i}^2) = 0, \quad i = 1, \cdots, n.$$

In our study, the singular values of K are given by σ_i $(i = 1, 2, \cdots, n)$. It follows that

$$\begin{aligned} [D_1]_{i,i} &= [2\alpha D^{-1} - I]_{i,i} = \frac{2\alpha^2}{\sigma_i^2 + \alpha^2} - 1 = \frac{\alpha^2 - \sigma_i^2}{\alpha^2 + \sigma_i^2}, \\ [D_2]_{i,i} &= [2\Sigma D^{-1}]_{i,i} = \frac{2\sigma_i \alpha}{\sigma_i^2 + \alpha^2}. \end{aligned} \tag{2.2}$$

As

$$[D_1]_{i,i}^2 + [D_2]_{i,i}^2 = \left(\frac{\alpha^2 - \sigma_i^2}{\alpha^2 + \sigma_i^2}\right)^2 + \left(\frac{2\sigma_i\alpha}{\sigma_i^2 + \alpha^2}\right)^2 = 1,$$

the quadratic equations in (ii) of Lemma 2.2 is reduced to

$$\lambda^2 - 2\left(\frac{\alpha^2 - \sigma_i^2}{\alpha^2 + \sigma_i^2}\right)\lambda + 1 = 0,$$

the corresponding discriminants are $-16\alpha^2\sigma_i^2\left(\alpha^2 + \sigma_i^2\right)^{-2}$, which is always negative because $\alpha > 0$ and $\sigma_i > 0$. Therefore, these eigenvalues are non-real and in conjugate pairs. Their real parts are given by

$$\frac{\alpha^2 - \sigma_i^2}{\alpha^2 + \sigma_i^2}. \tag{2.3}$$

Next we investigate the spectrum of $(\alpha I - \mathscr{H})(\alpha I + \mathscr{H})^{-1}$. Suppose an eigen-decomposition of W is given by

$$W = Q \, \mathrm{diag}(w_1, \cdots, w_m) \, Q^T,$$

where Q is orthogonal and $w_1, \cdots, w_m$ are the eigenvalues of W. From (1.3), we have

$$\begin{aligned} &(\alpha I - \mathscr{H})(\alpha I + \mathscr{H})^{-1} \\ &= \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \frac{\alpha - w_1}{\alpha + w_1} & & & \\ & \ddots & & 0 \\ & & \frac{\alpha - w_m}{\alpha + w_m} & \\ & 0 & & \frac{\alpha - \mu}{\alpha + \mu} I \end{bmatrix} \begin{bmatrix} Q^T & 0 \\ 0 & I \end{bmatrix}. \end{aligned}$$

It is clear that the spectral radius of $(\alpha I-\mathcal{H})(\alpha I+\mathcal{H})^{-1}(\alpha I-\mathcal{S})(\alpha I+\mathcal{S})^{-1}$ is bounded above by

$$w \equiv \max\left\{\left|\frac{\alpha-w_1}{\alpha+w_1}\right|,\cdots,\left|\frac{\alpha-w_m}{\alpha+w_m}\right|,\left|\frac{\alpha-\mu}{\alpha+\mu}\right|\right\} \tag{2.4}$$

regardless of the singular values of K. The eigenvalues of

$$(\alpha I-\mathcal{H})(\alpha I+\mathcal{H})^{-1}(\alpha I-\mathcal{S})(\alpha I+\mathcal{S})^{-1}$$

are then lying inside $\overline{\mathbf{D}(w)}$. A more refined region for the case $\mu=0$ is given as inside an annulus [11]. However, these results do not tell much how the eigenvalues lie inside $\overline{\mathbf{D}(w)}$.

Based the above decompositions of $(\alpha I-\mathcal{H})(\alpha I+\mathcal{H})^{-1}$ and $(\alpha I-\mathcal{S})(\alpha I+\mathcal{S})^{-1}$, we have the following result about the spectra of $\mathcal{P}^{-1}\mathcal{A}$.

Theorem 2.1. *Consider the linear system with coefficient matrix in* (1.2) *and its corresponding HSS preconditioner* $\mathcal{P}$ *in* (1.4). *Assume that* W *is symmetric and positive definite with eigenvalues* $w_1,\cdots,w_m$. *If either (i)* $\mu<\alpha<w_i$ *for all* i *or (ii)* $w_i<\alpha<\mu$ *for all* i, *then the eigenvalues of* $\mathcal{P}^{-1}\mathcal{A}$ *are real.*

Proof. Suppose (i) holds. Then $(\alpha I-W)(\alpha I+W)^{-1}$ is negative definite and $\frac{\alpha-\mu}{\alpha+\mu}I$ is positive definite. Therefore,

$$\begin{bmatrix} -(\alpha I-W)(\alpha I+W)^{-1} & 0 \\ 0 & \frac{\alpha-\mu}{\alpha+\mu}I \end{bmatrix}$$

is positive definite. Moreover, we have

$$\begin{aligned}
&(\alpha I-\mathcal{H})(\alpha I+\mathcal{H})^{-1}(\alpha I-\mathcal{S})(\alpha I+\mathcal{S})^{-1} \\
=&\begin{bmatrix} (\alpha I-W)(\alpha I+W)^{-1} & 0 \\ 0 & \frac{\alpha-\mu}{\alpha+\mu}I \end{bmatrix}\mathcal{V}\begin{bmatrix} 2\alpha D^{-1}-I & 0 & -2\Sigma D^{-1} \\ 0 & I & 0 \\ 2\Sigma D^{-1} & 0 & 2\alpha D^{-1}-I \end{bmatrix}\mathcal{V}^T \\
=-&\begin{bmatrix} -(\alpha I-W)(\alpha I+W)^{-1} & 0 \\ 0 & \frac{\alpha-\mu}{\alpha+\mu}I \end{bmatrix}\mathcal{V}\begin{bmatrix} 2\alpha D^{-1}-I & 0 & -2\Sigma D^{-1} \\ 0 & I & 0 \\ -2\Sigma D^{-1} & 0 & -(2\alpha D^{-1}-I) \end{bmatrix}\mathcal{V}^T.
\end{aligned}$$

Note that

$$\mathcal{V}\begin{bmatrix} 2\alpha D^{-1}-I & 0 & -2\Sigma D^{-1} \\ 0 & I & 0 \\ -2\Sigma D^{-1} & 0 & -(2\alpha D^{-1}-I) \end{bmatrix}\mathcal{V}^T$$

is symmetric, i.e., $(\alpha I-\mathcal{H})(\alpha I+\mathcal{H})^{-1}(\alpha I-\mathcal{S})(\alpha I+\mathcal{S})^{-1}$ is a product of a symmetric negative definite matrix and a symmetric matrix and so it must have a real spectrum. The result follows from Lemma 2.1. The proof for (ii) is similar. □

Remark 2.1. *Although a singular, symmetric positive semi-definite matrix does not have an invertible square root, we can still use continuity arguments about eigenvalues to conclude that the product of a semi-definite matrix and a symmetric matrix has a real spectrum. Therefore, we can further relax the assumptions in the statement of Theorem* 2.1 *where equality conditions are included to the strict inequalities.*

3. Spectral analysis of HSS preconditioned matrices

3.1. Mathematical tools

In this subsection, we present two main results for our further analysis.

Lemma 3.1. *Let*

$$U = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 \\ 0 & \rho \end{bmatrix},$$

where $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$, and $\rho \in \mathbb{R}$, $0 \le |\rho| \le 1$. The product PU has real eigenvalues if and only if either (i) $-1 \le \rho \le 0$ or (ii) $\rho > 0$ and $|a| \ge 2\sqrt{\rho}(\rho+1)^{-1}$. Moreover, if the eigenvalues of PU are non-real, then their moduli are $\sqrt{\rho}$ regardless of the values a and b of U.

Proof. As the characteristic polynomial of PU is

$$\det(PU - \lambda I) = \lambda^2 - (\rho + 1)a\lambda + \rho,$$

the discriminant of the characteristic equation is precisely

$$(\rho + 1)^2 a^2 - 4\rho,$$

which means the eigenvalues are real if and only if

$$(\rho + 1)^2 a^2 - 4\rho \ge 0.$$

Clearly this situation holds if $-1 \le \rho \le 0$ (i.e., when condition (i) is satisfied). Otherwise, if $\rho > 0$, then we need $|a| \ge 2\sqrt{\rho}(\rho+1)^{-1}$ (i.e., when condition (ii) is satisfied).

If the eigenvalues of PU are non-real, then they are in conjugate pair and they have the same moduli. Moreover, the product of them equals

$$\det(PU) = (\det P)(\det U) = \rho.$$

Hence, the moduli for non-real eigenvalues must be $\sqrt{\rho}$ regardless of the values a and b of U. □

The next lemma discusses the nilpotent part of the upper triangular matrix corresponding to the Schur's triangularization of the product PU. Denote by $\|\cdot\|_F$ the Frobenious norm of a matrix.

Lemma 3.2. *Let U and P be the matrices in Lemma* 3.1. *Suppose PU has Schur triangularization V^*TV where*

$$T = \begin{bmatrix} \lambda_1 & t \\ 0 & \lambda_2 \end{bmatrix}$$

is upper triangular and V is unitary. Then $|t| \leq \Big|\, 1-|\rho| \,\Big|$. In particular, if $0 \leq |\rho| \leq 1$ then we have $|t| \leq 1$.

Proof. As PU and T are similar to each other by a unitary matrix, they have the same determinant and the same Frobenius norm. Then,

$$\begin{aligned} 1+\rho^2 &= a^2+b^2+\rho^2 a^2+\rho^2 b^2 \\ &= \|PU\|_F^2 = \|T\|_F^2 = |\lambda_1|^2+|\lambda_2|^2+|t|^2 \end{aligned}$$

and

$$\rho = (\det P)(\det U) = \det(PU) = \det(T) = \lambda_1\lambda_2.$$

Hence,

$$\begin{aligned} |t|^2 &= 1+\rho^2-|\lambda_1|^2-|\lambda_2|^2 \leq 1+\rho^2-2|\lambda_1\lambda_2| \\ &= 1+\rho^2-2|\rho| = (1-|\rho|)^2, \end{aligned} \tag{3.1}$$

which gives $|t| \leq \Big|\, 1-|\rho| \,\Big|$. Hence, if $0 \leq |\rho| \leq 1$ then we have $|t| \leq 1$. □

According to the above result, if T is decomposed as $D+N$ where D is diagonal and

$$N = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$$

is nilpotent, then we have $\|N\| \leq 1$ for any usual matrix norm. In particular, if $\rho = \pm 1$ then PU is normal and from (3.1) we have $t = 0$, i.e. T is indeed diagonal.

3.2. Approximation of the spectra of HSS preconditioned matrices

In this subsection, we study an approximation of the spectra of HSS preconditioned matrices. We first note that the eigenvalues for $(\alpha I-\mathscr{H})(\alpha I+\mathscr{H})^{-1}$ and $(\alpha I-\mathscr{S})(\alpha I+\mathscr{S})^{-1}$ involve the key terms in the form

$$f_1(x) = \frac{\alpha-x}{\alpha+x} = 1-\frac{2}{\frac{\alpha}{x}+1} \quad \text{and} \quad f_2(y) = \frac{\alpha^2-y^2}{\alpha^2+y^2} = 1-\frac{2}{\left(\frac{\alpha}{y}\right)^2+1}$$

respectively (cf. (2.4) and (2.3)). Here x and y relate to the eigenvalues of W and the singular values of K respectively. Note that $f_1(x)$ is a decreasing function with respect to x. When α is fixed, we find that

$$f_1(x) \begin{cases} \to -1, & \text{if } \frac{\alpha}{x} \to 0, \\ = 0, & \text{if } \frac{\alpha}{x} = 1, \\ \to 1, & \text{if } \frac{\alpha}{x} \to \infty. \end{cases} \tag{3.2}$$

The function $f_2(\cdot)$ behaves similarly as $f_1(\cdot)$, but $f_2(\cdot)$ approaches the extreme values ± 1 much faster than $f_1(\cdot)$.

Lemma 3.3. *Let A and B be two normal matrices. If either A or B has eigenvalues being close to 1, then the spectrum of the product AB resembles to that of the other. In particular, if both A and B have eigenvalues being close to 1, then the product AB will also have eigenvalues being close to 1.*

Proof. We assume A to be the normal matrix having eigenvalues being close to 1. If $\|B\| = 0$, then B is the zero matrix and the result trivially holds. Here $\|\cdot\|$ refers to the spectral norm. Now suppose $\|B\| \neq 0$. Let

$$\epsilon_A = \|B\| \cdot \max_{\lambda \in \lambda(A)} |\lambda - 1|.$$

Hence, $\|A - I\| = \epsilon_A / \|B\|$ and we have

$$\|AB - B\| \leq \|A - I\| \, \|B\| = \epsilon_A.$$

This means AB differs from the normal matrix B with a matrix $AB - B$ which has a very small norm. According to Bauer-Fike theorem [8, p.321], we have

$$\min_{\lambda \in \lambda(AB)} |\lambda - \nu| \leq \epsilon_A$$

for any eigenvalue ν of B. The explanation to the case if B has eigenvalues being close to 1 is just the same. $\square$

With the above observations and Lemma 3.3, we consider an approximation to the matrix $(\alpha I - \mathscr{H})(\alpha I + \mathscr{H})^{-1}$. When the value of α lies away from the entries of W, we approximate the $(1,1)$-th block of $(\alpha I - \mathscr{H})(\alpha I + \mathscr{H})^{-1}$, i.e., $(\alpha I - W)(\alpha I + W)^{-1}$, by $\pm I_m$. The plus or minus sign depends on whether α is larger than or smaller than those non-negative entries of W respectively, as we can see from (3.2). We also note that when $x \ll \alpha$ or $\alpha \ll x$, $|(\alpha - x)/(\alpha + x)|$ tends to 1. Therefore, an approximation to the matrix $(\alpha I - \mathscr{H})(\alpha I + \mathscr{H})^{-1}$ takes the form

$$\begin{bmatrix} I_m & 0 \\ 0 & \rho I_n \end{bmatrix},$$

where $\rho \in \mathbb{R}$ and $|\rho| \leq 1$. Recall that if α is smaller than w_i's, the approximated $(1,1)$-th block is actually $-I_m$. In this case we simply multiply the whole matrix by $-I_{m+n}$. Hence, we can still have I_m where the corresponding sign of ρ will be adjusted and our analysis still applies. The $(2,2)$-th block of $(\alpha I - \mathscr{H})(\alpha I + \mathscr{H})^{-1}$ is $\frac{\alpha-\mu}{\alpha+\mu} I$, and therefore we take $\rho = \pm \frac{\alpha-\mu}{\alpha+\mu}$ where the minus sign is added if α is smaller than w_i's. Using this approximation and by (2.1), we obtain

$$
\begin{aligned}
&(\alpha I-\mathscr{H})(\alpha I+\mathscr{H})^{-1}(\alpha I-\mathscr{S})(\alpha I+\mathscr{S})^{-1}\\
\approx&\begin{bmatrix} I_n & & 0\\ & I_{n-m} & 0\\ 0 & 0 & \rho I\end{bmatrix}\mathscr{V}\begin{bmatrix} 2\alpha D^{-1}-I & 0 & -2\Sigma D^{-1}\\ 0 & I & 0\\ 2\Sigma D^{-1} & 0 & 2\alpha D^{-1}-I\end{bmatrix}\mathscr{V}^T\\
=&\,\mathscr{V}\begin{bmatrix} 2\alpha D^{-1}-I & 0 & -2\Sigma D^{-1}\\ 0 & I & 0\\ 2\rho\Sigma D^{-1} & 0 & \rho(2\alpha D^{-1}-I)\end{bmatrix}\mathscr{V}^T .
\end{aligned}
$$

For simplicity, we denote the above expression as $\mathscr{V}\mathscr{D}\mathscr{V}^T$. By using appropriate permutation matrix $\mathscr{P}$, it is straightforward to get

$$\mathscr{P}^T\mathscr{D}\mathscr{P}=\mathscr{F}=\ \mathrm{diag}(F_1,F_2,\cdots,F_n,I_{m-n})$$

and each F_i is of the form

$$\begin{bmatrix} [D_1]_{i,i} & -[D_2]_{i,i}\\ \rho[D_2]_{i,i} & \rho[D_1]_{i,i}\end{bmatrix}$$

and $|\rho|\le 1$. It follows that

$$(\alpha I-\mathscr{H})(\alpha I+\mathscr{H})^{-1}(\alpha I-\mathscr{S})(\alpha I+\mathscr{S})^{-1}\approx(\mathscr{V}\mathscr{P})\mathscr{F}(\mathscr{V}\mathscr{P})^T$$

which means the approximation of $(\alpha I-\mathscr{H})(\alpha I+\mathscr{H})^{-1}(\alpha I-\mathscr{S})(\alpha I+\mathscr{S})^{-1}$ is orthogonally similar to $\mathscr{F}$.

There are $m-n$ eigenvalues of $\mathscr{D}$ being 1. The remaining eigenvalues are clearly the eigenvalues of F_i's, which are all 2×2 blocks. Using Lemma 3.1 we can easily obtain the spectrum of $\mathscr{F}$. The blocks F_i's in $\mathscr{F}$ are precisely

$$\begin{bmatrix} \dfrac{\alpha^2-\sigma_i^2}{\alpha^2+\sigma_i^2} & -\dfrac{2\sigma_i\alpha}{\sigma_i^2+\alpha^2}\\ \rho\left(\dfrac{2\sigma_i\alpha}{\sigma_i^2+\alpha^2}\right) & \rho\left(\dfrac{\alpha^2-\sigma_i^2}{\alpha^2+\sigma_i^2}\right)\end{bmatrix},$$

where $\rho=\pm\frac{\alpha-\mu}{\alpha+\mu}$, which in turn implies the spectrum depends on the singular values of K, the preconditioning parameter α and the parameter μ. The relationship between σ_i and ρ will determine the spectral properties of the HSS preconditioned matrices. For example, if $\rho<0$, or if $\rho>0$ and

$$\left|\frac{\alpha^2-\sigma_i^2}{\alpha^2+\sigma_i^2}\right|>\frac{2\sqrt{\rho}}{\rho+1}\quad\text{for some }\ i=1,2,\cdots,n,$$

then the corresponding eigenvalues are real. Even if non-real eigenvalues occur, they will all be distributed on $\mathbf{R}(\sqrt{\rho})$. Hence, the approximated matrix $\mathscr{F}$ has eigenvalue 1 with multiplicity $m-n$ and n pairs (i.e., a total of $2n$) of eigenvalues, which may be all real or a combination of real and non-real corresponding to the n 2×2 blocks F_i's which can be determined explicitly.

After setting up the above argument, we assert that the approximated spectrum agrees with the actual spectrum to a certain level by setting a uniform upper bound for $\|\mathscr{E}\|$ where $\mathscr{E}$ is the perturbation induced. We have approximated $(\alpha I - W)(\alpha I + W)^{-1}$ by $\pm I$. In other words, the actual matrix can be regarded as perturbing the approximated matrix by

$$\mathscr{E} = \begin{bmatrix} (\alpha I - W)(\alpha I + W)^{-1} \mp I & 0 \\ 0 & 0 \end{bmatrix} (\alpha I - \mathscr{S})(\alpha I + \mathscr{S})^{-1}$$

and we denote $(\alpha I - W)(\alpha I + W)^{-1} \mp I$ by E, where the plus or minus sign depends on the relative values of α and w_i. We shall also denote $\min\{w_1, \cdots, w_m\}$ by $w_{\min}$, and $\max\{w_1, \cdots, w_m\}$ by $w_{\max}$.

In practice we usually have α smaller than w_i's, and thus $(\alpha I - W)(\alpha I + W)^{-1}$ is negative and we approximate it by $-I$. Hence, $E = (\alpha I - W)(\alpha I + W)^{-1} + I$. Moreover, we have

$$\|E\| = \max_{i=1,2,\cdots,m} \left\{ \left| \frac{\alpha - w_i}{\alpha + w_i} + 1 \right| \right\} = \frac{\alpha - w_{\min}}{\alpha + w_{\min}} + 1. \tag{3.3}$$

Although it is not very realistic, we also remark here the situation where α is larger than all w_i's, in which case $(\alpha I - W)(\alpha I + W)^{-1}$ is a positive definite matrix, with eigenvalues close to 1, and we shall approximate it by I. Hence, the perturbation matrix here is $E = (\alpha I - W)(\alpha I + W)^{-1} - I$. Moreover, we have

$$\|E\| = \max_{i=1,2,\cdots,m} \left\{ \left| \frac{\alpha - w_i}{\alpha + w_i} - 1 \right| \right\} = 1 - \frac{\alpha - w_{\max}}{\alpha + w_{\max}}.$$

The overall approximation differs from the HSS preconditioner by

$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} (\alpha I - \mathscr{S})(\alpha I + \mathscr{S})^{-1}.$$

Since $(\alpha I - \mathscr{S})(\alpha I + \mathscr{S})^{-1}$ is unitary, the norm of the difference $\mathscr{E}$ (which is essentially the perturbation) depends solely on E, i.e., $\|\mathscr{E}\| = \|E\|$. Recall that, if the perturbation from a matrix is not normal, then the nilpotent part of the upper triangular matrix obtained from the Schur's triangularization of the perturbation must need to be taken into consideration; see [8, p.321].

Lemma 3.4. *Let $Q^*AQ = D + N$ be a Schur decomposition of $A \in \mathbb{C}^{n\times n}$ where Q is unitary, D is diagonal and N is strictly upper triangular. If $\nu \in \lambda(A+E)$ and q is the smallest positive integer such that $N^q = 0$, then*

$$\min_{\lambda \in \lambda(A)} |\lambda - \nu| \le \max(\delta, \delta^{1/q}) \quad \text{where} \quad \delta = \|E\| \sum_{k=0}^{q-1} \|N\|^k.$$

By Lemma 3.2, each F_i is unitarily similar to an upper triangular matrix $T_i = D_i + N_i$ such that N_i has all entries zero except possibly the (1,2) entry which has absolute value at most 1. Hence, $\mathscr{F}$ is unitarily similar to $\mathscr{D} + \mathscr{N}$ where

$$\mathscr{D} = \begin{bmatrix} \lambda_{1,1} & & & & & \\ & \lambda_{1,2} & & & & \\ & & \ddots & & & 0 \\ & & & \lambda_{n,1} & & \\ & & & & \lambda_{n,2} & \\ & & 0 & & & I_{n-m} \end{bmatrix} \tag{3.4}$$

is diagonal, and

$$\mathscr{N} = \begin{bmatrix} 0 & t_1 & & & & \\ & 0 & & & & \\ & & \ddots & & & 0 \\ & & & 0 & t_n & \\ & & & & 0 & \\ & & 0 & & & 0_{n-m} \end{bmatrix}. \tag{3.5}$$

Clearly, $\mathscr{N}^2 = 0$ and

$$\|\mathscr{N}\| = \max_{i=1}^{n} |t_i| \le 1.$$

Combining (3.4) and (3.5), Lemma 3.4 and the fact that $\|\mathscr{E}\| = \|E\|$, the value δ in Lemma 3.4 is given by (here we have $q = 2$)

$$\delta = \|\mathscr{E}\| \sum_{k=0}^{q-1} \|\mathscr{N}\|^k = \|E\|(1 + \|\mathscr{N}\|) \le 2\|E\|. \tag{3.6}$$

Our discussion mainly deals with the cases where $\|E\|$ is small. Therefore, $2\|E\| < 1$ and we obtain the difference between the actual eigenvalue with the nearest approximated eigenvalue by $\sqrt{2\|E\|}$. A reminder here is that we only use $\sqrt{2\|E\|}$ when our approximation F_i's are not normal, i.e., for the cases where $\rho \neq \pm 1$. Recall that when $\rho = \pm 1$, those F_i's are still normal and we can use a more refined bound $\|E\|$ instead of $\sqrt{2\|E\|}$.

Concluding the above, provided that $\|E\|$ is small, the approximated spectrum can provide a rather good picture of the spectrum of the true preconditioned system. Although the actual eigenvalues cannot be given, and an accurate determination is impractical also, the approximation explains well the pattern and the distribution of them on $\overline{\mathbf{D}(w)}$.

4. Numerical examples

In this section, we provide some examples for illustrations of using the above arguments to explain the spectral properties. In many cases, it is found that the approximated spectra are very near to the true spectra. Recall that the important factors affecting the spectra are the interaction between the value

$$\rho = \pm\frac{\alpha-\mu}{\alpha+\mu} \quad \text{and} \quad \frac{\alpha^2-\sigma_i^2}{\alpha^2+\sigma_i^2}$$

(σ_i's are the singular values of K). Also the nearness between the approximated spectra and the true one depends on the perturbation E.

In our experiments, we test the systems arising from weighted Toeplitz least squares problems [5]

$$\min_z \|Az-b\|_2^2,$$

where the rectangular coefficient matrix A and the right-hand side vector b are of the form:

$$A = \begin{bmatrix} WK \\ \mu I \end{bmatrix} \in \mathbb{R}^{2n\times n} \quad \text{and} \quad b = \begin{bmatrix} Wf \\ 0 \end{bmatrix} \in \mathbb{R}^{2n}.$$

Here $W \in \mathbb{R}^{n\times n}$ is positive diagonal, $K \in \mathbb{R}^{n\times n}$ is a Toeplitz matrix (and therefore we have $m = n$). This least squares problem leads to a quadratic minimization which is equivalent to solving a linear system with coefficient matrix in the form of (1.2).

In our test cases we choose $n = 128$, $\kappa(W) \approx 10^6$, where $\kappa(\cdot)$ denotes the condition number of a matrix, and $\mu = 10^{-3}$. Our test cases mainly concern the cases where $\alpha < w_{\min}$ and so according to (3.3), we shall see how the values $(\alpha - w_{\min})/(\alpha + w_{\min})$ are near -1. The singular values σ_i's of K affect correspondingly those $(1,1)$ entries in the blocks F_i's and we know the relationship between a and ρ depends on whether F_i has real or non-real eigenvalues. Here we shall use two examples of K, one is well-conditioned and the other one is ill-conditioned. Numerically, the entries of the well-conditioned K are given by

$$K_{i,j}^{(1)} = \frac{1}{\sqrt{|i-j|+1}}$$

and those of the ill-conditioned K are given by (here we take $\sigma = 2$)

$$K_{i,j}^{(2)} = \frac{1}{\sqrt{2\pi}\sigma}\, e^{-|i-j|^2/(2\sigma^2)},$$

see [5]. Different preconditioning parameters are chosen with various orders and leading coefficients. Keep in mind that the spectra and the approximations taken are for the transformed matrices $I - \mathscr{P}^{-1}\mathscr{A}$. To obtain the original spectra we should transform back the eigenvalues correspondingly. In the following, we use an approximation of the spectra of HSS preconditioned matrices in Section 3 to illustrate and explain the spectra obtained from these numerical examples.

Fig. 1 shows the orders of the singular values of K. By looking at how many singular values are smaller or larger than the preconditioning parameter α and using f_2, we get a

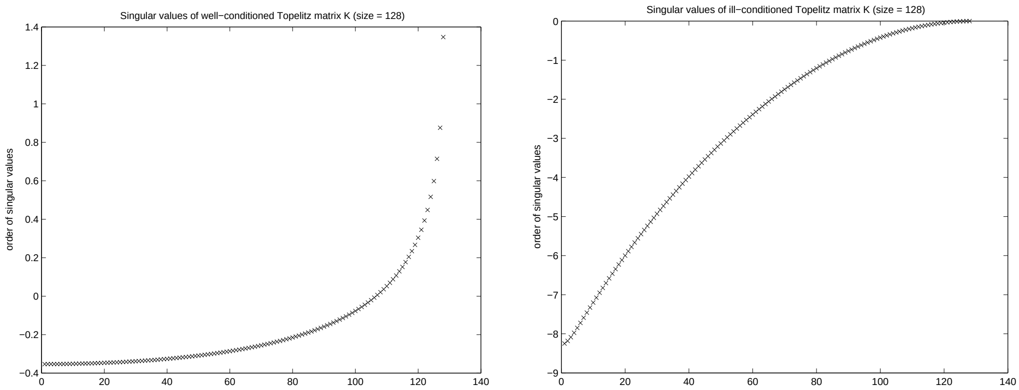

Figure 1: Plots (in order) of singular values of K (128×128). Left: well-conditioned; Right: ill-conditioned (note the difference in scales on the vertical axes).

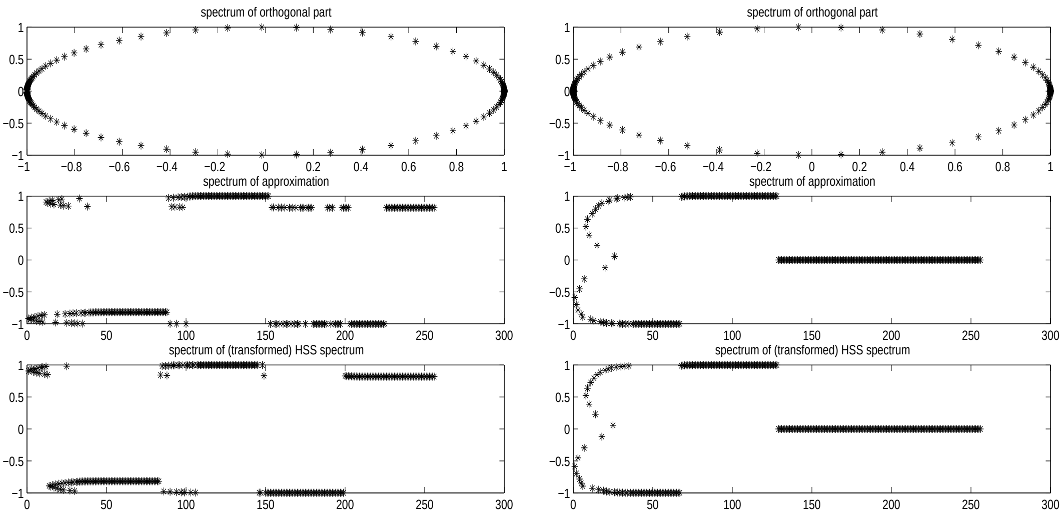

Figure 2: $w_i \in [1, 10^6]$; ill-conditioned K. Left: $\alpha = 0.01$; Right: $\alpha = 0.001$.

rough picture of the distribution of the eigenvalues of the orthogonal matrix $(\alpha I - \mathscr{S})(\alpha I + \mathscr{S})^{-1}$ on $\mathbf{R}(1)$. Note that

$$|f_2(x)| > 0.980 \quad \text{for} \quad \log\left|\frac{\alpha}{x}\right| > 1.$$

Except for the case of all the singular values are very near to α, otherwise a very large proportion of eigenvalues lie around ± 1.

Figs. 2-5 show the cases where the eigenvalues of W lie within $[1, 10^6]$. The uppermost figure shows the spectra of the corresponding orthogonal matrices $(\alpha I - \mathscr{S})(\alpha I + \mathscr{S})^{-1}$; the middle one shows the spectra of the approximated matrices $\mathscr{V}\mathscr{D}\mathscr{V}^T$ and the bottom figure shows the true spectra for the preconditioned systems. In all cases since $\alpha < w_{\min}$, we multiply -1 to ρ (see Section 3.2).

Figs. 2 and 3 correspond to the ill-conditioned K. We can see from each of the figures

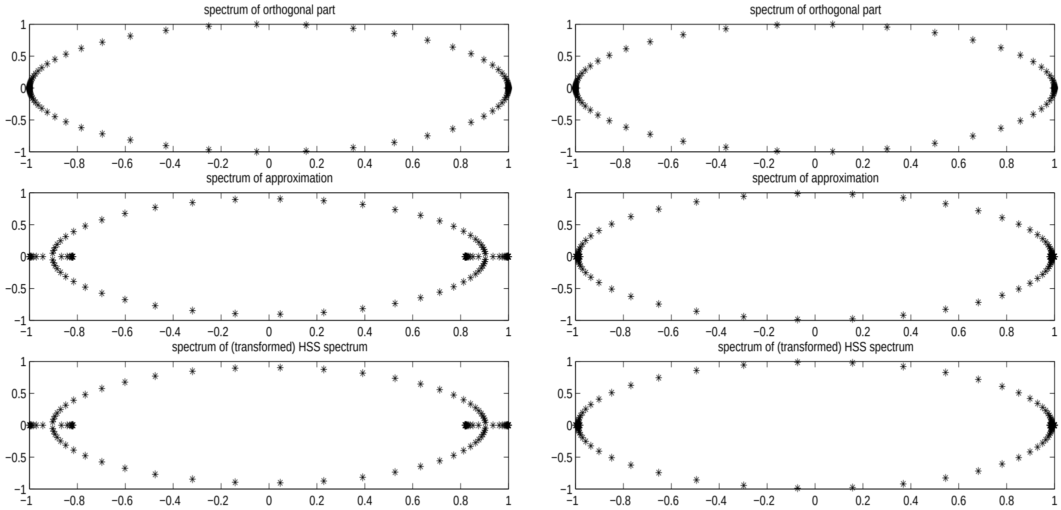

Figure 3: $w_i \in [1, 10^6]$; ill-conditioned K. Left: $\alpha = 10^{-4}$; Right: $\alpha = 10^{-5}$.

Table 1: Some parameters for Figs. 2-3.

	Fig. 2 (left)	Fig. 2 (right)	Fig. 3 (left)	Fig. 3 (right)
α	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$\rho = -\frac{\alpha-\mu}{\alpha+\mu}$	0.818	0	0.818	0.980
$\sqrt{\rho}$	N/A	N/A	0.904	0.990
$\|E\|$	2.00×10^{-2}	2.00×10^{-3}	2.00×10^{-4}	2.00×10^{-5}
$\sqrt{\theta}$	2.00×10^{-1}	6.32×10^{-2}	2.00×10^{-2}	6.32×10^{-3}

the eigenvalues of $(\alpha I - \mathscr{S})(\alpha I + \mathscr{S})^{-1}$ lie evenly on $\mathbf{R}(1)$ as the singular values of K lie within 10^{-9} and 1. Since α is smaller than w_i's, we approximate $(\alpha I - \mathscr{H})(\alpha I + \mathscr{H})^{-1}$ by

$$-\begin{bmatrix} I_m & 0 \\ 0 & \rho I_n \end{bmatrix}$$

and so

$$\rho = -\frac{\alpha - \mu}{\alpha + \mu}.$$

If $\rho < 0$ (i.e., $\alpha < \mu$), then real eigenvalues occur (see Lemma 2.3), and even when $\alpha \le \mu$ (see the remarks after Lemma 2.3). Otherwise if $\rho > 0$, then non-real eigenvalues exist and they lie on $\mathbf{R}(\sqrt{\rho})$. According to Lemma 3.1, real eigenvalues exist when

$$|a| \ge \frac{2\sqrt{\rho}}{\rho + 1}, \quad \text{where } a = \frac{\alpha^2 - \sigma_i^2}{\alpha^2 + \sigma_i^2}.$$

Figs. 4 and 5 correspond to the well-conditioned K. Here $(\alpha I - \mathscr{S})(\alpha I + \mathscr{S})^{-1}$ is very

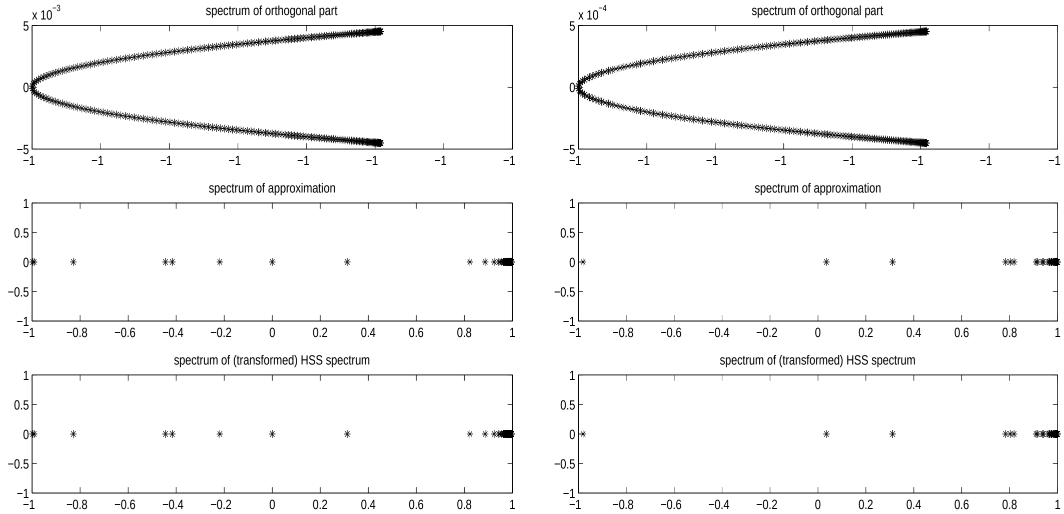

Figure 4: $w_i \in [1, 10^6]$; well-conditioned K. Left: $\alpha = 10^{-3}$; Right: $\alpha = 10^{-4}$.

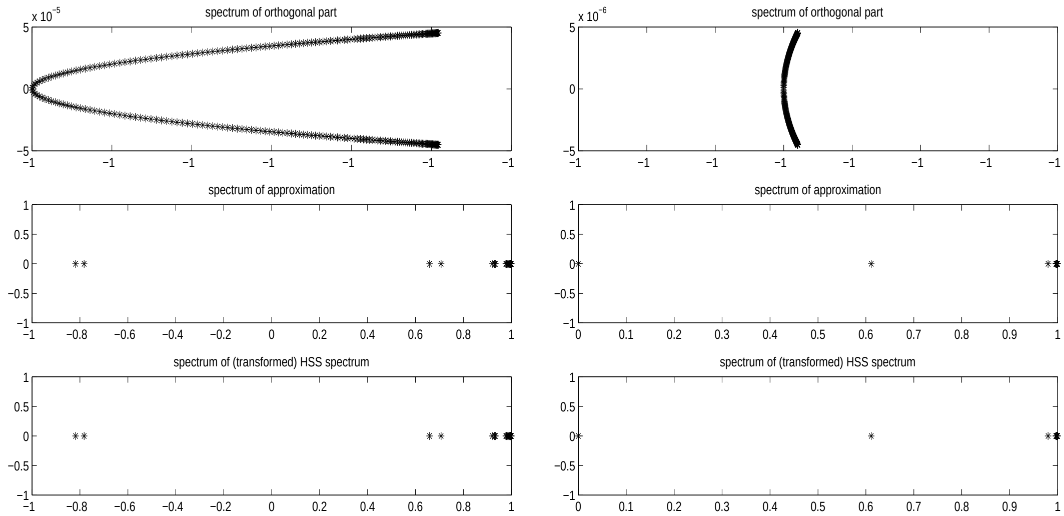

Figure 5: $w_i \in [1, 10^6]$; well-conditioned K. Left: $\alpha = 10^{-5}$; Right: $\alpha = 10^{-6}$.

near to $-I$ and all eigenvalues have real parts $(\alpha^2 - \sigma_i^2)/(\alpha^2 + \sigma_i^2)$ very near -1. Hence,

$$\left|\frac{\alpha^2 - \sigma_i^2}{\alpha^2 + \sigma_i^2}\right| > \frac{2\sqrt{\rho}}{\rho + 1} \quad \text{for all } i$$

and real spectrum occurs. In fact by Lemma 3.3 we can also see that the spectrum is close to that of $(\alpha I - \mathscr{H})(\alpha I + \mathscr{H})^{-1}$. Therefore, $(\alpha I - \mathscr{H})(\alpha I + \mathscr{H})^{-1}(\alpha I - \mathscr{S})(\alpha I + \mathscr{S})^{-1}$ is close to $(\alpha I - \mathscr{H})(\alpha I + \mathscr{H})^{-1}$. Since α is smaller than w_i's, the eigenvalues $(\alpha I - \mathscr{H})(\alpha I + \mathscr{H})^{-1}$ are close to 1 and $\rho = -(\alpha - \mu)/(\alpha + \mu)$.

Tables 1 and 2 summarize some important quantities mentioned above that are related

Table 2: Some parameters for Figs. 4-5.

	Fig. 4 (left)	Fig. 4 (right)	Fig. 5 (left)	Fig. 5 (right)
α	10^{-3}	10^{-4}	10^{-5}	10^{-6}
$\rho = -\frac{\alpha-\mu}{\alpha+\mu}$	0	0.818	0.980	0.998
$\|E\|$	2.00×10^{-3}	2.00×10^{-4}	2.00×10^{-5}	2.00×10^{-6}
$\sqrt{\theta}$	6.32×10^{-2}	2.00×10^{-2}	6.32×10^{-3}	2.00×10^{-3}

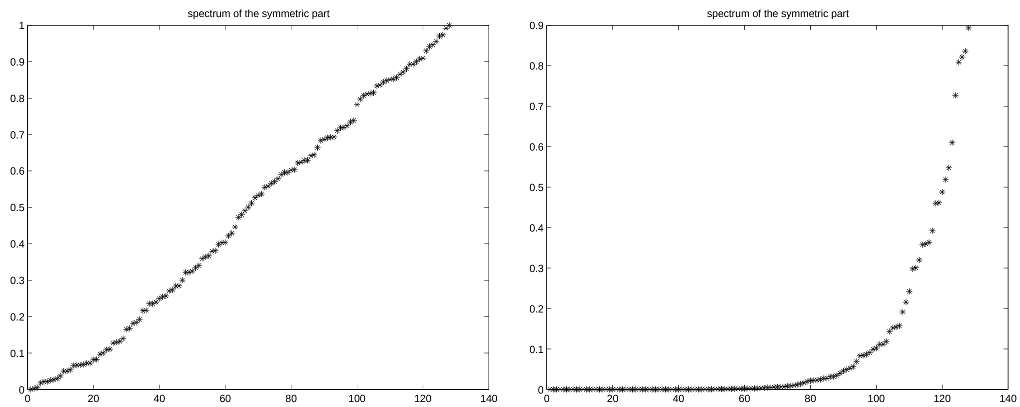

Figure 6: Plots (in order) the eigenvalues of W (in $[10^{-6}, 1]$): uniformly distributed (left) and log-uniformly distributed (right).

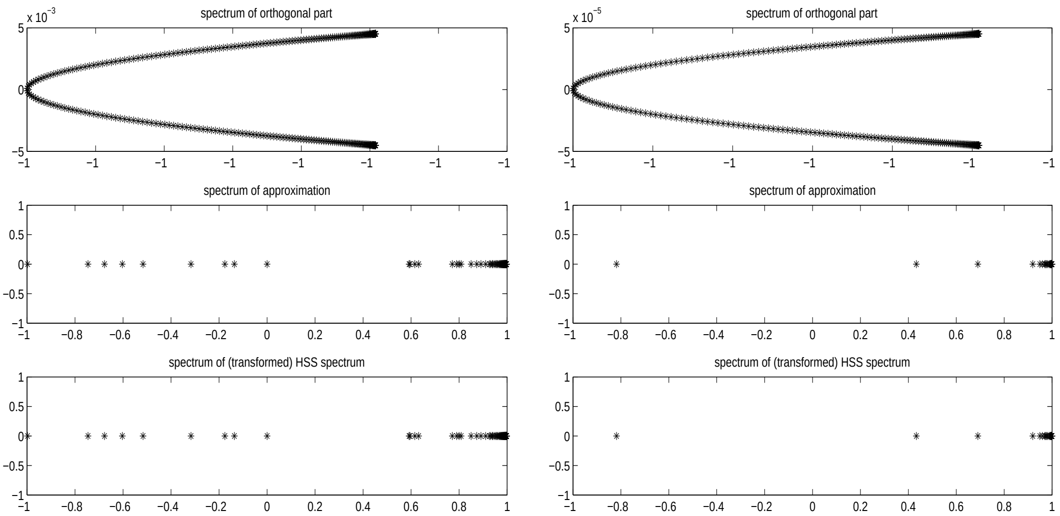

Figure 7: $w_i \in [10^{-6}, 1]$ (uniformly distributed); well-conditioned K. Left: $\alpha = 10^{-3}$; Right: $\alpha = 10^{-5}$.

to the spectra shown on Figs. 2-5. In the tables, ρ refers to the non-unit (real) eigenvalues of the approximated spectrum and $\sqrt{\rho}$ refers to the moduli of the non-real eigenvalues of the approximated spectrum. We also list the values of $\|E\|$ determined by (3.3) for each case which is equal to $\|\mathscr{E}\|$ that provides a bound on the difference between true and

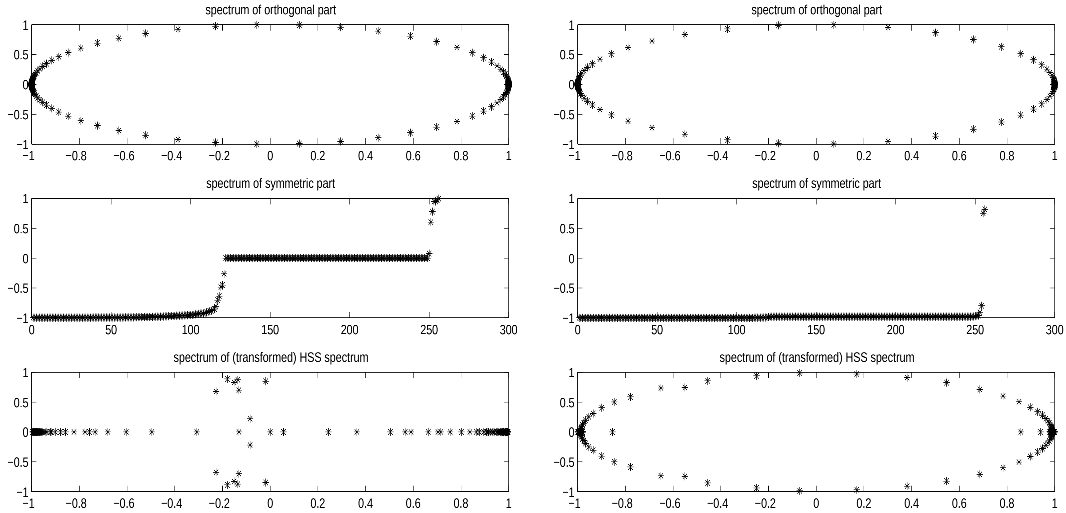

Figure 8: $w_i \in [10^{-6}, 1]$ (uniformly distributed); ill-conditioned K. Left: $\alpha = 10^{-3}$; Right: $\alpha = 10^{-5}$.

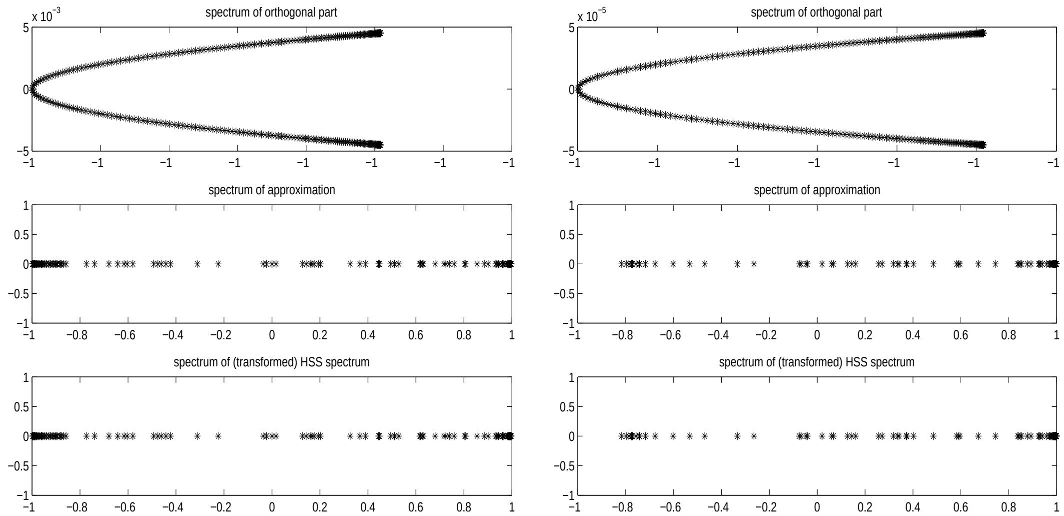

Figure 9: $w_i \in [10^{-6}, 1]$ (log-uniformly distributed); well-conditioned K. Left: $\alpha = 10^{-3}$; Right: $\alpha = 10^{-5}$.

approximated eigenvalues. Recall that unless $\rho = \pm 1$, we should use (3.6) for the purpose of calculating δ.

Figs. 7-10 show the cases with corresponding eigenvalues of W lying within $[10^{-6}, 1]$. Figs. 7 and 8 show the case where the eigenvalues are uniformly distributed while Figs. 9 and 10 show the case where the eigenvalues of W are log-uniformly distributed. A plot of uniformly distributed and log-uniformly distributed eigenvalues can be seen from Fig. 6. The upper figure still shows the spectra of the corresponding orthogonal matrices $(\alpha I - \mathscr{S})(\alpha I + \mathscr{S})^{-1}$ and the bottom figure shows the true spectra for the (transformed) preconditioned systems. The middle one of Figs. 7 and 9 are explained in the next paragraph.

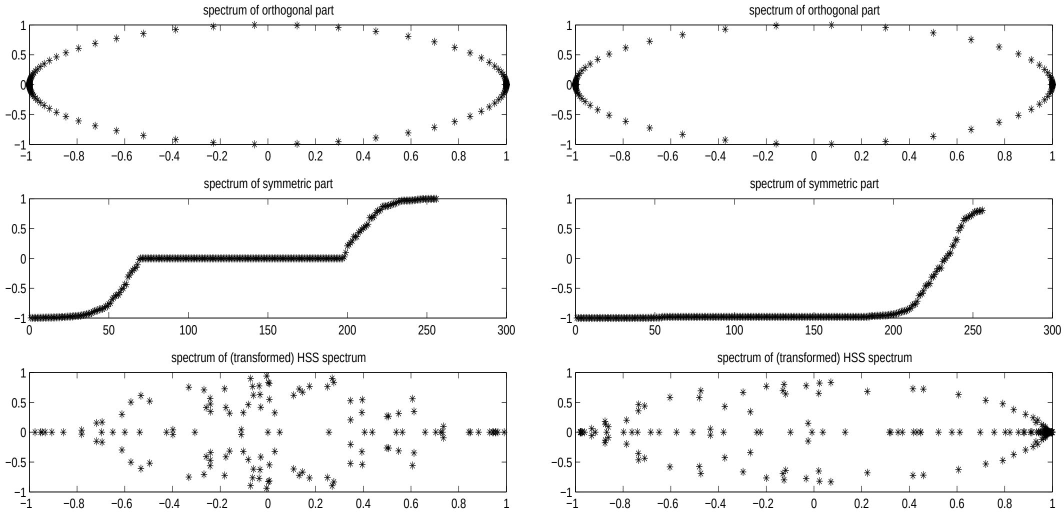

Figure 10: $w_i \in [10^{-6}, 1]$ (log-uniformly distributed); ill-conditioned K. Left: $\alpha = 10^{-3}$; Right: $\alpha = 10^{-5}$.

In general for well-conditioned K the spectra of $(\alpha I - \mathscr{S})(\alpha I + \mathscr{S})^{-1}$ is still clustered around -1. Similar to Figs. 4 and 5, we can still use Lemma 3.3 to explain spectrum of $(\alpha I - \mathscr{H})(\alpha I + \mathscr{H})^{-1}(\alpha I - \mathscr{S})(\alpha I + \mathscr{S})^{-1}$ looks like $-(\alpha I - \mathscr{H})(\alpha I + \mathscr{H})^{-1}$ (Figs. 7 and 9). The middle figures are now showing the spectra of $-(\alpha I - \mathscr{H})(\alpha I + \mathscr{H})^{-1}$. For ill-conditioned K (Figs. 8 and 10), the preconditioning parameter α lies inside the eigenvalues of W and we cannot make an approximation for estimating the eigenvalues. At this point we cannot tell too much about the spectra and we still cannot provide a general picture for such case. The explanations to this case may still require further effort.

5. Spectral analysis of HSS preconditioned system: the case $\mu = 0$

In the previous section we have provided some pictures for the preconditioned spectra corresponding to $\mu \neq 0$ with different preconditioning parameter α. Actually, the ideas presented fit well to explain the spectral properties corresponding to the particular case $\mu = 0$ also, which have been previously covered in [11]. When $\mu = 0$, the coefficient matrix corresponding to the saddle point problem is

$$\mathscr{A} = \begin{bmatrix} W & K \\ -K^T & 0 \end{bmatrix}, \tag{5.1}$$

where $\mathscr{H}$, $\mathscr{S}$ are defined as in (1.3) and the HSS preconditioner $\mathscr{P}$ is still in the form (1.4). Recall that $w_{\min} = \min\{w_1, \cdots, w_m\}$. It was mentioned in [11] that if $\alpha \leq \frac{1}{2} w_{\min}$ then the preconditioned system has a real spectrum. The explanations involved are mainly some conditions satisfied by the Rayleigh quotients corresponding to the coefficient matrices of the preconditioned systems. Since $\alpha > 0 = \mu$, by Lemma 2.1 the above condition can further be relaxed:

Theorem 5.1. *Consider the linear system with coefficient matrix* (5.1) *and its corresponding HSS preconditioner* $\mathscr{P}$. *Assume that W is symmetric and positive definite. If the preconditioning parameter satisfies* $\alpha < w_{\min}$, *then the eigenvalues of* $\mathscr{P}^{-1}\mathscr{A}$ *are real.*

Remark 5.1. *Following the remarks given after Lemma* 2.1, *we can still obtain a real spectrum if we have* $\alpha \le w_{\min}$ *only.*

A sufficient condition was also given when the preconditioned system has a clustered spectrum. Here we shall state the result and provide another new and simple proof.

Corollary 5.1. ([11]) *Consider the linear system with coefficient matrix* (5.1) *and its corresponding HSS preconditioner* $\mathscr{P}$. *Assume that W is symmetric and positive definite. For sufficiently small* $\alpha > 0$, *the eigenvalues of* $\mathscr{P}^{-1}\mathscr{A}$ *cluster near zero and* 2. *More precisely, for small* $\alpha > 0$,

$$\sigma(\mathscr{P}^{-1}\mathscr{A}) \subseteq (0, \epsilon_1) \cup (2 - \epsilon_2, 2)$$

with $\epsilon_1, \epsilon_2 > 0$ *and* $\epsilon_1, \epsilon_2 \to 0$, *as* $\alpha \to 0$.

Proof. If $\mu = 0$, then $\rho = (\alpha - \mu)/(\alpha + \mu) = 1$ regardless of what α is. On the other hand, as $\alpha \to 0$, then

$$\frac{\alpha - w_i}{\alpha + w_i} \to -1 \quad \text{and} \quad (\alpha I - W)(\alpha I + W)^{-1} \to -I.$$

Hence, the approximation $(\alpha I - \mathscr{H})(\alpha I + \mathscr{H})^{-1}$ gives us

$$\begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}.$$

To maintain the positivity of $(1,1)$-th block we multiply by $-I_{m+n}$ and the overall approximation now becomes

$$\mathscr{V} \begin{bmatrix} 2\alpha D^{-1} - I & 0 & 2\Sigma D^{-1} \\ 0 & I & 0 \\ -2\Sigma D^{-1} & 0 & -(2\alpha D^{-1} - I) \end{bmatrix} \mathscr{V}^T. \tag{5.2}$$

The above matrix have eigenvalues ± 1 (more precisely m eigenvalues being 1 and n eigenvalues being -1). We know that the preconditioned system must have a real spectrum by Lemma 5.1. Moreover, the overall matrix above is normal and so the difference between the true and the approximated eigenvalues due to perturbation is at most $\|E\|$. By (3.3), we have $\|E\| = (\alpha - w_{\min})/(\alpha + w_{\min}) + 1$. In this case since $\alpha \to 0$, $\|E\|$ also becomes very small and so we can ensure the true eigenvalues will be very near ± 1. We also know that the absolute values of the eigenvalues will not exceed 1 by the bound given by (2.4).

Combining the above, when $\mu = 0$ and $\alpha \to 0$, we have the eigenvalues (of $I - \mathscr{P}^{-1}\mathscr{A}$) lying within $(1 - \epsilon_1, 1) \cup (-1, -1 + \epsilon_2)$, which in turn implies that the eigenvalues for the preconditioned system lie within $(0, \epsilon_1) \cup (2 - \epsilon_2, 2)$. $\square$

Remark 5.2. *We can see from the above proof that* ϵ_1 *and* ϵ_2 *are bounded above by*

$$\|E\| = \frac{\alpha - w_{\min}}{\alpha + w_{\min}} + 1 = \frac{2\alpha}{\alpha + w_{\min}} = \frac{2}{1 + \frac{w_{\min}}{\alpha}} .$$

Therefore, $\epsilon_1, \epsilon_2 \to 0$ *as* $\alpha \to 0$.

From the proof of Corollary 5.2, we see that the matrix in (5.2) has, precisely, eigenvalues 1 of multiplicity m and -1 of multiplicity n. Recall that we have multiplied $-I$ to obtain this matrix, which means the approximation of

$$(\alpha I - \mathscr{H})(\alpha I + \mathscr{H})^{-1}(\alpha I - \mathscr{S})(\alpha I + \mathscr{S})^{-1}$$

has eigenvalues -1 of multiplicity m and 1 of multiplicity n. Finally, we know the preconditioned spectrum has m eigenvalues near 2 while n eigenvalues near 0.

6. Concluding remarks

Spectral analysis is a key issue for studying preconditioned systems. It is especially important for HSS preconditioner in the search of the optimal preconditioning parameter α. In this paper, we have extended previous results to the general situation of $\alpha \neq \mu$ by using an approximation scheme. In particular the approximated spectrum gives a picture which is very close to the true one when α lies away from the eigenvalues of W, which is rather usual when applying HSS preconditioner. Our analysis also covers the case $\mu = 0$ which was studied in [11]. Actually we can extend our result further to the case when the $(2,2)$-th block is a non-scalar positive semi-definite matrix C, i.e.,

$$\mathscr{A} = \begin{bmatrix} W & K \\ -K^T & C \end{bmatrix}.$$

In particular when the eigenvalues of C are clustered or the preconditioning parameter α can be set away from the eigenvalues of C, we can even approximate the $(2,2)$-th block of $(\alpha I - \mathscr{H})(\alpha I + \mathscr{H})^{-1}$ by a scalar times an identity matrix also and our analysis can still apply. Our arguments are elementary and simple enough to give a satisfactory picture.

We can see in most cases we cannot avoid a spectrum with very small eigenvalues, which correspond to those eigenvalues very near to 1 in

$$(\alpha I - \mathscr{H})(\alpha I + \mathscr{H})^{-1}(\alpha I - \mathscr{S})(\alpha I + \mathscr{S})^{-1}.$$

When α is smaller than the eigenvalues of W and the singular values of K, we can see

$$\frac{\alpha - w_i}{\alpha + w_i} \quad \text{and} \quad \frac{\alpha^2 - \sigma_i^2}{\alpha^2 + \sigma_i^2}$$

will both be very near to -1 and so the existence of eigenvalue 1 in $(\alpha I - \mathscr{H})(\alpha I + \mathscr{H})^{-1}(\alpha I - \mathscr{S})(\alpha I + \mathscr{S})^{-1}$ cannot be avoided. Therefore, the use of a small preconditioning parameter α is not beneficial in particular when K is well-conditioned. For an ill-conditioned K, HSS preconditioner may prevent the clustering of eigenvalues in $(\alpha I - \mathscr{S})(\alpha I + \mathscr{S})^{-1}$.

Nevertheless, we cannot rule out that under some special scenarios, it is possible to come up with a rather desirable (for example, cluster away from zero) spectrum for HSS preconditioned system. For instance if α happens to be bigger than the singular values of K but smaller than both the eigenvalues of W and the parameter μ, then

$$(\alpha I - \mathscr{S})(\alpha I + \mathscr{S})^{-1} \to I$$

while

$$(\alpha I - \mathscr{H})(\alpha I + \mathscr{H})^{-1} \to -I$$

and so

$$(\alpha I - \mathscr{H})(\alpha I + \mathscr{H})^{-1}(\alpha I - \mathscr{S})(\alpha I + \mathscr{S})^{-1} \to -I.$$

Finally the preconditioned system has eigenvalues near 2 which is beneficial for applying iterative methods.

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