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New Estimates for the Rate of Convergence of the Method of Subspace Corrections

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Abstract. We discuss estimates for the rate of convergence of the method of successive subspace corrections in terms of condition number estimate for the method of parallel subspace corrections. We provide upper bounds and in a special case, a lower bound for preconditioners defined via the method of successive subspace corrections.

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1. Introduction

The method of subspace corrections is a general iterative method for solving the linear system of equations arising from the variational formulation in a Hilbert space. The modern theory of the subspace correction methods has showed that the multigrid method and the domain decomposition method are systematically equivalent. In this paper, we study the method of subspace corrections. We refer readers to von Neumann [7], Bramble [1], Bramble and Zhang [2], Hackbusch [4], Griebel and Oswald [3], Trottenberg, Oosterlee and Schüller [5], Xu [8,9] and Xu and Zikatanov [10] for the method of subspace corrections.

One main focus in this paper is to provide an estimate for the rate of convergence of the method of successive subspace corrections (MSSC) in terms of the method of parallel subspace corrections (MPSC). This work can be considered as an extension and application of the convergence theory by Xu and Zikatanov [10]. Based on this framework, we obtain a formula for the convergence rate, which can be employed to derive many other estimates related to the method of subspace corrections. We then show how the convergence rate of the MSSC can be estimated in terms of the MPSC. Similar results and other approaches on deriving estimates relating MSSC and MPSC are also found in earlier works [2,3,8,9].

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The remainder of this paper is organized as follows. In Section 2, we present the variational problem in a Hilbert space and recall some notation and algorithms in [10]. In Section 3, we relate the convergence rate to the condition number of multiplicative preconditioner and derive a formula for this condition number. In Section 4, we present estimates for the convergence rate, using the formula for the condition number. We discuss the special cases of the subspace correction methods in Section 5.

2. Notation and preliminaries

Let V be a Hilbert space with an inner product $(\cdot,\cdot)$ and its induced norm $\|\cdot\|$ and let V^* be the dual space of V. We consider the variational problem:

Find $u \in V$ such that for any given $f \in V^*$

$$a(u,v) = \langle f, v\rangle, \qquad \forall\ v \in V. \tag{2.1}$$

Here, $a(\cdot,\cdot) : V \times V \mapsto \mathbb{R}$ is a continuous symmetric positive definite (SPD) bilinear form. Since $a(\cdot,\cdot)$ is a SPD, it introduces an inner product and a norm which we denote with $(\cdot,\cdot)_a$ and $\|\cdot\|_a$. In more classical notation, we define an operator $A : V \mapsto V$ by

$$(Au, v) = a(u,v) \quad \forall u \in V, \forall v \in V.$$

Following [10], we introduce some notation and the parallel and successive subspace correction algorithms. We first consider a collection of closed subspaces

$$V_k \subset V, \quad k = 1, \cdots, J,$$

such that

$$(\mathbf{A0}) \quad V = \sum_{k=1}^{J} V_k.$$

Associated with each subspace V_k, we define a continuous positive definite bilinear form $a_k(\cdot,\cdot)$ to be an approximation of $a(\cdot,\cdot)$ on V_k. We point out that in general $a_k(\cdot,\cdot)$ may not be symmetric. To assure the well-posedness of the subspace problems, we assume that the bilinear forms $a_k(\cdot,\cdot)$ satisfy appropriate inf-sup conditions.

The method of parallel subspace corrections (MPSC) is an iterative algorithm that correct the residual equations in parallel in each subspace. MPSC is described as follows.

Algorithm 2.1 (MPSC). *Let $u^0 \in V$ be given.*

for $\ell = 1, 2, \cdots$

  for $i = 1 : J$

    Let $e_i \in V_i$ solve

$$a_i(e_i, v_i) = f(v_i) - a(u^{\ell-1}, v_i) \quad \forall v_i \in V_i,$$

  endfor

  $u^\ell = u^{\ell-1} + \sum_{i=1}^{J} e_i,$

endfor

Next, we outline the method of successive subspace corrections (MSSC) which corrects successively the residual equations (sequentially in each subspace) as follows.

Algorithm 2.2 (MSSC). *Let $u^0 \in V$ be given.*

for $\ell = 1, 2, \cdots$

$u_0^{\ell-1} = u^{\ell-1}$

for $i = 1 : J$

Let $e_i \in V_i$ solve

$$a_i(e_i, v_i) = f(v_i) - a(u_{i-1}^{\ell-1}, v_i) \quad \forall v_i \in V_i,$$

$u_i^{\ell-1} = u_{i-1}^{\ell-1} + e_i,$

endfor

$u^\ell = u_J^{\ell-1},$

endfor

We introduce a class of linear operators $T_k : V \mapsto V_k$ for $k = 1, \cdots, J$, defined as

$$a_k(T_k v, v_k) = a(v, v_k), \qquad \forall\, v \in V,\ v_k \in V_k. \tag{2.2}$$

Thanks to the inf-sup conditions, T_k is well-defined and

(**A1**) $\mathscr{R}(T_k) = V_k$ and $T_k : V_k \mapsto V_k$ is isomorphic.

Moreover, we assume that T_k satisfies the following inequality:

(**A2**) $\|T_k v\|_a^2 \le \omega(T_k v, v)_a,\ \forall\, v \in V$ for some constant $\omega \in (0, 2)$.

In the special case that the subspace solvers are exact, we use a linear operator P_k, instead of T_k, satisfying

$$a(P_k v, v_k) = a(v, v_k), \quad v \in V,\ v_k \in V_k.$$

Associated with T_k, we will use the notation T_k^t and T_k^* as adjoint operators of T_k with respect to $(\cdot,\cdot)$ and $(\cdot,\cdot)_a$, respectively, and as its symmetrization,

$$\bar{T}_k = T_k + T_k^* - T_k^* T_k \quad \text{for} \quad k = 1, \cdots, J. \tag{2.3}$$

Two cases under consideration exist, depending on whether the subspace solvers are exact or inexact. When the subspace solvers are exact, a proof based on Theorem 2.1 of a result relating the MPSC and MSSC is given in [11]. Other proofs and different estimates can be found in [2,3,8,9]. In the paper, we focus only on the case of the inexact subspace solvers. In the successive subspace correction algorithm 2.2, the error propagation operator E_J is represented as

$$E_J = (I - T_J)(I - T_{J-1}) \cdots (I - T_1).$$

Once we have the form of E_J, then its symmetrized version is also observed, namely,

$$E_J^* E_J = (I - T_1^*)(I - T_2^*) \cdots (I - T_J^*)(I - T_J)(I - T_{J-1}) \cdots (I - T_1).$$

In what follow, we need an additive preconditioned operator T and a multiplicative preconditioned operator $\bar{T}$ such that

$$T = \sum_{k=1}^{J} \bar{T}_k \quad \text{and} \quad \bar{T} = I - E_J^* E_J,$$

respectively. We remark that $T = \sum_{k=1}^{J} \bar{T}_k$ can be viewed as the so-called symmetrized version of the parallel subspace correction algorithm. Also, we can define a SPD additive preconditioner $\bar{B}_{\mathrm{a}}$ and a SPD multiplicative preconditioner $\bar{B}_{\mathrm{m}}$ in more classical sense so that

$$\bar{B}_{\mathrm{a}} A = T \quad \text{and} \quad \bar{B}_{\mathrm{m}} A = \bar{T}, \tag{2.4}$$

respectively. As is well known (see [3, 9, 10]), an additive preconditioner satisfies:

$$(\bar{B}_{\mathrm{a}}^{-1} v, v) = (T^{-1} v, v)_a = \inf_{\sum_k v_k = v} \sum_{k=1}^{J} (\bar{T}_k^{-1} v_k, v_k)_a. \tag{2.5}$$

We recall the following results [10, Theorem 4.2 and Corollary 4.3]:

Theorem 2.1 (Theorem 4.2, [10])**.** *Under the assumptions* (**A0**), (**A1**) *and* (**A2**)*, the following identity holds:*

$$\|E_J\|_a^2 = \|(I - T_J)(I - T_{J-1}) \cdots (I - T_1)\|_a = 1 - \frac{1}{1 + c_0}, \tag{2.6}$$

where

$$c_0 = \sup_{\|v\|_a = 1} \inf_{\sum_k v_k = v} \sum_{k=1}^{J} \Big(T_k \bar{T}_k^{-1} T_k^* \big(\sum_{j=k}^{J} v_j - T_k^{-1} v_k \big), \sum_{j=k}^{J} v_j - T_k^{-1} v_k \Big)_a < \infty. \tag{2.7}$$

Corollary 2.1 (Corollary 4.3, [10])**.** *Suppose that the subspace solvers are exact. Under the assumption* (**A0**)*, the following identity holds:*

$$\|E_J\|_a^2 = \|(I - P_J)(I - P_{J-1}) \cdots (I - P_1)\|_a = 1 - \frac{1}{1 + c_0}, \tag{2.8}$$

where

$$c_0 = \sup_{\|v\|_a = 1} \inf_{\sum_k v_k = v} \sum_{i=1}^{J} \|P_k \sum_{j>k} v_j\|_a^2. \tag{2.9}$$

In the following, we will drop the subscript J. Unless stated otherwise, E denotes E_J.

3. MSSC and MPSC as preconditioners

From Theorem 2.1 it follows that the multiplicative (MSSC) iterations are convergent. If we now consider the preconditioner $\bar{B}_{\rm m}$ defined by the symmetrized MSSC iteration, then we have (for all $v \in V$)

$$\|v\|_a^2 \le (\bar{B}_{\rm m}^{-1} v, v) \le K \|v\|_a^2.$$

As a consequence from Theorem 2.1, the condition number of the preconditioned operator $\bar{B}_{\rm m} A$ is K, and also we have $K = c_0 + 1$. Clearly, estimating K is equivalent to estimating c_0. In some applications, it is more convenient to work with K, for example when the preconditioner is directly defined, without using the error propagation operator (see [6] for such approach). In this section we derive a formula for K, which is similar to the one that is used in case of additive preconditioner (2.5). We begin with stating and proving auxiliary results (Lemma 3.1–3.3 below).

Lemma 3.1. *The following identities hold for $k = 1, \dots, J$:*

$$I + T_k \bar{T}_k^{-1} T_k^* (I - T_k^{-1}) = T_k \bar{T}_k^{-1}, \tag{3.1}$$

$$((T_k^*)^{-1} - I) T_k \bar{T}_k^{-1} T_k^* (T_k^{-1} - I) = \bar{T}_k^{-1} - I. \tag{3.2}$$

Proof. Let $S_k = (T_k^*)^{-1} + T_k^{-1} - I$. It follows from the definition of $\bar{T}_k$ that $S_k^{-1} = T_k \bar{T}_k^{-1} T_k^*$. Then we have

$$\begin{aligned} & I + T_k \bar{T}_k^{-1} T_k^* (I - T_k^{-1}) = S_k^{-1} \left(S_k + I - T_k^{-1} \right) \\ = & S_k^{-1} \left((T_k^*)^{-1} + T_k^{-1} - I + I - T_k^{-1} \right) \\ = & S_k^{-1} (T_k^*)^{-1} = T_k \bar{T}_k^{-1}. \end{aligned}$$

This proves the first relation (3.1) stated in the lemma. The proof of (3.2) is found in [10, Lemma 4.9]. □

The formula for the condition number is given in Theorem 3.1 below. Its proof requires two technical results, Lemma 3.2 and Lemma 3.3, which are "local", in a sense that they hold for any fixed $v \in V$.

Lemma 3.2. *Given an arbitrary decomposition $v = \sum_{k=1}^J v_k$ of $v \in V$. Then we have*

$$\|v\|_a^2 + c_0(v) = K(v), \tag{3.3}$$

where

$$c_0(v) = \sum_{k=1}^J \left(T_k \bar{T}_k^{-1} T_k^* (w_k + (I - T_k^{-1}) v_k), (w_k + (I - T_k^{-1}) v_k) \right)_a,$$

$$K(v) = \sum_{k=1}^J (\bar{T}_k^{-1} (v_k + T_k^* w_k), (v_k + T_k^* w_k))_a,$$

with $w_k = \sum_{i=k+1}^J v_i$.

Proof. Direct calculations lead to

$$\begin{aligned}
&(v,v)_a+\sum_{k=1}^{J}\Big(T_k\bar{T}_k^{-1}T_k^*(w_k+(I-T_k^{-1})v_k),w_k+(I-T_k^{-1})v_k\Big)_a\\
=&(\sum_{k=1}^{J}v_k,\sum_{k=1}^{J}v_k)_a+\sum_{k=1}^{J}\Big(T_k\bar{T}_k^{-1}T_k^*(w_k+(I-T_k^{-1})v_k),w_k+(I-T_k^{-1})v_k\Big)_a\\
=&\sum_{k=1}^{J}\Big[(v_k,v_k)_a+2(v_k,w_k)_a+\Big(T_k\bar{T}_k^{-1}T_k^*(w_k+(I-T_k^{-1})v_k),w_k+(I-T_k^{-1})v_k\Big)_a\Big].
\end{aligned}$$

For each term in the sum above, we observe that

$$\begin{aligned}
&(v_k,v_k)_a+2(v_k,w_k)_a+\Big(T_k\bar{T}_k^{-1}T_k^*(w_k+(I-T_k^{-1})v_k),w_k+(I-T_k^{-1})v_k\Big)_a\\
=&(v_k,v_k)_a+(T_k\bar{T}_k^{-1}T_k^*(I-T_k^{-1})v_k,(I-T_k^{-1})v_k)_a\\
&+2\Big[(v_k,w_k)_a+(T_k\bar{T}_k^{-1}T_k^*(I-T_k^{-1})v_k,w_k)_a\Big]+(T_k\bar{T}_k^{-1}T_k^*w_k,w_k)_a.
\end{aligned}$$

We now use Lemma 3.1 to obtain

$$\begin{aligned}
&(v_k,v_k)_a+2(v_k,w_k)_a+\Big(T_k\bar{T}_k^{-1}T_k^*(w_k+(I-T_k^{-1})v_k),w_k+(I-T_k^{-1})v_k\Big)_a\\
=&(\bar{T}_k^{-1}v_k,v_k)_a+2((I+T_k\bar{T}_k^{-1}T_k^*(I-T_k^{-1}))v_k,w_k)_a+(\bar{T}_k^{-1}T_k^*w_k,T_k^*w_k)_a\\
=&(\bar{T}_k^{-1}v_k,v_k)_a+2(T_k\bar{T}_k^{-1}v_k,w_k)_a+(\bar{T}_k^{-1}T_k^*w_k,T_k^*w_k)_a.
\end{aligned}$$

Combining these results together results in

$$\begin{aligned}
&(v,v)_a+c_0(v)\\
=&\sum_{k=1}^{J}(\bar{T}_k^{-1}v_k,v_k)_a+2\sum_{k=1}^{J}(\bar{T}_k^{-1}v_k,T_k^*w_k)_a+\sum_{k=1}^{J}(\bar{T}_k^{-1}T_k^*w_k,T_k^*w_k)_a\\
=&\sum_{k=1}^{J}(\bar{T}_k^{-1}(v_k+T_k^*w_k),v_k+T_k^*w_k)_a=K(v).\quad\square
\end{aligned}$$

Next we prove a relation between the action of the multiplicative preconditioner $\bar{T}$ and $K(v)$, for any $v\in V$.

Lemma 3.3. *For $v\in V$, we have*

$$(\bar{B}_{\mathrm{m}}^{-1}v,v)=(\bar{T}^{-1}v,v)_a=\inf_{\sum_k v_k=v}\sum_{k=1}^{J}(\bar{T}_k^{-1}(v_k+T_k^*w_k),v_k+T_k^*w_k)_a,\tag{3.4}$$

where $w_k=\sum_{i>k}v_i$.

Proof. We will follow the procedure of the proof of [10, Theorem 4.2]. Let $\underset{\sim}{T}, \underset{\approx}{S}, \underset{\sim}{T}^*, \tilde{w}$ and $\tilde{\phi}$ be defined as in [10, Theorem 4.2]. We note that $I - E^*E = \bar{T}$ and

$$(\bar{T}v, v)_a = ((I - E^*E)v, v)_a = \|v\|_a^2 - \|Ev\|_a^2 = (\underset{\sim}{T}(\underset{\approx}{S} + \underset{\sim}{T}^*\underset{\sim}{T})^{-1}\underset{\sim}{T}^*v, v)_a,$$

which implies that

$$\bar{T} = \underset{\sim}{T}(\underset{\approx}{S} + \underset{\sim}{T}^*\underset{\sim}{T})^{-1}\underset{\sim}{T}^*.$$

For any $w \in V$, let

$$\tilde{w} = (\underset{\approx}{S} + \underset{\sim}{T}^*\underset{\sim}{T})^{-1}\underset{\sim}{T}^*w \text{ and } v = \underset{\sim}{T}\tilde{w}.$$

By writing

$$\underset{\approx}{S}\tilde{w} = (\underset{\approx}{S} + \underset{\sim}{T}^*\underset{\sim}{T})\tilde{w} - \underset{\sim}{T}^*\underset{\sim}{T}\tilde{w} = \underset{\sim}{T}^*w - \underset{\sim}{T}^*\underset{\sim}{T}\tilde{w},$$

we immediately see that for any $\tilde{\phi} \in \mathcal{N}(\underset{\sim}{T})$,

$$(\underset{\approx}{S}\tilde{w}, \tilde{\phi})_{V^J} = 0 \quad \text{and} \quad (\underset{\approx}{S}\tilde{w}, \tilde{w})_{V^J} \le (\underset{\approx}{S}(\tilde{w} + \tilde{\phi}), (\tilde{w} + \tilde{\phi}))_{V^J},$$

which yields

$$(\underset{\approx}{S}\tilde{w}, \tilde{w})_{V^J} = \inf_{\underset{\sim}{T}\tilde{v}=v} (\underset{\approx}{S}\tilde{v}, \tilde{v})_{V^J},$$

where $(\cdot,\cdot)_{V^J}$ is the usual inner product in the product space. It follows from the simple identity

$$\bar{T} = \bar{T}^2 + \underset{\sim}{T}(\underset{\approx}{S} + \underset{\sim}{T}^*\underset{\sim}{T})^{-1}\underset{\approx}{S}(\underset{\approx}{S} + \underset{\sim}{T}^*\underset{\sim}{T})^{-1}\underset{\sim}{T}^*$$

that

$$(\bar{T}^{-1}v, v)_a = (v, v)_a + (\underset{\approx}{S}\tilde{w}, \tilde{w})_{V^J}.$$

Mixing them together brings in

$$(\bar{T}^{-1}v, v)_a = (v, v)_a + \inf_{\underset{\sim}{T}\tilde{v}=v} (\underset{\approx}{S}\tilde{v}, \tilde{v})_{V^J} = \inf_{\underset{\sim}{T}\tilde{v}=v} K(v),$$

where Lemma 3.2 is used in the last equality. This completes the proof. □

As we pointed out, Lemma 3.2 and Lemma 3.3 imply that

$$(\bar{B}_{\mathrm{m}}^{-1}v, v) = (\bar{T}^{-1}v, v)_a = \inf_{\sum_k v_k = v} K(v),$$

for $v \in V$. From the relations that we have shown above, we obtain

Theorem 3.1. *Under the assumptions* (**A0**)*,* (**A1**) *and* (**A2**)*, the following identity holds*

$$\|E\|_a^2 = 1 - \frac{1}{K}, \tag{3.5}$$

where

$$K = \sup_{\substack{v \in V \\ \|v\|_a = 1}} \inf_{\sum_k v_k = v} K(v), \tag{3.6}$$

where $K(v)$ is defined as in Lemma 3.2*.*

4. On the relation between MSSC and MPSC

In this section, we give a new convergence estimate of MSSC in terms of the MPSC. In order to relate the convergence rate of MSSC to the MPSC, we need the following Lemma (proof can be found in Griebel and Oswald [3] or Bramble and Zhang [2]).

Lemma 4.1. ([2]) *Let $A(\cdot,\cdot)$ be a symmetric positive definite bilinear form on $M_J \times M_J$ and let $D_k : M_k \to M_k$ be symmetric and positive definite. Assume that*

$$\sum_{k,j=1}^{J} A(u_k, v_j) \le K \left(\sum_{k=1}^{J} (D_k u_k, u_k) \right)^{\frac{1}{2}} \left(\sum_{k=1}^{J} (D_k v_k, v_k) \right)^{\frac{1}{2}},$$

holds for any $u_k, v_k \in M_k$. Then

$$\sum_{k=1}^{J} \sum_{j>k}^{J} A(u_k, v_j) \le \frac{1}{2} \lceil \log_2 J \rceil K \left(\sum_{k=1}^{J} (D_k u_k, u_k) \right)^{\frac{1}{2}} \left(\sum_{k=1}^{J} (D_k v_k, v_k) \right)^{\frac{1}{2}},$$

holds for any $u_k, v_k \in M_k$.

4.1. Upper bounds

We first recall the parameter ω, which is defined on the assumption **(A2)**. Now we define a new parameter σ to be the smallest real number that satisfies the following inequality:

$$(\bar{T}_k^{-1} T_k^* v, T_k^* v) \le \sigma^2 (\bar{T}_k v, v), \quad \forall\ v \in V. \tag{4.1}$$

We would like to discuss more about the new parameter σ and, in particular, elaborate on its role. Since $T_k P_k = T_k$, $T_k^* P_k = T_k^*$ and $\bar{T}_k P_k = \bar{T}_k$, it suffices to consider the above inequality for $v \in V_k$. This means that σ^2 is the largest eigenvalue of the generalized eigenvalue problem

$$T_k \bar{T}_k^{-1} T_k^* x = \mu \bar{T}_k x.$$

If Y is an operator, defined as follows:

$$Y = \bar{T}_k^{-1/2} \left[T_k \bar{T}_k^{-1} T_k^* \right] \bar{T}_k^{-1/2} \left(= [\bar{T}_k^{-1/2} T_k \bar{T}_k^{-1/2}] \cdot [\bar{T}_k^{-1/2} T_k^* \bar{T}_k^{-1/2}] \right),$$

then σ^2 becomes the largest eigenvalue of Y. The definition of Y entails that

$$\sigma = \|\bar{T}_k^{-1/2} T_k \bar{T}_k^{-1/2}\|.$$

For instance, if T_k is self-adjoint, then we see that $\sigma = \frac{1}{2-\omega}$, whence $\sigma^2 = \frac{1}{(2-\omega)^2}$. Indeed, it follows from (**A2**) that

$$\begin{aligned}(\bar{T}_k v, v)_a &= ((2T_k - T_k^2)v, v)_a = 2(T_k v, v)_a - (T_k^2 v, v)_a \\ &\geq 2(T_k v, v)_a - \omega(T_k v, v)_a \\ &\geq (2-\omega)(T_k v, v)_a.\end{aligned}$$

When T_k is not self-adjoint, then $\frac{1}{2-\omega} \leq \sigma$. Indeed, it follows from [10, Lemma 4.1] † that

$$(\bar{T}_k v, v)_a = 2(T_k v, v)_a - \|T_k v\|_a^2 \geq 2(T_k v, v)_a - \frac{\omega}{2-\omega}(\bar{T}_k v, v)_a,$$

whence

$$(T_k v, v)_a \leq \frac{1}{2-\omega}(\bar{T}_k v, v)_a.$$

The desired inequality comes from a fact that

$$\rho(\bar{T}_k^{-1/2} T_k \bar{T}_k^{-1/2}) \leq \|\bar{T}_k^{-1/2} T_k \bar{T}_k^{-1/2}\|.$$

The following theorem is another main result in the paper, which shows that the convergence rate of MSSC can be estimated directly in terms of MPSC.

Theorem 4.1. *Assume that the subspace solvers are inexact and that T is the additive preconditioned operator, namely,*

$$T = \sum_{k=1}^{J} \bar{T}_k.$$

Then

$$\|E\|_a^2 = 1 - \frac{1}{K},$$

where K in (3.6) *is bounded above by*

$$K \leq \frac{1}{\lambda_{\min}(T)}\left[1 + \frac{\sigma(\log_2 J)\lambda_{\max}(T)}{2}\right]^2. \tag{4.2}$$

Proof. Let us fix $v \in V$ and take any decomposition $v = \sum_{k=1}^{J} v_k$ of v and set $w_k = \sum_{j>k} v_j$. We will here employ an inner product $(\cdot,\cdot)_{\bar{T}_k^{-1}}$ on V_k and its induced norm $\|\cdot\|_{\bar{T}_k^{-1}}$, defined by

$$(\cdot,\cdot)_{\bar{T}_k^{-1}} := (\bar{T}_k^{-1}\cdot,\cdot)_a \quad \text{and} \quad \|\cdot\|_{\bar{T}_k^{-1}} := (\cdot,\cdot)_{\bar{T}_k^{-1}}^{1/2},$$

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†Here, we need the following inequality given in [10, Lemma 4.1]: $\|T_k v\|_a^2 \leq \frac{\omega}{2-\omega}(\bar{T}_k v, v)_a, \quad \forall\, v \in V.$

respectively. It follows from several applications of the Cauchy-Schwarz inequality that

$$\begin{aligned}\sum_{k=1}^{J}\|v_k+T_k^*w_k\|^2_{\bar{T}_k^{-1}} &\le \sum_{k=1}^{J}\|v_k\|^2_{\bar{T}_k^{-1}}+2\sum_{k=1}^{J}\|v_k\|_{\bar{T}_k^{-1}}\|T_k^*w_k\|_{\bar{T}_k^{-1}}+\sum_{k=1}^{J}\|T_k^*w_k\|^2_{\bar{T}_k^{-1}}\\ &\le \sum_{k=1}^{J}\|v_k\|^2_{\bar{T}_k^{-1}}+2\sigma\sum_{k=1}^{J}\|v_k\|_{\bar{T}_k^{-1}}\|w_k\|_{\bar{T}_k}+\sigma^2\sum_{k=1}^{J}\|w_k\|^2_{\bar{T}_k}\\ &\le \left([S_1(v)]^{1/2}+\sigma[S_2(v)]^{1/2}\right)^2,\end{aligned}$$

where

$$S_1(v)=\sum_{k=1}^{J}\|v_k\|^2_{\bar{T}_k^{-1}},\quad \text{and}\quad S_2(v)=\sum_{k=1}^{J}\|w_k\|^2_{\bar{T}_k}.$$

It is readily to see that $\sup_{\|v\|_a=1}\inf_{\sum_k v_k=v}S_1(v)$ is equal to $[\lambda_{\min}(T)]^{-1}$. Therefore, the main part for proving this theorem is to get an estimate of the form

$$S_2(v)\le C^2S_1(v), \tag{4.3}$$

since taking the infimum on both sides of (4.3) over all decompositions $v=\sum_k v_k$ of v and the supremum over all $v\in V$ such that $\|v\|=1$ indicates

$$K\le[\lambda_{\min}(T)]^{-1}(1+\sigma C)^2.$$

Accordingly, we will focus on finding a constant C in (4.3). By the definition of T, we get that

$$[\lambda_{\max}(T)]^{-1}\|v\|_a^2\le(T^{-1}v,v)_a\le S_1(v).$$

Also the Cauchy-Schwarz inequality is applied to have

$$(u,v)_a=\sum_{j,k=1}^{J}(u_j,v_k)_a\le\|u\|_a\|v\|_a\le\lambda_{\max}(T)[S_1(u)]^{1/2}[S_1(v)]^{1/2}.$$

Lemma 4.1 implies that

$$\sum_{k=1}^{J}(u_k,w_k)_a=\sum_{k=1}^{J}(u_k,\sum_{j=k+1}^{J}v_j)_a\le\frac{1}{2}(\log_2 J)\lambda_{\max}(T)[S_1(u)]^{1/2}[S_1(v)]^{1/2}.$$

A simple replacement of u_k with $\bar{T}_kw_k$ in the above inequality leads to

$$S_2(v)\le\frac{1}{2}(\log_2 J)\lambda_{\max}(T)[S_2(v)]^{1/2}[S_1(v)]^{1/2}.$$

Dividing both sides by $[S_2(v)]^{1/2}$ and taking the square ultimately provide

$$C=\frac{1}{2}(\log_2 J)(\lambda_{\max}(T)),$$

in (4.3). This completes the proof. $\square$

4.2. Lower bound in a special case

We will now derive a lower bound in the case when $T_k = \omega P_k$ where $\omega \in (0,2)$. Setting $w_k = \sum_{i>k} v_i$, $k = 1, \cdots, J-1$. From (2.5) and Lemma 3.3

$$\begin{aligned}(\bar{B}_{\mathrm{a}}^{-1} v, v) &= \inf_{\sum_k v_k = v} \sum_{k=1}^{J} \frac{1}{\omega} \|v_k\|_a^2, \\ (\bar{B}_{\mathrm{m}}^{-1} v, v) &= \inf_{\sum_k v_k = v} \sum_{k=1}^{J} \frac{1}{2\omega - \omega^2} \|P_k(v_k + \omega w_k)\|_a^2.\end{aligned} \tag{4.4}$$

Assume that the actions of $\bar{B}_{\mathrm{m}}$ and $\bar{B}_{\mathrm{a}}$ are given by (4.4). We will prove the following inequality.

$$\frac{2-\omega}{4}(\bar{B}_{\mathrm{a}}^{-1} v, v) \leq (\bar{B}_{\mathrm{m}}^{-1} v, v). \tag{4.5}$$

Indeed, let $v \in V$ be such that $v = \sum_{k=1}^{J} v_k$ with $v_k \in V_k$. Applying some obvious identities and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sum_{k=1}^{J} \|v_k\|_a^2 &\leq \sum_{k=1}^{J} \|v_k\|_a^2 + \frac{\omega}{2-\omega}(v,v)_a = \sum_{k=1}^{J} \|v_k\|_a^2 + \frac{\omega}{2-\omega}(\sum_{k=1}^{J} v_k, \sum_{m=1}^{J} v_m)_a \\ &= \sum_{k=1}^{J} \|v_k\|_a^2 + \frac{\omega}{2-\omega} \sum_{k=1}^{J} \|v_k\|_a^2 + \frac{2}{2-\omega} \sum_{k=1}^{J} (v_k, \omega P_k w_k)_a \\ &= \left(1 + \frac{\omega}{2-\omega}\right) \sum_{k=1}^{J} \|v_k\|_a^2 + \frac{2}{2-\omega} \sum_{k=1}^{J} (v_k, \omega P_k w_k)_a \\ &= \frac{2}{2-\omega} \sum_{k=1}^{J} (v_k, P_k(v_k + \omega w_k))_a \\ &\leq \frac{2}{2-\omega} \left(\sum_{k=1}^{J} \|v_k\|_a^2\right)^{1/2} \left(\sum_{k=1}^{J} \|P_k(v_k + \omega w_k)\|_a^2\right)^{1/2}.\end{aligned}$$

5. Other estimates of K and $K(v)$

In this section, we discuss multilevel methods and we explain about how K in (3.6) can be estimated, when each of the T_k is itself a parallel (successive) sub-subspace solver. We further assume that the subspaces are nested:

$$V_1 \subset V_2 \subset \cdots \subset V_J,$$

and that there exist operators $\Pi_k : V \mapsto V_k$ satisfying

$$\mathscr{R}(\Pi_i - \Pi_{i-1}) \subset V_i, \quad \text{with} \quad \Pi_0 = 0, \quad \text{and} \quad \Pi_J = I.$$

We then have a telescopic decomposition

$$v=\sum_{k=1}^{J} v_k \text{ with } v_k=(\Pi_k-\Pi_{k-1})v, \tag{5.1}$$

then we have

$$K \le \sup_{\|v\|=1} \sum_{k=1}^{J} \|v_k+T_k^*(v-\Pi_k v)\|_{\bar{T}_k^{-1}}^2. \tag{5.2}$$

In the case that $\Pi_k=P_k$ is the a-orthogonal projection, then the right side of the inequality (5.2) can be reduced to

$$K \le \sup_{\|v\|=1} \sum_{k=1}^{J} \|v_k\|_{\bar{T}_k^{-1}}^2. \tag{5.3}$$

5.1. MPSC as sub-subspace solver

Let us assume that on each subspace V_k, T_k is given by

$$T_k=\sum_{i=1}^{n_k} T_{k,i},$$

which is an additive preconditioned operator on V_k. We then have

$$(T_k^{-1}v_k,v_k)_a=\inf_{\sum v_{k,i}=v_k} \sum_{i=1}^{n_k} (T_{k,i}^{-1}v_{k,i},v_{k,i})_a.$$

Also, we see that

$$\|v_k\|_{\bar{T}_k^{-1}}^2 \le (2-\omega)^{-1}(T_k^{-1}v_k,v_k)_a.$$

Combining the inequalities above with (5.3) yields

$$K \le (2-\omega)^{-1} \sup_{\|v\|_a=1} \sum_{k=1}^{J} \left(\inf_{\sum v_{k,i}=v_k} \sum_{i=1}^{n_k} (T_{k,i}v_{k,i},v_{k,i})_a \right),$$

which can be generalized to

$$K \le (2-\omega)^{-1} \sup_{\|v\|_a=1} \inf_{\sum_k v_k=v} \sum_{k=1}^{J} \left(\inf_{\sum v_{k,i}=v_k+T_k^*(\sum_{j>k} v_j)} \left(\sum_{i=1}^{n_k} (T_{k,i}v_{k,i},v_{k,i})_a \right) \right).$$

5.2. MSSC as a sub-subspace solver

We suppose that $\bar{T}_k$ is given by a multiplicative preconditioned operator on each subspace V_k. Then we have

$$(\bar{T}_k^{-1} v_k, v_k)_a = \inf_{\sum_i v_{k,i} = v_k} \sum_{i=1}^{n_k} (\bar{T}_{k,i}^{-1}(v_{k,i} + T_{k,i}^*(\sum_{j>i} v_{k,j})), v_{k,i} + T_{k,i}^*(\sum_{j>i} v_{k,j}))_a.$$

Consequently, we get

$$K \le \sup_{\|v\|_a=1} \inf_{\sum_k v_k = v} \sum_{k=1}^{J} \inf_{\sum_i v_{k,i} = v_k} \sum_{i=1}^{n_k} (\bar{T}_{k,i}^{-1}(v_{k,i} + T_{k,i}^*(\sum_{j>i} v_{k,j})), v_{k,i} + T_{k,i}^*(\sum_{j>i} v_{k,j}))_a.$$

6. Concluding remarks

The study presented here was motivated by the new representation for the convergence rate of multiplicative methods in Lemma 3.2. We have shown how this new representation can be used in deriving upper and lower bounds for the convergence rate of MSSC method.

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