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Mixed Spectral and Pseudospectral Methods for a Nonlinear Strongly Damped Wave Equation in an Exterior Domain

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Abstract. The aim of this paper is to develop the mixed spectral and pseudospectral methods for nonlinear problems outside a disc, using Fourier and generalized Laguerre functions. As an example, we consider a nonlinear strongly damped wave equation. The mixed spectral and pseudospectral schemes are proposed. The convergence is proved. Numerical results demonstrate the efficiency of this approach.

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Key words: Mixed spectral and pseudospectral methods, exterior problems, strongly damped wave equation.

1. Introduction

We consider the initial and boundary value problem of a nonlinear strongly damped wave equation outside a disc, with smooth boundary conditions,

$$\begin{cases} \partial_t^2 w(\rho,\theta,t) - \partial_t \Delta w - \Delta w + \varphi(w) = F(\rho,\theta,t), & \rho > 1,\ \theta \in \bar{I},\ t \in (0,T], \\ w(\rho,\theta+2\pi,t) = w(\rho,\theta,t), & \rho > 1,\ \theta \in \bar{I},\ t \in [0,T], \\ w(\rho,\theta,0) = w_0(\rho,\theta), \quad \partial_t w(\rho,\theta,0) = w_1(\rho,\theta), & \rho \geq 1,\ \theta \in \bar{I}, \\ w(1,\theta,t) = 0, \qquad \lim\limits_{\rho\to\infty} \rho^{\frac{3}{2}} w(\rho,\theta,t) = 0, & \theta \in \bar{I},\ t \in [0,T], \end{cases} \tag{1.1}$$

where $I = (0, 2\pi)$ and the Laplacian:

$$\Delta w(\rho,\theta,t) = \frac{\partial^2 w(\rho,\theta,t)}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial w(\rho,\theta,t)}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 w(\rho,\theta,t)}{\partial \theta^2}.$$

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As pointed out in [15], the above model describes the transversal vibrations of a homogeneous string and the longitudinal vibrations of a homogeneous bar, respectively, subject to viscous effects. The term $-\Delta\partial_t U$ indicates that the stress is proportional not only to the strain, as with the Hooke law, but also to the strain rate as in a linearized Kelvin-Voigt material. Ikehata [11] presented some uniform energy decay estimates of solutions to the linear wave equations with strong dissipation in the exterior domain case.

In this paper, we shall focus on developing the mixed spectral and pseudospectral methods for numerical simulation of problem (1.1), by using Fourier and generalized Laguerre function. Let

$$\mathscr{L}_l^{(\alpha,\beta)}(x)=\frac{1}{l!}x^{-\alpha}e^{\beta x}\partial_x^l(x^{l+\alpha}e^{-\beta x}),\qquad \alpha>-1,\quad \beta>0,\quad l=0,1,\cdots,$$

be the generalized Laguerre polynomials, which are mutually orthogonal in $(0,\infty)$, associated with the weight function $x^\alpha e^{-\beta x}$. The generalized Laguerre polynomials have been used extensively for numerical simulations of various problems in unbounded and exterior domains, see [2, 3, 6-9, 12-14, 19].

The generalized Laguerre functions are defined by (cf. [10]):

$$\widetilde{\mathscr{L}}_l^{(\alpha,\beta)}(x)=e^{-\frac{1}{2}\beta x}\mathscr{L}_l^{(\alpha,\beta)}(x)=\frac{1}{l!}x^{-\alpha}e^{\frac{1}{2}\beta x}\partial_x^l(x^{l+\alpha}e^{-\beta x}),\qquad \alpha>-1,\quad \beta>0, \tag{1.2}$$

which are mutually orthogonal with the weight function x^α. The generalized Laguerre functions are very suitable for numerical simulation of various problems in exterior domains. For instance, Wang, Guo and Wu [16] developed a pseudospectral method for symmetric solutions of partial differential equations outside a disc, by using the generalized Laguerre functions (1.2). Meanwhile, Wang, Guo and Zhang [17] proposed a mixed spectral method for three-dimensional exterior problems using spherical harmonic and generalized Laguerre functions. Moreover, Zhang, Wang and Guo [18] also presented the mixed spectral and pseudospectral methods for linear problems outside a disc, using Fourier and generalized Laguerre functions. But, in practice, it is more interesting and more challenging to consider nonlinear exterior problems. The aim of this paper is to develop the mixed spectral and pseudospectral methods for two-dimensional nonlinear exterior problem (1.1), by using Fourier and generalized Laguerre function.

This paper is organized as follows. In Section 2, we recall some basic results of the mixed approximation using Fourier and generalized Laguerre functions. In Section 3, we propose the mixed spectral and pseudospectral methods for problem (1.1). In Section 4, we also present some numerical results demonstrating the high efficiency of these methods. The final section is for concluding remarks.

2. Mixed Fourier and generalized Laguerre approximations

2.1. The generalized Laguerre function approximation

We first recall some results on the generalized Laguerre function approximation. Let $\Lambda = (0, \infty)$ and $\chi(x)$ be a certain weight function in the usual sense. We define

$$L^2_\chi(\Lambda) = \Big\{ v \;\Big|\; v \text{ is measurable on } \Lambda \text{ and } \|v\|_{\chi,\Lambda} < \infty \Big\},$$

with the following inner product and norm,

$$(u,v)_{\chi,\Lambda} = \int_\Lambda u(x)v(x)\chi(x)dx, \quad \|v\|_{\chi,\Lambda} = (v,v)^{\frac{1}{2}}_{\chi,\Lambda}, \quad \forall u,v \in L^2_\chi(\Lambda).$$

For simplicity, we denote $\frac{d^k}{dx^k}v(x)$ by $\partial^k_x v(x)$, $k \geq 1$. For any integer $m \geq 0$,

$$H^m_\chi(\Lambda) = \Big\{ v \;\Big|\; \partial^k_x v \in L^2_\chi(\Lambda), \quad 0 \leq k \leq m \Big\},$$

with the following inner product, semi-norm and norm,

$$(u,v)_{m,\chi,\Lambda} = \sum_{k=0}^m (\partial^k_x u, \partial^k_x v)_{\chi,\Lambda}, \qquad |v|_{m,\chi,\Lambda} = \|\partial^m_x v\|_{\chi,\Lambda}, \qquad \|v\|_{m,\chi,\Lambda} = (v,v)^{\frac{1}{2}}_{m,\chi,\Lambda}.$$

For any $r > 0$, we define the space $H^r_\chi(\Lambda)$ and its norm $\|v\|_{r,\chi,\Lambda}$ by space interpolation as in [1]. For $\chi(x) \equiv 1$, we drop the subscript χ in the notations as usual. In particular,

$${}_0H^1_\chi(\Lambda) = \Big\{ v \;\Big|\; v \in H^1_\chi(\Lambda), \quad v(0) = 0 \Big\}.$$

The generalized Laguerre functions (1.2) satisfy the following recurrence relations (cf. [10,16,17]):

$$\partial_x \widetilde{\mathscr{L}}^{(\alpha,\beta)}_l(x) = -\beta \widetilde{\mathscr{L}}^{(\alpha+1,\beta)}_{l-1}(x) - \frac{1}{2}\beta \widetilde{\mathscr{L}}^{(\alpha,\beta)}_l(x), \tag{2.1}$$

$$(l+1)\widetilde{\mathscr{L}}^{(\alpha,\beta)}_{l+1}(x) + (\beta x - 2l - \alpha - 1)\widetilde{\mathscr{L}}^{(\alpha,\beta)}_l(x) + (l+\alpha)\widetilde{\mathscr{L}}^{(\alpha,\beta)}_{l-1}(x) = 0, \tag{2.2}$$

$$\partial_x \widetilde{\mathscr{L}}^{(\alpha,\beta)}_l(x) - \partial_x \widetilde{\mathscr{L}}^{(\alpha,\beta)}_{l+1}(x) = \frac{1}{2}\beta(\widetilde{\mathscr{L}}^{(\alpha,\beta)}_l(x) + \widetilde{\mathscr{L}}^{(\alpha,\beta)}_{l+1}(x)), \tag{2.3}$$

$$\widetilde{\mathscr{L}}^{(\alpha,\beta)}_l(x) = \widetilde{\mathscr{L}}^{(\alpha+1,\beta)}_l(x) - \widetilde{\mathscr{L}}^{(\alpha+1,\beta)}_{l-1}(x), \tag{2.4}$$

$$\begin{aligned} -x\partial_x \widetilde{\mathscr{L}}^{(\alpha,\beta)}_l(x) &= (l+\alpha)\widetilde{\mathscr{L}}^{(\alpha,\beta)}_{l-1}(x) + (\frac{1}{2}\beta x - l)\widetilde{\mathscr{L}}^{(\alpha,\beta)}_l(x) \\ &= \frac{1}{2}(l+\alpha)\widetilde{\mathscr{L}}^{(\alpha,\beta)}_{l-1}(x) + \frac{1}{2}(\alpha+1)\widetilde{\mathscr{L}}^{(\alpha,\beta)}_l(x) - \frac{1}{2}(l+1)\widetilde{\mathscr{L}}^{(\alpha,\beta)}_{l+1}(x). \end{aligned} \tag{2.5}$$

Let $\widetilde{\omega}_\alpha = x^\alpha$. The set of $\widetilde{\mathscr{L}}^{(\alpha,\beta)}_l(x)$ is a complete $L^2_{\widetilde{\omega}_\alpha}(\Lambda)$-orthogonal system, namely,

$$(\widetilde{\mathscr{L}}^{(\alpha,\beta)}_l, \widetilde{\mathscr{L}}^{(\alpha,\beta)}_m)_{\widetilde{\omega}_\alpha,\Lambda} = \gamma^{(\alpha,\beta)}_l \delta_{l,m}, \tag{2.6}$$

where δ_{lm} is the Kronecker symbol, and $\gamma_l^{(\alpha,\beta)}=\frac{\Gamma(l+\alpha+1)}{\beta^{\alpha+1}l!}$. Hence, for any $v\in L^2_{\widetilde{\omega}_\alpha}(\Lambda)$,

$$v(x)=\sum_{l=0}^{\infty}\widetilde{v}_l^{(\alpha,\beta)}\widetilde{\mathscr{L}}_l^{(\alpha,\beta)}(x),\qquad \widetilde{v}_l^{(\alpha,\beta)}=\frac{1}{\gamma_l^{(\alpha,\beta)}}(v,\widetilde{\mathscr{L}}_l^{(\alpha,\beta)})_{\widetilde{\omega}_\alpha,\Lambda}. \tag{2.7}$$

Next, let N be any positive integer and $\mathscr{P}_N(\Lambda)$ be the set of all algebraic polynomials of degree at most N. Denote by

$$\mathscr{Q}_{N,\beta}(\Lambda)=\left\{\, e^{-\frac{1}{2}\beta x}\psi \;\middle|\; \psi\in\mathscr{P}_N(\Lambda)\right\},\qquad {}_0\mathscr{Q}_{N,\beta}(\Lambda)=\left\{\, v\in\mathscr{Q}_{N,\beta}(\Lambda)\;\middle|\; v(0)=0\right\}.$$

For description of approximation results, we introduce the non-uniformly weighted Sobolev space $\widetilde{A}^r_{\alpha,\beta}(\Lambda)$,

$$\widetilde{A}^r_{\alpha,\beta}(\Lambda)=\left\{\, v \;\middle|\; v \text{ is measurable on } \Lambda \text{ and } \|v\|_{\widetilde{A}^r_{\alpha,\beta},\Lambda}<\infty\right\},$$

equipped with the following semi-norm and norm,

$$|v|_{\widetilde{A}^r_{\alpha,\beta},\Lambda}=\left(\int_\Lambda (\partial_x^r(e^{\frac{1}{2}\beta x}v(x)))^2x^{\alpha+r}e^{-\beta x}dx\right)^{\frac{1}{2}},\quad \|v\|_{\widetilde{A}^r_{\alpha,\beta},\Lambda}=\left(\sum_{k=0}^{r}|v|^2_{\widetilde{A}^k_{\alpha,\beta},\Lambda}\right)^{\frac{1}{2}}.$$

For any $r>0$, the space $\widetilde{A}^r_{\alpha,\beta}(\Lambda)$ is defined by space interpolation as in [1].

We next recall some results on the generalized Laguerre-Gauss-type interpolation. Let $\widetilde{\xi}^{(\alpha,\beta)}_{G,N,j}$ and $\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j}$, $0\le j\le N$, be the zeros of the functions $\widetilde{\mathscr{L}}^{(\alpha,\beta)}_{N+1}(x)$ and $x\partial_x(e^{\frac{1}{2}\beta x}\widetilde{\mathscr{L}}^{(\alpha,\beta)}_{N+1}(x))$, respectively. They are arranged in ascending order. Denote by $\widetilde{\omega}^{(\alpha,\beta)}_{Z,N,j}$, $0\le j\le N$, $Z=G,R$, the corresponding Christoffel numbers such that for any $\phi\in\mathscr{Q}_{m,\beta}(\Lambda),\psi\in\mathscr{Q}_{n,\beta}(\Lambda)$ and $m+n\le 2N+\lambda_Z$ (cf. [16]),

$$(\phi,\psi)_{\widetilde{\omega}_\alpha,\Lambda}=\sum_{j=0}^{N}\phi(\widetilde{\xi}^{(\alpha,\beta)}_{Z,N,j})\psi(\widetilde{\xi}^{(\alpha,\beta)}_{Z,N,j})\widetilde{\omega}^{(\alpha,\beta)}_{Z,N,j}, \tag{2.8}$$

where $\lambda_G=1$ and $\lambda_R=0$.

We introduce the following discrete inner product and norm,

$$(u,v)_{Z,N,\alpha,\beta,\Lambda}=\sum_{j=0}^{N}u(\widetilde{\xi}^{(\alpha,\beta)}_{Z,N,j})v(\widetilde{\xi}^{(\alpha,\beta)}_{Z,N,j})\widetilde{\omega}^{(\alpha,\beta)}_{Z,N,j},\quad \|v\|_{Z,N,\alpha,\beta,\Lambda}=(v,v)^{\frac{1}{2}}_{Z,N,\alpha,\beta,\Lambda},\quad Z=G,R.$$

Thanks to the exactness of (2.8), for any $\phi\in\mathscr{Q}_{m,\beta}(\Lambda)$, $\psi\in\mathscr{Q}_{n,\beta}(\Lambda)$ and $m+n\le 2N+\lambda_Z$,

$$(\phi,\psi)_{Z,N,\alpha,\beta,\Lambda}=(\phi,\psi)_{\widetilde{\omega}_\alpha,\Lambda},\quad Z=G,R. \tag{2.9}$$

In particular, $\|\phi\|_{Z,N,\alpha,\beta,\Lambda}=\|\phi\|_{\widetilde{\omega}_\alpha,\Lambda}$ for any $\phi\in\mathscr{Q}_{N,\beta}(\Lambda)$, $Z=G,R$.

The generalized Laguerre-Gauss-type interpolation $\widetilde{\mathscr{I}}_{Z,N,\alpha,\beta}v \in \mathscr{Q}_{N,\beta}(\Lambda)$ is determined uniquely by

$$\widetilde{\mathscr{I}}_{Z,N,\alpha,\beta}v(\widetilde{\xi}^{(\alpha,\beta)}_{Z,N,j}) = v(\widetilde{\xi}^{(\alpha,\beta)}_{Z,N,j}), \quad 0 \le j \le N, \quad Z = G, R.$$

$\widetilde{\mathscr{I}}_{G,N,\alpha,\beta}v$ and $\widetilde{\mathscr{I}}_{R,N,\alpha,\beta}v$ are called as the generalized Laguerre-Gauss interpolation and the generalized Laguerre-Gauss-Radau interpolation, respectively. In the sequel, we denote by c a generic positive constant.

Lemma 2.1. (*see Lemma* 3.2 *of* [16]). *For any $\phi \in \mathscr{Q}_{N,\beta}(\Lambda)$ and $r \ge 0$,*

$$\|\phi\|_{r,\widetilde{\omega}_\alpha,\Lambda} \le c(\beta N)^r \|\phi\|_{\widetilde{\omega}_\alpha,\Lambda}.$$

2.2. The Fourier approximation

We next recall some results on the Fourier approximation. Let $H^r(I)$ be the Sobolev space with norm $\|\cdot\|_{r,I}$ and semi-norm $|\cdot|_{r,I}$. For any non-negative integer m, $H^m_p(I)$ denotes the subspace of $H^m(I)$, consisting of all functions whose derivatives of order up to $m-1$ are periodic with the period 2π. For any $r > 0$, the space $H^r_p(I)$ is defined by space interpolation as in [1]. In particular, $L^2_p(I) = H^0_p(I)$. Let M be any positive integer, and $\widetilde{V}_M(I) = \text{span}\{\ e^{il\theta} \mid |l| \le M\}$. We denote by $V_M(I)$ the subset of $\widetilde{V}_M(I)$ consisting of all real-valued functions.

Let $\theta_{M,k} = \frac{2\pi k}{2M+1}, k = 0, 1, \cdots, 2M$. For any $v \in C(\bar{I})$, the interpolant $\mathscr{I}_M v \in \widetilde{V}_M(I)$ is defined by

$$\mathscr{I}_M v(\theta_{M,k}) = v(\theta_{M,k}), \quad 0 \le k \le 2M.$$

We define the discrete inner product and the norm as

$$(u,v)_{M,I} = \frac{2\pi}{2M+1}\sum_{k=0}^{2M} u(\theta_{M,k})\bar{v}(\theta_{M,k}), \qquad \|v\|_{M,I} = (v,v)^{\frac{1}{2}}_{M,I}.$$

Then

$$(\phi,\psi)_{M,I} = (\phi,\psi)_I, \qquad \forall \phi,\psi \in \widetilde{V}_M(I). \tag{2.10}$$

Lemma 2.2. (*see Theorem* 5 *of* [5]). *For any $\phi \in \widetilde{V}_M(I)$ and $r \ge 0$,*

$$\|\phi\|_{r,I} \le cM^r \|\phi\|_I.$$

2.3. The mixed approximation using Fourier and generalized Laguerre functions

We now recall the results on the mixed approximation using Fourier and generalized Laguerre functions. For this purpose, let $\Omega = I \times \Lambda$, $\widetilde{\eta}(x) = (x+1)^2$ and define the spaces

$${}_0H^1_p(\Omega) = \Big\{\ v \in H^1(\Lambda, L^2(I)) \ \Big|\ v(x,\theta+2\pi) = v(x,\theta) \text{ and } v(0,\theta) = 0\Big\},$$

and

$${}_0H^1_{p,\widetilde{\eta}}(\Omega)=\Big\{\ v\ \Big|\ v\in {}_0H^1_p(\Omega) \text{ and } \|v\|_{1,\widetilde{\eta},\Omega}<\infty\Big\},$$

with the semi-norm and norm

$$|v|_{1,\widetilde{\eta},\Omega}=(\|\partial_x v\|^2_{L^2_{\widetilde{\eta}}(\Lambda,L^2(I))}+\|\partial_\theta v\|^2_{L^2(\Lambda,L^2(I))})^{\frac{1}{2}},\qquad \|v\|_{1,\widetilde{\eta},\Omega}=(|v|^2_{1,\widetilde{\eta},\Omega}+\|v\|^2_{L^2_{\widetilde{\eta}}(\Lambda,L^2(I))})^{\frac{1}{2}}.$$

Moreover

$$(u,v)_{\chi,\Omega}=\int_\Omega u(x,\theta)v(x,\theta)\chi(x)dxd\theta,\quad \|v\|_{\chi,\Omega}=(v,v)^{\frac{1}{2}}_{\chi,\Omega}.$$

For $\chi(x)\equiv 1$, we drop the subscript χ in the notations for simplicity.

Now let $V_{M,N,\beta}(\Omega)=V_M(I)\otimes\mathcal{Q}_{N,\beta}(\Lambda)$ and ${}_0V_{M,N,\beta}(\Omega)=V_M(I)\otimes{}_0\mathcal{Q}_{N,\beta}(\Lambda)$. For $\mu>0$, the orthogonal projection ${}_0\widetilde{P}^1_{M,N,\mu,\beta}$: ${}_0H^1_{p,\widetilde{\eta}}(\Omega)\to{}_0V_{M,N,\beta}(\Omega)$ is defined by

$$\begin{aligned}(\partial_x({}_0\widetilde{P}^1_{M,N,\mu,\beta}v-v),\partial_x\phi)_{\widetilde{\eta},\Omega}&+(\partial_\theta({}_0\widetilde{P}^1_{M,N,\mu,\beta}v-v),\partial_\theta\phi)_\Omega\\&+\mu({}_0\widetilde{P}^1_{M,N,\mu,\beta}v-v,\phi)_{\widetilde{\eta},\Omega}=0,\qquad\forall\phi\in{}_0V_{M,N,\beta}(\Omega).\end{aligned}\tag{2.11}$$

For simplicity of statements, we introduce the non-isotropic space

$$\widetilde{\mathcal{B}}^{r,s}_\beta(\Omega)=L^2(\Lambda,H^s_p(I))\cap H^1_{\widetilde{\eta}}(\Lambda,H^{s-1}_p(I))\cap\widetilde{A}^r_{0,\beta}(\Lambda,H^1_p(I))$$

equipped with the norm

$$\|v\|_{\widetilde{\mathcal{B}}^{r,s}_\beta(\Omega)}=\Big(\|v\|^2_{L^2(\Lambda,H^s(I))}+\|v\|^2_{H^1_{\widetilde{\eta}}(\Lambda,H^{s-1}(I))}+\|v\|^2_{\widetilde{A}^r_{0,\beta}(\Lambda,H^1(I))}\Big)^{\frac{1}{2}}.$$

For description of approximation error, we shall use the following notation with integers $r\geq 2$ and $s\geq 1$,

$$\widetilde{\mathcal{D}}^{r,s}_\beta(v)=|v|_{L^2(\Lambda,H^s(I))}+|v|_{L^2_{\widetilde{\eta}}(\Lambda,H^{s-1}(I))}+|v|_{H^1_{\widetilde{\eta}}(\Lambda,H^{s-1}(I))}+|v|_{\widetilde{A}^r_{0,\beta}(\Lambda,L^2(I))}+|v|_{\widetilde{A}^r_{0,\beta}(\Lambda,H^1(I))}.$$

Lemma 2.3. (*see Theorem* 2.1 *of* [18]). *For any* $v\in\widetilde{\mathcal{B}}^{r,s}_\beta(\Omega)\cap{}_0H^1_{p,\widetilde{\eta}}(\Omega)$*, integers* $r\geq 2$ *and* $s\geq 1$*,*

$$\|v-{}_0\widetilde{P}^1_{M,N,\mu,\beta}v\|_{1,\widetilde{\eta},\Omega}\leq c(M^{1-s}+(\beta+\tfrac{1}{\beta})\beta^{-\frac{r}{2}}N^{1-\frac{r}{2}})\widetilde{\mathcal{D}}^{r,s}_\beta(v).\tag{2.12}$$

We now turn to the mixed interpolation by using the generalized Laguerre functions. We introduce the following discrete inner product and norm,

$$\begin{aligned}(u,v)_{Z,M,N,\alpha,\beta,\Omega}&=\frac{2\pi}{2M+1}\sum_{j=0}^{N}\sum_{k=0}^{2M}u(\widetilde{\xi}^{(\alpha,\beta)}_{Z,N,j},\theta_{M,k})v(\widetilde{\xi}^{(\alpha,\beta)}_{Z,N,j},\theta_{M,k})\widetilde{\omega}^{(\alpha,\beta)}_{Z,N,j},\\ \|v\|_{Z,M,N,\alpha,\beta,\Omega}&=(v,v)^{\frac{1}{2}}_{Z,M,N,\alpha,\beta,\Omega},\qquad Z=G,R.\end{aligned}$$

By virtue of (2.9) and (2.10), we observe that for any $\phi\in V_{m,n,\beta}(\Omega)$, $\psi\in V_{p,q,\beta}(\Omega)$, with $m,\ p\leq M$ and $n+q\leq 2N+\lambda_Z$,

$$(\phi,\psi)_{Z,M,N,\alpha,\beta,\Omega}=(\phi,\psi)_{\widetilde{\omega}_\alpha,\Omega}.\tag{2.13}$$

The mixed interpolation $\widetilde{\mathscr{I}}_{Z,M,N,\alpha,\beta}v \in V_{M,N,\beta}(\Omega)$ is determined uniquely by

$$\widetilde{\mathscr{I}}_{Z,M,N,\alpha,\beta}v(\widetilde{\xi}^{(\alpha,\beta)}_{Z,N,j},\theta_{M,k}) = v(\widetilde{\xi}^{(\alpha,\beta)}_{Z,N,j},\theta_{M,k}), \qquad 0 \le j \le N, \quad 0 \le k \le 2M, \quad Z = G,R.$$

Obviously, $\widetilde{\mathscr{I}}_{Z,M,N,\alpha,\beta}v(x,\theta) = \widetilde{\mathscr{I}}_{Z,N,\alpha,\beta}\mathscr{I}_M v(x,\theta)$. Moreover, by (2.25) of [18], for any $v \in C(\overline{\Omega})$ with $v(0,\theta) = 0$,

$$\widetilde{\mathscr{I}}_{R,M,N,\alpha,\beta}v(x,\theta) = x\widetilde{\mathscr{I}}_{G,M,N-1,\alpha+1,\beta}(x^{-1}v(x,\theta)). \tag{2.14}$$

For simplicity of statement, we shall use the notation

$$\begin{aligned}\widetilde{E}^r_{\alpha,\beta}(v) =&\beta^{-1}|v|_{\widetilde{A}^r_{\alpha-1,\beta}(\Lambda,L^2(I))} + \beta^{-1}M^{-1}|v|_{\widetilde{A}^r_{\alpha-1,\beta}(\Lambda,H^1(I))}\\ &+(1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}}|v|_{\widetilde{A}^r_{\alpha,\beta}(\Lambda,L^2(I))} + (1+\beta^{-\frac{1}{2}})M^{-1}(\ln N)^{\frac{1}{2}}|v|_{\widetilde{A}^r_{\alpha,\beta}(\Lambda,H^1(I))}.\end{aligned}$$

Lemma 2.4. (*see Theorem* 2.2 *of* [18]). *For any $v \in L^2_{\widetilde{\omega}_\alpha}(\Lambda, H^s_p(I)) \cap \widetilde{A}^r_{\alpha-1,\beta}(\Lambda, H^1(I)) \cap \widetilde{A}^r_{\alpha,\beta}(\Lambda, H^1(I))$, integer $r \ge 1$ and $s > \frac{1}{2}$,*

$$\|v - \widetilde{\mathscr{I}}_{G,M,N,\alpha,\beta}v\|_{L^2_{\widetilde{\omega}_\alpha}(\Lambda,L^2(I))} \le c(\beta N)^{\frac{1}{2}-\frac{r}{2}}\widetilde{E}^r_{\alpha,\beta}(v) + cM^{-s}|v|_{L^2_{\widetilde{\omega}_\alpha}(\Lambda,H^s(I))}. \tag{2.15}$$

Lemma 2.5. (*see Theorem* 2.3 *of* [18]). *For any $v \in L^2_{\widetilde{\omega}_\alpha}(\Lambda, H^s_p(I)) \cap \widetilde{A}^r_{\alpha-1,\beta}(\Lambda, H^1(I)) \cap \widetilde{A}^r_{\alpha,\beta}(\Lambda, H^1(I))$, integer $r \ge 1$, $r > \alpha + 1$ and $s > \frac{1}{2}$,*

$$\|v - \widetilde{\mathscr{I}}_{R,M,N,\alpha,\beta}v\|_{L^2_{\widetilde{\omega}_\alpha}(\Lambda,L^2(I))} \le c(\beta N)^{\frac{1}{2}-\frac{r}{2}}\widetilde{E}^r_{\alpha,\beta}(v) + cM^{-s}|v|_{L^2_{\widetilde{\omega}_\alpha}(\Lambda,H^s(I))}. \tag{2.16}$$

If, in addition, $|\alpha| < 1$, then the above result holds for all integer $r \ge 1$.

The following two lemmas will be used in the error analysis of the mixed pseudospectral method.

Lemma 2.6. *For any $\phi(\cdot,\theta) \in V_M(I)$, $\phi(0,\theta) = 0$ and $\psi(x,\cdot) \in {}_0\mathscr{Q}_{N,\beta}(\Lambda)$, we have that*

$$\begin{aligned}&\|\phi\psi\|^2_{R,M,N,2,\beta,\Omega} \le c\big(\|\phi\|^2_{\widetilde{\omega}_1,\Omega} + \|\phi\|_{\widetilde{\omega}_2,\Omega}\|\partial_x\phi\|_{\widetilde{\omega}_2,\Omega}\big)\big(\|\psi\|_\Omega\|\partial_\theta\psi\|_\Omega + \|\psi\|^2_\Omega\big),\\ &\|\phi\psi\|^2_{R,M,N,1,\beta,\Omega} \le c\big(\|\phi\|^2_\Omega + \|\phi\|_{\widetilde{\omega}_1,\Omega}\|\partial_x\phi\|_{\widetilde{\omega}_1,\Omega}\big)\big(\|\psi\|_\Omega\|\partial_\theta\psi\|_\Omega + \|\psi\|^2_\Omega\big),\\ &\|\phi\psi\|^2_{R,M,N,0,\beta,\Omega} \le c\|\phi\|_\Omega\|\partial_x\phi\|_\Omega\big(\|\psi\|_\Omega\|\partial_\theta\psi\|_\Omega + \|\psi\|^2_\Omega\big).\end{aligned}$$

In particular,

$$\|\phi^2\|^2_{R,M,N,2,\beta,\Omega} + \|\phi^2\|^2_{R,M,N,1,\beta,\Omega} + \|\phi^2\|^2_{R,M,N,0,\beta,\Omega} \le c\|\phi\|^4_{1,\widetilde{\eta},\Omega}.$$

Proof. It is clear that for $\alpha \geq 0$,

$$
\begin{aligned}
(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j})^{\alpha}\phi^2(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j},\theta_{M,k}) =& 2\int_0^{\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j}} x^{\frac{\alpha}{2}}\phi(x,\theta_{M,k})\partial_x(x^{\frac{\alpha}{2}}\phi(x,\theta_{M,k}))dx \\
\leq& 2\int_\Lambda |x^{\frac{\alpha}{2}}\phi(x,\theta_{M,k})\partial_x(x^{\frac{\alpha}{2}}\phi(x,\theta_{M,k}))|dx,
\end{aligned} \tag{2.17}
$$

and

$$
\begin{aligned}
\psi^2(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j},\theta_{M,k}) =& 2\int_0^{\theta_{M,k}} \psi(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j},\theta)\partial_\theta\psi(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j},\theta)d\theta + \psi^2(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j},0) \\
\leq& 2\int_I |\psi(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j},\theta)\partial_\theta\psi(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j},\theta)|d\theta + \psi^2(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j},0).
\end{aligned} \tag{2.18}
$$

On the other hand, by the trace theorem,

$$
\begin{aligned}
\psi^2(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j},0) \leq& c\|\psi(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j},\cdot)\|^2_{H^{\frac{1}{2}}(I)} \leq c\|\psi(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j},\cdot)\|_{H^1(I)}\|\psi(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j},\cdot)\|_{L^2(I)} \\
\leq& c\|\partial_\theta\psi(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j},\cdot)\|_{L^2(I)}\|\psi(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j},\cdot)\|_{L^2(I)} + c\|\psi(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j},\cdot)\|^2_{L^2(I)}.
\end{aligned} \tag{2.19}
$$

Therefore, by (2.17)-(2.19), for $\alpha \geq 0$,

$$
\begin{aligned}
&\|\phi\psi\|^2_{R,M,N,\alpha,\beta,\Omega} \\
=&\frac{2\pi}{2M+1}\sum_{j=0}^{N}\sum_{k=0}^{2M}(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j})^{\alpha}\phi^2(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j},\theta_{M,k})(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j})^{-\alpha}\psi^2(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j},\theta_{M,k})\widetilde{\omega}^{(\alpha,\beta)}_{R,N,j} \\
\leq& c\int_\Lambda \frac{2\pi}{2M+1}\sum_{k=0}^{2M}|x^{\frac{\alpha}{2}}\phi(x,\theta_{M,k})\partial_x(x^{\frac{\alpha}{2}}\phi(x,\theta_{M,k}))|dx \\
&\times\sum_{j=0}^{N}(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j})^{-\alpha}\Big(\int_I|\psi(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j},\theta)\partial_\theta\psi(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j},\theta)|d\theta \\
&+\|\partial_\theta\psi(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j},\cdot)\|_{L^2(I)}\|\psi(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j},\cdot)\|_{L^2(I)}+\|\psi(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j},\cdot)\|^2_{L^2(I)}\Big)\widetilde{\omega}^{(\alpha,\beta)}_{R,N,j}.
\end{aligned}
$$

Since $\psi(x,\cdot) \in {}_0\mathcal{Q}_{N,\beta}(\Lambda)$, we have that $x^{-1}\psi(x,\cdot) \in \mathcal{Q}_{N-1,\beta}(\Lambda)$. Hence, for $\alpha = 0$, 1, 2, we derive from (2.9), (2.10) and the Cauchy-Schwatz inequality that

$$
\begin{aligned}
&\|\phi\psi\|^2_{R,M,N,\alpha,\beta,\Omega} \\
\leq& c\int_\Lambda\int_I|x^{\frac{\alpha}{2}}\phi(x,\theta)\partial_x(x^{\frac{\alpha}{2}}\phi(x,\theta))|d\theta dx\Big(\int_\Lambda\int_I|\psi(x,\theta)\partial_\theta\psi(x,\theta)|d\theta dx \\
&+(\int_\Lambda\int_I(\partial_\theta\psi(x,\theta))^2d\theta dx)^{\frac{1}{2}}(\int_\Lambda\int_I\psi^2(x,\theta)d\theta dx)^{\frac{1}{2}}+\int_\Lambda\int_I\psi^2(x,\theta)d\theta dx\Big)
\end{aligned}
$$

$$\leq \begin{cases} c(\|\phi\|^2_{\tilde{\omega}_1,\Omega} + \|\phi\|_{\tilde{\omega}_2,\Omega}\|\partial_x\phi\|_{\tilde{\omega}_2,\Omega})(\|\psi\|_\Omega\|\partial_\theta\psi\|_\Omega + \|\psi\|^2_\Omega), & \text{for} \quad \alpha = 2, \\ c(\|\phi\|^2_\Omega + \|\phi\|_{\tilde{\omega}_1,\Omega}\|\partial_x\phi\|_{\tilde{\omega}_1,\Omega})(\|\psi\|_\Omega\|\partial_\theta\psi\|_\Omega + \|\psi\|^2_\Omega), & \text{for} \quad \alpha = 1, \\ c\|\phi\|_\Omega\|\partial_x\phi\|_\Omega(\|\psi\|_\Omega\|\partial_\theta\psi\|_\Omega + \|\psi\|^2_\Omega), & \text{for} \quad \alpha = 0. \end{cases}$$

□

Next let $\Lambda_1 = (0,1)$, $\Lambda_2 = [1,\infty)$, $\Omega_1 = I \times \Lambda_1$ and $\Omega_2 = I \times \Lambda_2$.

Lemma 2.7. *For any $\phi \in {}_0V_{M,N,\beta}(\Omega)$ and $\varepsilon > 0$, we have that*

$$\|x\phi\|_{L^\infty(\Omega_1)} \leq c(\beta N + M^2)^\varepsilon \|\phi\|_{1,\tilde{\eta},\Omega},$$
$$\|\phi\|_{L^\infty(\Omega_2)} \leq c(\beta N + M)^\varepsilon \|\phi\|_{1,\tilde{\eta},\Omega}.$$

Proof. By the imbedding theorem and the Gagliardo-Nirenberg-type inequality, we deduce that

$$\begin{aligned}\|x\phi\|_{L^\infty(\Omega_1)} \leq& c\|x\phi\|_{H^{1+\varepsilon}(\Omega_1)} \leq c\|x\phi\|^{1-\varepsilon}_{H^1(\Omega_1)}\|x\phi\|^\varepsilon_{H^2(\Omega_1)} \\ =& c\|x\phi\|^{1-\varepsilon}_{H^1(\Omega_1)}\Big(\|x\phi\|^2_{H^1(\Omega_1)} + |x\phi|^2_{H^2(\Omega_1)}\Big)^{\frac{\varepsilon}{2}}. \end{aligned} \tag{2.20}$$

Similarly

$$\|\phi\|_{L^\infty(\Omega_2)} \leq c\|\phi\|^{1-\varepsilon}_{H^1(\Omega_2)}\Big(\|\phi\|^2_{H^1(\Omega_2)} + |\phi|^2_{H^2(\Omega_2)}\Big)^{\frac{\varepsilon}{2}}. \tag{2.21}$$

Next let ξ and η be the Cartesian coordinates, $\xi = x\cos\theta$ and $\eta = x\sin\theta$. Then by a direct calculation, we derive that

$$\begin{aligned}\|x\phi\|^2_{H^1(\Omega_1)} =& \int_{\Omega_1}\Big((\partial_x(x\phi(x,\theta))\frac{\partial x}{\partial\xi} + \partial_\theta(x\phi(x,\theta))\frac{\partial\theta}{\partial\xi})^2 + (\partial_x(x\phi(x,\theta))\frac{\partial x}{\partial\eta} \\ &+ \partial_\theta(x\phi(x,\theta))\frac{\partial\theta}{\partial\eta})^2 + x^2\phi^2(x,\theta)\Big)xdxd\theta \\ \leq& c\int_{\Omega_1}((\partial_x\phi(x,\theta))^2x^3 + (\partial_\theta\phi(x,\theta))^2x + \phi^2(x,\theta)x)dxd\theta \\ \leq& c\int_{\Omega_1}((\partial_x\phi(x,\theta))^2x^2 + (\partial_\theta\phi(x,\theta))^2 + \phi^2(x,\theta)x)dxd\theta \\ \leq& c\|\phi\|^2_{1,\tilde{\eta},\Omega}. \end{aligned} \tag{2.22}$$

Similarly

$$\begin{aligned}\|\phi\|^2_{H^1(\Omega_2)} \leq& c\int_{\Omega_2}((\partial_x\phi(x,\theta))^2x + x^{-1}(\partial_\theta\phi(x,\theta))^2 + \phi^2(x,\theta)x)dxd\theta \\ \leq& c\int_{\Omega_2}((\partial_x\phi(x,\theta))^2x + (\partial_\theta\phi(x,\theta))^2 + \phi^2(x,\theta)x)dxd\theta \\ \leq& c\|\phi\|^2_{1,\tilde{\eta},\Omega}. \end{aligned} \tag{2.23}$$

Moreover, by the Hardy inequality and Lemmas 2.1 and 2.2, a direct calculation leads to that for any $\phi \in {}_0V_{M,N,\beta}(\Omega)$,

$$
\begin{aligned}
|x\phi|^2_{H^2(\Omega_1)} \le & c\int_{\Omega_1}\Big((\partial_x^2\phi(x,\theta))^2x^3+(\partial_x\phi(x,\theta))^2x+(\partial_x\partial_\theta\phi(x,\theta))^2x \\
& \quad + x^{-1}(\partial_\theta^2\phi(x,\theta))^2+x^{-1}(\partial_\theta\phi(x,\theta))^2+x^{-1}\phi^2(x,\theta)\Big)dxd\theta \\
\le & c\int_{\Omega}\Big((\partial_x^2\phi(x,\theta))^2x^2+(\partial_x\phi(x,\theta))^2x+(\partial_x\partial_\theta\phi(x,\theta))^2 \\
& \quad + x^{-2}(\partial_\theta^2\phi(x,\theta))^2+x^{-2}(\partial_\theta\phi(x,\theta))^2+x^{-2}\phi^2(x,\theta)\Big)dxd\theta \\
\le & c\int_{\Omega}\Big((\beta N)^2(\partial_x\phi(x,\theta))^2x^2+(\partial_x\phi(x,\theta))^2x+(\beta N)^2(\partial_\theta\phi(x,\theta))^2 \\
& \quad + M^4x^{-2}\phi^2(x,\theta)+M^2x^{-2}\phi^2(x,\theta)+x^{-2}\phi^2(x,\theta)\Big)dxd\theta \\
\le & c\int_{\Omega}\Big((\beta N)^2(\partial_x\phi(x,\theta))^2x^2+(\partial_x\phi(x,\theta))^2x+(\beta N)^2(\partial_\theta\phi(x,\theta))^2 \\
& \quad + M^4(\partial_x\phi(x,\theta))^2+M^2(\partial_x\phi(x,\theta))^2+(\partial_x\phi(x,\theta))^2\Big)dxd\theta \\
\le & c(\beta N+M^2)^2\|\phi\|^2_{1,\widetilde{\eta},\Omega},
\end{aligned} \tag{2.24}
$$

and

$$
\begin{aligned}
|\phi|^2_{H^2(\Omega_2)} \le & c\int_{\Omega_2}\Big((\partial_x^2\phi(x,\theta))^2x+x^{-1}(\partial_x\partial_\theta\phi(x,\theta))^2 \\
& \quad + x^{-1}(\partial_x\phi(x,\theta))^2+x^{-3}(\partial_\theta^2\phi(x,\theta))^2+x^{-3}(\partial_\theta\phi(x,\theta))^2\Big)dxd\theta \\
\le & c\int_{\Omega}\Big((\partial_x^2\phi(x,\theta))^2x+(\partial_x\partial_\theta\phi(x,\theta))^2 \\
& \quad + (\partial_x\phi(x,\theta))^2+(\partial_\theta^2\phi(x,\theta))^2+(\partial_\theta\phi(x,\theta))^2\Big)dxd\theta \\
\le & c\int_{\Omega}\Big((\beta N)^2(\partial_x\phi(x,\theta))^2x+(\beta N)^2(\partial_\theta\phi(x,\theta))^2 \\
& \quad + (\partial_x\phi(x,\theta))^2+M^2(\partial_\theta\phi(x,\theta))^2+(\partial_\theta\phi(x,\theta))^2\Big)dxd\theta \\
\le & c(\beta N+M)^2\|\phi\|^2_{1,\widetilde{\eta},\Omega}.
\end{aligned} \tag{2.25}
$$

A combination of (2.20)-(2.25) leads to the desired results.

3. Spectral and pseudospectral methods for problem (1.1)

In this section, we shall construct the mixed spectral and pseudospectral methods for exterior problem (1.1). For simplicity, let $\varphi(w)=w+w^2$.

We make the variable transformation

$$\rho = x+1, \quad U(x,\theta,t) = w(\rho,\theta,t), \quad U_0(x,\theta) = w_0(\rho,\theta),$$
$$U_1(x,\theta) = w_1(\rho,\theta), \quad f(x,\theta,t) = \rho^2 F(\rho,\theta,t).$$

Then (1.1) is changed to

$$\begin{cases} (x+1)^2\partial_t^2 U - (x+1)^2\partial_t\partial_x^2 U - (x+1)\partial_t\partial_x U - \partial_t\partial_\theta^2 U - (x+1)^2\partial_x^2 U, \\ -(x+1)\partial_x U - \partial_\theta^2 U + (x+1)^2 U + (x+1)^2 U^2 = f, & x>0,\ \theta\in\bar{I},\ t\in(0,T], \\ U(x,\theta+2\pi,t) = U(x,\theta,t), & x>0,\ \theta\in\bar{I},\ t\in[0,T], \\ U(x,\theta,0) = U_0(x,\theta), \quad \partial_t U(x,\theta,0) = U_1(x,\theta), & x\ge 0,\ \theta\in\bar{I}, \\ U(0,\theta,t) = 0, \quad \lim\limits_{x\to\infty} x^{\frac{3}{2}} U(x,\theta,t) = 0, & \theta\in\bar{I},\ t\in[0,T]. \end{cases} \tag{3.1}$$

Let $\widetilde{\Omega} = \{\,(y,z) \mid y^2+z^2>1\}$. For any function $u(y,z)$ defined on $\widetilde{\Omega}$, we set $v(x,\theta) = (x+1)^{-\frac{1}{p}} u((x+1)\cos\theta,(x+1)\sin\theta)$ defined on Ω. By a direct calculation, we can prove that the norm $\|u\|_{L^p(\widetilde{\Omega})}$ is equivalent to the norm $\|v\|_{L^p_{\widetilde{\eta}}(\Lambda,L^p(I))}$, and for $p\ge 2$, $\|u\|_{H^1(\widetilde{\Omega})} \le \|v\|_{1,\widetilde{\eta},\Omega}$. Therefore, by the imbedding theory,

$$\|v\|_{L^p_{\widetilde{\eta}}(\Lambda,L^p(I))} \le c\|v\|_{1,\widetilde{\eta},\Omega}, \qquad 2\le p<\infty. \tag{3.2}$$

Let $\widetilde{\xi}(x) = x+1$. By multiplying (3.1) by $v(x,\theta,t)$ and integrating the resulting equation by parts, we derive a weak formulation. It is to find $U(t)\in {}_0H^1_{p,\widetilde{\eta}}(\Omega)$ such that

$$(\partial_t^2 U, v)_{\widetilde{\eta},\Omega} + (\partial_t\partial_x U, \partial_x v)_{\widetilde{\eta},\Omega} + (\partial_t\partial_x U, v)_{\widetilde{\xi},\Omega} + (\partial_t\partial_\theta U, \partial_\theta v)_\Omega + (\partial_x U, \partial_x v)_{\widetilde{\eta},\Omega}$$
$$+ (\partial_x U, v)_{\widetilde{\xi},\Omega} + (\partial_\theta U, \partial_\theta v)_\Omega + (U,v)_{\widetilde{\eta},\Omega} + (U^2,v)_{\widetilde{\eta},\Omega} = (f,v)_\Omega, \tag{3.3a}$$
$$U(x,\theta,0) = U_0(x,\theta), \qquad \partial_t U(x,\theta,0) = U_1(x,\theta). \tag{3.3b}$$

3.1. Mixed spectral scheme

The mixed spectral scheme for (3.3) is to seek $u_{M,N}(x,\theta,t)\in {}_0V_{M,N,\beta}(\Omega)$ such that

$$(\partial_t^2 u_{M,N}, \phi)_{\widetilde{\eta},\Omega} + (\partial_t\partial_x u_{M,N}, \partial_x\phi)_{\widetilde{\eta},\Omega} + (\partial_t\partial_x u_{M,N}, \phi)_{\widetilde{\xi},\Omega} + (\partial_t\partial_\theta u_{M,N}, \partial_\theta\phi)_\Omega$$
$$+ (\partial_x u_{M,N}, \partial_x\phi)_{\widetilde{\eta},\Omega} + (\partial_x u_{M,N}, \phi)_{\widetilde{\xi},\Omega} + (\partial_\theta u_{M,N}, \partial_\theta\phi)_\Omega + (u_{M,N},\phi)_{\widetilde{\eta},\Omega}$$
$$+ (u_{M,N}^2,\phi)_{\widetilde{\eta},\Omega} = (f,\phi)_\Omega, \qquad \forall\phi\in {}_0V_{M,N,\beta}(\Omega), \tag{3.4a}$$
$$u_{M,N}(x,\theta,0) = {}_0\widetilde{P}^1_{M,N,1,\beta} U_0(x,\theta), \qquad \partial_t u_{M,N}(x,\theta,0) = {}_0\widetilde{P}^1_{M,N,1,\beta} U_1(x,\theta). \tag{3.4b}$$

We now analyze the numerical error of scheme (3.4). Let $U_{M,N} = {}_0\widetilde{P}^1_{M,N,1,\beta}U$. According to the definition of ${}_0\widetilde{P}^1_{M,N,1,\beta}U$ in (2.11), we obtain from (3.3) that

$$(\partial_t^2 U_{M,N}, \phi)_{\widetilde{\eta},\Omega} + (\partial_t\partial_x U_{M,N}, \partial_x\phi)_{\widetilde{\eta},\Omega} + (\partial_t\partial_x U_{M,N}, \phi)_{\widetilde{\xi},\Omega} + (\partial_t\partial_\theta U_{M,N}, \partial_\theta\phi)_\Omega$$
$$+ (\partial_x U_{M,N}, \partial_x\phi)_{\widetilde{\eta},\Omega} + (\partial_x U_{M,N}, \phi)_{\widetilde{\xi},\Omega} + (\partial_\theta U_{M,N}, \partial_\theta\phi)_\Omega + (U_{M,N},\phi)_{\widetilde{\eta},\Omega}$$
$$+ (U_{M,N}^2,\phi)_{\widetilde{\eta},\Omega} + \sum_{j=1}^{5} G_j(t,\phi) = (f,\phi)_\Omega, \qquad \forall\phi\in {}_0V_{M,N,\beta}(\Omega), \tag{3.5}$$

where

$$G_1(t,\phi)=(\partial_t^2(U(t)-U_{M,N}(t)),\phi)_{\widetilde{\eta},\Omega},\qquad G_2(t,\phi)=(\partial_t(U_{M,N}(t)-U(t)),\phi)_{\widetilde{\eta},\Omega},$$
$$G_3(t,\phi)=(\partial_t\partial_x(U(t)-U_{M,N}(t)),\phi)_{\widetilde{\xi},\Omega},\qquad G_4(t,\phi)=(\partial_x(U(t)-U_{M,N}(t)),\phi)_{\widetilde{\xi},\Omega},$$
$$G_5(t,\phi)=(U^2(t)-U_{M,N}^2(t),\phi)_{\widetilde{\eta},\Omega}.$$

Further, let $\widetilde{U}_{M,N}(t)=u_{M,N}(t)-U_{M,N}(t)$. Subtracting (3.5) from (3.4) yields

$$\begin{aligned}
&(\partial_t^2\widetilde{U}_{M,N},\phi)_{\widetilde{\eta},\Omega}+(\partial_t\partial_x\widetilde{U}_{M,N},\partial_x\phi)_{\widetilde{\eta},\Omega}+(\partial_t\partial_x\widetilde{U}_{M,N},\phi)_{\widetilde{\xi},\Omega}+(\partial_t\partial_\theta\widetilde{U}_{M,N},\partial_\theta\phi)_\Omega\\
&\quad+(\partial_x\widetilde{U}_{M,N},\partial_x\phi)_{\widetilde{\eta},\Omega}+(\partial_x\widetilde{U}_{M,N},\phi)_{\widetilde{\xi},\Omega}+(\partial_\theta\widetilde{U}_{M,N},\partial_\theta\phi)_\Omega+(\widetilde{U}_{M,N},\phi)_{\widetilde{\eta},\Omega}\\
=&\sum_{j=1}^{6}G_j(t,\phi),\qquad \forall\phi\in{}_0V_{M,N,\beta}(\Omega),
\end{aligned}\tag{3.6}$$

where

$$G_6(t,\phi)=(U_{M,N}^2(t)-u_{M,N}^2(t),\phi)_{\widetilde{\eta},\Omega}.$$

Taking $\phi=2\partial_t\widetilde{U}_{M,N}(t)$ in (3.6), we obtain that

$$\begin{aligned}
&\partial_t\|\partial_t\widetilde{U}_{M,N}\|_{\widetilde{\eta},\Omega}^2+2|\partial_t\widetilde{U}_{M,N}|_{1,\widetilde{\eta},\Omega}^2+\partial_t\|\widetilde{U}_{M,N}\|_{1,\widetilde{\eta},\Omega}^2\\
\le&2\sum_{j=1}^{6}|G_j(t,\partial_t\widetilde{U}_{M,N})|+2|(\partial_t\partial_x\widetilde{U}_{M,N},\partial_t\widetilde{U}_{M,N})_{\widetilde{\xi},\Omega}|+2|(\partial_x\widetilde{U}_{M,N},\partial_t\widetilde{U}_{M,N})_{\widetilde{\xi},\Omega}|.
\end{aligned}\tag{3.7}$$

Obviously, by virtue of Lemma 2.3, we deduce that

$$\begin{aligned}
&2\sum_{j=1}^{4}|G_j(t,\partial_t\widetilde{U}_{M,N})|\\
\le&\|\partial_t\widetilde{U}_{M,N}\|_{\widetilde{\eta},\Omega}^2+c\left(M^{1-s}+\left(\beta+\frac{1}{\beta}\right)\beta^{-\frac{r}{2}}N^{1-\frac{r}{2}}\right)^2\sum_{j=0}^{2}(\widetilde{\mathscr{D}}_\beta^{r,s}(\partial_t^jU))^2.
\end{aligned}\tag{3.8}$$

Further, let $\|U(t)\|_\infty=\operatorname{ess\,sup}_{(x,\theta)\in\Omega}|U(x,\theta,t)|$. Then we use Lemma 2.3 and imbedding inequality (3.2) to derive that

$$\begin{aligned}
&\left|G_5(t,\partial_t\widetilde{U}_{M,N})\right|\\
=&\left|2\int_\Omega(U-U_{M,N})U\partial_t\widetilde{U}_{M,N}\widetilde{\eta}(x)dxd\theta-\int_\Omega(U-U_{M,N})^2\partial_t\widetilde{U}_{M,N}\widetilde{\eta}(x)dxd\theta\right|\\
\le&\|\partial_t\widetilde{U}_{M,N}\|_{\widetilde{\eta},\Omega}^2+c\|U\|_\infty^2\|U-U_{M,N}\|_{\widetilde{\eta},\Omega}^2+c\|U-U_{M,N}\|_{L^4_{\widetilde{\eta}}(\Lambda,L^4(I))}^4\\
\le&\|\partial_t\widetilde{U}_{M,N}\|_{\widetilde{\eta},\Omega}^2+c\|U\|_\infty^2\|U-U_{M,N}\|_{\widetilde{\eta},\Omega}^2+c\|U-U_{M,N}\|_{1,\widetilde{\eta},\Omega}^4\\
\le&\|\partial_t\widetilde{U}_{M,N}\|_{\widetilde{\eta},\Omega}^2+c\|U\|_\infty^2\left(M^{1-s}+\left(\beta+\frac{1}{\beta}\right)\beta^{-\frac{r}{2}}N^{1-\frac{r}{2}}\right)^2(\widetilde{\mathscr{D}}_\beta^{r,s}(U))^2\\
&\quad+c\left(M^{1-s}+\left(\beta+\frac{1}{\beta}\right)\beta^{-\frac{r}{2}}N^{1-\frac{r}{2}}\right)^4(\widetilde{\mathscr{D}}_\beta^{r,s}(U))^4.
\end{aligned}\tag{3.9}$$

Similarly, we get that for certain constant $c^* > 0$,

$$
\begin{aligned}
&\left|G_6(t,\partial_t\widetilde{U}_{M,N})\right| = \left|((U_{M,N}+u_{M,N})\widetilde{U}_{M,N},\partial_t\widetilde{U}_{M,N})_{\widetilde{\eta},\Omega}\right| \\
=&\left|(\widetilde{U}^2_{M,N},\partial_t\widetilde{U}_{M,N})_{\widetilde{\eta},\Omega}+2(U\widetilde{U}_{M,N},\partial_t\widetilde{U}_{M,N})_{\widetilde{\eta},\Omega}+2((U_{M,N}-U)\widetilde{U}_{M,N},\partial_t\widetilde{U}_{M,N})_{\widetilde{\eta},\Omega}\right| \\
\le&\|\partial_t\widetilde{U}_{M,N}\|_{\widetilde{\eta},\Omega}\|\widetilde{U}_{M,N}\|^2_{L^4_{\widetilde{\eta}}(\Lambda,L^4(I))}+\|U\|^2_\infty\|\widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega}+2\|\partial_t\widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega} \\
&\quad+\|U-U_{M,N}\|^2_{L^4_{\widetilde{\eta}}(\Lambda,L^4(I))}\|\widetilde{U}_{M,N}\|^2_{L^4_{\widetilde{\eta}}(\Lambda,L^4(I))} \\
\le&c^*\|\partial_t\widetilde{U}_{M,N}\|_{\widetilde{\eta},\Omega}\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega}+\|U\|^2_\infty\|\widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega}+2\|\partial_t\widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega} \\
&\quad+c\|U-U_{M,N}\|^2_{1,\widetilde{\eta},\Omega}\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega} \\
\le&c^*\|\partial_t\widetilde{U}_{M,N}\|_{\widetilde{\eta},\Omega}\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega}+\|U(t)\|^2_\infty\|\widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega}+2\|\partial_t\widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega} \\
&\quad+c\left(M^{1-s}+\left(\beta+\frac{1}{\beta}\right)\beta^{\frac{r}{2}}N^{1-\frac{r}{2}}\right)^2(\widetilde{\mathscr{D}}^{r,s}_\beta(U))^2\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega}. \qquad (3.10)
\end{aligned}
$$

If $\widetilde{D}^{r,s}_\beta(U(t))$ is bounded above, then for larger N and M, and $r>2$, $s>1$,

$$
\begin{aligned}
|G_6(t,\partial_t\widetilde{U}_{M,N})|\le\, &c^*\|\partial_t\widetilde{U}_{M,N}\|_{\widetilde{\eta},\Omega}\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega}+\|U\|^2_\infty\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega} \\
&+2\|\partial_t\widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega}+\frac{1}{2}\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega}. \qquad (3.11)
\end{aligned}
$$

Moreover, it clear that

$$2|(\partial_t\partial_x\widetilde{U}_{M,N},\partial_t\widetilde{U}_{M,N})_{\widetilde{\xi},\Omega}|\le|\partial_t\widetilde{U}_{M,N}|^2_{1,\widetilde{\eta},\Omega}+\|\partial_t\widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega}, \qquad (3.12)$$

$$2|(\partial_x\widetilde{U}_{M,N},\partial_t\widetilde{U}_{M,N})_{\widetilde{\xi},\Omega}|\le|\widetilde{U}_{M,N}|^2_{1,\widetilde{\eta},\Omega}+\|\partial_t\widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega}. \qquad (3.13)$$

Therefore, by substituting (3.8)-(3.9) and (3.11)-(3.13) into (3.7), we derive that

$$
\begin{aligned}
&\partial_t\|\partial_t\widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega}+|\partial_t\widetilde{U}_{M,N}|^2_{1,\widetilde{\eta},\Omega}+\partial_t\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega} \\
\le&9\|\partial_t\widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega}+(2\|U\|^2_\infty+2)\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega}+2c^*\|\partial_t\widetilde{U}_{M,N}\|_{\widetilde{\eta},\Omega}\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega} \\
&\quad+c\left(M^{1-s}+\left(\beta+\frac{1}{\beta}\right)\beta^{\frac{r}{2}}N^{1-\frac{r}{2}}\right)^2\widetilde{\mathscr{F}}^{r,s}_\beta(U), \qquad (3.14)
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{\mathscr{F}}^{r,s}_\beta(U)=&(\widetilde{\mathscr{D}}^{r,s}_\beta(\partial_t^2U))^2+(\widetilde{\mathscr{D}}^{r,s}_\beta(\partial_tU))^2+(\|U\|^2_\infty+1)(\widetilde{\mathscr{D}}^{r,s}_\beta(U))^2 \\
&+\left(M^{1-s}+\left(\beta+\frac{1}{\beta}\right)\beta^{-\frac{r}{2}}N^{1-\frac{r}{2}}\right)^2(\widetilde{\mathscr{D}}^{r,s}_\beta(U))^4.
\end{aligned}
$$

Integrating (3.14) with respect to t, we deduce that

$$\begin{aligned}
&\|\partial_t \widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega} + \|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega} + \int_0^t |\partial_\xi \widetilde{U}_{M,N}(\xi)|^2_{1,\widetilde{\eta},\Omega} d\xi \\
&\quad + \int_0^t (1 - 2c^* \|\partial_\xi \widetilde{U}_{M,N}(\xi)\|_{\widetilde{\eta},\Omega}) \|\widetilde{U}_{M,N}(\xi)\|^2_{1,\widetilde{\eta},\Omega} d\xi \\
\le & 9 \int_0^t \|\partial_\xi \widetilde{U}_{M,N}(\xi)\|^2_{\widetilde{\eta},\Omega} d\xi + \int_0^t (2\|U(\xi)\|^2_\infty + 3) \|\widetilde{U}_{M,N}(\xi)\|^2_{1,\widetilde{\eta},\Omega} d\xi \\
&\quad + c\left(M^{1-s} + \left(\beta + \frac{1}{\beta}\right)\beta^{-\frac{r}{2}} N^{1-\frac{r}{2}}\right)^2 \int_0^t \widetilde{\mathscr{F}}^{r,s}_\beta (U(\xi)) d\xi. \qquad (3.15)
\end{aligned}$$

We shall use the following lemma.

Lemma 3.1. (*See Lemma* 3.1 *of* [4]). *Assume that*

- *the constants* $b_1 > 0$, $b_2 \ge 0$, $b_3 \ge 0$ *and* $d \ge 0$,
- $Z(t)$ *and* $A(t)$ *are non-negative functions of* t,
- $d \le \frac{b_1^2}{b_2^2} e^{-b_3 t_1}$ *for certain* $t_1 > 0$,
- *for all* $t \le t_1$,

$$Z(t) + \int_0^t (b_1 - b_2 Z^{\frac{1}{2}}(\eta)) A(\eta) d\eta \le d + b_3 \int_0^t Z(\eta) d\eta.$$

Then for all $t \le t_1$, *we have* $Z(t) \le d e^{b_3 t}$.

We now take in Lemma 3.1,

$$\begin{aligned}
&Z(t) = \|\partial_t \widetilde{U}_{M,N}(t)\|^2_{\widetilde{\eta},\Omega} + \|\widetilde{U}_{M,N}(t)\|^2_{1,\widetilde{\eta},\Omega} + \int_0^t |\partial_\xi \widetilde{U}_{M,N}(\xi)|^2_{1,\widetilde{\eta},\Omega} d\xi, \\
&A(t) = \|\widetilde{U}_{M,N}(t)\|^2_{1,\widetilde{\eta},\Omega}, \quad d = c\left(M^{1-s} + \left(\beta + \frac{1}{\beta}\right)\beta^{-\frac{r}{2}} N^{1-\frac{r}{2}}\right)^2 \int_0^T \widetilde{\mathscr{F}}^{r,s}_\beta (U(t)) dt, \\
&b_1 = 1, \quad b_2 = 2c^*, \quad b_3 = 2|||U|||^2_\infty + 9, \quad |||U|||_\infty = \sup_{0\le t\le T} \|U(t)\|_\infty. \qquad (3.16)
\end{aligned}$$

Moreover, if $r > 2$, $s > 1$ and the norms mentioned in d are bounded above, then $d \to 0$ as $M, N \to \infty$. Therefore, applying Lemma 3.1 to (3.15), we obtain that for any $0 \le t \le T$,

$$\begin{aligned}
&\|\partial_t \widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega} + \|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega} + \int_0^t |\partial_\xi \widetilde{U}_{M,N}(\xi)|^2_{1,\widetilde{\eta},\Omega} d\xi \\
\le & c\left(M^{2-2s} + \left(\beta + \frac{1}{\beta}\right)^2 \beta^{-r} N^{2-r}\right) e^{(2|||U|||^2_\infty + 9)T} \int_0^T \widetilde{\mathscr{F}}^{r,s}_\beta (U) dt. \qquad (3.17)
\end{aligned}$$

Finally, a combination of (3.17) and Lemma 2.3 leads to the following conclusion.

Theorem 3.1. *If the norms mentioned in the notation d of* (3.16) *are bounded uniformly for* $0 \le t \le T$*, integer* $r > 2$*,* $s > 1$*, then for suitably larger* M *and* N *and any* $0 \le t \le T$*,*

$$
\begin{aligned}
&\|\partial_t(U-u_{M,N})\|^2_{\widetilde{\eta},\Omega} + \|U-u_{M,N}\|^2_{1,\widetilde{\eta},\Omega} + \int_0^t |\partial_\xi(U(\xi)-u_{M,N}(\xi))|^2_{1,\widetilde{\eta},\Omega} d\xi \\
\le & c\left(M^{2-2s} + \left(\beta+\frac{1}{\beta}\right)^2 \beta^{-r} N^{2-r}\right) \\
&\times \left(e^{(2|||U|||^2_\infty+9)T} \int_0^T \widetilde{\mathscr{F}}^{r,s}_\beta(U)dt + \sup_{0\le t\le T} \sum_{j=0}^{1} (\widetilde{\mathscr{D}}^{r,s}_\beta(\partial_t^j U))^2\right).
\end{aligned}
$$

3.2. Mixed pseudospectral scheme

The mixed pseudospectral scheme for (3.3) is to seek $u_{M,N}(x,\theta,t) \in {}_0V_{M,N,\beta}(\Omega)$ such that

$$
\begin{cases}
(\partial_t^2 u_{M,N},\phi)_{\widetilde{\eta},\Omega} + (\partial_t\partial_x u_{M,N},\partial_x\phi)_{\widetilde{\eta},\Omega} + (\partial_t\partial_x u_{M,N},\phi)_{\widetilde{\xi},\Omega} + (\partial_t\partial_\theta u_{M,N},\partial_\theta\phi)_\Omega \\
\quad +(\partial_x u_{M,N},\partial_x\phi)_{\widetilde{\eta},\Omega} + (\partial_x u_{M,N},\phi)_{\widetilde{\xi},\Omega} + (\partial_\theta u_{M,N},\partial_\theta\phi)_\Omega + (u_{M,N},\phi)_{\widetilde{\eta},\Omega} \\
\quad +(u^2_{M,N},\phi)_{R,M,N,2,\beta,\Omega} + 2(u^2_{M,N},\phi)_{R,M,N,1,\beta,\Omega} + (u^2_{M,N},\phi)_{R,M,N,0,\beta,\Omega} \\
= (f,\phi)_{R,M,N,0,\beta,\Omega}, \qquad \forall\phi \in {}_0V_{M,N,\beta}(\Omega), \\
u_{M,N}(x,\theta,0) = {}_0\widetilde{P}^1_{M,N,1,\beta}U_0(x,\theta), \quad \partial_t u_{M,N}(x,\theta,0) = {}_0\widetilde{P}^1_{M,N,1,\beta}U_1(x,\theta).
\end{cases}
\tag{3.18}
$$

We now analyze the numerical error of scheme (3.18). Let $U_{M,N} = {}_0\widetilde{P}^1_{M,N,1,\beta}U$. According to the definition of ${}_0\widetilde{P}^1_{M,N,1,\beta}U$ in (2.11), we obtain from (3.3) that

$$
\begin{aligned}
&(\partial_t^2 U_{M,N},\phi)_{\widetilde{\eta},\Omega} + (\partial_t\partial_x U_{M,N},\partial_x\phi)_{\widetilde{\eta},\Omega} + (\partial_t\partial_x U_{M,N},\phi)_{\widetilde{\xi},\Omega} + (\partial_t\partial_\theta U_{M,N},\partial_\theta\phi)_\Omega \\
&\quad + (\partial_x U_{M,N},\partial_x\phi)_{\widetilde{\eta},\Omega} + (\partial_x U_{M,N},\phi)_{\widetilde{\xi},\Omega} + (\partial_\theta U_{M,N},\partial_\theta\phi)_\Omega + (U_{M,N},\phi)_{\widetilde{\eta},\Omega} \\
&\quad + (U^2_{M,N},\phi)_{R,M,N,2,\beta,\Omega} + 2(U^2_{M,N},\phi)_{R,M,N,1,\beta,\Omega} + (U^2_{M,N},\phi)_{R,M,N,0,\beta,\Omega} \\
&\quad + \sum_{j=1}^{8} G_j(t,\phi) = (f,\phi)_{R,M,N,0,\beta,\Omega}, \qquad \forall\phi \in {}_0V_{M,N,\beta}(\Omega),
\end{aligned}
\tag{3.19}
$$

where

$$
\begin{aligned}
&G_1(t,\phi) = (\partial_t^2(U-U_{M,N}),\phi)_{\widetilde{\eta},\Omega}, && G_5(t,\phi) = (U^2,\phi)_{\widetilde{\omega}_2,\Omega} - (U^2_{M,N},\phi)_{R,M,N,2,\beta,\Omega}, \\
&G_2(t,\phi) = (\partial_t(U_{M,N}-U),\phi)_{\widetilde{\eta},\Omega}, && G_6(t,\phi) = 2(U^2,\phi)_{\widetilde{\omega}_1,\Omega} - 2(U^2_{M,N},\phi)_{R,M,N,1,\beta,\Omega}, \\
&G_3(t,\phi) = (\partial_t\partial_x(U-U_{M,N}),\phi)_{\widetilde{\xi},\Omega}, && G_7(t,\phi) = (U^2,\phi)_\Omega - (U^2_{M,N},\phi)_{R,M,N,0,\beta,\Omega}, \\
&G_4(t,\phi) = (\partial_x(U-U_{M,N}),\phi)_{\widetilde{\xi},\Omega}, && G_8(t,\phi) = (f,\phi)_{R,M,N,0,\beta,\Omega} - (f,\phi)_\Omega.
\end{aligned}
$$

Further, let $\widetilde{U}_{M,N}(t)=u_{M,N}(t)-U_{M,N}(t)$. Subtracting (3.22) from (3.18) yields

$$\begin{aligned}&(\partial_t^2\widetilde{U}_{M,N},\phi)_{\widetilde{\eta},\Omega}+(\partial_t\partial_x\widetilde{U}_{M,N},\partial_x\phi)_{\widetilde{\eta},\Omega}+(\partial_t\partial_x\widetilde{U}_{M,N},\phi)_{\widetilde{\xi},\Omega}+(\partial_t\partial_\theta\widetilde{U}_{M,N},\partial_\theta\phi)_{\Omega}\\&\quad+(\partial_x\widetilde{U}_{M,N},\partial_x\phi)_{\widetilde{\eta},\Omega}+(\partial_x\widetilde{U}_{M,N},\phi)_{\widetilde{\xi},\Omega}+(\partial_\theta\widetilde{U}_{M,N},\partial_\theta\phi)_{\Omega}+(\widetilde{U}_{M,N},\phi)_{\widetilde{\eta},\Omega}\\&=\sum_{j=1}^{11}G_j(t,\phi),\qquad\forall\phi\in{}_0V_{M,N,\beta}(\Omega),\end{aligned}\tag{3.20}$$

where

$$\begin{aligned}&G_9(t,\phi)=(U_{M,N}^2-u_{M,N}^2,\phi)_{R,M,N,2,\beta,\Omega},\qquad G_{10}(t,\phi)=2(U_{M,N}^2-u_{M,N}^2,\phi)_{R,M,N,1,\beta,\Omega},\\&G_{11}(t,\phi)=(U_{M,N}^2-u_{M,N}^2,\phi)_{R,M,N,0,\beta,\Omega}.\end{aligned}$$

Taking $\phi=2\partial_t\widetilde{U}_{M,N}(t)$ in (3.20), we obtain that

$$\begin{aligned}&\partial_t\|\partial_t\widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega}+2|\partial_t\widetilde{U}_{M,N}|^2_{1,\widetilde{\eta},\Omega}+\partial_t\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega}\\&\le2\sum_{j=1}^{11}\left|G_j(t,\partial_t\widetilde{U}_{M,N})\right|+2\left|(\partial_t\partial_x\widetilde{U}_{M,N},\partial_t\widetilde{U}_{M,N})_{\widetilde{\xi},\Omega}\right|+2\left|(\partial_x\widetilde{U}_{M,N},\partial_t\widetilde{U}_{M,N})_{\widetilde{\xi},\Omega}\right|.\end{aligned}\tag{3.21}$$

Obviously, by virtue of Lemma 2.3, we deduce that

$$\begin{aligned}&2\sum_{j=1}^{4}|G_j(t,\partial_t\widetilde{U}_{M,N})|\\&\le\|\partial_t\widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega}+c\left(M^{2-2s}+\left(\beta+\frac{1}{\beta}\right)^2\beta^{-r}N^{2-r}\right)\sum_{j=0}^{2}(\widetilde{\mathscr{D}}_\beta^{r,s}(\partial_t^jU))^2.\end{aligned}\tag{3.22}$$

Moreover, we have from (2.13) that

$$\begin{aligned}\left|G_j(t,\partial_t\widetilde{U}_{M,N})\right|\le&\left|(U^2-\widetilde{\mathscr{I}}_{R,M,N,\alpha,\beta}U^2,\partial_t\widetilde{U}_{M,N})_{\widetilde{\omega}_\alpha,\Omega}\right|\\&+\left|(\widetilde{\mathscr{I}}_{R,M,N,\alpha,\beta}U^2-U_{M,N}^2,\partial_t\widetilde{U}_{M,N})_{R,M,N,\alpha,\beta,\Omega}\right|,\qquad j=5,6,7,\end{aligned}$$

where $\alpha=2$ for $j=5$, $\alpha=1$ for $j=6$ and $\alpha=0$ for $j=7$. Next, according to Lemma 2.5, it is easy to derive that

$$\begin{aligned}&\left|(U^2-\widetilde{\mathscr{I}}_{R,M,N,\alpha,\beta}U^2,\partial_t\widetilde{U}_{M,N})_{\widetilde{\omega}_\alpha,\Omega}\right|\\&\le\|\partial_t\widetilde{U}_{M,N}\|^2_{\widetilde{\omega}_\alpha,\Omega}+\|U^2-\widetilde{\mathscr{I}}_{R,M,N,\alpha,\beta}U^2\|^2_{\widetilde{\omega}_\alpha,\Omega}\\&\le\|\partial_t\widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega}+c(\beta N)^{1-r}(\widetilde{E}^r_{\alpha,\beta}(U^2))^2+cM^{-2s}|U^2|^2_{L^2_{\widetilde{\omega}_\alpha}(\Lambda,H^s(I))}.\end{aligned}\tag{3.23}$$

Further, by (2.13) and (2.14),

$$
\begin{aligned}
&\left|(\widetilde{\mathscr{I}}_{R,M,N,\alpha,\beta}U^2 - U_{M,N}^2, \partial_t \widetilde{U}_{M,N})_{R,M,N,\alpha,\beta,\Omega}\right| \\
=&\left|((\widetilde{\mathscr{I}}_{R,M,N,\alpha,\beta}U + U_{M,N})(\widetilde{\mathscr{I}}_{R,M,N,\alpha,\beta}U - U_{M,N}), \partial_t \widetilde{U}_{M,N})_{R,M,N,\alpha,\beta,\Omega}\right| \\
\leq&\frac{1}{4}\|\partial_t \widetilde{U}_{M,N}\|^2_{\widetilde{\omega}_\alpha,\Omega} + c\|U\|^2_\infty \|\widetilde{\mathscr{I}}_{R,M,N,\alpha,\beta}U - U_{M,N}\|^2_{\widetilde{\omega}_\alpha,\Omega} \\
&+ \frac{2\pi}{2M+1}\left(\sum_{\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j}\geq 1} + \sum_{\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j}<1}\right)\sum_{k=0}^{2M}|U_{M,N}(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j}, \theta_{M,k}, t)(\widetilde{\mathscr{I}}_{R,M,N,\alpha,\beta}U(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j}, \theta_{M,k}, t) \\
&- U_{M,N}(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j}, \theta_{M,k}, t))\partial_t \widetilde{U}_{M,N}(\widetilde{\xi}^{(\alpha,\beta)}_{R,N,j}, \theta_{M,k}, t)\widetilde{\omega}^{(\alpha,\beta)}_{R,N,j}| \\
\leq&\frac{1}{2}\|\partial_t \widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega} + c(\|U\|^2_\infty + \|U_{M,N}\|^2_{L^\infty(\Omega_2)})\|\widetilde{\mathscr{I}}_{R,M,N,\alpha,\beta}U - U_{M,N}\|^2_{\widetilde{\omega}_\alpha,\Omega} \\
&+ \frac{1}{2}\|\partial_t \widetilde{U}_{M,N}\|^2_{\widetilde{\omega}_{\alpha-1},\Omega} + c\|xU_{M,N}\|^2_{L^\infty(\Omega_1)}\|\widetilde{\mathscr{I}}_{R,M,N,\alpha,\beta}U - U_{M,N}\|^2_{\widetilde{\omega}_{\alpha-1},\Omega} \\
\leq&\|\partial_t \widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega} + c(\|U\|^2_\infty + \|U_{M,N}\|^2_{L^\infty(\Omega_2)})\|\widetilde{\mathscr{I}}_{R,M,N,\alpha,\beta}U - U_{M,N}\|^2_{\widetilde{\omega}_\alpha,\Omega} \\
&+ c\|xU_{M,N}\|^2_{L^\infty(\Omega_1)}\left(\|\widetilde{\mathscr{I}}_{G,M,N-1,\alpha+1,\beta}(x^{-1}U) - x^{-1}U\|^2_{\widetilde{\omega}_{\alpha+1}} + \|U - U_{M,N}\|^2_{\widetilde{\eta},\Omega}\right).
\end{aligned}
$$

Thus by Lemmas 2.3-2.5 and 2.7, we get that for integers $r > 3$, $s > 1$ and $\alpha = 1,\ 2$,

$$
\begin{aligned}
&|(\widetilde{\mathscr{I}}_{R,M,N,\alpha,\beta}U^2 - U_{M,N}^2, \partial_t \widetilde{U}_{M,N})_{R,M,N,\alpha,\beta,\Omega}| \\
\leq&\|\partial_t \widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega} + c\left(\|U\|^2_\infty + (\beta N + M)^{2\varepsilon}\|U_{M,N}\|^2_{1,\widetilde{\eta},\Omega}\right)\bigg((\beta N)^{1-r}(\widetilde{E}^r_{\alpha,\beta}(U))^2 \\
&+ \left(M^{2-2s} + (\beta + \frac{1}{\beta})^2\beta^{-r}N^{2-r})(\widetilde{\mathscr{D}}^{r,s}_\beta(U)\right)^2\bigg) + c(\beta N + M^2)^{2\varepsilon}\|U_{M,N}\|^2_{1,\widetilde{\eta},\Omega} \\
&\times\left((\beta N)^{1-r}(\widetilde{E}^r_{\alpha+1,\beta}(x^{-1}U))^2 + \left(M^{2-2s} + (\beta + \frac{1}{\beta})^2\beta^{-r}N^{2-r}\right)(\widetilde{\mathscr{D}}^{r,s}_\beta(U))^2\right). \qquad (3.24)
\end{aligned}
$$

Thanks to Lemma 2.3,

$$
\|U_{M,N}\|^2_{1,\widetilde{\eta},\Omega} \leq \|U\|^2_{1,\widetilde{\eta},\Omega} + c\left(M^{2-2s} + \left(\beta + \frac{1}{\beta}\right)^2\beta^{-r}N^{2-r}\right)(\widetilde{\mathscr{D}}^{r,s}_\beta(U))^2.
$$

Consequently, we deduce from (3.24) that for integers $r > 3$, $s > 1$ and $\alpha = 1,\ 2$,

$$
\begin{aligned}
&|(\widetilde{\mathscr{I}}_{R,M,N,\alpha,\beta}U^2 - U_{M,N}^2, \partial_t \widetilde{U}_{M,N})_{R,M,N,\alpha,\beta,\Omega}| \\
\leq&\|\partial_t \widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega} + c_0(\beta N + M^2)^{2\varepsilon}\bigg((\beta N)^{1-r}(\widetilde{E}^r_{\alpha,\beta}(U) + \widetilde{E}^r_{\alpha+1,\beta}(x^{-1}U))^2 \\
&+ \left(M^{2-2s} + (\beta + \frac{1}{\beta})^2\beta^{-r}N^{2-r}\right)(\widetilde{\mathscr{D}}^{r,s}_\beta(U))^2\bigg), \qquad (3.25)
\end{aligned}
$$

where c_0 is related to the norms $\sup_{0\leq t\leq T}\|U(t)\|_\infty$, $\sup_{0\leq t\leq T}\|U(t)\|_{1,\widetilde{\eta},\Omega}$ and $\sup_{0\leq t\leq T}\widetilde{\mathscr{D}}^{r,s}_\beta(U(t))$ for certain $T > 0$. Furthermore, by the Hardy inequality and a similar argument as in (3.24), we deduce that for $\alpha=0$,

$$\begin{aligned}&(\widetilde{\mathscr{I}}_{R,M,N,0,\beta}U^2-U^2_{M,N},\partial_t\widetilde{U}_{M,N})_{R,M,N,0,\beta}|\le\|\partial_t\widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega}+\frac{1}{16}\|\partial_t\widetilde{U}_{M,N}\|^2_{\widetilde{\omega}_{-2},\Omega}\\&\quad+c\Big(\|U\|^2_\infty+\|U_{M,N}\|^2_{L^\infty(\Omega_2)}+\|xU_{M,N}\|^2_{L^\infty(\Omega_1)}\Big)\|\widetilde{\mathscr{I}}_{R,M,N,0,\beta}U-U_{M,N}\|^2_\Omega\\&\le\|\partial_t\widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega}+\frac{1}{4}\|\partial_t\partial_x\widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega}+c_0(\beta N+M^2)^{2\varepsilon}\Big((\beta N)^{1-r}(\widetilde{E}^r_{0,\beta}(U))^2\\&\quad+\Big(M^{2-2s}+(\beta+\frac{1}{\beta})^2\beta^{-r}N^{2-r}\Big)(\widetilde{\mathscr{D}}^{r,s}_\beta(U))^2\Big).\end{aligned}\tag{3.26}$$

A combination of (3.23)-(3.26) leads to that

$$\begin{aligned}&2\sum_{j=5}^7|G_j(t,\partial_t\widetilde{U}_{M,N})|\le12\|\partial_t\widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega}+\frac{1}{2}|\partial_t\widetilde{U}_{M,N}|^2_{1,\widetilde{\eta},\Omega}\\&\quad+c(\beta N)^{1-r}\sum_{\alpha=0}^2(\widetilde{E}^r_{\alpha,\beta}(U^2))^2+cM^{-2s}\sum_{\alpha=0}^2|U^2|^2_{L^2_{\widetilde{\omega}_\alpha}(\Lambda,H^s(I))}\\&\quad+c_0(\beta N+M^2)^{2\varepsilon}\Big((\beta N)^{1-r}\sum_{\alpha=0}^2(\widetilde{E}^r_{\alpha,\beta}(U))^2+(\beta N)^{1-r}\sum_{\alpha=1}^2(\widetilde{E}^r_{\alpha+1,\beta}(x^{-1}U))^2\\&\quad+\Big(M^{2-2s}+(\beta+\frac{1}{\beta})^2\beta^{-r}N^{2-r}\Big)(\widetilde{\mathscr{D}}^{r,s}_\beta(U))^2\Big).\end{aligned}\tag{3.27}$$

Further, by (2.13) and Lemma 2.5, we get that for integer $r_1\ge1$ and $s_1>\frac{1}{2}$,

$$|2G_8(t,\partial_t\widetilde{U}_{M,N})|\le\|\partial_t\widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega}+c(\beta N)^{1-r_1}(\widetilde{E}^{r_1}_{0,\beta}(f))^2+cM^{-2s_1}|f|^2_{L^2(\Lambda,H^{s_1}(I))}.\tag{3.28}$$

Moreover, it is clear that

$$\begin{aligned}&G_9(t,\partial_t\widetilde{U}_{M,N})=\frac{2\pi}{2M+1}\sum_{j=0}^N\sum_{k=0}^{2M}\Big(\widetilde{U}^2_{M,N}(\widetilde{\xi}^{(2,\beta)}_{R,N,j},\theta_{M,k},t)\\&\quad+2U(\widetilde{\xi}^{(2,\beta)}_{R,N,j},\theta_{M,k},t)\widetilde{U}_{M,N}(\widetilde{\xi}^{(2,\beta)}_{R,N,j},\theta_{M,k},t)+2(U_{M,N}(\widetilde{\xi}^{(2,\beta)}_{R,N,j},\theta_{M,k},t)\\&\quad-\widetilde{\mathscr{I}}_{R,M,N,2,\beta}U(\widetilde{\xi}^{(2,\beta)}_{R,N,j},\theta_{M,k},t))\widetilde{U}_{M,N}(\widetilde{\xi}^{(2,\beta)}_{R,N,j},\theta_{M,k},t)\Big)\partial_t\widetilde{U}_{M,N}(\widetilde{\xi}^{(2,\beta)}_{R,N,j},\theta_{M,k},t)\widetilde{\omega}^{(2,\beta)}_{R,N,j}.\end{aligned}$$

Hence, by (2.13), Lemmas 2.3, 2.5 and 2.6, and the Cauchy-Schwatz inequality, we deduce that for certain constant $c^*>0$,

$$\begin{aligned}&|G_9(t,\partial_t\widetilde{U}_{M,N})|\le\|\partial_t\widetilde{U}_{M,N}\|_{\widetilde{\omega}_2,\Omega}\|\widetilde{U}^2_{M,N}\|_{R,M,N,2,\beta,\Omega}+\|\partial_t\widetilde{U}_{M,N}\|^2_{\widetilde{\omega}_2,\Omega}\\&\quad+\|U\|^2_\infty\|\widetilde{U}_{M,N}\|^2_{\widetilde{\omega}_2,\Omega}+2\|U_{M,N}-\widetilde{\mathscr{I}}_{R,M,N,2,\beta}U\|_{\widetilde{\omega}_2,\Omega}\|\widetilde{U}_{M,N}\partial_t\widetilde{U}_{M,N}\|_{R,M,N,2,\beta,\Omega}\end{aligned}$$

$$
\begin{aligned}
\le & \frac{c^*}{6}\|\partial_t \widetilde{U}_{M,N}\|_{\widetilde{\omega}_2,\Omega}\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega} + \|\partial_t \widetilde{U}_{M,N}\|^2_{\widetilde{\omega}_2,\Omega} + \|U\|^2_\infty \|\widetilde{U}_{M,N}\|^2_{\widetilde{\omega}_2,\Omega} \\
& + 2\|U_{M,N} - \widetilde{\mathscr{I}}_{R,M,N,2,\beta} U\|_{\widetilde{\omega}_2,\Omega}(\|\widetilde{U}_{M,N}\|^2_{\widetilde{\omega}_1,\Omega} + \|\widetilde{U}_{M,N}\|_{\widetilde{\omega}_2,\Omega}\|\partial_x \widetilde{U}_{M,N}\|_{\widetilde{\omega}_2,\Omega})^{\frac{1}{2}} \\
& \times (\|\partial_t \widetilde{U}_{M,N}\|_\Omega \|\partial_t\partial_\theta \widetilde{U}_{M,N}\|_\Omega + \|\partial_t \widetilde{U}_{M,N}\|^2_\Omega)^{\frac{1}{2}} \\
\le & \frac{c^*}{6}\|\partial_t \widetilde{U}_{M,N}\|_{\widetilde{\eta},\Omega}\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega} + \|\partial_t \widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega} + \|U(t)\|^2_\infty \|\widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega} \\
& + c\bigg(\Big(M^{1-s} + (\beta + \frac{1}{\beta})\beta^{-\frac{r}{2}}N^{1-\frac{r}{2}}\Big)\widetilde{\mathscr{D}}^{r,s}_\beta(U) + (\beta N)^{\frac{1}{2}-\frac{r}{2}}\widetilde{E}^r_{2,\beta}(U)\bigg) \\
& \times (\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega} + \|\partial_t \widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega}).
\end{aligned} \tag{3.29}
$$

Similarly,

$$
\begin{aligned}
|G_j(t,\partial_t \widetilde{U}_{M,N})| \le & \frac{c^*}{6}\|\partial_t \widetilde{U}_{M,N}\|_{\widetilde{\eta},\Omega}\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega} + \|\partial_t \widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega} \\
& + \|U\|^2_\infty\|\widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega} + c\bigg(\Big(M^{1-s} + (\beta + \frac{1}{\beta})\beta^{-\frac{r}{2}}N^{1-\frac{r}{2}}\Big)\widetilde{\mathscr{D}}^{r,s}_\beta(U) \\
& + (\beta N)^{\frac{1}{2}-\frac{r}{2}}\widetilde{E}^r_{\alpha,\beta}(U)\bigg)\Big(\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega} + \|\partial_t \widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega}\Big), \qquad j = 10,\ 11,
\end{aligned} \tag{3.30}
$$

where $\alpha = 1$ for $j = 10$ and $\alpha = 0$ for $j = 11$. Hence, for larger M and N, integers $r > 3$ and $s > 1$,

$$
\begin{aligned}
2\sum_{j=9}^{11}|G_j(t,\partial_t \widetilde{U}_{M,N})| \le & c^*\|\partial_t \widetilde{U}_{M,N}\|_{\widetilde{\eta},\Omega}\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega} + 6\|\partial_t \widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega} \\
& + 6\|U\|^2_\infty\|\widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega} + \frac{1}{4}\Big(\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega} + \|\partial_t \widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega}\Big).
\end{aligned} \tag{3.31}
$$

It is clear that

$$
2|(\partial_t\partial_x \widetilde{U}_{M,N}, \partial_t \widetilde{U}_{M,N})_{\widetilde{\xi},\Omega}| \le |\partial_t \widetilde{U}_{M,N}|^2_{1,\widetilde{\eta},\Omega} + \|\partial_t \widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega}, \tag{3.32}
$$

$$
2|(\partial_x \widetilde{U}_{M,N}(t), \partial_t \widetilde{U}_{M,N})_{\widetilde{\xi},\Omega}| \le |\widetilde{U}_{M,N}|^2_{1,\widetilde{\eta},\Omega} + \|\partial_t \widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega}. \tag{3.33}
$$

Therefore, substituting (3.22), (3.27), (3.28) and (3.31)-(3.33) into (3.21), we obtain that

$$
\begin{aligned}
& \partial_t\|\partial_t \widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega} + \partial_t\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega} + \frac{1}{4}|\partial_t \widetilde{U}_{M,N}|^2_{1,\widetilde{\eta},\Omega} \\
\le & c\|\partial_t \widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega} + c(\|U\|^2_\infty + 1)\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega} + c^*\|\partial_t \widetilde{U}_{M,N}\|_{\widetilde{\eta},\Omega}\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega} \\
& + c\left(M^{2-2s} + \left(\beta + \frac{1}{\beta}\right)^2\beta^{-r}N^{2-r}\right)\sum_{j=0}^{2}(\widetilde{\mathscr{D}}^{r,s}_\beta(\partial_t^j U))^2 + c(\beta N)^{1-r}\sum_{\alpha=0}^{2}(\widetilde{E}^r_{\alpha,\beta}(U^2))^2 \\
& + cM^{-2s}\sum_{\alpha=0}^{2}|U^2|^2_{L^2_{\widetilde{\omega}_\alpha}(\Lambda,H^s(I))} + c_0(\beta N + M^2)^{2\varepsilon}\bigg((\beta N)^{1-r}\sum_{\alpha=0}^{2}(\widetilde{E}^r_{\alpha,\beta}(U))^2
\end{aligned}
$$

$$
+(\beta N)^{1-r}\sum_{\alpha=1}^{2}(\widetilde{E}^{r}_{\alpha+1,\beta}(x^{-1}U))^2+\Big(M^{2-2s}+(\beta+\frac{1}{\beta})^2\beta^{-r}N^{2-r}\Big)(\widetilde{\mathscr{D}}^{r,s}_{\beta}(U))^2\Big)
+c(\beta N)^{1-r_1}(\widetilde{E}^{r_1}_{0,\beta}(f))^2+cM^{-2s_1}|f|^2_{L^2(\Lambda,H^{s_1}(I))}. \tag{3.34}
$$

Let $|||U|||_\infty=\sup_{0\le t\le T}\|U\|_\infty$. Integrating (3.34) with respect to t, we deduce that

$$
\begin{aligned}
&\|\partial_t\widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega}+\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega}+\frac{1}{4}\int_0^t|\partial_\xi\widetilde{U}_{M,N}|^2_{1,\widetilde{\eta},\Omega}d\xi\\
\le& c\int_0^t\|\partial_\xi\widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega}d\xi+c(|||U|||^2_\infty+1)\int_0^t\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega}d\xi+c^*\int_0^t\|\partial_\xi\widetilde{U}_{M,N}\|_{\widetilde{\eta},\Omega}\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega}d\xi\\
&+c(M^{2-2s}+(\beta+\frac{1}{\beta})^2\beta^{-r}N^{2-r})\sum_{j=0}^{2}\int_0^t(\widetilde{\mathscr{D}}^{r,s}_{\beta}(\partial^j_\xi U))^2d\xi+c(\beta N)^{1-r}\sum_{\alpha=0}^{2}\int_0^t(\widetilde{E}^{r}_{\alpha,\beta}(U^2))^2d\xi\\
&+cM^{-2s}\sum_{\alpha=0}^{2}\int_0^t|U^2|^2_{L^2_{\widetilde{\omega}_\alpha}(\Lambda,H^s(I))}d\xi+c_0(\beta N+M^2)^{2\varepsilon}\Bigg((\beta N)^{1-r}\sum_{\alpha=0}^{2}\int_0^t(\widetilde{E}^{r}_{\alpha,\beta}(U))^2d\xi\\
&+(\beta N)^{1-r}\sum_{\alpha=1}^{2}\int_0^t(\widetilde{E}^{r}_{\alpha+1,\beta}(x^{-1}U))^2d\xi+\left(M^{2-2s}+\left(\beta+\frac{1}{\beta}\right)^2\beta^{-r}N^{2-r}\right)\int_0^t(\widetilde{\mathscr{D}}^{r,s}_{\beta}(U))^2d\xi\Bigg)\\
&+c(\beta N)^{1-r_1}\int_0^t(\widetilde{E}^{r_1}_{0,\beta}(f))^2d\xi+cM^{-2s_1}\int_0^t|f|^2_{L^2(\Lambda,H^{s_1}(I))}d\xi.
\end{aligned}\tag{3.35}
$$

We now take in Lemma 3.1,

$$
\begin{aligned}
&Z(t)=\|\partial_t\widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega}+\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega}+\frac{1}{4}\int_0^t|\partial_\xi\widetilde{U}_{M,N}(\xi)|^2_{1,\widetilde{\eta},\Omega}d\xi,\\
&A(t)=\|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega},\quad b_1=1,\quad b_2=c^*,\quad b_3=1+c(|||U|||^2_\infty+1),\\
&d=d_1+d_2,
\end{aligned}\tag{3.36}
$$

and

$$
\begin{aligned}
d_1=&c(M^{2-2s}+(\beta+\frac{1}{\beta})^2\beta^{-r}N^{2-r})\sum_{j=0}^{2}\int_0^T(\widetilde{\mathscr{D}}^{r,s}_{\beta}(\partial^j_\xi U))^2d\xi\\
&+c(\beta N)^{1-r}\sum_{\alpha=0}^{2}\int_0^T(\widetilde{E}^{r}_{\alpha,\beta}(U^2))^2d\xi+cM^{-2s}\sum_{\alpha=0}^{2}\int_0^T|U^2|^2_{L^2_{\widetilde{\omega}_\alpha}(\Lambda,H^s(I))}d\xi,\\
d_2=&c_0(\beta N+M^2)^{2\varepsilon}\Bigg((\beta N)^{1-r}\sum_{\alpha=0}^{2}\int_0^T(\widetilde{E}^{r}_{\alpha,\beta}(U))^2d\xi+(\beta N)^{1-r}\sum_{\alpha=1}^{2}\int_0^T(\widetilde{E}^{r}_{\alpha+1,\beta}(x^{-1}U))^2d\xi\\
&+\left(M^{2-2s}+\left(\beta+\frac{1}{\beta}\right)^2\beta^{-r}N^{2-r}\right)\int_0^T(\widetilde{\mathscr{D}}^{r,s}_{\beta}(U))^2d\xi\Bigg)\\
&+c(\beta N)^{1-r_1}\int_0^T(\widetilde{E}^{r_1}_{0,\beta}(f))^2d\xi+cM^{-2s_1}\int_0^T|f|^2_{L^2(\Lambda,H^{s_1}(I))}d\xi.
\end{aligned}
$$

Clearly, if $r > 3$, $s > 1$, $r_1 \geq 1$, $s_1 > \frac{1}{2}$ and the norms mentioned in the notation d are bounded above, then $d \to 0$ as $M, N \to \infty$. Therefore, applying Lemma 3.1 to (3.35), we obtain that for any $0 \leq t \leq T$,

$$\|\partial_t \widetilde{U}_{M,N}\|^2_{\widetilde{\eta},\Omega} + \|\widetilde{U}_{M,N}\|^2_{1,\widetilde{\eta},\Omega} + \frac{1}{4}\int_0^t |\partial_\xi \widetilde{U}_{M,N}(\xi)|^2_{1,\widetilde{\eta},\Omega} d\xi \leq d e^{b_3 T}. \tag{3.37}$$

Finally, a combination of (3.37) and Lemma 2.3 leads to the following conclusion.

Theorem 3.2. *If the norms mentioned in the notation d of* (3.36) *are bounded uniformly for $0 \leq t \leq T$ and integers $r > 3$, $s > 1$, $r_1 \geq 1$ and $s_1 > \frac{1}{2}$, then for suitably larger N, M and any $0 \leq t \leq T$,*

$$\begin{aligned}
&\|\partial_t(U - u_{M,N})\|^2_{\widetilde{\eta},\Omega} + \|U - u_{M,N}\|^2_{1,\widetilde{\eta},\Omega} + \frac{1}{4}\int_0^t \left|\partial_\xi\big(U(\xi) - u_{M,N}(\xi)\big)\right|^2_{1,\widetilde{\eta},\Omega} d\xi \\
\leq &\left(d + c\left(M^{2-2s} + \left(\beta + \frac{1}{\beta}\right)^2 \beta^{-r} N^{2-r}\right) \sup_{0\leq t\leq T} \sum_{j=0}^{1} \big(\widetilde{\mathscr{D}}^{r,s}_\beta(\partial_t^j U)\big)^2 \right) e^{b_3 T}.
\end{aligned}$$

Remark 3.1. The result of Theorem 3.2 still holds even for real numbers $r > 3$, $s > 1$, $r_1 \geq 1$ and $s_1 > \frac{1}{2}$, but in this case, the semi-norms mentioned in the notation d should be replaced by the corresponding norms.

4. Numerical results

In this section, we describe the numerical implementations and present some numerical results. Let $\psi_k(x) = \widetilde{\mathscr{L}}^{(0,\beta)}_{k-1}(x) - \widetilde{\mathscr{L}}^{(0,\beta)}_k(x), 1 \leq k \leq N$, and $\widetilde{\mathscr{L}}^{(1,\beta)}_{-1}(x) = 0$. Denote by τ the mesh size in time t. Then by (2.2) and (2.4),

$$\psi_k(x) = 2\widetilde{\mathscr{L}}^{(1,\beta)}_{k-1}(x) - \widetilde{\mathscr{L}}^{(1,\beta)}_{k-2}(x) - \widetilde{\mathscr{L}}^{(1,\beta)}_k(x) = \frac{\beta}{k} x \widetilde{\mathscr{L}}^{(1,\beta)}_{k-1}(x), \qquad 1 \leq k \leq N.$$

We take the basis functions as

$$\begin{aligned}
\phi^1_{k,m}(x,\theta) &= \frac{1}{\sqrt{2\pi}}\psi_k(x)\sin(m\theta), \qquad 1 \leq k \leq N, \quad 1 \leq m \leq M, \\
\phi^2_{k,m}(x,\theta) &= \frac{1}{\sqrt{2\pi}}\psi_k(x)\cos(m\theta), \qquad 1 \leq k \leq N, \quad 0 \leq m \leq M.
\end{aligned}$$

The numerical solution is expanded as

$$u_{M,N}(x,\theta,t) = \sum_{1\leq k\leq N}\sum_{1\leq m\leq M} u^1_{k,m}(t)\phi^1_{k,m}(x,\theta) + \sum_{1\leq k\leq N}\sum_{0\leq m\leq M} u^2_{k,m}(t)\phi^2_{k,m}(x,\theta).$$

4.1. Scheme (3.4)

The fully discrete scheme for (3.4) is as follows,

$$\begin{aligned}
&(2+\tau^2)(u_{M,N}(t+\tau),\phi)_{\widetilde{\eta}}+(\tau+\tau^2)(\partial_x u_{M,N}(t+\tau),\partial_x\phi)_{\widetilde{\eta}}\\
&\quad+(\tau+\tau^2)(\partial_x u_{M,N}(t+\tau),\phi)_{\widetilde{\xi}}+(\tau+\tau^2)(\partial_\theta u_{M,N}(t+\tau),\partial_\theta\phi)\\
=&4(u_{M,N}(t),\phi)_{\widetilde{\eta}}-(2+\tau^2)(u_{M,N}(t-\tau),\phi)_{\widetilde{\eta}}+(\tau-\tau^2)(\partial_x u_{M,N}(t-\tau),\partial_x\phi)_{\widetilde{\eta}}\\
&\quad+(\tau-\tau^2)(\partial_x u_{M,N}(t-\tau),\phi)_{\widetilde{\xi}}+(\tau-\tau^2)(\partial_\theta u_{M,N}(t-\tau),\partial_\theta\phi)\\
&\quad-2\tau^2(u^2_{M,N}(t),\phi)_{\widetilde{\eta}}+\tau^2(f(t+\tau)+f(t-\tau),\phi),\qquad \forall\phi\in{}_0V_{M,N,\beta}(\Omega).
\end{aligned}\tag{4.1}$$

Take $\phi=\phi^q_{j,l}$, $q=1,2$ in (4.1). By the orthogonality of the triangle functions, we deduce that for $1\le j\le N$ and $q=1,2$,

$$\begin{aligned}
&\sum_{k=1}^{N}\Big((2+\tau^2)\int_\Lambda(x+1)^2\psi_k\psi_j dx+(\tau+\tau^2)\int_\Lambda(x+1)^2\partial_x\psi_k\partial_x\psi_j dx\\
&\qquad+(\tau+\tau^2)\int_\Lambda(x+1)\partial_x\psi_k\psi_j dx+(\tau+\tau^2)l^2\int_\Lambda\psi_k\psi_j dx\Big)\widetilde{u}^q_{k,l}(t+\tau)\\
=&\sum_{k=1}^{N}\Big(4\int_\Lambda(x+1)^2\psi_k\psi_j dx\Big)\widetilde{u}^q_{k,l}(t)+\sum_{k=1}^{N}\Big(-(2+\tau^2)\int_\Lambda(x+1)^2\psi_k\psi_j dx\\
&\qquad+(\tau-\tau^2)\int_\Lambda(x+1)^2\partial_x\psi_k\partial_x\psi_j dx+(\tau-\tau^2)\int_\Lambda(x+1)\partial_x\psi_k\psi_j dx\\
&\qquad+(\tau-\tau^2)l^2\int_\Lambda\psi_k\psi_j dx\Big)\widetilde{u}^q_{k,l}(t-\tau)+d(l,q)\widetilde{g}^q_{j,l}(t),
\end{aligned}\tag{4.2}$$

where $d(0,2)=1$, $d(l,q)=2$ otherwise, and

$$\widetilde{g}^q_{j,l}(t)=\tau^2\int_\Omega(f(x,\theta,t+\tau)+f(x,\theta,t-\tau))\phi^q_{j,l}(x,\theta)dxd\theta-2\tau^2(u^2_{M,N}(t),\phi^q_{j,l})_{\widetilde{\eta},\Omega}.$$

Next, we introduce the matrices $A=(a_{j,k})$, $B=(b_{j,k})$, $C=(c_{j,k})$, $D=(d_{j,k})$ and $E=(e_{j,k})$, with the following entries:

$$\begin{aligned}
a_{j,k}&=\int_\Lambda(x+1)^2\partial_x\psi_k(x)\partial_x\psi_j(x)dx, & b_{j,k}&=\int_\Lambda x^2\psi_k(x)\psi_j(x)dx,\\
c_{j,k}&=\int_\Lambda x\psi_k(x)\psi_j(x)dx, & d_{j,k}&=\int_\Lambda\psi_k(x)\psi_j(x)dx,\\
e_{j,k}&=\int_\Lambda(x+1)\partial_x\psi_k(x)\psi_j(x)dx.
\end{aligned}$$

With the aid of (2.1), (2.4) and (2.5), a direct calculation shows that the corresponding matrices have the following structures (see [18]):

$$\begin{aligned} &a_{jk} := 0, \quad \text{if} \quad |j-k| > 3, && b_{jk} := 0, \quad \text{if} \quad |j-k| > 3,\\ &c_{jk} := 0, \quad \text{if} \quad |j-k| > 2, && d_{jk} := 0, \quad \text{if} \quad |j-k| > 1,\\ &e_{jk} := 0, \quad \text{if} \quad |j-k| > 2. \end{aligned}$$

Furthermore, we set

$$\widetilde{X}_l^q(t) = (\widetilde{u}_{1,l}^q(t), \widetilde{u}_{2,l}^q(t), \cdots, \widetilde{u}_{N,l}^q(t))^T, \quad \widetilde{G}_l^q(t) = (\widetilde{g}_{1,l}^q(t), \widetilde{g}_{2,l}^q(t), \cdots, \widetilde{g}_{N,l}^q(t))^T, \quad q = 1, 2.$$

Then we have from (4.2) that

$$\begin{aligned} &[(\tau+\tau^2)A + (2+\tau^2)B + (4+2\tau^2)C + (2+\tau^2+\tau l^2+\tau^2 l^2)D + (\tau+\tau^2)E]\widetilde{X}_l^q(t+\tau)\\ =&(4B+8C+4D)\widetilde{X}_l^q(t) + [(\tau-\tau^2)A - (2+\tau^2)B - (4+2\tau^2)C\\ &+(-2-\tau^2+\tau l^2-\tau^2 l^2)D + (\tau-\tau^2)E]\widetilde{X}_l^q(t-\tau) + d(l,q)\widetilde{G}_l^q(t). \end{aligned} \tag{4.3}$$

4.2. Scheme (3.18)

The fully discrete scheme for (3.18) is as follows,

$$\begin{aligned} &(2+\tau^2)(u_{M,N}(t+\tau),\phi)_{\widetilde{\eta}} + (\tau+\tau^2)(\partial_x u_{M,N}(t+\tau),\partial_x\phi)_{\widetilde{\eta}}\\ &\quad + (\tau+\tau^2)(\partial_x u_{M,N}(t+\tau),\phi)_{\widetilde{\xi}} + (\tau+\tau^2)(\partial_\theta u_{M,N}(t+\tau),\partial_\theta\phi)\\ =&4(u_{M,N}(t),\phi)_{\widetilde{\eta}} - (2+\tau^2)(u_{M,N}(t-\tau),\phi)_{\widetilde{\eta}}\\ &\quad + (\tau-\tau^2)(\partial_x u_{M,N}(t-\tau),\partial_x\phi)_{\widetilde{\eta}} + (\tau-\tau^2)(\partial_x u_{M,N}(t-\tau),\phi)_{\widetilde{\xi}}\\ &\quad + (\tau-\tau^2)(\partial_\theta u_{M,N}(t-\tau),\partial_\theta\phi) - 2\tau^2(u_{M,N}^2(t),\phi)_{R,M,N,2,\beta,\Omega}\\ &\quad - 4\tau^2(u_{M,N}^2(t),\phi)_{R,M,N,1,\beta,\Omega} - 2\tau^2(u_{M,N}^2(t),\phi)_{R,M,N,0,\beta,\Omega}\\ &\quad + \tau^2(f(t+\tau)+f(t-\tau),\phi)_{R,M,N,0,\beta,\Omega}, \qquad \forall\phi\in {}_0V_{M,N,\beta}. \end{aligned} \tag{4.4}$$

Take $\phi = \phi_{j,l}^q(x,\theta)$, $q = 1$, 2 in (4.4). By the orthogonality of the triangle functions, we deduce that for $1 \le j \le N$ and $q = 1, 2$,

$$\begin{aligned} &\sum_{k=1}^{N}\bigg((2+\tau^2)\int_\Lambda (x+1)^2\psi_k\psi_j dx + (\tau+\tau^2)\int_\Lambda (x+1)^2\partial_x\psi_k\partial_x\psi_j dx\\ &\quad + (\tau+\tau^2)\int_\Lambda (x+1)\partial_x\psi_k\psi_j dx + (\tau+\tau^2)l^2\int_\Lambda \psi_k\psi_j dx\bigg)\widetilde{u}_{k,l}^q(t+\tau)\\ =&\sum_{k=1}^{N}\bigg(4\int_\Lambda (x+1)^2\psi_k\psi_j dx\bigg)\widetilde{u}_{k,l}^q(t) + \sum_{k=1}^{N}\bigg(-(2+\tau^2)\int_\Lambda (x+1)^2\psi_k\psi_j dx\\ &\quad + (\tau-\tau^2)\int_\Lambda (x+1)^2\partial_x\psi_k\partial_x\psi_j dx + (\tau-\tau^2)\int_\Lambda (x+1)\partial_x\psi_k\psi_j dx\\ &\quad + (\tau-\tau^2)l^2\int_\Lambda \psi_k\psi_j dx\bigg)\widetilde{u}_{k,l}^q(t-\tau) + d(l,q)\widetilde{g}_{j,l}^q(t), \end{aligned} \tag{4.5}$$

where $d(l,q)$ is the same as in (4.2), and

$$\begin{aligned}\bar{g}^q_{j,l}(t) =&\tau^2\int_\Omega (f(x,\theta,t+\tau)+f(x,\theta,t-\tau))\phi^q_{j,l}(x,\theta)dxd\theta\\ &-2\tau^2(u^2_{M,N}(t),\phi^q_{j,l})_{R,M,N,2,\beta,\Omega}-4\tau^2(u^2_{M,N}(t),\phi^q_{j,l})_{R,M,N,1,\beta,\Omega}\\ &-2\tau^2(u^2_{M,N}(t),\phi^q_{j,l})_{R,M,N,0,\beta,\Omega}, \qquad q=1,2.\end{aligned}$$

Furthermore, we set

$$\bar{X}^q_l(t)=(\bar{u}^q_{1,l}(t),\bar{u}^q_{2,l}(t),\cdots,\bar{u}^q_{N,l}(t))^T,\quad \bar{G}^q_l(t)=(\bar{g}^q_{1,l}(t),\bar{g}^q_{2,l}(t),\cdots,\bar{g}^q_{N,l}(t))^T.$$

Then, we have from (4.5) that

$$\begin{aligned}&[(\tau+\tau^2)A+(2+\tau^2)B+(4+2\tau^2)C+(2+\tau^2+\tau l^2+\tau^2 l^2)D+(\tau+\tau^2)E]\bar{X}^q_l(t+\tau)\\ =&(4B+8C+4D)\bar{X}^q_l(t)+[(\tau-\tau^2)A-(2+\tau^2)B-(4+2\tau^2)C+d(l,q)\bar{G}^q_l(t)\\ &+(-2-\tau^2+\tau l^2-\tau^2 l^2)D+(\tau-\tau^2)E]\bar{X}^q_l(t-\tau), \end{aligned} \tag{4.6}$$

where the matrices A, B, C, D, E are the same as before.

4.3. Numerical results

In the end of this section, we present some numerical results. In actual computations, we take $N=4M$. The numerical errors are measured by

$$\begin{aligned}&E_{M,N,1}(t)\\ =&\Bigg(\frac{2\pi}{2M+1}\sum_{j=0}^{N}\sum_{k=0}^{2M}(U(\widetilde{\xi}^{(0,\beta)}_{R,N,j},\theta_{M,k},t)-u_{M,N}(\widetilde{\xi}^{(0,\beta)}_{R,N,j},\theta_{M,k},t))^2(\widetilde{\xi}^{(0,\beta)}_{R,N,j}+1)\widetilde{\omega}^{(0,\beta)}_{R,N,j}\Bigg)^{\frac{1}{2}}\\ \sim&\Bigg(\int_\Omega (U(x,\theta,t)-u_{M,N}(x,\theta,t))^2(x+1)dxd\theta\Bigg)^{\frac{1}{2}},\\ &E_{M,N,2}(t)=\max_{0\le j\le N,0\le k\le 2M}|U(\widetilde{\xi}^{(0,\beta)}_{R,N,j},\theta_{M,k},t)-u_{M,N}(\widetilde{\xi}^{(0,\beta)}_{R,N,j},\theta_{M,k},t)|.\end{aligned}$$

We now present some numerical results. We take the test function

$$U(x,\theta,t)=\frac{x\sin(\theta+x)}{(x+1)(1+e^{-t})}e^{-\frac{1}{2}x}.$$

We use (4.1) to solve (3.3). In Fig. 1, we plot $\log_{10}E_{N,M,1}(1)$ and $\log_{10}E_{N,M,2}(1)$ vs. M, with $\beta=1.5$ and $\tau=0.01,\ 0.001$, respectively. Clearly, the errors decay fast as M and N increase and τ decreases. It is seen from the left of Fig. 1 that for fixed $\tau=0.01$, $\beta=1.5$ and the mode $M\le 12$, the total numerical errors are dominated by the approximation errors in the space and so decay fast as M increases. But for $M>12$, the total numerical errors are dominated by the approximation errors in time t. Thus, the numerical results keep the same accuracy, even if M and N increase again. A similar situation happens in other cases, see Figs. 1, 2, 4 and 5. The above facts coincide very well with theoretical analysis. In particular, they indicate the spectral accuracy in the space of scheme (4.1).

In Fig. 2, we plot $\log_{10} E_{N,M,1}(1)$ and $\log_{10} E_{N,M,2}(1)$ vs. M, with $\tau = 0.001$ and $\beta = 1,\ 1.5$. We observe that the numerical results with $\beta = 1.5$ are better than the numerical results with $\beta = 1$. This fact demonstrates that a suitable choice of parameter β could raise the numerical accuracy. In Fig. 3, we plot the errors vs. t with $\beta = 1.5$,

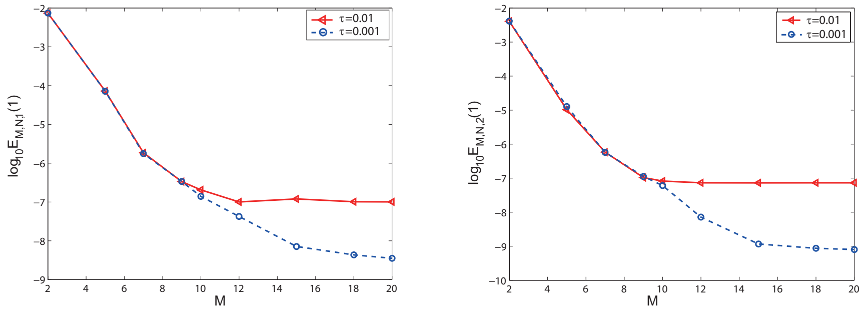

Figure 1: Convergence rates with $N = 4M$. Left: $E_{N,M,1}(1)$, and Right: $E_{N,M,2}(1)$.

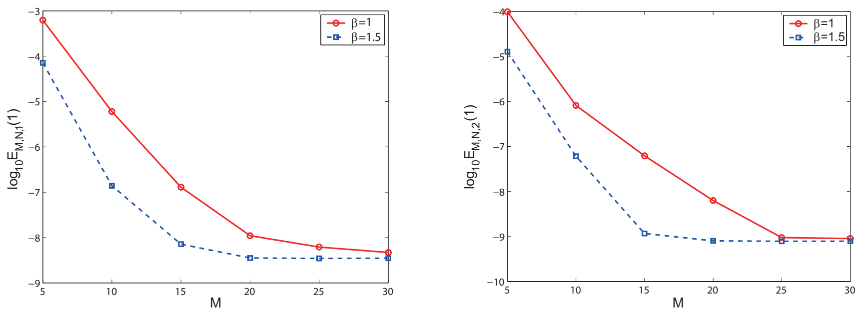

Figure 2: Convergence rates with $N = 4M$. Left: $E_{N,M,1}(1)$, and Right: $E_{N,M,2}(1)$.

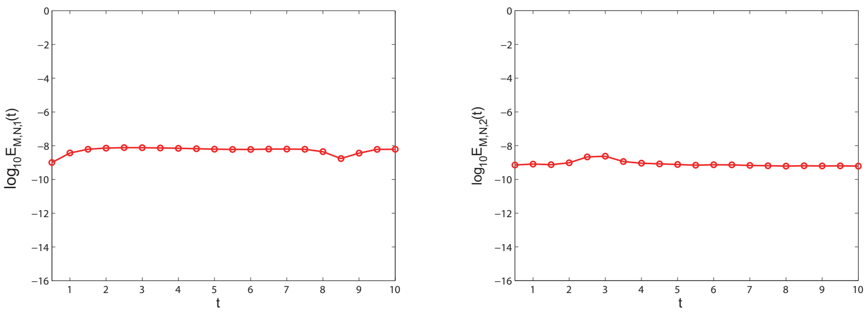

Figure 3: Stability of scheme (4.1). Left: $E_{N,M,1}(t)$, and Right: $E_{N,M,2}(t)$.

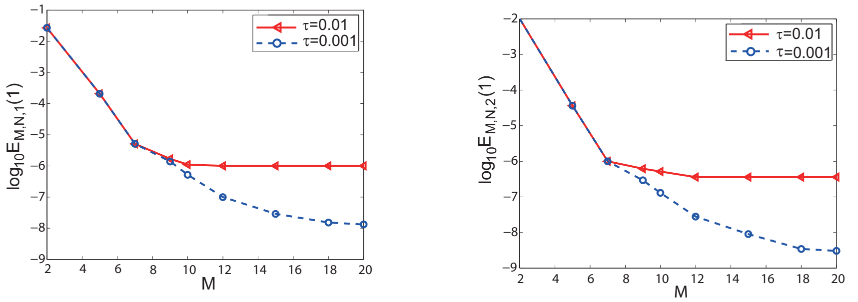

Figure 4: Convergence rates with $N = 4M$. Left: $E_{N,M,1}(1)$, and Right: $E_{N,M,2}(1)$.

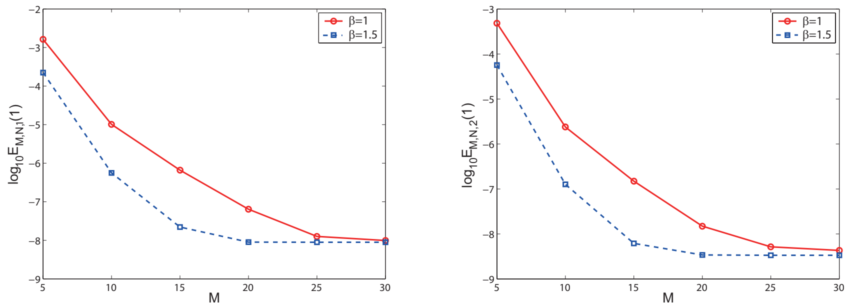

Figure 5: Convergence rates with $N = 4M$. Left: $E_{N,M,1}(1)$, and Right: $E_{N,M,2}(1)$.

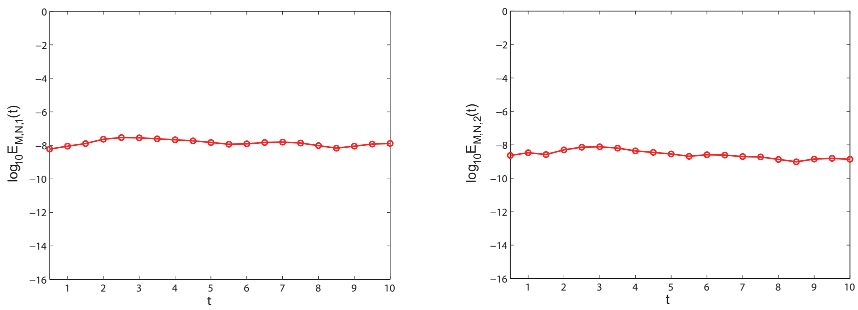

Figure 6: Stability of scheme (4.4). Left: $E_{N,M,1}(t)$, and Right: $E_{N,M,2}(t)$.

$\tau = 0.001$ and $N = 4M = 100$, which demonstrate the stability of scheme (4.1) for long time calculation.

We also use (4.4) to solve (3.3). In Fig. 4, we plot $\log_{10} E_{N,M,1}(1)$ and $\log_{10} E_{N,M,2}(1)$ vs. M with $\beta = 1.5$ and $\tau = 0.01$, 0.001. In Figs. 5, we plot $\log_{10} E_{N,M,1}(1)$ and $\log_{10} E_{N,M,2}(1)$ vs. M with $\tau = 0.001$ and $\beta = 1$, 1.5. They also indicate the convergence and the spectral accuracy of numerical solutions. In particular, by using the mixed pseudospectral method, we save much work and still obtain accurate numerical results. In Figs. 6, we plot the errors vs. t with $\beta = 1.5$, $\tau = 0.001$ and $N = 4M = 100$, which demonstrate the stability of scheme (4.4) for long time calculation.

5. Concluding remarks

In this paper, we proposed the mixed spectral and pseudospectral schemes for a nonlinear strongly damped wave equation outside a disc. The numerical results demonstrated the high efficiency of the proposed methods. These approaches have several advantages:

• By taking the generalized Laguerre functions as the base functions, we can approximate exterior problems directly, without any variable transformation. Therefore, it is much easier to be implemented. Moreover, the numerical solutions keep the same properties as the exact solutions.

• By choosing a set of suitable base functions, we are able to construct an efficient numerical algorithm in which the discrete systems are sparse, and hence can be efficiently solved.

• The adjustable parameter β offers a great flexibility for matching the asymptotic behaviors of the exact solutions at infinity.

In particular, the suggested methods and techniques are also applicable to various nonlinear problems outside a disc in fluid dynamics and electromagnetics.

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