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A Finite Difference Scheme for Blow-Up Solutions of Nonlinear Wave Equations

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Abstract. We consider a finite difference scheme for a nonlinear wave equation, whose solutions may lose their smoothness in finite time, i.e., blow up in finite time. In order to numerically reproduce blow-up solutions, we propose a rule for a time-stepping, which is a variant of what was successfully used in the case of nonlinear parabolic equations. A numerical blow-up time is defined and is proved to converge, under a certain hypothesis, to the real blow-up time as the grid size tends to zero.

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1. Introduction

In some evolution equations, a singularity appears in a solution spontaneously, although everything prescribed is smooth. Here singularity implies a discontinuity of a solution, of its derivative, or of its higher-order derivatives. In this broad sense, appearance of shock waves in compressible fluid motion is included in blow-up phenomena. But many researchers use blow-up in a little narrower sense. For instance, blow-up very often refers to a singular behavior of solutions of

$$\frac{\partial u}{\partial t} = \Delta u + f(u), \quad \text{or} \quad \frac{\partial^2 u}{\partial t^2} = \Delta u + f(u),$$

or of similar equations. Here $f(u)$ is the nonlinear term such as $f(u) = u^p$ with $p > 1$, or $f(u) = e^u$. The singularity appearing in these equations are different from the shock waves of fluid motion in that a time-global weak solution does not exist in nonlinear heat equation above (Baras & Cohen [5]), while a weak solution past a blow-up time is proved

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to exist in the case of compressible fluid motion. In the case of compressible fluid, we are interested in computing solutions after the occurrence of a shock wave. On the other hand, computation up to the blow-up time is the issue in the case of blow-up problem. We do not know whether a global weak solution exists in the case of nonlinear wave equations. This problem seems to be an open problem.

The purpose of the present paper is to consider finite difference approximations for blow-up problems appearing in the nonlinear wave equation of one spatial variable:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + u^2. \tag{1.1}$$

The nonlinear term is assumed to be u^2 for the sake of simplicity. But our method is applicable, if suitably modified, to other nonlinearity. The present paper can be considered to be a sequel to [8,9], where a similar study on the nonlinear heat equation was carried out. In the fluid dynamics contexts, the most famous blow-up problem would be that of the 3D Euler equations for incompressible inviscid fluid (see [7,23]). Numerical computations on this problem and those related to it are abundant but we only cite [13, 15, 21]. The problem is notoriously difficult and the occurrence of blow-up is yet to be decided. We therefore think it worthwhile to develop a mathematical theory for a less difficult problem. One of our purpose is to show that approximation for nonlinear partial differential equations of hyperbolic type is considerably more difficult than that for PDEs of parabolic type. Accordingly, we point out mathematical issues which we cannot resolve, and we would like to invite readers to numerical analysis of blow-up problems.

The present paper consists of five sections. We discuss the Constantin-Lax-Majda equation in Section 2. The nonlinear wave equation is considered from Section 3 to Section 5, where ideas are explained in Section 3 and mathematical analysis is laid down in Sections 4 and 5. In Section 4, we consider a semi-discrete scheme whose solution blows up in finite time and show the convergence of the numerical solution and the numerical blow-up time. Then we apply the idea introduced in [8] and consider a full discrete scheme for the nonlinear wave equation in Section 5. Several numerical examples are also shown there.

2. CLM equation

In order to show mathematical difficulty in hyperbolic or fluid-mechanical blow-up problem, we here warm up ourselves by a simple model equation proposed by [10].

The Constantin-Lax-Majda equation, which we abbreviate to CLM, is the following equation:

$$\frac{\partial u}{\partial t} = u \cdot Hu, \qquad u(0,x) = u_0(x), \tag{2.1}$$

where H denotes the Hilbert transform. Let us restrict ourselves in the case of the periodic boundary condition. H is then given as

$$Hu(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cot\left(\frac{x-y}{2}\right) u(y)\mathrm{d}y \qquad (-\pi < x < \pi),$$

where the integral denotes Cauchy's principal part. If we use a theorem in [10] we have

Theorem 2.1. *For all $u_0 \in H^1(-\pi,\pi)$ with $\int_{-\pi}^{\pi} u_0(x)\mathrm{d}x = 0$ and $u_0 \not\equiv 0$, the solutions of* (2.1) *blow up in finite time.*

We now consider a semi-discrete approximation of CLM by the spectral method. Let N be a positive integer. For

$$u(t,x) = \sum_{n=1}^{\infty} \left(a_n(t)\sin nx + b_n(t)\cos nx\right),$$

we define

$$P_N u(t,x) = \sum_{n=1}^{N} \left(a_n(t)\sin nx + b_n(t)\cos nx\right).$$

Then we consider the following spectral approximation:

$$\frac{\partial u_N}{\partial t} = P_N\left(u_N \cdot Hu_N\right).$$

For the sake of simplicity, let us consider the case of odd functions:

$$u(0,x) = \sum_{n=1}^{\infty} a_n^0 \sin nx,$$

whence

$$u(t,x) = \sum_{n=1}^{\infty} a_n(t)\sin nx.$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}a_1(t) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}a_2(t) = -\frac{1}{2}a_1^2, \qquad \frac{\mathrm{d}}{\mathrm{d}t}a_3(t) = -a_1a_2,$$

and, for $k \geq 1$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}a_{2k+2}(t) &= -\frac{1}{2}a_{k+1}^2 - a_1a_{2k+1} - a_2a_{2k} - \cdots - a_ka_{k+2},\\
\frac{\mathrm{d}}{\mathrm{d}t}a_{2k+1}(t) &= -a_1a_{2k} - a_2a_{2k-1} - \cdots - a_ka_{k+1}.
\end{aligned}$$

Therefore, $a_1(t) \equiv a_1^0, a_2(t) = a_2^0 - (a_1^0)^2t/2$ and so on. We find that $a_m(t)$ is a polynomial in t of order $m-1$. This means that $u_N(t,x)$ exists for all t and N. Therefore the individual semi-discrete solution never blows up. The blow-up in (2.1) can be observed only if we extrapolate from $\{u_N\}_{1\leq N}$.

Similar phenomena can be observed in the case of the finite difference approximation. To see this, remember that for all $x \in [-\pi,\pi]$

$$\limsup_{t\to T} u(t,x) < \infty, \tag{2.2}$$

although

$$\limsup_{t\to T} \|u(t,\cdot)\|_\infty = +\infty.$$

This can be seen from the following explicit formula given in [10]:

$$u(t,x) = \frac{2u_0(x)}{\big(2 - tHu_0(x)\big)^2 + t^2 u_0(x)^2}.$$

Note that if $\xi(t)$ denotes the point where $\|u(t,\cdot)\|_\infty$ is attained, then $\xi(t)$ moves towards one of the zeros of u_0. (2.2) suggests that if only a finite number of spatial grid points are available, the blow-up is not reproduced.

We now set

$$\zeta(t,x) = u(t,x) + \mathrm{i}Hu(t,x).$$

It then satisfies (see [10])

$$\frac{\partial \zeta}{\partial t} = \frac{1}{2\mathrm{i}}\zeta^2.$$

Suppose that we employ a finite difference approximation in such a way that $\zeta_n(t)$ approximates $\zeta(t, 2\pi n/N - \pi)$ $(1 \le n \le N)$. Here, Hu_n is, by definition, the imaginary part of ζ_n. We then see that

$$\zeta_n(t) = \frac{2\mathrm{i}\zeta_n(0)}{2\mathrm{i} - \zeta_n(0)t}$$

is a solution. For any $n = 1, 2, \cdots, N$ such that $\zeta_n(0)$ is not pure imaginary, $\zeta_n(t)$ does not blow up. For those n such that $\zeta_n(0)$ is pure imaginary, $u_n(t) \equiv 0$. Therefore it does not blow up, either. In fact it is not difficult to see that the solution is bounded in $0 \le t < \infty$ for any $n = 1, 2, \cdots, N$.

The results above do not imply that blow-up is impossible to numerically observe for CLM. In fact, if we perform extrapolation from several numerical data, then we can guess occurrence of a blow-up. Our result shows only that no single numerical experiment guarantees the existence of blow-up. However, it would be worthwhile to note this difficulty.

3. A nonlinear wave equation

In view of the difficulties we saw in the last section, we start with a rather modest goal. We here consider a one-dimensional nonlinear wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + u^2 \qquad (0 < x < 1,\ 0 < t). \tag{3.1}$$

We consider it with the Dirichlet boundary condition $u(t,0) = u(t,1) = 0$ and the initial condition:

$$u(0,x) = f(x), \quad u_t(0,x) = g(x). \tag{3.2}$$

We hereafter consider the initial-boundary value problem (3.1)-(3.2).

Existence of solution local in time is known, see for instance, [19] or [26]. If we use one of their theorems, we have

Theorem 3.1. *For all* $f \in H^2(0,1)\cap H_0^1(0,1)$ *and* $g \in H_0^1(0,1)$*, there exists a* $T>0$ *such that a solution of* (3.1) *and* (3.2) *exists uniquely in* $0 \le t < T$.

Here and hereafter, H^n denotes the usual Sobolev space. Proof is a simple application of Theorem 1 of [26].

Blow-up in nonlinear wave equations is studied in [12, 16, 18, 22]. There are also nice surveys [4, 17, 20, 26, 30, 31]. But those references are concerned mostly with the Cauchy problem. The mixed problems, i.e., initial-boundary value problems, do not seem to be completely studied. Exceptions are [19, 20, 26], where local existence theory is developed for initial-boundary value problems. Many sufficient conditions for a solution to blow up are known. But here we only quote a theorem in Levine [22]. If his Theorem I is used, we have the following theorem.

Theorem 3.2. *Suppose that* $f \in H^2(0,1)\cap H_0^1(0,1)$ *and* $g \in H_0^1(0,1)$. *If*

$$\frac{1}{3}\int_0^1 f(x)^3 \mathrm{d}x > \frac{1}{2}\int_0^1 \left[\left(f'(x)\right)^2 + g(x)^2\right]\mathrm{d}x,$$

then the solution blows up in finite time.

Namely a sufficiently large initial data makes the solution blow up. There is, however, a striking difference between parabolic and hyperbolic blow-up problems. Caffarelli and Friedman [6] found that there exists the so-called blow-up curve $t = B(x)$ such that the solution $u(t,x)$ satisfies $|u(t,x)| < \infty$ if and only if $t < B(x)$. The blow-up time is therefore $\inf_x B(x)$, but in some part of the space-time region, the solution exists beyond the blow-up time. We do not know, however, whether a weak solution exists in $t > B(x)$.

A blow-up criterion different from Theorem 3.2 is obtained as follows: Let $u(t,x)$ be a solution of (3.1)-(3.2) and put $\varphi(t) = \frac{\pi}{2}\int_0^1 u(t,x)\sin \pi x \mathrm{d}x$. We then prove the following theorem, which neither contains Theorem 3.2 nor is contained in it.

Theorem 3.3. *Assume that the initial data satisfy*

$$\int_0^1 f(x)\sin \pi x \mathrm{d}x > 2\pi \quad \textit{and} \quad \int_0^1 g(x)\sin \pi x \mathrm{d}x \ge 0. \tag{3.3}$$

Then $\varphi(t)$ *blows up in finite time. That is, there exists a finite* $T>0$ *such that* $\lim_{t\to T}\varphi(t) = \infty$.

Proof. The argument below is known as a convexity method, and the idea is not a new one. However, as we will do a similar analysis later in this paper for a finite difference scheme, we give here a proof as a warming-up for the discrete case. Observe first that

$$\begin{aligned}\varphi''(t) &= \frac{\pi}{2}\int_0^1 u_{tt}(t,x)\sin(\pi x)\mathrm{d}x \\ &= \frac{\pi}{2}\int_0^1 u_{xx}(t,x)\sin(\pi x)\mathrm{d}x + \frac{\pi}{2}\int_0^1 u^2(t,x)\sin(\pi x)dx.\end{aligned}$$

Since

$$\int_0^1 u_{xx}\sin(\pi x)\mathrm{d}x = -\pi^2\int_0^1 u(t,x)\sin(\pi x)\mathrm{d}x$$

and

$$\int_0^1 u^2\sin\pi x\mathrm{d}x \geq \frac{1}{\int_0^1\sin\pi x\mathrm{d}x}\left(\int_0^1 |u|\sin\pi x\mathrm{d}x\right)^2 \geq \frac{\pi}{2}\left(\int_0^1 u\sin\pi x\mathrm{d}x\right)^2,$$

we have

$$\varphi''(t) \geq -\pi^2\varphi(t) + \left(\varphi(t)\right)^2. \tag{3.4}$$

Note that (3.3) implies that $\varphi(0) > \pi^2, \varphi'(0) \geq 0$. It is then easy to prove that

$$\varphi(t) > \pi^2, \quad \varphi'(t) > 0 \tag{3.5}$$

for all $t > 0$ as far as $\varphi(t)$ exists. By virtue of (3.4)-(3.5), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi'(t)^2 - \frac{2}{3}\varphi(t)^3 + \pi^2\varphi(t)^2\right)\\ &= 2\varphi'(t)\left(\varphi'' - \varphi^2 + \pi^2\varphi\right) \geq 0.\end{aligned}$$

Consequently,

$$\varphi'(t)^2 \geq \frac{2}{3}(\varphi(t))^3 - \pi^2\varphi(t)^2 - \frac{2}{3}\varphi(0)^3 + \pi^2\varphi(0)^2 + \varphi'(0)^2.$$

Put $K = -\frac{2}{3}\varphi(0)^3 + \pi^2\varphi(0)^2 + \varphi'(0)^2$. The property (3.5) ensures that the right hand side of the inequality above is always positive, hence we have

$$\varphi'(t) \geq \left(\frac{2}{3}\varphi(t)^3 - \pi^2\varphi(t)^2 + K\right)^{1/2}.$$

This implies that $\varphi(t) \to \infty$ in finite time T, where

$$T \leq \int_{\varphi(0)}^{\infty}\left(\frac{2}{3}x^3 - \pi^2x^2 + K\right)^{-1/2}\mathrm{d}x.$$

This completes the proof of this theorem. □

4. A semi-discrete finite difference scheme

A blow-up convergent finite difference scheme, discretized only in the space variable, for nonlinear heat equation

$$u_t = u_{xx} + u^p \qquad (0 < x < 1,\ t > 0)$$

was considered by several authors, for instance see [1,2,14]. In this section, we are going to consider an analogue for the nonlinear wave equation (3.1). That is, we consider the following semi-discrete scheme

$$\frac{d^2}{dt^2}u_j(t)=\frac{u_{j+1}(t)-2u_j(t)+u_{j-1}(t)}{h^2}+(u_j(t))^2 \qquad (j=1,\cdots,N-1), \tag{4.1}$$

where $u_j(t)$ denotes the approximation of $u(t,x_j)$, $Nh=1$, and $x_j=jh,\ j=0,\cdots,N$. The corresponding initial data are given by

$$u_j(0)=f(x_j), \qquad \frac{d}{dt}u_j(0)=g(x_j) \qquad (j=1,\cdots,N-1),$$

while the discrete boundary condition is given by

$$u_0(t)=u_N(t)=0 \qquad (t\geq 0). \tag{4.2}$$

4.1. Blow-up of the numerical solution

Let

$$\psi_h(t)=\frac{1}{\sigma(h)}\sum_{j=1}^{N-1}hu_j(t)\sin(\pi x_j), \tag{4.3}$$

where

$$\sigma(h)=\sum_{j=1}^{N-1}h\sin(\pi x_j). \tag{4.4}$$

We assume that, corresponding to the assumption of Theorem 3.3, the initial data satisfy

$$\psi_h(0)>\pi^2, \qquad \psi_h'(0)\geq 0. \tag{4.5}$$

Then we have the following.

Theorem 4.1. *Let* $\{u_j(t)\}$ *be the solution of* (4.1)-(4.2). *Assume that the initial data satisfy* (4.5). *Then* $\psi_h(t)$ *blows up in finite time. That is, there exists a finite* $T_h>0$ *such that* $\lim_{t\to T_h}\psi_h(t)=\infty$.

Proof. Calculating directly, we have

$$\begin{aligned}&\frac{1}{\sigma(h)}\sum_{j=1}^{N-1}h\sin(\pi x_j)\frac{u_{j+1}(t)-2u_j(t)+u_{j-1}(t)}{h^2}\\&=-\left(\sin\frac{\pi h}{2}\right)^2\frac{4}{h^2}\psi_h(t)\geq-\pi^2\psi_h(t),\end{aligned}$$

as long as $\psi_h(t)\geq 0$.

On the other hand, by the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}\frac{1}{\sigma(h)}\sum_{j=1}^{N-1} h\sin(\pi x_j)(u_j(t))^2 &\geq \left(\frac{1}{\sigma(h)}\sum_{j=1}^{N-1} h\sin(\pi x_j)\left|u_j(t)\right|\right)^2\\ &\geq \left(\frac{1}{\sigma(h)}\sum_{j=1}^{N-1} h\sin(\pi x_j)u_j(t)\right)^2 = \left(\psi_h(t)\right)^2.\end{aligned}$$

Combining the two inequalities we have

$$\frac{d^2}{dt^2}\psi_h(t) \geq -\pi^2\psi_h(t) + \left(\psi_h(t)\right)^2, \tag{4.6}$$

so long as $\psi_h(t) \geq 0$. In fact, by (4.5) and (4.6), it is easy to prove that

$$\psi_h(t) > \pi^2, \qquad \psi_h'(t) > 0,$$

for all $t > 0$ as long as $\psi_h(t)$ exists. This implies that (4.6) holds for all $t > 0$ as long as $\psi_h(t)$ exists. Then, the same argument used in the proof of Theorem 3.3 yields that $\psi_h(t) \to \infty$ in finite time T_h, where

$$T_h \leq \int_{\psi_h(0)}^{\infty} \left(\frac{2}{3}x^3 - \pi^2 x^2 + \bar{K}\right)^{-1/2} \mathrm{d}x.$$

Here $\bar{K}$ is a constant given by

$$\bar{K} = -\frac{2}{3}\psi_h(0)^3 + \pi^2\psi_h(0)^2 + \psi_h'(0)^2.$$

This completes the proof of the theorem. □

4.2. Convergence of the numerical solution

Theorem 4.2. *Let $\{u_j(t)\}$ be the solution of (4.1)-(4.2) and T be the blow-up time of the solution of (3.1). Assume that the initial-boundary value problem (3.1)-(3.2) has a solution $u \in C^{2,4}([0,T) \times [0,1])$. Let $T_0 < T$. Then, for all $t \leq T_0$, we have*

$$\max_{j=1,\cdots,N-1} |u_j(t) - u(t,x_j)| \leq Ch,$$

for h sufficiently small. Here C is a constant depending only on the initial data and T_0.

Proof. Let

$$\max_{(t,x)\in[0,T_0]\times[0,1]} |u(t,x)| = R, \qquad \max_{(t,x)\in[0,T_0]\times[0,1]} |u_{xxxx}(t,x)| = 12Q.$$

Put $e_j(t) = u_j(t) - u(t, x_j)$. Then, by Taylor's expansion, there exist $0 \leq \theta_1, \theta_2 \leq 1$ such that

$$\begin{aligned}
&\frac{d^2}{dt^2} e_j(t) - \frac{e_{j+1}(t) - 2e_j(t) + e_{j-1}(t)}{h^2} \\
&= \big(u_j(t) + u(t, x_j)\big) e_j(t) + \frac{h^2}{24}\Big(u_{xxxx}(t, x_j + \theta_1 h) + u_{xxxx}(t, x_j - \theta_2 h)\Big) \\
&= \big(2u(t, x_j) + e_j(t)\big) e_j(t) + \frac{h^2}{24}\Big(u_{xxxx}(t, x_j + \theta_1 h) + u_{xxxx}(t, x_j - \theta_2 h)\Big). \qquad (4.7)
\end{aligned}$$

We need the following two lemmas:

Lemma 4.1. *It holds that*

$$\sum_{j=1}^{N-1} e_j'(t) \frac{e_{j+1}(t) - 2e_j(t) + e_{j-1}(t)}{h^2} = -\frac{1}{2}\frac{d}{dt}\sum_{j=1}^{N}\left(\frac{e_j(t) - e_{j-1}(t)}{h}\right)^2.$$

The proof of this lemma is elementary and thus we omit it.

Lemma 4.2. *It holds that*

$$E(t) \leq \left(\sum_{j=1}^{N} h \left(\frac{e_j(t) - e_{j-1}(t)}{h}\right)^2\right)^{1/2}, \qquad (4.8)$$

where $E(t) = \max_{j=1,\cdots,N-1} |e_j(t)|$.

Proof. It follows directly from the inequality

$$\begin{aligned}
|e_j(t)| &= \left|\sum_{i=1}^{j} h \frac{e_i(t) - e_{i-1}(t)}{h}\right| \leq \sum_{i=1}^{j} h \left|\frac{e_i(t) - e_{i-1}(t)}{h}\right| \\
&\leq \left(\sum_{i=1}^{N} h \left(\frac{e_i(t) - e_{i-1}(t)}{h}\right)^2\right)^{1/2}.
\end{aligned}$$

This completes the proof of the lemma. □

Now, we are in a position to prove Theorem 4.2. Multiplying (4.7) by $he_j'(t)$ and summing from $j = 1$ to $N - 1$, by Lemma 4.1, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\sum_{j=1}^{N-1} h(e_j'(t))^2 + \sum_{j=1}^{N} h\left(\frac{e_j(t) - e_{j-1}(t)}{h}\right)^2\right) \\
&= \sum_{j=1}^{N-1} h\big(2u(t, x_j) + e_j(t)\big) e_j(t) e_j'(t) \\
&\quad + \sum_{j=1}^{N-1} h \frac{h^2}{24}\Big(u_{xxxx}(t, x_j + \theta_1 h) + u_{xxxx}(t, x_j - \theta_2 h)\Big) e_j'(t).
\end{aligned}$$

Putting

$$\Phi_h(t) = \sum_{j=1}^{N-1} h(e_j'(t))^2 + \sum_{j=1}^{N} h\left(\frac{e_j(t)-e_{j-1}(t)}{h}\right)^2,$$

then we have $\Phi_h(0)=0$ and, as long as $\Phi_h(t)<1$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Phi_h(t) &\le (2R+1)\sum_{j=1}^{N-1} h|e_j(t)e_j'(t)| + Qh^2\left(\sum_{j=1}^{N-1} h(e_j'(t))^2\right)^{1/2} \\
&\le (2R+1)E(t)\left(\sum_{j=1}^{N-1} h(e_j'(t))^2\right)^{1/2} + Qh^2 \\
&\le \frac{2R+1}{2}\left((E(t))^2 + \sum_{j=1}^{N-1} h(e_j'(t))^2\right) + Qh^2 \\
&\le \frac{2R+1}{2}\Phi_h(t) + Qh^2.
\end{aligned}$$

Here, use has been made of Lemma 4.2. Thus, $\Phi_h(t)$ satisfies

$$\Phi_h(t) \le \left(\exp((2R+1)t)-1\right)\left(\frac{2Q}{2R+1}h^2\right), \tag{4.9}$$

as long as $\Phi_h(t)<1$. Since, for h sufficiently small, the right-hand side of (4.9) is always less than one for all $t \le T_0$, this tells that (4.9) holds for all $t \le T_0$. Putting

$$C^2 = \left(\exp((2R+1)T_0)-1\right)\frac{2Q}{2R+1},$$

by Lemma 4.2, we have the desired error estimate. □

4.3. Convergence of the numerical blow-up time

Theorem 4.3. *Let T and T_h denote the blow-up time of the solutions of* (3.1) *and* (4.1)*, respectively. Then, we have*

$$\lim_{h\to 0} T_h = T. \tag{4.10}$$

Proof. First, we assume that $T_* = \liminf_{h\to 0} T_h < T$ and derive a contradiction. If $T_* < T$, then there exists a sequence $\{h_i\}$ such that $h_i \to 0$ as $i\to\infty$ and that

$$T_{h_i} < (T+T_*)/2 < T.$$

Let $\{u_k^{h_i}(t)\}$ be the corresponding solutions. Thus, by Theorem 4.2, we have a contradiction since

$$\max_{j=1,\cdots,N-1} u_k^{h_i}(t) \to \infty \qquad \text{as} \quad t \to T_{h_i} < \frac{T+T_*}{2},$$

while $\max_{x\in(0,1)} u(t,x)$ remains bounded for all $t \le (T+T_*)/2$.

Next, we assume that

$$T^* = \limsup_{h\to 0} T_h > T.$$

Then there exists a sequence $\{h_i'\}$ with $\lim_{i\to\infty} h_i' = 0$ such that $T_{h_i'} > (T+T^*)/2 > T$. Let $\delta = (T^*-T)/2$. Since

$$\int_{\psi_h(0)}^{\infty} \left(\frac{2}{3}s^3 - \pi^2 s^2 + \bar{K}\right)^{-1/2} \mathrm{d}s < \infty,$$

there exists R such that

$$\int_{y}^{\infty} \left(\frac{2}{3}s^3 - \pi^2 s^2 + \bar{K}\right)^{-1/2} \mathrm{d}s < \frac{\delta}{2}, \qquad \text{for all } \ y \in [R,\infty). \tag{4.11}$$

On the other hand, we note that there exist $0 < T_0 < T$ such that

$$\varphi(t) = \frac{\pi}{2}\int_0^1 u(t,x)\sin(\pi x)dx \ge 2R, \qquad \text{for all } \ t \in [T_0, T).$$

Then, by virtue of Theorem 4.2, for any $t \in \big[T_0, (T_0+T)/2\big]$, there exists a subsequence $\{h_{n_j}'\}$ of $\{h_i'\}$ such that

$$\psi_{h_{n_j}'}(t) \ge \varphi(t) - R > R, \qquad \text{for all } \ t \in \left[T_0, \frac{T_0+T}{2}\right].$$

Therefore, we have

$$T_{h_{n_j}'} \le T_0 + \int_{\psi_{h_{n_j}'}(T_0)}^{\infty} \left(\frac{2}{3}s^3 - \pi^2 s^2 + \bar{K}\right)^{-1/2} \mathrm{d}s < T + \frac{\delta}{2} < T_{h_{n_j}'}. \tag{4.12}$$

This is a contradiction. Thus, we have

$$\liminf_{h\to 0} T_h \ge T \qquad \text{and} \qquad \limsup_{h\to 0} T_h \le T,$$

which yields the desired result. □

5. A full-discrete finite difference scheme

Let N be a positive integer and let $h = 1/N$. We define the grid points x_j by $x_j = jh$ for $j = 0,1,2,\cdots,N$. In this section, we consider the following difference scheme for (3.1)-(3.2):

$$\begin{cases} \dfrac{1}{\tau_n}\left(\dfrac{u_j^{n+1}-u_j^n}{\Delta t_n} - \dfrac{u_j^n - u_j^{n-1}}{\Delta t_{n-1}}\right) = \dfrac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} + \left(u_j^n\right)^2, \\ u_0^n = u_N^n = 0 \qquad (n = 0,1,2,\cdots), \end{cases} \tag{5.1}$$

with the initial condition

$$u_j^0 = f(x_j), \quad u_j^1 = f(x_j) + \Delta t_0 g(x_j) \qquad (j = 1, \cdots, N-1),$$

where u_j^n denotes the approximation for $u(t_n, x_j)$. Δt_n is defined by

$$\Delta t_n = \tau \min\left\{1, \frac{1}{H(\|u^n\|_p)}\right\}, \tag{5.2}$$

where $\|u^n\|_p$ denotes

$$\|u^n\|_p = \left(\sum_{j=1}^{N-1} h|u_j^n|^p\right)^{1/p}$$

if $1 \le p < \infty$, and

$$\|u^n\|_\infty = \max_{j=1,\cdots,N-1} |u_j^n|.$$

τ is a prescribed constant, and $H(s)$ is a prescribed function satisfying the following conditions (a)-(c):

(a) H is monotone increasing and $\lim_{s\to\infty} H(s) = \infty$;

(b) the mapping $s \longmapsto s + \tau \frac{G(s)}{H(s)}$ is monotone increasing;

(c) $\int_{H^{-1}(1)}^{\infty} \frac{G'(z)dz}{G(z)H(z)} < \infty$, where $G(z)$ will be defined in (5.6).

A typical example is $H(s) = \sqrt{s}$. $\tau_n = \frac{1}{2}(\Delta t_n + \Delta t_{n-1})$. We define

$$t_n = \begin{cases} 0, & \text{if } n = 0, \\ t_{n-1} + \Delta t_{n-1}, & \text{if } n > 0. \end{cases}$$

The definition of Δt_n implies that while u^n is small, Δt_n is equal to τ and that once u^n becomes large, Δt_n becomes smaller. How rapidly it decreases is measured by the function H and its choice is a crucial issue. On the other hand, the spatial grid x_j are fixed.

Remark 5.1. In the computation of the Euler equations and the nonlinear Schrödinger equation, adaptive spatial grid points are often used, and we know that they are effective. See, for instance [13, 27]. However, rigorous error analysis is very difficult for such methods. We therefore do not touch those problems.

The definition (5.2) is originally due to Nakagawa [25] and was used successfully for parabolic blow-up problems, see [3, 8, 9]. In what follows we will demonstrate that it works also in the case of hyperbolic blow-up problems.

Definition 5.1. *We define*

$$T(\tau, h) = \sum_{k=0}^{\infty} \Delta t_k = \lim_{n\to\infty} t_n$$

and call it a numerical blow-up time.

At this stage there is no guarantee that $T(\tau,h)$ is finite. Rather, a construction of a scheme which guarantees $T(\tau,h) < \infty$ is an issue. In fact, if $\Delta t_n \equiv \tau$ is employed, then obviously $T(\tau,h) = \infty$. Therefore the choice of the function H is important. Note that, if the solution blows up and $\|u^n\|_p$ tends to infinity, then $\Delta t_n = \tau/H(\|u^n\|_p)$ tends to zero.

5.1. Local convergence

The following assumption seems to hold true.

Assumption (C) *Let* $\{u_j^n\}$ *be the solution of* (5.1). *Let an arbitrary* $T_0 < T$ *be given, where* T *denotes the blow-up time of* (3.1)-(3.2). *Suppose that* $\tau \le h$. *Then, for sufficiently smooth initial data,*

$$\lim_{h\to 0} \max_{j=1,\cdots,N-1} |u_j^n - u(t_n, x_j)| = 0$$

holds so far as $t_n \le T_0$.

To prove this, we need some a priori estimates or stability in some norms, which are available if Δt_n is independent of n. We found that the proof is quite difficult, since we are dealing with the scheme where Δt_n decreases indefinitely. In fact the stability in the case of variable Δt_n is not completely well-known even in the case of linear wave equations. $[11, 24, 28]$ propose stable schemes but they either do not admit decreasing Δt_n or is implicit, meaning that we have to solve a nonlinear problem in each time step. In particular, we could not prove the stability in the case where $\Delta t_n \downarrow 0$ as $n \to \infty$. We therefore leave Assumption (C) to the future work and assume its validity in the present paper. Later in this paper we show some numerical examples which seem to support our assumption.

Suppose that the initial data is large in the sense that the solution blows up in finite time, but it is not so large as $H(\|u^0\|_p) > 1$. Then $\Delta t_n \equiv \tau$ until $H(\|u^n\|_p) > 1$. At the time when Δt_n is changed from τ to $\tau/H(\|u^n\|_p)$, by Assumption (C), the error $\|u(t_n,\cdot) - u^n\|$ can be controlled as small as we want by choosing a sufficient small h. With this 'fact' in mind, we may not lose generality if we assume that the initial data is large enough so that $\Delta t_n = \tau/H(\|u^n\|_p)$ for all $n = 0, 1, 2, \cdots$. We henceforth assume this.

5.2. Convergence of numerical blow-up time

With Assumption (C) at hand, we are now going to prove that $T(\tau,h)$ is finite and converges to T. Let $\{u_j^n\}$ be a solution of (5.1). We first define φ_h by

$$\varphi_h(t_n) = \frac{1}{\sigma(h)} \sum_{j=1}^{N-1} h \sin(\pi x_j) u_j^n,$$

where $\sigma(h)$ is given by (4.4). Assume that, corresponding to the assumptions in Theorem 3.3, the initial data satisfies

$$\varphi_h(t_1) \ge \varphi_h(t_0) > \pi^2. \tag{5.3}$$

Lemma 5.1. *Let* $\{u_j^n\}$ *be the solution of* (5.1) *with time interval defined by* (5.2). *Assume that* (5.3) *holds. Then* $\varphi(t_n)$ *is increasing in n and it holds that*

$$\frac{\varphi_h(t_{n+1})-\varphi_h(t_n)}{\Delta t_n} \geq \frac{\varphi_h(t_n)-\varphi_(t_{n-1})}{\Delta t_{n-1}} + \tau_n \left(\varphi_h(t_n)-\pi^2\right)\varphi_h(t_n) \tag{5.4}$$

for all $n=1,2,\cdots$.

Proof. By (5.1), we have

$$\frac{1}{\tau_1}\left(\frac{\varphi_h(t_2)-\varphi_h(t_1)}{\Delta t_1} - \frac{\varphi_h(t_1)-\varphi_h(t_0)}{\Delta t_0}\right)$$
$$= \frac{1}{\sigma(h)}\left(\sum_{j=1}^{N-1} h\sin(\pi x_j)\frac{u_{j+1}^1-2u_j^1+u_{j-1}^1}{h^2} + \sum_{j=1}^{N-1} h\sin(\pi x_j)(u_j^1)^2\right).$$

By a direct calculation, we obtain

$$\begin{aligned}
&\frac{1}{\sigma(h)}\sum_{j=1}^{N-1} h\sin(\pi x_j)\frac{u_{j+1}^1-2u_j^1+u_{j-1}^1}{h^2}\\
&= -\frac{1}{\sigma(h)}\left(\sin\frac{\pi h}{2}\right)^2 \frac{4}{h^2}\sum_{j=1}^{N-1} h\sin(\pi x_j)u_j^1\\
&\geq -\pi^2\left(\frac{1}{\sigma(h)}\sum_{j=1}^{N-1} h\sin(\pi x_j)u_j^1\right) = -\pi^2\varphi_h(t_1).
\end{aligned}$$

On the other hand, by the Cauchy-Schwarz inequality, it follows that

$$\frac{1}{\sigma(h)}\sum_{j=1}^{N-1} h\sin(\pi x_j)(u_j^1)^2$$
$$\geq \left(\frac{1}{\sigma(h)}\sum_{j=1}^{N-1} h\sin(\pi x_j)|u_j^1|\right)^2 \geq \left(\frac{1}{\sigma(h)}\sum_{j=1}^{N-1} h\sin(\pi x_j)u_j^1\right)^2 = \left(\varphi_h(t_1)\right)^2.$$

Thus, we have

$$\frac{1}{\tau_1}\left(\frac{\varphi_h(t_2)-\varphi_h(t_1)}{\Delta t_1} - \frac{\varphi_h(t_1)-\varphi_h(t_0)}{\Delta t_0}\right) \geq -\pi^2\varphi_h(t_1) + \left(\varphi_h(t_1)\right)^2,$$

the right hand side of which is positive by our assumption (5.3). This implies that

$$\varphi_h(t_2) \geq \varphi_h(t_1)\ (> \pi^2).$$

We now repeat this argument to obtain

$$\frac{1}{\tau_n}\left(\frac{\varphi_h(t_{n+1})-\varphi_h(t_n)}{\Delta t_n}-\frac{\varphi_h(t_n)-\varphi_h(t_{n-1})}{\Delta t_{n-1}}\right)\geq \varphi_h(t_n)\left(\varphi_h(t_n)-\pi^2\right)$$

and $\varphi_h(t_{n+1})\geq\varphi_h(t_n)>\pi^2$. □

Lemma 5.2. *For $n\geq 1$, we have*

$$\frac{\varphi_h(t_{n+1})-\varphi_h(t_n)}{\Delta t_n}\geq\sqrt{\frac{1}{6}\varphi_h(t_n)^3-\frac{\pi^2}{4}\varphi_h(t_n)^2+K_0},\tag{5.5}$$

where

$$K_0=-\frac{1}{6}\varphi_h(t_0)^3+\frac{\pi^2}{4}\varphi_h(t_0)^2+\left(\frac{\varphi_h(t_1)-\varphi_h(t_0)}{\Delta t_0}\right)^2.$$

Proof. By Lemma 5.1, we have

$$\begin{aligned}&\left(\frac{\varphi_h(t_{n+1})-\varphi_h(t_n)}{\Delta t_n}\right)^2\\ \geq&\frac{\varphi_h(t_n)-\varphi_h(t_{n-1})}{\Delta t_{n-1}}\left(\frac{\varphi_h(t_n)-\varphi_h(t_{n-1})}{\Delta t_{n-1}}+\tau_n\left(\varphi_h(t_n)-\pi^2\right)\varphi(t_n)\right)\\ \geq&\left(\frac{\varphi_h(t_n)-\varphi_h(t_{n-1})}{\Delta t_{n-1}}\right)^2+\frac{1}{2}\left(\varphi_h(t_n)-\varphi_h(t_{n-1})\right)\left(\varphi_h(t_n)-\pi^2\right)\varphi(t_n).\end{aligned}$$

Therefore,

$$\begin{aligned}&\left(\frac{\varphi_h(t_{n+1})-\varphi_h(t_n)}{\Delta t_n}\right)^2-\left(\frac{\varphi_h(t_1)-\varphi_h(t_0)}{\Delta t_0}\right)^2\\ \geq&\frac{1}{2}\sum_{k=1}^{n}\left(\varphi_h(t_k)-\varphi_h(t_{k-1})\right)\left(\varphi(t_k)-\pi^2\right)\varphi_h(t_k)\geq\frac{1}{2}\int_{\varphi_h(t_0)}^{\varphi_h(t_n)}x(x-\pi^2)\mathrm{d}x\\ =&\frac{1}{6}\left[\varphi_h(t_n)^3-\varphi_h(t_0)^3\right]-\frac{\pi^2}{4}\left[\varphi_h(t_n)^2-\varphi_h(t_0)^2\right].\end{aligned}$$

Thus, (5.5) holds true. □

Next, we define the function $G(z)$ by

$$G(z)=\sqrt{\frac{z^3}{6}-\frac{\pi^2z^2}{4}+K_0}.\tag{5.6}$$

Note that G is increasing in $z\in[\varphi_h(t_0),\infty)$. We now consider the finite difference equation

$$\frac{v^{n+1}-v^n}{\Delta s_n}=G(v^n)\quad(n=0,1,2,\cdots),\qquad v^0=\varphi_h(t_0).\tag{5.7}$$

Here, Δs_n is defined by

$$\Delta s_n = \tau \cdot \min\left\{1, \frac{1}{H(v^n)}\right\},$$

Recall that we are dealing with a large data such that $H(v^n) > 1$, whence $\Delta s_n = \tau/H(v^n)$ for all n.

It was shown in Theorem 2.1 of [8] that $\lim_{n\to\infty} v^n = \infty$. Moreover, we have

$$\sum_{k=0}^{\infty} \Delta s_k \le \int_{\varphi_h(t_0)}^{\infty} \frac{dz}{G(z)} + C\tau, \tag{5.8}$$

where C is a constant independent of τ.

Lemma 5.3. *Let $\{u_j^n\}$ be the solution of* (5.1) *with initial data satisfying* (5.3) *and $p = \infty$. Then we have*

$$T(\tau,h) = \sum_{k=0}^{\infty} \Delta t_k \le 2\left(\int_{\varphi_h(t_0)}^{\infty} \frac{dz}{G(z)} + C\tau\right).$$

In particular, $T(\tau,h) < \infty$.

Proof. By the monotonicity of H and the fact that $\|u^0\|_p \ge \varphi_h(t_0) = v^0$, one obtains

$$\Delta t_0 = \frac{\tau}{H(\|u^0\|_p)} \le \frac{\tau}{H(v_0)} = \Delta s_0.$$

We are going to prove that

$$\sum_{k=0}^{\infty} \Delta t_n \le 2\sum_{k=0}^{\infty} \Delta s_n.$$

To this end, we first note that either

$$\sum_{k=0}^{\infty} \Delta t_k \le \Delta s_0$$

or there exists an n_1 such that

$$\sum_{k=0}^{n_1} \Delta t_k \le \Delta s_0 \qquad \text{and} \qquad \Delta s_0 < \sum_{k=0}^{n_1+1} \Delta t_k.$$

The proof ends in the first case. In the second case, we have by (5.5), (5.7), and the monotonicity of G,

$$\begin{aligned}
\varphi_h(t_{n_1+2}) &\geq \varphi_h(t_{n_1+1}) + \Delta t_{n_1+1} G(\varphi_h(t_{n_1+1})) \\
&\geq \varphi_h(t_{n_1}) + \Delta t_{n_1} G(\varphi_h(t_{n_1})) + \Delta t_{n_1+1} G(\varphi_h(t_{n_1+1})) \\
&\geq \varphi_h(t_{n_1}) + (\Delta t_{n_1} + \Delta t_{n_1+1}) G(\varphi_h(t_{n_1})) \\
&\vdots \\
&\geq \varphi_h(t_0) + \left(\sum_{k=0}^{n_1+1} \Delta t_k \right) G(\varphi_h(t_0)) \\
&\geq v^0 + \Delta s_0 G(v^0) = v^1,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{n_1+1} \Delta t_k &= \sum_{k=0}^{n_1} \Delta t_k + \Delta t_{n_1+1} \leq \Delta s_0 + \frac{\tau}{H(\|u^{n_1+1}\|_p)} \\
&\leq \Delta s_0 + \frac{\tau}{H(\varphi_h(t_{n_1+1}))} \leq \Delta s_0 + \frac{\tau}{H(\varphi_h(t_0))} = 2\Delta s_0.
\end{aligned}$$

Similarly, either

$$\sum_{k=n_1+2}^{\infty} \Delta t_k \leq \Delta s_1$$

holds or there exists an n_2 such that

$$\sum_{k=n_1+2}^{n_2} \Delta t_k \leq \Delta s_1 \qquad \text{and} \qquad \Delta s_1 < \sum_{k=n_1+2}^{n_2+1} \Delta t_k.$$

Again, it is sufficient to consider the second case. By means of (5.5) and (5.7), we obtain

$$\begin{aligned}
\varphi_h(t_{n_2+2}) &\geq \varphi_h(t_{n_2+1}) + \Delta t_{n_2+1} G(\varphi_h(t_{n_2+1})) \\
&\geq \varphi_h(t_{n_2}) + \Delta t_{n_2} G(\varphi_h(t_{n_2})) + \Delta t_{n_2+1} G(\varphi_h(t_{n_2+1})) \\
&\geq \varphi_h(t_{n_2}) + (\Delta t_{n_2} + \Delta t_{n_2+1}) G(\varphi_h(t_{n_2})) \\
&\vdots \\
&\geq \varphi_h(t_{n_1+2}) + \left(\sum_{k=n_1+2}^{n_2+1} \Delta t_k \right) G(\varphi_h(t_{n_1+2})) \\
&\geq v^1 + \Delta s_1 G(v^1) = v^2,
\end{aligned}$$

and

$$\begin{aligned}\sum_{k=n_1+2}^{n_2+1} \Delta t_k &= \sum_{k=n_1+2}^{n_2} \Delta t_k + \Delta t_{n_2+1} \\ &\le \Delta s_1 + \frac{\tau}{H(\|u^{n_2+1}\|_p)} \le \Delta s_1 + \frac{\tau}{H(\varphi_h(t_{n_2+1}))} \\ &\le \Delta s_1 + \frac{\tau}{H(\varphi_h(t_{n_1+2}))} \le \Delta s_1 + \frac{\tau}{H(\nu^1)} = 2\Delta s_1.\end{aligned}$$

If we repeat this process, we can find $\{n_j\}$ satisfying $\varphi_h(t_{n_j+2}) \ge \nu^j$ and

$$\sum_{k=n_j+2}^{n_{j+1}+1} \Delta t_k \le 2\Delta s_j,$$

for all $j \ge 1$. Thus,

$$\begin{aligned}\sum_{k=0}^{\infty} \Delta t_k &= \lim_{r\to\infty} \left(\sum_{k=0}^{n_1+1} \Delta t_k + \sum_{k=n_1+2}^{n_2+1} \Delta t_k + \cdots + \sum_{k=n_{r-1}+2}^{n_r+1} \Delta t_k \right) \\ &\le 2 \lim_{r\to\infty} \sum_{k=0}^{r-1} \Delta s_k \\ &\le 2 \left(\int_{\varphi_h(t_0)}^{\infty} \frac{dz}{G(z)} + C\tau \right),\end{aligned}$$

where use has been made of (5.8). This yields the desired result. □

Now we are in a position to prove the convergence of the blow-up time.

Theorem 5.1. *Let u be a solution of* (3.1)-(3.2) *which blows up in finite time, and let T denote its blow-up time. Let $\tau/h < 1$ and $T(\tau,h)$ be the discrete blow-up time. Assume that Δt_n is defined by* (5.2) *with $p = \infty$. Assume finally that Assumption* (C) *holds true. Then we have*

$$\lim_{h\to 0} T(\tau,h) = T.$$

Proof. Note first that, if $h \to 0$, then τ, too, tends to zero because of the stability condition $\tau/h < 1$.

We now assume that

$$T_* = \liminf_{h\to 0} T(\tau,h) < T.$$

Then there exists a subsequence $\{\tau_i\}$ and $\{h_i\}$ such that $\tau_i \to 0$ as $i \to \infty$ and that

$$T(\tau_i,h_i) \le T_* + \delta < T,$$

where $\delta = \frac{1}{2}(T - T_*)$. Let $\{u_j^n(\tau_i, h_i)\}$ be the solution corresponding to the parameter τ_i and h_i. Thus, we have

$$t_n(\tau_i, h_i) \le T_* + \delta < T \qquad (0 \le n),$$

while $\|u^n(\tau_i, h_i)\|_p \to \infty$ as $n \to \infty$. This contradicts to Assumption (C). That is,

$$\liminf_{h\to 0} T(\tau, h) \ge T. \tag{5.9}$$

Next, we assume that

$$T^* = \limsup_{h\to 0} T(\tau, h) > T.$$

Then for all $0 < \varepsilon < \frac{1}{2}(T^* - T)$, there exists a subsequence $\{\tau_i'\}$ and $\{h_i'\}$ such that

$$T(\tau_i', h_i') > T + \varepsilon,$$

with $\tau_i'/h_i' < 1$ and $h_i \to 0$ as $i \to \infty$. Since $\varphi(t)$ blows up in finite time T, for all $K > 0$ there exists $t' < T$ such that $\varphi(t') > 2K$. Let $\theta < \frac{1}{2}(T - t')$. By virtue of Assumption (C), there exists subsequences $\{\tau_{n_i}'\}$ and $\{h_{n_i}'\}$ of $\{\tau_i'\}$ and $\{h_i'\}$, respectively, such that there exists a positive $k(n_i) > 0$, corresponding to τ_{n_i}', satisfying that

$$t' \le t_{k(n_i)} < T, \qquad \varphi_h(t_{k(n_i)}) \ge \varphi(t_{k(n_i)}) - K \ge K.$$

Thus, we have

$$T(\tau_{n_i}', h_{n_i}') = t_{k(n_i)} + \sum_{n=k(n_i)}^{\infty} \Delta t_n < T + 2\left(\int_K^\infty \frac{dz}{G(z)} + C\tau_{n_i}'\right).$$

Since K can be taken so large and τ_{n_i}' can be chosen so small that

$$2\left(\int_K^\infty \frac{dz}{G(z)} + C\tau_{n_i}'\right) < \varepsilon,$$

we have $T(\tau_{n_i}', h_{n_i}') < T + \varepsilon$. This is a contradiction. Thus,

$$\limsup_{h\to 0} T(\tau, h) \le T. \tag{5.10}$$

By (5.9) and (5.10), we have $\lim_{h\to 0} T(\tau, h) = T$. □

Remark 5.2. For $1 \le p < \infty$, we may consider instead of $\varphi_h(t_n)$ the discrete functional

$$\bar{\varphi}_h(t_n) = \sigma(h)\varphi_h(t_n) = \sum_{j=1}^{N-1} h \sin(\pi x_j) u_j^n,$$

and assume corresponding to Theorem 3.3 that the initial data satisfy $\bar{\varphi}_h(t_1) \geq \bar{\varphi}_h(t_0) > 2\pi$. Then the convergence of the numerical blow-up time, i.e. $\lim_{h\to 0} T(\tau,h) = T$, follows by the same arguments given above with $G(z)$ in (5.6) being replaced by

$$\bar{G}(z) = \sqrt{\frac{\pi z^3}{12} - \frac{\pi^2 z^2}{4} + K_1},$$

where

$$K_1 = -\frac{\pi}{12}\bar{\varphi}_h(t_0)^3 + \frac{\pi^2}{4}\bar{\varphi}_h(t_0)^2 + \left(\frac{\bar{\varphi}_h(t_1) - \bar{\varphi}_h(t_0)}{\Delta t_0}\right)^2.$$

Remark 5.3. The reader may wonder if we could obtain an a priori error bound such as $|T(\tau,h) - T| \leq c\tau$. We found that this is a difficult problem. In fact no error bound has been obtained even in the case of parabolic equations, see [8].

5.3. Numerical examples

We now present some numerical examples. First example is as follows. The initial data is $f(x) = 30\sin(\pi x)$ and $g(x) \equiv 0$. The function H in (5.2) is $H(s) = s^{1/2}$. $h = 0.002, \tau = 0.0016$. As Fig. 1 shows the computation proceeds stably until $\max_j u_j^n$ reaches 12000.

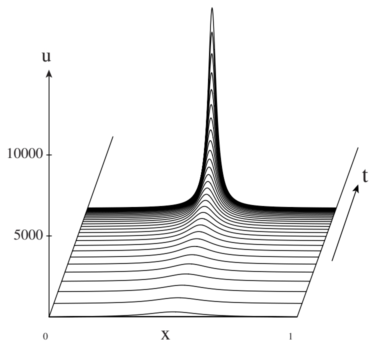

Figure 1: Numerical example. $f(x) = 30\sin(\pi x)$, $g(x) \equiv 0$.

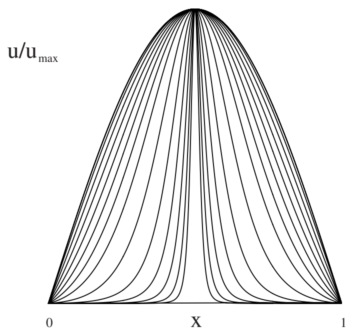

Figure 2: Plot of $u_k^n/\max_j u_j^n$. $f(x) = 30\sin(\pi x)$, $g(x) \equiv 0$.

With the same initial data, we now set $N = 1000$ and $\tau/h = 0.02$. We then plot

$$\frac{u_k^n}{\max_j u_j^n} \qquad (k = 0, 1, \cdots, N).$$

The maximum is one and is always taken at $x = 0.5$ as is shown in Fig. 2. At the largest n, $\max_j u_j^n$ is approximately 9.5×10^6. With this large solution, the computation is stable.

Fig. 3 (left) shows the case where $f(x) = 10\sin(\pi x)$ and $g(x) = -200\sin(2\pi x)$. Here, neither the assumption of Theorem 3.2 nor (3.3) is satisfied, but the solution seems to blow up. In Fig. 3 (right), we consider $u_{tt} = u_{xx} + u^3$ with $f(x) = 30\sin(2\pi x)$, $g(x) \equiv 0$. Here $H(s) = s$. This example shows that our idea can be applied to not only the nonlinearity u^2, but also other nonlinearities.

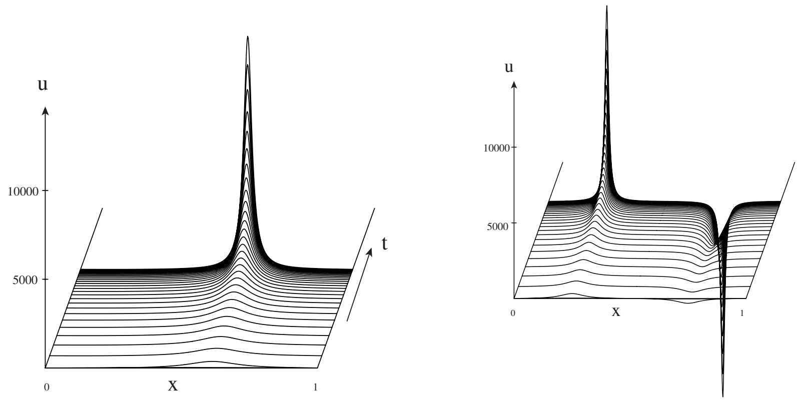

Figure 3: $f(x) = 10\sin(\pi x)$, $g(x) = -200\sin(2\pi x)$ (left). $f(x) = 30\sin(2\pi x)$, $g(x) = 0$ (right). $N = 500$, $\tau/h = 0.8$.

Table 1: $h(=1/N)$ vs. $T(\tau,h)$ with some initial data.

	$T(\tau,h)$	
N	$f(x)=300\sin(\pi x)$ $g(x)=0$	$f(x)=500\sin(\pi x)$ $g(x)=20\sin(2\pi x)$
16	0.171645722923	0.131784930496
32	0.171474088158	0.131643996036
64	0.171383857597	0.131571709667
128	0.171339126667	0.131535079994
256	0.171315681332	0.131516641296
512	0.171303492901	0.131507390839

Next, we consider the dependence of $T(\tau,h)$ on τ and h. We put $\tau/h = 0.2$. The function H is given by $H(s) = s^{1/2}$. The computation stops when $\max_j u_j^n$ attains 10^6. From Table 1, we observe that $T(\tau,h)$ monotonically decreases as $\tau \downarrow 0$, while the numerical blow-up time for semi-linear heat equation increases as $\tau \downarrow 0$ (see [8]). Moreover, the numerical results seem to suggest that the convergence order of the numerical blow-up time is $\mathcal{O}(\tau)$. (see Fig. 4). However, this problem still remains open.

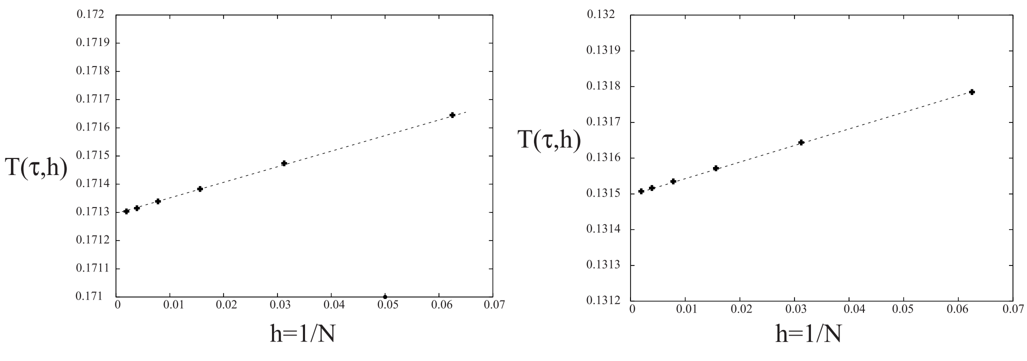

Figure 4: $f(x) = 300\sin(\pi x)$, $g(x) = 0$ (left). $f(x) = 500\sin(\pi x)$, $g(x) = 20\sin(2\pi x)$ (right).

5.4. Discussion

In this section, we proposed a finite difference scheme (5.1) that reproduces the blow-up for the nonlinear wave equation (3.1). Moreover, if we assume the validity of Assumption (C), namely, the convergence of our finite difference solution, we can show the convergence of the numerical blow-up time (Theorem 5.1). However, the verification of Assumption (C) is difficult to us.

Recalling the convergence results given in Section 4 and the discussion in Section 5.1, we could say that the difficulty in proving Assumption (C) lies in the non-uniformity of the time mesh. Thus, as a support of Assumption (C), we consider the following second order ODE problem

$$u''(t) = u^2, \qquad (u(0) = a_0 > 0,\ u'(0) = a_1 > 0), \tag{5.11}$$

and its difference analogue

$$\frac{1}{\tau_n}\left(\frac{u^{n+1} - u^n}{\Delta t_n} - \frac{u^n - u^{n-1}}{\Delta t_{n-1}}\right) = (u^n)^2, \tag{5.12}$$

where $\tau_n = \frac{1}{2}(\Delta t_n + \Delta t_{n-1})$ and Δt_n is given by

$$\Delta t_n = \tau \cdot \min\left\{1, \frac{1}{H(u^n)}\right\}.$$

Here, the function H is defined in a similar way as we did in the nonlinear wave equation. That is, H satisfies: (a) H is monotone increasing and $\lim_{s\to\infty} H(s) = \infty$; (b) the mapping $s \longmapsto s + \tau\frac{F(s)}{H(s)}$ is monotone increasing; and (c)

$$\int_{H^{-1}(1)}^{\infty} \frac{F'(s)ds}{F(s)H(s)} < \infty,$$

where $F(s)$ is given by

$$F(s) = \left(\frac{2}{3}s^3 + a_1^2 - \frac{2}{3}a_0^3\right)^{1/2}.$$

It is not difficult to show that the solution of (5.11) blows up in finite time $T = \int_{a_0}^{\infty}\frac{ds}{F(s)}$. Moreover, one can prove mathematically that the finite difference solution (5.12) also blows up in finite time $T(\tau) = \sum_{n=0}^{\infty}\Delta t_n < \infty$ and that the numerical blow-up time $T(\tau)$ converges to the real blow-up time T as $\tau \downarrow 0$. We remark that in the second order ODE case, we can not only prove the convergence of the numerical solution and the numerical blow-up time, but also show that the convergence order of the numerical blow-up time is $\mathcal{O}(\tau)$ for certain choices of the function H. The related results will be published elsewhere.

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