Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2009.m88032

Vol. **2**, No. 3, pp. 326-340
August 2009

A Fourth-Order Modified Method for the Cauchy Problem of the Modified Helmholtz Equation

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Received 20 October 2008; Accepted (in revised version) 25 March 2009

Abstract. This paper is concerned with the Cauchy problem for the modified Helmholtz equation in an infinite strip domain $0 < x \leq 1$, $y \in \mathbb{R}$. The Cauchy data at $x = 0$ is given and the solution is then sought for the interval $0 < x \leq 1$. This problem is highly ill-posed and the solution (if it exists) does not depend continuously on the given data. In this paper, we propose a fourth-order modified method to solve the Cauchy problem. Convergence estimates are presented under the suitable choices of regularization parameters and the *a priori* assumption on the bounds of the exact solution. Numerical implementation is considered and the numerical examples show that our proposed method is effective and stable.

AMS subject classifications: 65M32

Key words: Cauchy problem for the modified Helmholtz equation, ill-posed problem, fourth-order modified method.

1. Introduction

The modified Helmholtz equation arises in many areas, especially in practical physical applications, such as implicit marching schemes for the heat equation, Debye-Huckel theory and the linearization of the Poisson-Boltzmann equation, see, e.g., [1,2,8,11,12]. The direct problems, i.e., the Dirichlet, Neumann or mixed boundary value problems for the modified Helmholtz equation have been studied extensively in the past century. However, in some practical problems, the boundary data on the whole boundary cannot be obtained. We only know the noisy data on a part of boundary or at some interior points of the concerned domain. This is called an inverse problem. The Cauchy problem (a function that satisfies a partial differential equation and the Dirichlet and Neumann boundary conditions which are given on a part of the boundary) for the modified Helmholtz equation is an inverse problem and is severely ill-posed, which means the solution does not depend continuously on the given Cauchy data (the given Dirichlet and Neumann data) and any small change in the given data may cause large change to the solution [7,22]. Several numerical

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methods have been proposed to solve this problem, such as the alternating iterative algorithm based on the Landweber method in conjunction with the boundary element method (BEM) [13], the conjugate gradient method with the BEM [14], the method of fundamental solution [10,15,24]. In [9], the boundary knot method was applied to solve the Cauchy problem for the inhomogeneous Helmholtz equation.

In this paper, we will consider the Cauchy problem of the modified Helmholtz equation in an infinite strip domain $0 < x < 1$, $y \in \mathbb{R}$ as follows

$$-\triangle u(x,y) + k^2 u(x,y) = 0, \quad x \in (0,1), \quad y \in \mathbb{R}, \tag{1.1}$$

$$u(0,y) = \varphi(y), \quad y \in \mathbb{R}, \tag{1.2}$$

$$u_x(0,y) = 0, \qquad y \in \mathbb{R}. \tag{1.3}$$

The problem (1.1)-(1.3) is ill-posed, see Section 2 below and [6, 7]. One kind of regularization methods is to cut off high frequencies, see, e.g., [16, 20].

In this paper, we use a modified method to solve the Cauchy problem for the modified Helmholtz equation (1.1)-(1.3). That is, we modify the original equation (1.1) to give the following fourth-order equation,

$$-\Delta u(x,y) + k^2 u(x,y) + \mu^2 u_{xxyy}(x,y) = 0. \tag{1.4}$$

The basic idea originated from Weber's paper [23] and then Elden's paper [3] in which the authors used a similar method to solve a standard inverse heat conduction problem. This kind of method has been used to solve a wide range of ill-posed problems. For example, the Cauchy problem for the Laplace equation, the backward heat conduction problem and the sideways heat equation, see, e.g., [4, 17–19, 21].

The paper is organized as follows. In Section 2, we consider the ill-posedness of the proposed problem and propose a fourth-order modified method. In Section 3, the convergence estimates under the suitable choices of regularization parameters are established. The numerical implementation is discussed in Section 4. Some conclusions are given in Section 5.

2. Ill-posedness and a modified method

In this section, we will analyze the ill-posedness of the Cauchy problem (1.1)-(1.3) in the frequency space and give a modified method for obtaining a stable approximate solution.

Define the Fourier transform of a function as follows,

$$\hat{\varphi}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(y) \exp(-i\xi y) dy.$$

To solve the Cauchy problem (1.1)-(1.3), we take the Fourier transform with respect to variable y to Eq. (1.1) and boundary conditions (1.2)-(1.3). Then the Cauchy problem can be formulated in the frequency space as follows

$$-\hat{u}_{xx}+\xi^2\hat{u}+k^2\hat{u}=0,\quad x\in(0,1),\;\;\xi\in\mathbb{R}, \tag{2.1}$$

$$\hat{u}(0,\xi)=\hat{\varphi}(\xi),\quad \xi\in\mathbb{R}, \tag{2.2}$$

$$\hat{u}_x(0,\xi)=0,\qquad \xi\in\mathbb{R}. \tag{2.3}$$

It is easy to verify that the solution of problem (2.1)-(2.3) is

$$\hat{u}(x,\xi)=\hat{\varphi}(\xi)\cosh(\sqrt{\xi^2+k^2}x). \tag{2.4}$$

Note that the function $\cosh(\sqrt{\xi^2+k^2}x)$ in (2.4) is unbounded as $|\xi|$ tends to infinity for $0<x\le 1$. If $\hat{u}(x,\xi)$ is a function in $L^2(\mathbb{R})$ with respect to ξ, the exact data function $\hat{\varphi}(\xi)$ must decay rapidly as $|\xi|\to\infty$. Small errors in high-frequency components can lead to solution blow up and completely destroy the solution for $0<x\le 1$. That means, the solution $\hat{u}(x,\xi)$ does not depend continuously on the data $\hat{\varphi}(\xi)$ and the Cauchy problem (1.1)-(1.3) is ill-posed in the Hadamard sense. Now we modify Eq. (1.1) by adding a fourth-order mixed derivative term and consider the following problem

$$-\triangle v+k^2v+\mu^2v_{xxyy}=0,\quad x\in(0,1),\;\;y\in\mathbb{R}, \tag{2.5}$$

$$v(0,y)=\varphi_\delta(y),\quad y\in\mathbb{R}, \tag{2.6}$$

$$v_x(0,y)=0,\qquad y\in\mathbb{R}, \tag{2.7}$$

where φ_δ is an approximate function of φ and μ is a regularization parameter. Since the data $\varphi(\cdot)$ is based on physical observations, there is a measurement error in φ.

In this paper, we assume the measured data function $\varphi_\delta(\cdot)\in L^2(\mathbb{R})$ satisfies

$$\|\varphi(\cdot)-\varphi_\delta(\cdot)\|\le\delta, \tag{2.8}$$

where $\|\cdot\|$ denotes the L^2-norm and the constant $\delta>0$ represents a bound on the measurement error. It is not difficult to show that the solution of problem (2.5)-(2.7) in frequency space is

$$\hat{v}(x,\xi)=\hat{\varphi}_\delta(\xi)\cosh\Big(\sqrt{\frac{\xi^2+k^2}{1+\mu^2\xi^2}}x\Big). \tag{2.9}$$

For simplicity, denote

$$s=\sqrt{\xi^2+k^2}, \tag{2.10}$$

and

$$\tau=\sqrt{\frac{\xi^2+k^2}{1+\mu^2\xi^2}}. \tag{2.11}$$

It is observed that if μ is chosen sufficiently small, then for small $|\xi|$, τ is close to s. Further, as $|\xi|$ tends to infinity, τ tends to $1/\mu$. Thus τ is bounded. The function $\cosh(\tau x)$ is also bounded for $0<x\le 1$. Therefore, by such modification, the high frequencies have been damped out. The key point of the proposed method is to choose an appropriate parameter

μ, which serves as a regularization parameter, such that the convergence result can be obtained.

Assume that $u(x,\cdot)\in L^2(\mathbb{R})$ and an *a priori* bound exists

$$\| u(1,\cdot) \| \leq E, \tag{2.12}$$

where E is a finite positive constant.

In the following section, we will prove that the solution of problem (2.5)-(2.7) convergent to the solution of problem (1.1)-(1.3) in the L^2-norm when the noise level δ tends to zero.

3. Convergence estimate

In this section, convergence estimates for the cases of $0 < x < 1$ and $x = 1$ will be given, respectively.

Theorem 3.1. *Let u be the solution of the Cauchy problem* (1.1)-(1.3) *and v be the solution of modified problem* (2.5)-(2.7)*. The regularization parameter μ is taken as*

$$\mu = \sqrt{2}\Big(\ln\frac{E}{\delta}\Big)^{-1}. \tag{3.1}$$

Assume the measured data function φ_δ satisfies (2.8) *and the exact solution $u(1,y)$ satisfies* (2.12)*. Then we have the following convergence estimates:*

(i) If $\frac{\delta}{E}\geq e^{-3}$, then

$$\begin{aligned}&\|u(x,\cdot)-v(x,\cdot)\|\\ &\leq \delta e^{\sqrt{2}|k|x} + \Big(2+\ln\Big(\frac{E}{\delta}\Big)\Big)E^x\delta^{1-x}, \quad \textit{for}\ \ 0<x<1.\end{aligned} \tag{3.2}$$

(ii) If $\frac{\delta}{E}< e^{-3}$, then

$$\begin{aligned}&\|u(x,\cdot)-v(x,\cdot)\|\\ &\leq \begin{cases} \delta e^{\sqrt{2}|k|x} + 2E^x\delta^{1-x} + \Big(\frac{3}{(1-x)e}\Big)^3\frac{E}{(\ln\frac{E}{\delta})^2}, \\ \qquad\qquad \textit{for}\ 0<x<1-3(\ln\frac{E}{\delta})^{-1}; \\ \delta e^{\sqrt{2}|k|x} + \Big(2+\ln\frac{E}{\delta}\Big)E^x\delta^{1-x}, \\ \qquad\qquad \textit{for}\ 1-3(\ln(\frac{E}{\delta}))^{-1}\leq x<1.\end{cases}\end{aligned} \tag{3.3}$$

Proof. It follows from (2.4) and (2.9) and the Parseval's equality that

$$
\begin{aligned}
&\|u(x,\cdot)-v(x,\cdot)\| = \|\hat{u}(x,\cdot)-\hat{v}(x,\cdot)\| \\
=&\left\|\hat{\varphi}(\xi)\cosh(\sqrt{\xi^2+k^2}x)-\hat{\varphi}_\delta(\xi)\cosh\left(\sqrt{\frac{\xi^2+k^2}{1+\mu^2\xi^2}}x\right)\right\| \\
\le&\left\|\hat{\varphi}(\xi)\Big[\cosh(\sqrt{\xi^2+k^2}x)-\cosh\left(\sqrt{\frac{\xi^2+k^2}{1+\mu^2\xi^2}}x\right)\Big]\right\| \\
&+\left\|\Big(\hat{\varphi}(\xi)-\hat{\varphi}_\delta(\xi)\Big)\cosh\left(\sqrt{\frac{\xi^2+k^2}{1+\mu^2\xi^2}}x\right)\right\|.
\end{aligned}
$$

Consider the assumptions (2.8) and (2.12). By (2.4), it is not difficult to yield

$$
\|u(x,\cdot)-v(x,\cdot)\| \le E\sup_{\xi\in\mathbb{R}} A(\xi)+\delta\sup_{\xi\in\mathbb{R}} B(\xi), \tag{3.4}
$$

where

$$
A(\xi):=\left|\frac{\cosh\left(\sqrt{\xi^2+k^2}x\right)-\cosh\left(\sqrt{\frac{\xi^2+k^2}{1+\mu^2\xi^2}}x\right)}{\cosh\left(\sqrt{\xi^2+k^2}\right)}\right|, \tag{3.5}
$$

$$
B(\xi):=\left|\cosh\left(\sqrt{\frac{\xi^2+k^2}{1+\mu^2\xi^2}}x\right)\right|. \tag{3.6}
$$

Firstly, we establish an upper bound for $B(\xi)\delta$. Since μ is given by (3.1), when $|\xi|<|k|$, we have

$$
B(\xi)\delta \le \delta\cosh(\sqrt{2}|k|x) \le \delta e^{\sqrt{2}|k|x}. \tag{3.7}
$$

For $|\xi|\ge|k|$, we have

$$
B(\xi)\delta \le \delta\cosh\left(\sqrt{2}\mu^{-1}x\right) \le \delta e^{\sqrt{2}\mu^{-1}x} = E^x\delta^{1-x}. \tag{3.8}
$$

Consequently,

$$
\sup_{\xi\in\mathbb{R}} B(\xi)\delta \le \delta e^{\sqrt{2}|k|x}+E^x\delta^{1-x}. \tag{3.9}
$$

Next, we estimate the bound of $A(\xi)$. It follows from (2.10) and (2.11) that, for $s>\tau$,

$$
\begin{aligned}
A(\xi)=&\frac{\cosh(sx)-\cosh(\tau x)}{\cosh(s)} \\
=&\frac{e^{sx}-e^{\tau x}+e^{-sx}-e^{-\tau x}}{e^s+e^{-s}} \\
\le&\frac{(e^{sx}-e^{\tau x})-(e^{sx}-e^{\tau x})/e^{(s+\tau)x}}{e^s} \\
\le&\frac{e^{sx}-e^{\tau x}}{e^s}=e^{-s(1-x)}\left(1-e^{-(s-\tau)x}\right).
\end{aligned} \tag{3.10}
$$

Using the inequality $1-e^{-r}\le r$ for $r\ge 0$ and (3.10), it is easy to verify that

$$A(\xi)\le (s-\tau)e^{-s(1-x)}. \tag{3.11}$$

Further, using $\sqrt{1+\mu^2\xi^2}\le 1+\frac{1}{2}\mu^2\xi^2$ gives

$$\begin{aligned} s-\tau &= \sqrt{\xi^2+k^2}-\sqrt{\frac{\xi^2+k^2}{1+\mu^2\xi^2}} \\ &= \sqrt{\xi^2+k^2}\left(1-\frac{1}{\sqrt{1+\mu^2\xi^2}}\right) \\ &\le \frac{1}{2}\mu^2\xi^2\sqrt{\xi^2+k^2}. \end{aligned} \tag{3.12}$$

Consequently,

$$A(\xi)\le \frac{1}{2}\mu^2\xi^2\sqrt{\xi^2+k^2}e^{-\sqrt{\xi^2+k^2}(1-x)}. \tag{3.13}$$

Denote $M:=\sqrt{\xi^2+k^2}$. Then $\xi^2=M^2-k^2$ and

$$A(\xi)\le \frac{1}{2}\mu^2(M^2-k^2)Me^{-M(1-x)}\le \frac{1}{2}\mu^2M^3e^{-M(1-x)}. \tag{3.14}$$

Define a function $H(M):=M^3e^{-M(1-x)}$. It is easy to prove that its maximum is

$$H_{\max}(M)=\left(\frac{3}{(1-x)e}\right)^3$$

and maximizer is

$$M_*=\frac{3}{1-x}.$$

To get the convergence results, we need to consider the following two cases:

1. For a large value of ξ such that $M\ge M_0:=\ln(E/\delta)$, note that $s\ge\tau$, inequality (3.10) becomes

$$A(\xi)E\le e^{-M(1-x)}E\le e^{-M_0(1-x)}E=E^x\delta^{1-x},\quad \text{for } 0<x<1. \tag{3.15}$$

2. For a small value of ξ such that $M<M_0=\ln(E/\delta)$, we have
   (a) When $M_*=3/(1-x)<M_0=\sqrt{2}/\mu$, from (3.14), we have

$$A(\xi)E\le \frac{1}{2}\mu^2\Big(\frac{3}{(1-x)e}\Big)^3E=\Big(\frac{3}{(1-x)e}\Big)^3\frac{E}{(\ln\frac{E}{\delta})^2}. \tag{3.16}$$

   (b) If $3/(1-x)\ge\sqrt{2}/\mu$, then $H(M)$ attains its maximum at $M=M_0$. For this case,

$$\begin{aligned} A(\xi)E &\le \frac{1}{2}\mu^2\Big(\frac{\sqrt{2}}{\mu}\Big)^3e^{-\frac{\sqrt{2}}{\mu}(1-x)}E \\ &= \Big(\ln\frac{E}{\delta}\Big)^{-2}\Big(\ln\frac{E}{\delta}\Big)^3\Big(\frac{E}{\delta}\Big)^{x-1}E \\ &= \ln\Big(\frac{E}{\delta}\Big)E^x\delta^{1-x}. \end{aligned} \tag{3.17}$$

In the following, we summary the convergence estimates for various cases:

(i) Since $x \in (0,1)$, $3/(1-x) > 3$. When $\delta/E \geq e^{-3}$, i.e., $\ln\frac{E}{\delta} \leq 3$, we have $3/(1-x) \geq \sqrt{2}/\mu = M_0$. Thus for any $x \in (0,1)$, estimate (3.17) holds. Consequently, combining (3.9), (3.15) and (3.17) gives

$$\|u(x,\cdot)-v(x,\cdot)\| \leq \delta e^{\sqrt{2}|k|x} + \Big(2+\ln\frac{E}{\delta}\Big)E^x\delta^{1-x}, \quad \text{for } 0<x<1. \tag{3.18}$$

(ii) When $\delta/E < e^{-3}$, for $3/(1-x) < M_0$, i.e., $0 < x < 1-3(\ln\frac{E}{\delta})^{-1}$, it follows from (3.9), (3.15) and (3.16) that

$$\|u(x,\cdot)-v(x,\cdot)\| \leq \delta e^{\sqrt{2}|k|x} + 2E^x\delta^{1-x} + \Big(\frac{3}{(1-x)e}\Big)^3\frac{E}{(\ln\frac{E}{\delta})^2}. \tag{3.19}$$

Moreover, for $1-3(\ln(\frac{E}{\delta}))^{-1} \leq x < 1$, by (3.9), (3.15) and (3.17), we have

$$\|u(x,\cdot)-v(x,\cdot)\| \leq \delta e^{\sqrt{2}|k|x} + \Big(2+\ln\frac{E}{\delta}\Big)E^x\delta^{1-x}. \tag{3.20}$$

The proof of the theorem is complete. □

Note that the convergence estimates in Theorem 3.1 are only valid for $0 < x < 1$ and invalid at $x = 1$. To restore the convergence of the approximate solution $v(x,y)$ at $x = 1$, we introduce a stronger priori assumption

$$\|u(1,\cdot)\|_p \leq E, \quad p > 1, \tag{3.21}$$

where $\|\cdot\|_p$ denotes the norm in Sobolev space $H^p(\mathbb{R})$ defined by

$$\|u(1,\cdot)\|_p := \left(\int_{-\infty}^{+\infty}(1+\xi^2)^p|\hat{u}(1,\cdot)|^2d\xi\right)^{\frac{1}{2}}.$$

In this case, we choose μ as

$$\mu = \frac{\sqrt{2}}{\ln\Big(\frac{E}{\delta}(\ln\frac{E}{\delta})^{-2p}\Big)}. \tag{3.22}$$

Theorem 3.2. *Let $u(x,y)$ be the solution of* (1.1)-(1.3) *and let $v(x,y)$ be the solution of* (2.5)-(2.7)*. Assume that μ is given by* (3.22)*. If the measured data $\varphi_\delta(y)$ satisfies* (2.8) *and the exact solution $u(1,y)$ satisfies* (3.21)*, then we have*

$$\begin{aligned}&\|u(1,\cdot)-v(1,\cdot)\|\\ \leq\, & E\Big(\ln\frac{E}{\delta}\Big)^{-2p} + \delta e^{\sqrt{2}|k|} + \Big(\frac{\sqrt{2}\mu}{2}\Big)^pE\\ &+\max\Big\{2^{\frac{1-p}{2}}|k|\mu^p, \frac{\sqrt{2}}{2}|k|\mu^2\Big\}E + \max\Big\{2^{\frac{2-p}{2}}\mu^{p-1}, \frac{\sqrt{2}}{2}\mu^2\Big\}E, \quad p>1.\end{aligned} \tag{3.23}$$

Proof. Taking a similar process of Theorem 3.1, we have

$$\begin{aligned}&\|u(1,\cdot)-v(1,\cdot)\|\\ \leq&\left\|(1+\xi^2)^{\frac{p}{2}}\hat{u}(1,\xi)\tilde{A}(\xi)\right\|+\left\|\left[\hat{\varphi}(\xi)-\hat{\varphi}_\delta(\xi)\right]\tilde{B}(\xi)\right\|,\end{aligned} \tag{3.24}$$

where

$$\tilde{A}(\xi)=\frac{\cosh\sqrt{\xi^2+k^2}-\cosh\sqrt{\frac{\xi^2+k^2}{1+\mu^2\xi^2}}}{\cosh\sqrt{\xi^2+k^2}}(1+\xi^2)^{-\frac{p}{2}}, \tag{3.25}$$

$$\tilde{B}(\xi)=\cosh\sqrt{\frac{\xi^2+k^2}{1+\mu^2\xi^2}}. \tag{3.26}$$

By using the conditions (2.8) and (3.21), we can obtain

$$\|u(1,\cdot)-v(1,\cdot)\|\leq\sup_{\xi\in\mathbb{R}}|\tilde{A}(\xi)|E+\sup_{\xi\in\mathbb{R}}|\tilde{B}(\xi)|\delta. \tag{3.27}$$

We first estimate the second term on the right hand side of (3.27). For $|\xi|<|k|$, we have

$$|\tilde{B}(\xi)|\delta=\delta\cosh\sqrt{\frac{\xi^2+k^2}{1+\mu^2\xi^2}}\leq\delta\cosh\sqrt{2}|k|\leq\delta e^{\sqrt{2}|k|}. \tag{3.28}$$

For the case of $|\xi|\geq|k|$, we have

$$|\tilde{B}(\xi)|\delta=\delta\cosh\sqrt{\frac{\xi^2+k^2}{1+\mu^2\xi^2}}\leq\delta\cosh\frac{\sqrt{2}}{\mu}\leq\delta e^{\frac{\sqrt{2}}{\mu}}=E\left(\ln\frac{E}{\delta}\right)^{-2p}. \tag{3.29}$$

Therefore, we have the following estimate,

$$\sup_{\xi\in\mathbb{R}}|\tilde{B}(\xi)|\delta\leq\delta e^{\sqrt{2}|k|}+E\left(\ln\frac{E}{\delta}\right)^{-2p}. \tag{3.30}$$

In the following, we estimate $\tilde{A}(\xi)$. Taking a procedure similar to (3.10), we have

$$|\tilde{A}(\xi)|\leq(1-e^{-(s-\tau)})(1+\xi^2)^{-\frac{p}{2}}. \tag{3.31}$$

Two cases will be considered. Consider the case of large values of $|\xi|$ such that $|\xi|\geq\xi_0:=\sqrt{2}/\mu$. Note that $s\geq\tau$, from (3.31), we have

$$|\tilde{A}(\xi)|\leq(1+\xi^2)^{-\frac{p}{2}}\leq|\xi|^{-p}\leq\xi_0^{-p}=\left(\frac{\sqrt{2}\mu}{2}\right)^p. \tag{3.32}$$

For small values of $|\xi|$ satisfies $|\xi|<\xi_0$, from (3.31), we have

$$|\tilde{A}(\xi)|\leq(s-\tau)\left(1+\xi^2\right)^{-\frac{p}{2}}\leq\frac{1}{2}\mu^2\xi^2\sqrt{\xi^2+k^2}\left(1+\xi^2\right)^{-\frac{p}{2}}. \tag{3.33}$$

Case 1. For the case of $|\xi| > |k|$, we have

$$|\tilde{A}(\xi)| \leq \frac{\sqrt{2}}{2}\mu^2|\xi|^3\left(1+\xi^2\right)^{-\frac{p}{2}}.$$

We now consider two possibilities on p. If $1 < p < 3$, then

$$|\tilde{A}(\xi)| \leq \frac{\sqrt{2}}{2}\mu^2|\xi|^{3-p} \leq \frac{\sqrt{2}}{2}\mu^2\xi_0^{3-p} = 2^{\frac{2-p}{2}}\mu^{p-1};$$

and if $p \geq 3$, then

$$|\tilde{A}(\xi)| \leq \begin{cases} \frac{\sqrt{2}}{2}\mu^2|\xi|^{3-p} \leq \frac{\sqrt{2}}{2}\mu^2, & |\xi| \geq 1, \\ \frac{\sqrt{2}}{2}\mu^2(1+\xi^2)^{-\frac{p}{2}} \leq \frac{\sqrt{2}}{2}\mu^2, & |\xi| < 1. \end{cases}$$

Thus, in **Case 1**, we have

$$|\tilde{A}(\xi)|E \leq \max\left\{2^{\frac{2-p}{2}}\mu^{p-1}, \frac{\sqrt{2}}{2}\mu^2\right\}E, \quad p > 1, \quad \xi_0 > |\xi| > |k|. \tag{3.34}$$

Case 2. For the case of $|\xi| < |k|$, it follows from (3.33) that

$$|\tilde{A}(\xi)| \leq \frac{\sqrt{2}}{2}\mu^2|k|\xi^2\left(1+\xi^2\right)^{-\frac{p}{2}}.$$

Again, we consider two possibilities on p. If $0 < p < 2$, then

$$|\tilde{A}(\xi)| \leq \frac{\sqrt{2}}{2}\mu^2|k||\xi|^{2-p} \leq \frac{\sqrt{2}}{2}\mu^2|k||\xi_0|^{2-p} = 2^{\frac{1-p}{2}}|k|\mu^p;$$

and if $p \geq 2$, we have

$$|\tilde{A}(\xi)| \leq \begin{cases} \frac{\sqrt{2}}{2}\mu^2|k||\xi|^{2-p} \leq \frac{\sqrt{2}}{2}\mu^2|k|, & |\xi| \geq 1, \\ \frac{\sqrt{2}}{2}\mu^2|k|(1+\xi^2)^{-\frac{p}{2}} \leq \frac{\sqrt{2}}{2}\mu^2|k|, & |\xi| < 1. \end{cases}$$

Thus for **Case 2**, we have

$$|\tilde{A}(\xi)|E \leq \max\left\{2^{\frac{1-p}{2}}|k|\mu^p, \frac{\sqrt{2}}{2}|k|\mu^2\right\}E, \quad p > 0, \quad |\xi| < |k|. \tag{3.35}$$

Combining (3.32), (3.34) and (3.35) gives

$$\begin{aligned} \sup_{\xi\in\mathbb{R}}|\tilde{A}(\xi)|E \leq \left(\frac{\sqrt{2}\mu}{2}\right)^p E + \max\left\{\, 2^{\frac{2-p}{2}}\mu^{p-1},\ \frac{\sqrt{2}}{2}\mu^2\right\}E \\ + \max\left\{\, 2^{\frac{1-p}{2}}|k|\mu^p,\ \frac{\sqrt{2}}{2}|k|\mu^2\right\}E, \quad p > 1. \end{aligned}$$

This completes the proof of Theorem 3.2. □

We close this section by making the following remark. For the Cauchy problem with inhomogeneous Neumann boundary condition:

$$
\begin{aligned}
&-\triangle u(x,y)+k^2u(x,y)=0, && x\in(0,1),\ \ y\in\mathbb{R}, && \text{(3.36a)}\\
&u(0,y)=\psi(y), && y\in\mathbb{R}, && \text{(3.36b)}\\
&u_x(0,y)=g(y), && y\in\mathbb{R}, && \text{(3.36c)}
\end{aligned}
$$

we can not use the proposed method to analyze it directly. However, we can define the solution u of problem (3.36) as $u = u_1 + u_2$, where u_1 satisfies (3.36a), (3.36c) and $u(1, y) = 0$, and u_2 satisfies (3.36a) with boundary conditions $u(0, y) = \psi(y) - u_1(0, y)$ and $u_x(0, y) = 0$. Then problem (3.36) is divided into a well-posed problem for u_1 and an ill-posed problem for u_2 which can be solved by our proposed method.

4. Numerical implementation

In this section, we will give a numerical method to solve problem (2.5)-(2.7) in the finite region $0 \le x \le 1$, $0 \le y \le 1$. In fact, the numerical implementation can also be extended to other regions, such as $0 \le x \le 1$, $L \le y \le M$ for any finite constants L and M. The basic idea comes from the method of lines, see, e.g., [5].

Consider the following Cauchy problem in a finite domain:

$$
\begin{aligned}
&-\triangle v+k^2v+\mu^2v_{xxyy}=0, && x\in(0,1),\ \ y\in(0,1), && \text{(4.1)}\\
&v(0,y)=\varphi_\delta(y), && y\in[0,1], && \text{(4.2)}\\
&v_x(0,y)=0, && y\in[0,1], && \text{(4.3)}
\end{aligned}
$$

where μ is chosen by (3.1) or (3.22).

Suppose that the second-order derivative of a function $f(y)$ is approximated by the standard central-difference, i.e.,

$$
\frac{d^2f(y)}{dy^2}\approx\frac{f(y+h)-2f(y)+f(y-h)}{(h)^2}. \tag{4.4}
$$

Taking an equidistant grid $0 = y_1 < \cdots < y_n = 1$, where $y_i = (i-1)h$, $h = 1/(n-1)$, $i = 1, \cdots, n$, we discretize Eq. (4.1) with respect to the variable y, and leave the variable x continuous. Then we obtain the following linear system of ordinary differential equations (ODEs),

$$
-V_{xx}-\frac{DV}{h^2}+k^2V+\frac{\mu^2}{h^2}DV_{xx}=0. \tag{4.5}
$$

Furthermore, we have

$$
V_{xx}=-\Big(I-\frac{\mu^2}{h^2}D\Big)^{-1}\Big(\frac{D}{h^2}-k^2I\Big)V=:MV, \tag{4.6}
$$

where

$$D = \begin{pmatrix} 2 & -5 & 4 & -1 & & \\ 1 & -2 & 1 & & & \\ 0 & 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & -1 & 4 & -5 & 2 & \end{pmatrix}_{n\times n}, \quad V = \begin{pmatrix} v(x,y_1) \\ v(x,y_2) \\ v(x,y_3) \\ \vdots \\ v(x,y_n) \end{pmatrix}. \tag{4.7}$$

By (4.2)-(4.3), we can transform the second-order ODEs (4.6) into the first-order ODEs with initial condition as follows:

$$\begin{pmatrix} V \\ V_x \end{pmatrix}_x = \begin{pmatrix} 0 & I \\ M & 0 \end{pmatrix} \begin{pmatrix} V \\ V_x \end{pmatrix}, \tag{4.8}$$

$$\begin{pmatrix} V(0) \\ V_x(0) \end{pmatrix} = \begin{pmatrix} \Phi_\delta \\ 0 \end{pmatrix}, \tag{4.9}$$

where $\Phi_\delta = (\varphi_\delta(y_i)) \in \mathbb{R}^n$.

Now we explain the first row of the matrix D; and the last row can be deduced by the same approach. By (4.4), for $y = y_1$,

$$v_{yy}(x,y_1) \approx \frac{v(x,y_2) - 2v(x,y_1) + v(x,y_0)}{h^2}.$$

Since the value of $v(x,y_0)$ is unavailable, the function of $v_{yy}(x,y_1)$ is approximated by the following formula:

$$\begin{aligned} v_{yy}(x,y_1) &\approx 2v_{yy}(x,y_2) - v_{yy}(x,y_3) \\ &= 2\frac{v(x,y_3) - 2v(x,y_2) + v(x,y_1)}{h^2} - \frac{v(x,y_4) - 2v(x,y_3) + v(x,y_2)}{h^2} \\ &= \frac{1}{h^2}(2,\ -5,\ 4,\ -1) \begin{pmatrix} v(x,y_1) \\ v(x,y_2) \\ v(x,y_3) \\ v(x,y_4) \end{pmatrix}. \end{aligned}$$

In our computations, to solve the problem of ODEs (4.8)-(4.9), we use an explicit fourth-order Runge-Kutta method whose accuracy is set to be 10^{-4}. In the numerical implement, the MATLAB solver ode45 is used.

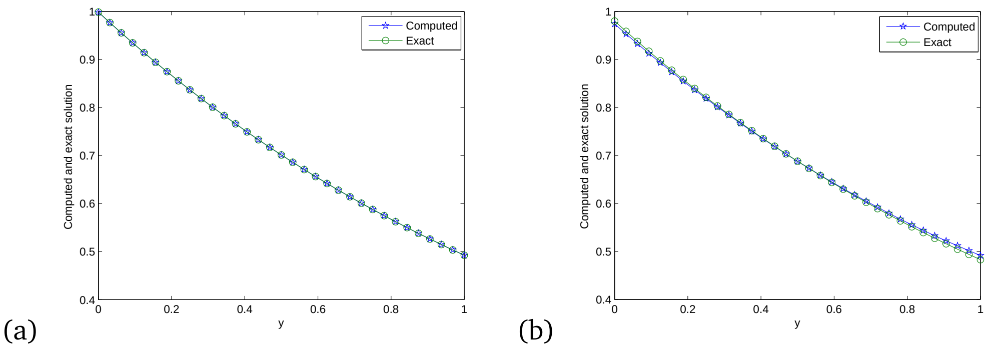
In the following, two numerical examples will be given to demonstrate the effectiveness of our proposed method. First, it is easy to verify that the function

$$u(x,y) = \cos(kx)e^{-\sqrt{2}k|y|}, \tag{4.10}$$

is an exact solution of Eq. (1.1) with Cauchy data

$$u(0,y) = \varphi(y) = e^{-\sqrt{2}k|y|}, \tag{4.11}$$

$$\frac{\partial u}{\partial x}(0,y) = 0, \tag{4.12}$$



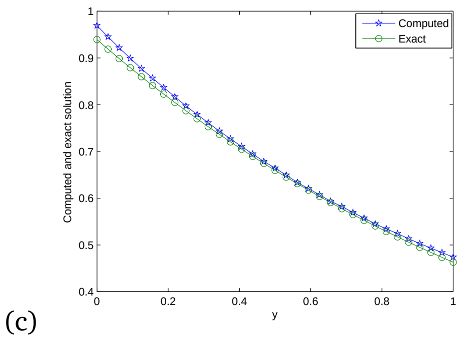


Figure 1: $u(x,\cdot)$ and $v(x,\cdot)$ with $k=0.5$, $\varepsilon=10^{-4}$ and $E=1.05$. (a) $x=0.1$; (b) $x=0.4$; (c) $x=0.7$.

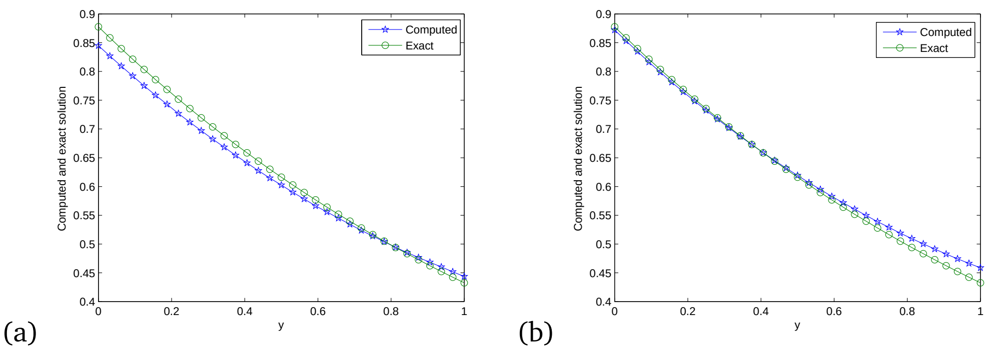


Figure 2: $u(1,\cdot)$ and $v(1,\cdot)$ with $k=0.5$, $\varepsilon=10^{-4}$, and (a) $(E,p)=(3.36,1.1)$; (b) $(E,p)=(4.86,1.4)$.

where k is a positive real number. We add a random perturbation to the exact data φ for generating noisy data φ_δ as follows:

$$\varphi_\delta = \varphi + \varepsilon rand(size(\varphi)), \tag{4.13}$$

where ε is a noise level, the function of $rand(size(\varphi))$ produces a random vector with the same size of φ.

Denote

$$\delta \equiv \Big(\frac{1}{n}\sum_{i=1}^{n}|\varphi_\delta(y_i)-\varphi_\delta(y_i)|^2\Big)^{1/2}. \tag{4.14}$$

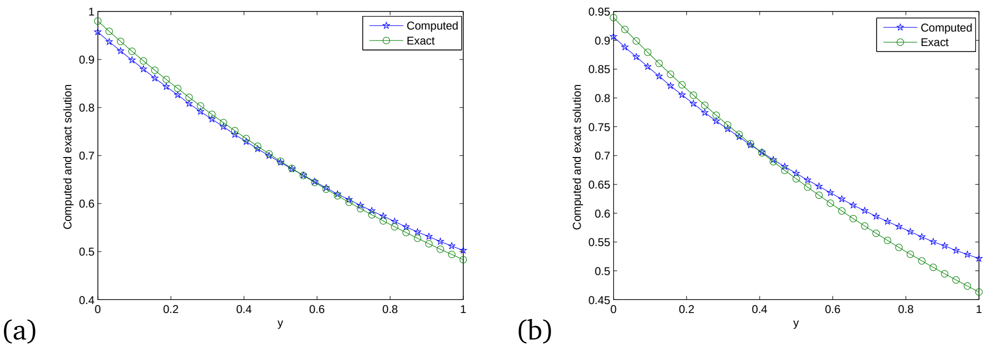


Figure 3: $u(x,\cdot)$ and $v(x,\cdot)$ with $k=0.5$, $\varepsilon=10^{-3}$ and $E=1.05$. (a) $x=0.4$; (b) $x=0.7$.

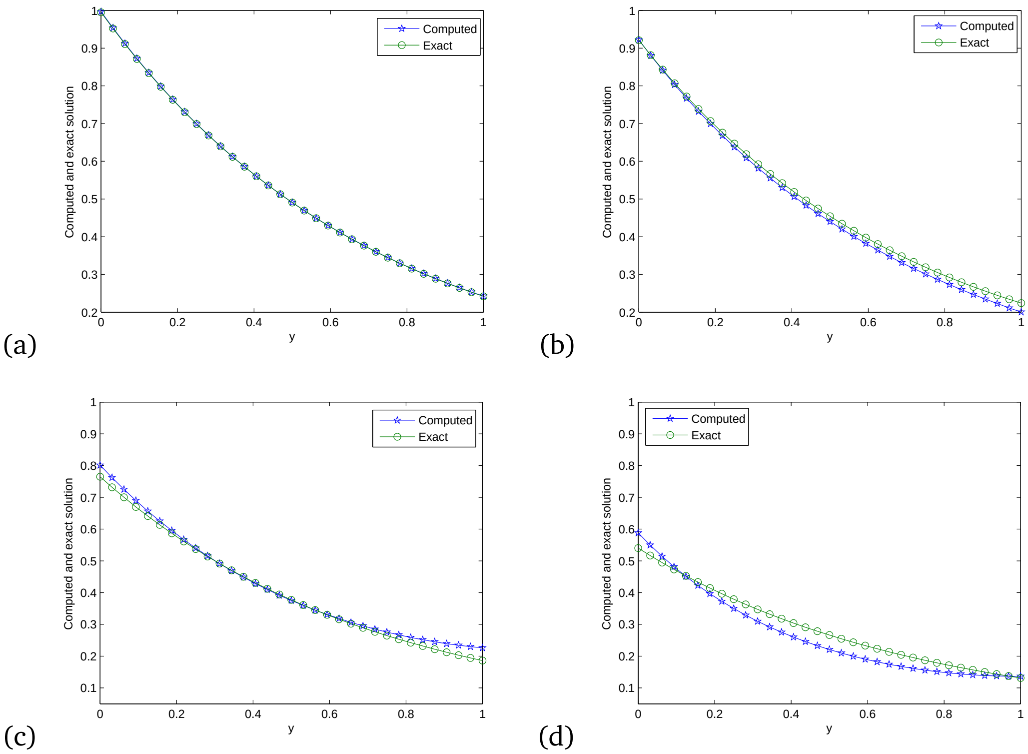


Figure 4: $u(x,\cdot)$ and $v(x,\cdot)$ with $k=1$ and $\varepsilon=10^{-4}$. (a) $x=0.1$ and $E=0.46$; (b) $x=0.4$ and $E=0.46$; (c) $x=0.7$ and $E=0.46$; (d) $x=1$, $E=2.23$ and $p=1.1$.

In the following numerical tests, the *a priori* bound $E \approx \| u(1,\cdot) \|$ or $\|u(1,\cdot)\|_p$, and μ is chosen by (3.1) or (3.22). Moreover, we always choose $n=33$.

In Figs. 1 and 2, we choose $k=0.5$ with the error level $\varepsilon=10^{-4}$. In Fig. 1, we give the numerical results for the exact solution u and the computed solution v at $x=0.1,\ 0.4,\ 0.7$. In Fig. 2, we give the numerical results for $u(1,\cdot)$ and $v(1,\cdot)$ with $p=1.1$ and $p=1.4$.

From Fig. 2, we note that the bigger the value of p, the better the numerical result, which is in good agreement with our theoretical result. The numerical results for $u(x,\cdot)$ and $v(x,\cdot)$ at $x = 0.4,\ 0.7$ with $k = 0.5$ and $\varepsilon = 10^{-3}$ are given in Fig. 3. From Figs. 1(b), 1(c), 3(b) and 3(c), we find that the numerical results are less encouraging with the increase of the noise level.

The numerical results for $u(x,\cdot)$ and $v(x,\cdot)$ at $x = 0.1,\ 0.4,\ 0.7$ and $x = 1$ with $p = 1.1$ are shown in Fig. 4 in which we choose $k = 1$ and $\varepsilon = 10^{-4}$. From Figs. 1 and 4, we find that the numerical results become discouraging when x approaches 1, which also agrees with the theory prediction. Meanwhile, we note that the bigger the value of k, the worse the numerical results, which indicates that the proposed regularization method is only valid for small values of k.

5. Conclusion

The Cauchy problem for the modified Helmholtz equation is a severely ill-posed problem. In this paper, we proposed an efficient regularization method to solve it. Under suitable choices of the regularization parameters, the logarithmic type error estimates are obtained. Finally, the numerical examples are presented to verify our theoretical results.

Acknowledgments The authors would like to thank the referees for the valuable comments and suggestions that improve the quality of our paper. The work described in this paper was supported by the NSF of China (10571079, 10671085) and the program of NCET.

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