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Preconditioned Iterative Methods for Algebraic Systems from Multiplicative Half-Quadratic Regularization Image Restorations

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Abstract. Image restoration is often solved by minimizing an energy function consisting of a data-fidelity term and a regularization term. A regularized convex term can usually preserve the image edges well in the restored image. In this paper, we consider a class of convex and edge-preserving regularization functions, i.e., multiplicative half-quadratic regularizations, and we use the Newton method to solve the correspondingly reduced systems of nonlinear equations. At each Newton iterate, the preconditioned conjugate gradient method, incorporated with a constraint preconditioner, is employed to solve the structured Newton equation that has a symmetric positive definite coefficient matrix. The eigenvalue bounds of the preconditioned matrix are deliberately derived, which can be used to estimate the convergence speed of the preconditioned conjugate gradient method. We use experimental results to demonstrate that this new approach is efficient, and the effect of image restoration is reasonably well.

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1. Introduction

In image restoration, the restored image $\hat{\mathbf{x}} \in \mathscr{R}^p$ is estimated based upon a degraded data vector $\mathbf{b} \in \mathscr{R}^q$ by minimizing an energy function $\mathbf{J} : \mathscr{R}^p \to \mathscr{R}^q$, and the function $\mathbf{J}$

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consists of a data-fidelity term and a weighted regularization term Φ. Thus, it holds that

$$\hat{\mathbf{x}} = \min_{\mathbf{x}\in\mathscr{R}^p} \mathbf{J}(\mathbf{x}),$$
$$\mathbf{J}(\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 + \beta\Phi(\mathbf{x}),$$

where $\beta > 0$ is a regularization parameter. The data-fidelity term given above assumes that $\mathbf{b}$ and $\mathbf{x}$ satisfy an approximate linear relation $\mathbf{A}\mathbf{x} \approx \mathbf{b}$, but that $\mathbf{b}$ is contaminated by noise. The treatment of using such a data-fidelity term is popular in computations of many inverse problems such as seismic imaging, non-destructive evaluation, and x-ray tomography; see, e.g., [5,8]. Here, we consider regularization terms Φ of the form

$$\Phi(\mathbf{x}) = \sum_{i=1}^{r} \phi(\mathbf{g}_i^T \mathbf{x}), \tag{1.1}$$

where $\mathbf{g}_i^T : \mathscr{R}^p \to \mathscr{R}$, $i = 1, \cdots, r$, are linear operators. Typically, $\{\mathbf{g}_i^T\mathbf{x}\}_{i=1}^r$ are the first- or the second-order differences between neighboring samples in $\mathbf{x}$. For example, if $\mathbf{x}$ is a one-dimensional signal, then it usually holds that $\mathbf{g}_i^T\mathbf{x} = x_i - x_{i+1}$, $i = 1, \cdots, p-1$, where $\mathbf{x} = (x_1, x_2, \cdots, x_p)^T \in \mathscr{R}^p$. Let $\mathbf{G}$ denote the $r \times p$ matrix whose ith row is $\mathbf{g}_i^T$, $i = 1, \cdots, r$, such that

$$\phi \not\equiv 0 \quad \text{and} \quad \ker(\mathbf{A}^T\mathbf{A}) \cap \ker(\mathbf{G}^T\mathbf{G}) = \{\mathbf{0}\}, \tag{1.2}$$

where $\ker(\cdot)$ denotes the kernel space of the corresponding matrix.

In this paper, we will focus on convex, edge-preserving potential functions $\phi : \mathscr{R} \to \mathscr{R}$ employed in (1.1), because they give rise to image and signal estimates of high quality involving edges and homogeneous regions. Examples of such functions (see, e.g., [5,13]) are listed as follows:

$$\phi_1(t) = |t| - \alpha\log(1 + |t|/\alpha), \tag{1.3}$$
$$\phi_2(t) = \sqrt{\alpha + t^2} - \sqrt{\alpha}, \tag{1.4}$$
$$\phi_3(t) = \log(\cosh(\alpha t))/\alpha, \tag{1.5}$$
$$\phi_4(t) = \begin{cases} t^2/(2\alpha), & \text{if } |t| \le \alpha, \\ |t| - \alpha/2, & \text{if } |t| > \alpha, \end{cases} \tag{1.6}$$

where $\alpha > 0$ is a prescribed parameter. In general, we assume that ϕ is convex, even, $\mathscr{C}^2$, and satisfies

$$\mathbf{A}^T\mathbf{A} \text{ is invertible} \quad \text{and/or} \quad \phi''(t) > 0, \quad \forall t \in \mathscr{R}, \tag{1.7}$$

where $\phi''(t)$ denotes the second-order derivative of the function $\phi(t)$ with respect to t. It is easy to see that the assumptions in (1.7) and (1.2) guarantee that, for every $\mathbf{y} \in \mathscr{R}^p$, the function $\mathbf{J}$ has a unique global minimum point.

However, the minimizers $\hat{\mathbf{x}}$ of the cost functions $\mathbf{J}$ involving edge-preserving regularization terms are nonlinear with respect to $\mathbf{x}$. Hence, their computations are quite complicated and costly. To simplify such computations, a multiplicative half-quadratic reformulation of

$\mathbf{J}$ was proposed in [10] and [11], whose basic idea is as follows: Construct an augmented cost function $\tilde{\mathbf{J}} : \mathscr{R}^p \times \mathscr{R}^r \to \mathscr{R}$ involving an auxiliary variable $\mathbf{z} = (z_1, z_2, \cdots, z_r)^T \in \mathscr{R}^r$, i.e.,

$$\begin{aligned} \tilde{\mathbf{J}}(\mathbf{x}, \mathbf{z}) &:= \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 + \beta \sum_{i=1}^{r} \left(\frac{z_i}{2} (\mathbf{g}_i^T \mathbf{x})^2 + \psi(z_i) \right) \\ &= \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 + \frac{\beta}{2} (\mathbf{G}\mathbf{x})^T \operatorname{diag}(\mathbf{z}) \, (\mathbf{G}\mathbf{x}) + \beta \sum_{i=1}^{r} \psi(z_i), \end{aligned} \tag{1.8}$$

where $\operatorname{diag}(\mathbf{z})$ is a diagonal matrix with the diagonal entries being equal to $\{z_i\}_{i=1}^r$, and

$$\phi(t) = \min_{s \in \mathscr{R}} \left\{ \frac{1}{2} t^2 s + \psi(s) \right\}, \quad \forall t \in \mathscr{R}. \tag{1.9}$$

Such a dual potential function ψ is then determined by using the theory of convex conjugacy. The condition (1.9) ensures that

$$\mathbf{J}(\mathbf{x}) = \min_{\mathbf{z} \in \mathscr{R}^r} \tilde{\mathbf{J}}(\mathbf{x}, \mathbf{z}), \quad \forall \mathbf{x} \in \mathscr{R}^p.$$

The minimizer $(\hat{\mathbf{x}}, \hat{\mathbf{z}})$ of the augmented cost function $\tilde{\mathbf{J}}$ is calculated by making use of an alternating minimization technique. That is to say, from the solution $(\mathbf{x}^{(k-1)}, \mathbf{z}^{(k-1)})$ at iterate $k-1$, we compute the solution $(\mathbf{x}^{(k)}, \mathbf{z}^{(k)})$ at iterate k through finding

$$\begin{array}{lll} \mathbf{z}^{(k)} & \text{such that} & \tilde{\mathbf{J}}(\mathbf{x}^{(k-1)}, \mathbf{z}^{(k)}) \le \tilde{\mathbf{J}}(\mathbf{x}^{(k-1)}, \mathbf{z}), \quad \forall \mathbf{z} \in \mathscr{R}^r, \\ \mathbf{x}^{(k)} & \text{such that} & \tilde{\mathbf{J}}(\mathbf{x}^{(k)}, \mathbf{z}^{(k)}) \le \tilde{\mathbf{J}}(\mathbf{x}, \mathbf{z}^{(k)}), \qquad \forall \mathbf{x} \in \mathscr{R}^p. \end{array}$$

The major cost at each iterate of this approach is in computing $\mathbf{x}^{(k)}$, which requires to solve the system of linear equations

$$(2\mathbf{A}^T\mathbf{A} + \beta \mathbf{G}^T \operatorname{diag}(\mathbf{z}^{(k-1)}) \, \mathbf{G}) \mathbf{x}^{(k)} = 2\mathbf{A}^T \mathbf{b}. \tag{1.10}$$

In the spatial-invariant blurring, the matrix $\mathbf{A}$ is often a Toeplitz-like matrix [14]. In the regularization term, $\mathbf{G}$ is usually the discretization matrix of the first-order difference operator. Thus, the system of linear equations (1.10) can be solved fast and accurately by an iterative method [13]. Numerical results have shown that the minimization using a half-quadratic regularization can substantially accelerate the computations. However, the main drawbacks of this approach are that the convergence rate is only linear [13] and the cost for solving the linear system (1.10) is expensive.

In order to speed up the overall convergence rate of this approach, we may adopt the Newton method to compute the minimizer of the augmented cost function $\tilde{\mathbf{J}}(\mathbf{x}, \mathbf{z})$ in (1.8), as the Newton method preserves quadratic convergence rate when the nonlinear function is smooth enough and the initial point is close to the exact solution. Note that the Hessian matrix of $\tilde{\mathbf{J}}(\mathbf{x}, \mathbf{z})$ is given by

$$\mathbf{H}(\mathbf{x}, \mathbf{z}) = \begin{bmatrix} 2\mathbf{A}^T\mathbf{A} + \beta \mathbf{G}^T \operatorname{diag}(\mathbf{z}) \, \mathbf{G} & \beta \mathbf{G}^T \operatorname{diag}(\mathbf{G}\mathbf{x}) \\ \beta \operatorname{diag}(\mathbf{G}\mathbf{x}) \, \mathbf{G} & \frac{\beta}{2} \operatorname{diag}(\psi''(\mathbf{z})) \end{bmatrix}, \tag{1.11}$$

where $\mathrm{diag}(\psi''(\mathbf{z}))$ is a diagonal matrix with the diagonal entries being given by $\{\psi''(z_i)\}_{i=1}^{r}$. At each Newton step, we need to solve a structured linear system of the form

$$\mathbf{H}(\mathbf{x},\mathbf{z})\mathbf{d} = \mathbf{r}. \tag{1.12}$$

In this paper, we employ the *preconditioned conjugate gradient* (**PCG**) method, incorporated with a constraint preconditioner $\mathbf{M} := \mathbf{M}(\mathbf{x},\mathbf{z})$ for $\mathbf{H} := \mathbf{H}(\mathbf{x},\mathbf{z})$, to solve the Newton equation (1.12). This results in an inexact Newton method; see [1,2,9]. From both theoretical analysis and numerical experiments we will show that this constraint preconditioner is of high quality, and the resulting PCG is very effective for solving the edge-preserving signal and image restoration problems.

The rest of the paper is outlined as follows. In Section 2, we construct the constraint preconditioners and estimate eigenvalue bounds of the corresponding preconditioned matrices. In Section 3, we present two special cases of the constraint preconditioning matrix and estimate eigenvalue bounds of the corresponding preconditioned matrices, too. Experimental results are presented in Section 4 to demonstrate the performance of the PCG method incorporated with the constraint preconditioner as well as the Newton method. Finally, in Section 5 we give some brief concluding remarks.

2. The constraint preconditioning

In this section, we will construct and analyze a constraint preconditioner for the Hessian matrix $\mathbf{H} := \mathbf{H}(\mathbf{x},\mathbf{z})$ defined in (1.11), i.e.,

$$\mathbf{H} := \mathbf{H}(\mathbf{x},\mathbf{z}) = \begin{bmatrix} \mathbf{H}_{11}(\mathbf{z}) & \mathbf{H}_{12}(\mathbf{x}) \\ \mathbf{H}_{21}(\mathbf{x}) & \mathbf{H}_{22}(\mathbf{z}) \end{bmatrix},$$

where

$$\begin{cases} \mathbf{H_{11}} := \mathbf{H_{11}}(\mathbf{z}) = 2\mathbf{A}^T\mathbf{A} + \beta\mathbf{G}^T \ \mathrm{diag}(\mathbf{z})\ \mathbf{G}, \\ \mathbf{H_{12}} := \mathbf{H_{12}}(\mathbf{x}) = \beta\mathbf{G}^T \ \mathrm{diag}(\mathbf{G}\mathbf{x}), \\ \mathbf{H_{21}} := \mathbf{H_{21}}(\mathbf{x}) = \beta \ \mathrm{diag}(\mathbf{G}\mathbf{x})\ \mathbf{G}, \\ \mathbf{H_{22}} := \mathbf{H_{22}}(\mathbf{z}) = \frac{\beta}{2} \ \mathrm{diag}(\psi''(\mathbf{z})). \end{cases}$$

Note that both $\mathbf{H}_{11}$ and $\mathbf{H}_{22}$ are symmetric positive definite matrices, with $\mathbf{H}_{22}$ being diagonal.

The following theorem describes the positive definiteness of the symmetric matrix $\mathbf{H}$.

Theorem 2.1. *Let* $\mathbf{H} := \mathbf{H}(\mathbf{x},\mathbf{z})$ *be the Hermitian matrix defined in* (1.11) *with* $\psi''(t) > 0$ $(\forall\ t \in \mathscr{R})$. *Then under the condition* (1.2)*, the matrix* $\mathbf{H}$ *is symmetric positive definite, provided*

$$\mathbf{z}_i\psi''(\mathbf{z}_i) > 2(\mathbf{g}_i^T\mathbf{x})^2, \quad 1 \le i \le r. \tag{2.1}$$

Proof. It follows immediately from the expression of the Schur complement $\mathbf{S_H}$ of the matrix $\mathbf{H}$ with respect to its (1,1) block:

$$\begin{aligned}
\mathbf{S_H} &= \mathbf{H}_{11} - \mathbf{H}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{21} \\
&= \left(2\,\mathbf{A}^T\mathbf{A} + \beta\,\mathbf{G}^T\,\mathrm{diag}(\mathbf{z})\,\mathbf{G}\right) - 2\,\beta\,\mathbf{G}^T\,\mathrm{diag}(\mathbf{Gx})\,\mathrm{diag}(\psi''(\mathbf{z}))^{-1}\,\mathrm{diag}(\mathbf{Gx})\,\mathbf{G} \\
&= 2\,\mathbf{A}^T\mathbf{A} + \beta\,\mathbf{G}^T\,\left[\mathrm{diag}(\mathbf{z}) - 2\,\mathrm{diag}(\mathbf{Gx})\,\mathrm{diag}(\psi''(\mathbf{z}))^{-1}\,\mathrm{diag}(\mathbf{Gx})\right]\,\mathbf{G} \\
&= 2\,\mathbf{A}^T\mathbf{A} + \beta\,\mathbf{G}^T\,\mathrm{diag}(\psi''(\mathbf{z}))^{-1}\left[\mathrm{diag}(\mathbf{z})\,\mathrm{diag}(\psi''(\mathbf{z})) - 2\,(\mathrm{diag}(\mathbf{Gx}))^2\right]\,\mathbf{G}.
\end{aligned}$$

This completes the proof of the theorem. □

We consider the constraint preconditioner

$$\mathbf{M} := \begin{bmatrix} \mathbf{M}_{11}(\mathbf{z}) & \mathbf{H}_{12}(\mathbf{x}) \\ \mathbf{H}_{21}(\mathbf{x}) & \mathbf{H}_{22}(\mathbf{z}) \end{bmatrix} := \begin{bmatrix} \mathbf{M}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix} \tag{2.2}$$

for the Hessian matrix $\mathbf{H}$, where $\mathbf{M}_{11}$ is an approximation to the matrix block $\mathbf{H}_{11}$; see [3,4,6,7,12].

The eigenvalues and eigenvectors of the preconditioned matrix $\mathbf{M}^{-1}\mathbf{H}$ are precisely described in the following theorem.

Theorem 2.2. *Let* $\mathbf{H} := \mathbf{H}(\mathbf{x},\mathbf{z})$ *be the Hessian matrix defined in* (1.11) *satisfying the conditions* (1.2) *and* (2.1)*, with* $\psi''(t) > 0$ $(\forall\ t \in \mathscr{R})$*. Let* $\mathbf{M}$ *be the constraint preconditioner defined in* (2.2) *for* $\mathbf{H}$*, with* $\mathbf{M}_{11}$ *being an approximating matrix to* $\mathbf{H}_{11}$*. Denote by*

$$\mathbf{N}_{11} = \mathbf{M}_{11} - \mathbf{H}_{11} \quad \textit{and} \quad \mathbf{S} = \mathbf{M}_{11} - \mathbf{H}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{21}. \tag{2.3}$$

Then the eigenvalues of the preconditioned matrix $\mathbf{M}^{-1}\mathbf{H}$ *are* 1 *with multiplicity* r*, and* $1 - \lambda(\mathbf{S}^{-1}\mathbf{N}_{11})$*. Here and in the sequel,* $\lambda(\cdot)$ *denotes an eigenvalue of the corresponding matrix.*

In addition, the eigenvectors of $\mathbf{M}^{-1}\mathbf{H}$ *associated with the eigenvalue* 1 *are given by* $[\tilde{\mathbf{u}}\ \mathbf{v}]^T$*, with* $\tilde{\mathbf{u}} \in \ker(\mathbf{N}_{11})$ *and* $\mathbf{v} \in \mathscr{R}^r$*, and those associated with the eigenvalues* $\lambda \neq 1$ *are given by*

$$\begin{bmatrix} \tilde{\mathbf{u}} \\ -\mathbf{H}_{22}^{-1}\mathbf{H}_{21}\tilde{\mathbf{u}} \end{bmatrix}, \tag{2.4}$$

with $\tilde{\mathbf{u}}$ *being such that* $-\mathbf{S}^{-1}\mathbf{N}_{11}\tilde{\mathbf{u}} = (\lambda - 1)\tilde{\mathbf{u}}$*.*

Proof. Because $\mathbf{H} = \mathbf{M} + (\mathbf{H} - \mathbf{M})$, we have

$$\mathbf{M}^{-1}\mathbf{H} = \mathbf{I} + \mathbf{M}^{-1}(\mathbf{H} - \mathbf{M}) = \mathbf{I} + \mathbf{M}^{-1}\begin{bmatrix} -\mathbf{N}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

We can factorize the preconditioning matrix $\mathbf{M}$ into the block-triangular product form

$$\mathbf{M} = \begin{bmatrix} \mathbf{I} & \mathbf{H}_{12}\mathbf{H}_{22}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{22} \end{bmatrix}\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{H}_{22}^{-1}\mathbf{H}_{21} & \mathbf{I} \end{bmatrix},$$

where

$$\mathbf{S} = \mathbf{M}_{11} - \mathbf{H}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{21}$$

is the Schur complement of $\mathbf{M}$ with respect to its (1,1) block $\mathbf{M}_{11}$. It is easy to obtain

$$\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{H}_{22}^{-1}\mathbf{H}_{21} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{S}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{H}_{12}\mathbf{H}_{22}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix},$$

$$\mathbf{M}^{-1} \begin{bmatrix} -\mathbf{N}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} -\mathbf{S}^{-1}\mathbf{N}_{11} & \mathbf{0} \\ \mathbf{H}_{22}^{-1}\mathbf{H}_{21}\mathbf{S}^{-1}\mathbf{N}_{11} & \mathbf{0} \end{bmatrix}.$$

Therefore,

$$\mathbf{M}^{-1}\mathbf{H} = \begin{bmatrix} \mathbf{I} - \mathbf{S}^{-1}\mathbf{N}_{11} & \mathbf{0} \\ \mathbf{H}_{22}^{-1}\mathbf{H}_{21}\mathbf{S}^{-1}\mathbf{N}_{11} & \mathbf{I} \end{bmatrix}.$$

It then follows that the eigenvalues of $\mathbf{M}^{-1}\mathbf{H}$ are 1 with multiplicity r, and $1 - \lambda(\mathbf{S}^{-1}\mathbf{N}_{11})$.

We now discuss the eigenvectors of the preconditioned matrix $\mathbf{M}^{-1}\mathbf{H}$. To this end, let λ be an eigenvalue of $\mathbf{M}^{-1}\mathbf{H}$ and $\mathbf{w} = [\mathbf{u}\ \mathbf{v}]^T$ be a corresponding eigenvector, i.e., $\mathbf{M}^{-1}\mathbf{H}\mathbf{w} = \lambda\mathbf{w}$, or

$$\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} + \begin{bmatrix} -\mathbf{S}^{-1}\mathbf{N}_{11} & \mathbf{0} \\ \mathbf{H}_{22}^{-1}\mathbf{H}_{21}\mathbf{S}^{-1}\mathbf{N}_{11} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}.$$

Then, it holds that

$$-\mathbf{S}^{-1}\mathbf{N}_{11}\mathbf{u} = (\lambda - 1)\mathbf{u}$$

and

$$\mathbf{H}_{22}^{-1}\mathbf{H}_{21}\mathbf{S}^{-1}\mathbf{N}_{11}\mathbf{u} = (\lambda - 1)\mathbf{v}.$$

So, if $\lambda = 1$, the eigenvectors are

$$\begin{bmatrix} \tilde{\mathbf{u}} \\ \mathbf{v} \end{bmatrix}, \quad \text{with } \tilde{\mathbf{u}} \in \ker(\mathbf{N}_{11}),\ \forall\ \mathbf{v} \in \mathscr{R}^r;$$

and if $\lambda \neq 1$, the eigenvectors are (2.4), with $\tilde{\mathbf{u}}$ being such that $-\mathbf{S}^{-1}\mathbf{N}_{11}\tilde{\mathbf{u}} = (\lambda - 1)\tilde{\mathbf{u}}$. $\square$

3. Two specific choices

In actual applications, we may choose the approximating $(1,1)$-block matrix $\mathbf{M}_{11}$ according to the following two cases:

(a) $\mathbf{M}_{11} = 0$;

(b) $\mathbf{M}_{11} = 2\mathbf{A}^T\mathbf{A} + \beta\tau\mathbf{G}^T\mathbf{G}$,

where τ is a prescribed nonnegative constant. Note that for each case the resulting constraint preconditioning matrix $\mathbf{M}$ is symmetric, and it is indefinite for Case (a), and positive definite for Case (b) under certain conditions.

Theorem 3.1. *Let* $\mathbf{M}$ *be the constraint preconditioner defined in* (2.2)*, with*

$$\mathbf{M}_{11} = 2\mathbf{A}^T\mathbf{A} + \beta\tau\mathbf{G}^T\mathbf{G},$$

where τ *is a prescribed nonnegative constant. Then* $\mathbf{M}$ *is symmetric positive definite, provided the conditions* (1.2) *and* (2.1) *are satisfied,* $\psi''(t) > 0$ $(\forall\ t \in \mathscr{R})$*, and* $\tau\psi''(\mathbf{z}_i) > 2(\mathbf{g}_i^T\mathbf{x})^2$ $(1 \le i \le r)$*.*

Proof. It follows straightforwardly from the positive definiteness of the Schur complement of the matrix $\mathbf{M}$ with respect to its (1,1) block. □

For the eigenvalues of the preconditioned matrix $\mathbf{M}^{-1}\mathbf{H}$ corresponding to these two special choices, we have the following conclusion.

Theorem 3.2. *Let* $\mathbf{H} := \mathbf{H}(\mathbf{x}, \mathbf{z})$ *be the Hessian matrix defined in* (1.11) *satisfying the conditions* (1.2) *and* (2.1)*, with* $\psi''(t) > 0$ $(\forall\ t \in \mathscr{R})$*. Let* $\mathbf{M}$ *be the constraint preconditioner defined in* (2.2) *for* $\mathbf{H}$*, with* $\mathbf{M}_{11}$ *being an approximating matrix to* $\mathbf{H}_{11}$*. Assume that the vector* $\mathbf{x}$ *is such that all entries of* $\mathbf{G}\mathbf{x}$ *are nonzero. Denote by*

$$\zeta_{\min} = \min_{1\le i\le r}\{z_i\}, \qquad \zeta_{\max} = \max_{1\le i\le r}\{z_i\},$$

$$\sigma^2_{\min} = \inf_{\mathbf{u}\neq\mathbf{0}} \frac{\beta\mathbf{u}^T\mathbf{G}^T\mathbf{G}\mathbf{u}}{\mathbf{u}^T(2\mathbf{A}^T\mathbf{A} + \beta\mathbf{G}^T\mathbf{G})\mathbf{u}}, \qquad \sigma^2_{\max} = \sup_{\mathbf{u}\neq\mathbf{0}} \frac{\beta\mathbf{u}^T\mathbf{G}^T\mathbf{G}\mathbf{u}}{\mathbf{u}^T(2\mathbf{A}^T\mathbf{A} + \beta\mathbf{G}^T\mathbf{G})\mathbf{u}},$$

$$\delta_{\min} = \frac{2}{\max\{1,\zeta_{\max}\}} \cdot \min_{1\le i\le r}\left\{\frac{\omega_i^2}{\gamma_i}\right\}, \qquad \delta_{\max} = \frac{2}{\min\{1,\zeta_{\min}\}} \cdot \max_{1\le i\le r}\left\{\frac{\omega_i^2}{\gamma_i}\right\},$$

where $\gamma_i = \psi''(\mathbf{z}_i)$ *and* $\omega_i = \mathbf{g}_i^T\mathbf{x}$, $1 \le i \le r$*. Then*

(a) *when* $\mathbf{M}_{11} = 0$*, one eigenvalue of the preconditioned matrix* $\mathbf{M}^{-1}\mathbf{H}$ *is* 1 *with multiplicity* r*, and the other* p *eigenvalues are included in the interval*

$$\left[1 - \frac{1}{\delta_{\min}\sigma^2_{\min}},\ 1 - \frac{1}{\delta_{\max}\sigma^2_{\max}}\right];$$

(b) *when* $\mathbf{M}_{11} = 2\mathbf{A}^T\mathbf{A} + \beta\tau\mathbf{G}^T\mathbf{G}$*, with* τ *a prescribed nonnegative constant satisfying*

$$\tau \ge \max\left\{\zeta_{\max},\ 2\cdot\max_{1\le i\le r}\left\{\omega_i^2/\gamma_i\right\}\right\},$$

one eigenvalue of the preconditioned matrix $\mathbf{M}^{-1}\mathbf{H}$ *is 1 with multiplicity* r*, and the other* p *eigenvalues are included in the interval* $[1-\theta_{\max}, 1-\theta_{\min}]$*, where*

$$\theta_{\min} = \frac{(\tau - \zeta_{\max})\sigma^2_{\min}\min\{1,\zeta_{\min}\}}{\max\{1,\zeta_{\max}\}\max\left\{1, \tau - 2\min_{1\le i\le r}\left\{\omega_i^2/\gamma_i\right\}\right\}},$$

$$\theta_{\max} = \frac{(\tau - \zeta_{\min})\sigma^2_{\max}\max\{1,\zeta_{\max}\}}{\min\{1,\zeta_{\min}\}\min\left\{1, \tau - 2\max_{1\le i\le r}\left\{\omega_i^2/\gamma_i\right\}\right\}}.$$

Proof. According to Theorem 2.2, we only need to discuss the eigenvalues of the matrix $\mathbf{S}^{-1}\mathbf{N}_{11}$. To this end, we denote by $\mathbf{N}_{11} = \mathbf{M}_{11} - \mathbf{H}_{11}$ and $\mathbf{S} = \mathbf{M}_{11} - \mathbf{H}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{21}$.

(a) When $\mathbf{M}_{11} = 0$, we see that $\mathbf{N}_{11} = -\mathbf{H}_{11}$ is nonsingular. Hence, it holds that

$$\mathbf{S}^{-1}\mathbf{N}_{11} = (\mathbf{N}_{11}^{-1}\mathbf{S})^{-1} = (\mathbf{N}_{11}^{-1}\mathbf{M}_{11} - \mathbf{N}_{11}^{-1}\mathbf{H}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{21})^{-1} = (\mathbf{H}_{11}^{-1}\mathbf{H}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{21})^{-1}.$$

It follows that all eigenvalues of the matrix $\mathbf{S}^{-1}\mathbf{N}_{11}$ are positive. Write

$$\boldsymbol{\Omega} := \mathrm{diag}(\omega_i) = \begin{bmatrix} \mathbf{g}_1^T\mathbf{x} & & \\ & \ddots & \\ & & \mathbf{g}_r^T\mathbf{x} \end{bmatrix}, \quad \boldsymbol{\Gamma} := \mathrm{diag}(\gamma_i) = \begin{bmatrix} \psi''(\mathbf{z}_1) & & \\ & \ddots & \\ & & \psi''(\mathbf{z}_r) \end{bmatrix}.$$

Then we have

$$\lambda(\mathbf{H}_{11}^{-1}\mathbf{H}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{21}) = \lambda(\mathbf{H}_{11}^{-\frac{1}{2}}\mathbf{H}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{21}\mathbf{H}_{11}^{-\frac{1}{2}}).$$

Consequently,

$$\begin{aligned}
\max\left\{\lambda\left((\mathbf{S}^{-1}\mathbf{N}_{11})^{-1}\right)\right\} &= \sup_{\mathbf{u}\neq\mathbf{0}} \frac{\mathbf{u}^T\mathbf{H}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{21}\mathbf{u}}{\mathbf{u}^T\mathbf{H}_{11}\mathbf{u}} \\
&= \sup_{\mathbf{u}\neq\mathbf{0}} \frac{\mathbf{u}^T\cdot\beta\mathbf{G}^T\boldsymbol{\Omega}\cdot(\frac{1}{2}\beta\boldsymbol{\Gamma})^{-1}\cdot\beta\boldsymbol{\Omega}\mathbf{G}\mathbf{u}}{\mathbf{u}^T(2\mathbf{A}^T\mathbf{A}+\beta\mathbf{G}^T\mathrm{diag}(\mathbf{z})\mathbf{G})\mathbf{u}} \\
&= \sup_{\mathbf{u}\neq\mathbf{0}} \left\{2\beta\cdot\frac{\mathbf{u}^T\mathbf{G}^T\boldsymbol{\Omega}\boldsymbol{\Gamma}^{-1}\boldsymbol{\Omega}\mathbf{G}\mathbf{u}}{\mathbf{u}^T(2\mathbf{A}^T\mathbf{A}+\beta\mathbf{G}^T\mathrm{diag}(\mathbf{z})\mathbf{G})\mathbf{u}}\right\} \\
&\le \sup_{\mathbf{u}\neq\mathbf{0}} \left\{2\beta\cdot\max_{1\le i\le r}\left\{\frac{\omega_i^2}{\gamma_i}\right\}\cdot\frac{\mathbf{u}^T\mathbf{G}^T\mathbf{G}\mathbf{u}}{\mathbf{u}^T(2\mathbf{A}^T\mathbf{A}+\beta\zeta_{\min}\mathbf{G}^T\mathbf{G})\mathbf{u}}\right\} \\
&\le \sup_{\mathbf{u}\neq\mathbf{0}} \left\{2\beta\cdot\max_{1\le i\le r}\left\{\frac{\omega_i^2}{\gamma_i}\right\}\cdot\frac{1}{\min\{1,\zeta_{\min}\}}\cdot\frac{\mathbf{u}^T\mathbf{G}^T\mathbf{G}\mathbf{u}}{\mathbf{u}^T(2\mathbf{A}^T\mathbf{A}+\beta\mathbf{G}^T\mathbf{G})\mathbf{u}}\right\} \\
&= \delta_{\max}\sigma_{\max}^2.
\end{aligned}$$

Analogously, we can obtain,

$$\begin{aligned}
\min\left\{\lambda\left((\mathbf{S}^{-1}\mathbf{N}_{11})^{-1}\right)\right\} &\ge \inf_{\mathbf{u}\neq 0} \left\{2\beta\cdot\min_{1\le i\le r}\left\{\frac{\omega_i^2}{\gamma_i}\right\}\cdot\frac{\mathbf{u}^T\mathbf{G}^T\mathbf{G}\mathbf{u}}{\mathbf{u}^T(2\mathbf{A}^T\mathbf{A}+\beta\zeta_{\max}\mathbf{G}^T\mathbf{G})\mathbf{u}}\right\} \\
&\ge \inf_{u\neq 0} \left\{2\beta\cdot\min_{1\le i\le r}\left\{\frac{\omega_i^2}{\gamma_i}\right\}\cdot\frac{1}{\max\{1,\zeta_{\max}\}}\cdot\frac{\mathbf{u}^T\mathbf{G}^T\mathbf{G}\mathbf{u}}{\mathbf{u}^T(2\mathbf{A}^T\mathbf{A}+\beta\mathbf{G}^T\mathbf{G})\mathbf{u}}\right\} \\
&= \delta_{\min}\sigma_{\min}^2.
\end{aligned}$$

Now, from Theorem 2.2 we know that one eigenvalue of the preconditioned matrix $\mathbf{M}^{-1}\mathbf{H}$ is 1 with multiplicity r, and the other p eigenvalues are included in the interval

$$\left[1-\frac{1}{\delta_{\min}\sigma_{\min}^2}, \quad 1-\frac{1}{\delta_{\max}\sigma_{\max}^2}\right].$$

(b) As $\mathbf{M}_{11} = 2\mathbf{A}^T\mathbf{A} + \beta\tau\mathbf{G}^T\mathbf{G}$ is nonsingular, we see that

$$\mathbf{N}_{11} = \mathbf{M}_{11} - \mathbf{H}_{11} = \beta\mathbf{G}^T(\tau\mathbf{I} - \mathrm{diag}(\mathbf{z}))\mathbf{G}$$

is symmetric positive semidefinite and $\mathbf{S}^{-1}\mathbf{N}_{11}$ is similar to $\mathbf{S}^{-\frac{1}{2}}\mathbf{N}_{11}\mathbf{S}^{-\frac{1}{2}}$. Hence, all eigenvalues of $\mathbf{S}^{-1}\mathbf{N}_{11}$ are nonnegative reals. It follows that

$$\max\{\lambda(\mathbf{S}^{-1}\mathbf{N}_{11})\} = \sup_{\mathbf{u}\neq\mathbf{0}} \frac{\mathbf{u}^T\mathbf{N}_{11}\mathbf{u}}{\mathbf{u}^T\mathbf{S}\mathbf{u}} \leq \sup_{\mathbf{u}\neq\mathbf{0}} \frac{\mathbf{u}^T\mathbf{N}_{11}\mathbf{u}}{\mathbf{u}^T\mathbf{H}_{11}\mathbf{u}} \cdot \sup_{\mathbf{u}\neq\mathbf{0}} \frac{\mathbf{u}^T\mathbf{H}_{11}\mathbf{u}}{\mathbf{u}^T\mathbf{S}\mathbf{u}},$$

where

$$\begin{aligned}
\sup_{\mathbf{u}\neq\mathbf{0}} \frac{\mathbf{u}^T\mathbf{N}_{11}\mathbf{u}}{\mathbf{u}^T\mathbf{H}_{11}\mathbf{u}} &= \sup_{\mathbf{u}\neq\mathbf{0}} \frac{\beta\mathbf{u}^T\mathbf{G}^T(\tau\mathbf{I} - \mathrm{diag}(\mathbf{z}))\mathbf{G}\mathbf{u}}{\mathbf{u}^T(2\mathbf{A}^T\mathbf{A} + \beta\mathbf{G}^T\mathrm{diag}(\mathbf{z})\mathbf{G})\mathbf{u}} \\
&\leq \frac{\tau - \zeta_{\min}}{\min\{1, \zeta_{\min}\}} \cdot \sup_{\mathbf{u}\neq\mathbf{0}} \frac{\beta\mathbf{u}^T\mathbf{G}^T\mathbf{G}\mathbf{u}}{\mathbf{u}^T(2\mathbf{A}^T\mathbf{A} + \beta\mathbf{G}^T\mathbf{G})\mathbf{u}} \\
&\leq \frac{(\tau - \zeta_{\min})\sigma_{\max}^2}{\min\{1, \zeta_{\min}\}}
\end{aligned}$$

and

$$\begin{aligned}
\inf_{\mathbf{u}\neq\mathbf{0}} \frac{\mathbf{u}^T\mathbf{S}\mathbf{u}}{\mathbf{u}^T\mathbf{H}_{11}\mathbf{u}} &= \inf_{\mathbf{u}\neq\mathbf{0}} \frac{\mathbf{u}^T(\mathbf{M}_{11} - \mathbf{H}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{21})\mathbf{u}}{\mathbf{u}^T\mathbf{H}_{11}\mathbf{u}} \\
&= \inf_{\mathbf{u}\neq\mathbf{0}} \frac{\mathbf{u}^T(2\mathbf{A}^T\mathbf{A} + \beta\mathbf{G}^T(\tau\mathbf{I} - 2\Gamma^{-1}\Omega^2)\mathbf{G})\mathbf{u}}{\mathbf{u}^T(2\mathbf{A}^T\mathbf{A} + \beta\mathbf{G}^T\mathrm{diag}(\mathbf{z})\mathbf{G})\mathbf{u}} \\
&\geq \frac{\min\left\{1, \tau - 2\max\limits_{1\leq i\leq r}\left\{\omega_i^2/\gamma_i\right\}\right\}}{\max\{1, \zeta_{\max}\}}.
\end{aligned}$$

Thus, it holds that

$$\max\{\lambda(\mathbf{S}^{-1}\mathbf{N}_{11})\} \leq \frac{(\tau - \zeta_{\min})\sigma_{\max}^2 \max\{1, \zeta_{\max}\}}{\min\{1, \zeta_{\min}\}\min\left\{1, \tau - 2\max\limits_{1\leq i\leq r}\left\{\omega_i^2/\gamma_i\right\}\right\}} = \theta_{\max}.$$

Similarly, we can obtain

$$\min\{\lambda(\mathbf{S}^{-1}\mathbf{N}_{11})\} \geq \frac{(\tau - \zeta_{\max})\sigma_{\min}^2 \min\{1, \zeta_{\min}\}}{\max\{1, \zeta_{\max}\}\max\left\{1, \tau - 2\min\limits_{1\leq i\leq r}\left\{\omega_i^2/\gamma_i\right\}\right\}} = \theta_{\min}.$$

Now, from Theorem 2.2 we know that one eigenvalue of $\mathbf{M}^{-1}\mathbf{H}$ is 1 with multiplicity r, and the other p eigenvalues are included in the interval $[1 - \theta_{\max}, 1 - \theta_{\min}]$. □

We remark that all eigenvalues of the preconditioned matrix $\mathbf{M}^{-1}\mathbf{H}$ are positive when $\delta_{\min}\sigma_{\min}^2 > 1$ holds for Case (a), and when $\theta_{\max} < 1$ for Case (b), respectively.

4. Numerical results

In this section, we present experimental results to demonstrate the effectiveness of the PCG method for solving the image restoration problem, in which the multiplicative half-quadratic regularization technique is applied.

The Newton iteration is of the form

$$\begin{bmatrix} \mathbf{x}^{(k+1)} \\ \mathbf{z}^{(k+1)} \end{bmatrix} = \begin{bmatrix} \mathbf{x}^{(k)} \\ \mathbf{z}^{(k)} \end{bmatrix} - \mu_k \mathbf{H}(\mathbf{x}^{(k)}, \mathbf{z}^{(k)})^{-1} \nabla \tilde{\mathbf{J}}(\mathbf{x}^{(k)}, \mathbf{z}^{(k)})^T,$$

where μ_k is the step-size determined by a line-search procedure of lower computational cost. The PCG iteration method, incorporated with the constraint preconditioner $\mathbf{M}$ defined in (2.2), is applied to solve the sub-system of linear equations $\mathbf{H}(\mathbf{x}, \mathbf{z})\mathbf{d} = \mathbf{r}$. In general, the Newton method is more efficient than the alternating iteration method proposed in [13] when an accurate solution is desired.

In the Newton method, the initial vector $\mathbf{x}^{(\mathbf{0})}$ is set to be the observed image and $\mathbf{z}^{(\mathbf{0})}$ is set to be a constant vector, the iteration is stopped once the current residual satisfies

$$\frac{\|\nabla \tilde{\mathbf{J}}(\mathbf{x}^{(k)}, \mathbf{z}^{(k)})\|_2}{\|\nabla \tilde{\mathbf{J}}(\mathbf{x}^{(0)}, \mathbf{z}^{(0)})\|_2} \leq 10^{-6},$$

and the PCG iteration at the k-th Newton iterate is stopped once the current residual, say, $\mathbf{r}^{(k,\ell)}$, satisfies

$$\frac{\|\mathbf{r}^{(k,\ell)}\|_2}{\|\mathbf{r}^{(k,0)}\|_2} \leq 10^{-6},$$

where $\mathbf{r}^{(k,\ell)}$ represents the ℓ-th residual vector generated at the ℓ-th PCG iterate, with $\mathbf{r}^{(k,0)}$ the initial residual vector. In addition, all codes are written in MATLAB 7.01 and run on a personal computer with 1.86GHz central processing unit and 512M memory.

We show the restorations for the different degraded images with different blurs. The “Cameraman” and “Shapes” images are used in the experiments. The original “Cameraman” image is shown in Fig. 1 (left), the contaminated image blurred by an averaging function [15] and added a Gaussian white noise with the standard deviation 0.005 is shown in Fig. 2 (left), and the contaminated image blurred by a two-dimensional Gaussian function, say,

$$h(i,j) = e^{-2(i/3)^2 - 2(j/3)^2},$$

with size of 7×7 and added a Gaussian white noise with standard deviation 0.005 is shown in Fig. 3 (left).

In actual computations, we take

$$\tau = \mathbf{ave}(\mathbf{z}) := \frac{1}{r} \sum_{j=1}^{r} z_j$$

Figure 1: The original "Cameraman" image (left) and "Shapes" image (right).

Figure 2: The blurred and noisy image by average blur (left) and the restored image (right).

Figure 3: The blurred and noisy image by Gaussian blur (left) and the restored image (right).

to be the average value of the vector **z**, and

$$\mathbf{M}_{11} = 2\mathbf{A}^T\mathbf{A} + \beta\tau\mathbf{G}^T\mathbf{G}.$$

With this choice, we obtain

$$\mathbf{S} = 2\mathbf{A}^T\mathbf{A} + \eta\mathbf{G}^T\mathbf{G},$$

with

$$\eta = \beta\left(\tau - \beta \cdot \mathbf{ave}(\mathbf{Gx})\right).$$

Fig. 2 (right) shows the restored image using the regularization function ϕ_1 in (1.3). We see from this figure that the edge-preserving solution tends to sharpen the edges. The first two lines of Table 1 summarize the computing results corresponding to this image

Table 1: Numerical results for the restorations of images "Cameraman" and "Shapes".

		I		**M**		
Images	Blurs	IT	PCG	IT	PCG	CPU(s)
"Cameraman"	Average	11	> 1000	11	37	194.7
	Gaussian	11	> 1000	11	38	14.6
"Shapes"	Average	11	> 1000	11	51	172.5
	Gaussian	11	> 1000	11	53	171.9

restoration, where the column labelled with "**I**" represents the results without using a preconditioner, while the column labelled with "**M**" represents the results using the constraint preconditioner **M**. In addition, we use "IT" to denote the number of iteration steps of the Newton method, "PCG" the average number of iteration steps of the PCG method, and "CPU" the total computing time for the overall iteration process. From Table 1 we see that the average number of PCG iteration steps is greater than 1000 when no preconditioner is applied, and it is less than 40, however, when the constraint preconditioner is used. The computing efficiency can be improved up to about 25 times when using the constraint preconditioner. We remark that the image restoration effect and computing results of using the other regularization functions ϕ_2, ϕ_3 and ϕ_4 defined by (1.4)-(1.6), respectively, are about the same.

Next, we present experimental results for the original "Shapes" image shown in Fig. 1 (right). Fig. 5 (left) shows the contaminated image blurred by the averaging function and added a Gaussian white noise with the standard deviation 0.005, and Fig. 6 (left) shows the contaminated image by the Gaussian function and added a Gaussian white noise with standard deviation 0.005. Here, the averaging blur and the Gaussian blur added to image "Shapes" are the same as those added to the "Cameraman" image. Again, we take

$$\tau = \mathbf{ave}(\mathbf{z}) \quad \text{and} \quad \mathbf{M}_{11} = 2\mathbf{A}^T\mathbf{A} + \beta\tau\mathbf{G}^T\mathbf{G}.$$

The restored image using the edge-preserving regularization function ϕ_1 in (1.3) is shown in Figs. 5 (right) and 6 (right). The last two lines of Table 1 list the computing results corresponding to this image restoration. The computing efficiency for this example can be improved up to about 20 times when using the constraint preconditioner. Hence, we can conclude that the constraint preconditioner shows about the same efficiency for different image restorations degraded by different blurs.

Fig. 4 depicts the spectral distribution of the original matrix **H** and the preconditioned matrix $\mathbf{M}^{-1}\mathbf{H}$ when the Newton method is applied to restore the "Cameraman" image by averaging blur. This figure clearly shows that the matrices without preconditioning are very ill-conditioned and, therefore, the corresponding conjugate gradient method may be convergent slowly; the matrices with preconditioning are, however, well-conditioned as they have tightly clustered eigenvalues and, therefore, the corresponding PCG method converges faster.

Numerical implementations also indicate that using the other regularization functions ϕ_2, ϕ_3 and ϕ_4 defined by (1.4)-(1.6), respectively, and using different parameters α and β, lead to similar numerical results.

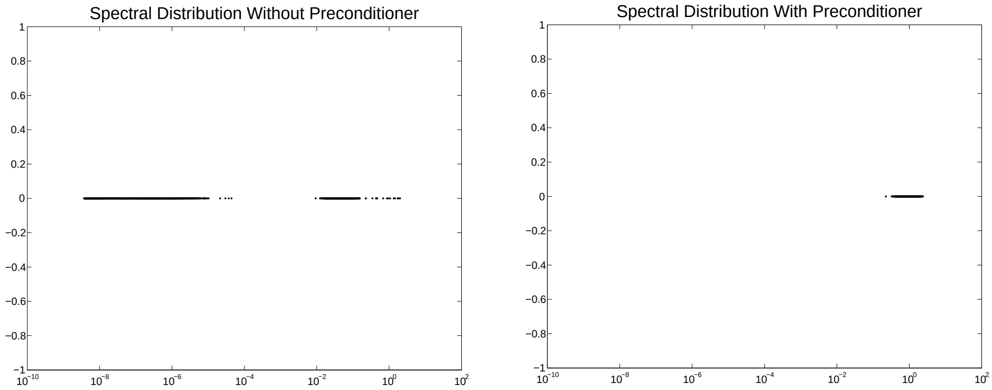

Figure 4: The spectral distributions without preconditioner (left) and with the constraint preconditioner (right) for the "Cameraman" restoration by averaging blur.

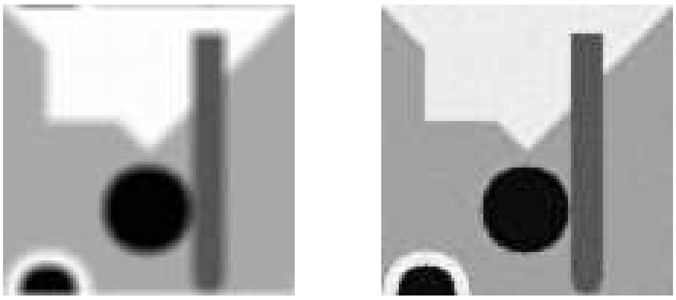

Figure 5: The blurred and noisy image by the average blur (left) and the restored image (right).

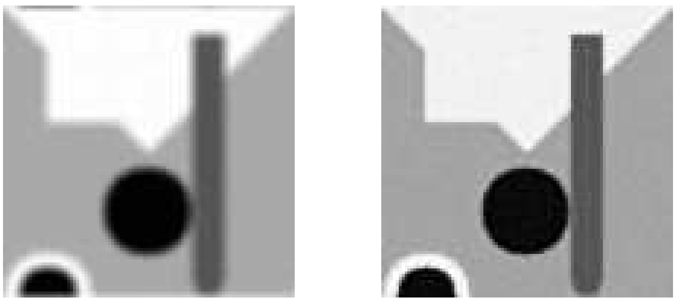

Figure 6: The blurred and noisy image by Gaussian blur (left) and the restored image (right).

5. Concluding remarks

We have discussed the multiplicative half-quadratic edge-preserving regularization techniques for image restoration problems and solved them by the Newton method incorporated with the preconditioned conjugate gradient process with the constraint preconditioner. Both theoretical analysis and experimental results have shown that this approach is more feasible and effective than the alternating iteration method studied in [13].

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