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Spectral Petrov-Galerkin Methods for the Second Kind Volterra Type Integro-Differential Equations

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Abstract. This work is to provide general spectral and pseudo-spectral Jacobi-Petrov-Galerkin approaches for the second kind Volterra integro-differential equations. The Gauss-Legendre quadrature formula is used to approximate the integral operator and the inner product based on the Jacobi weight is implemented in the weak formulation in the numerical implementation. For some spectral and pseudo-spectral Jacobi-Petrov-Galerkin methods, a rigorous error analysis in both $L^2_{\omega^{\alpha,\beta}}$ and L^∞ norms is given provided that both the kernel function and the source function are sufficiently smooth. Numerical experiments validate the theoretical prediction.

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1. Introduction

This paper is concerned with the following second-kind Volterra integro-differential equation with initial condition, i.e.,

$$\begin{cases} u'(x) + \int_{-1}^{x} k(x,s)u(s)ds = g(x), \qquad x \in [-1,1], \\ u(-1) = 0, \end{cases} \tag{1.1}$$

where the kernel function $k(x,s)$ and the source function $g(x)$ are given smooth functions, $u(x)$ is the unknown function.

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Actually any second-kind Volterra integro-differential equation with smooth kernel and initial condition can be transformed into (1.1) by a simple linear transformation used in [12]. As a result, our approach can be generalized to the second-kind Volterra integro-differential equations with initial condition defined in any interval, where the kernel is smooth. We will consider the case that the solutions of (1.1) are sufficiently smooth. Consequently it is natural to implement very high-order numerical methods such as spectral methods for the solutions of (1.1). It is known that there are many numerical approaches for solving (1.1), such as collocation methods, finite element methods, see, e.g., [1] and references therein. Nevertheless, few works touched the spectral approximations to (1.1). In [9], Chebyshev spectral methods are proposed to solve nonlinear Volterra-Hammerstein integral equations. Then Chebyshev spectral methods are investigated in [10] for the first kind Fredholm integral equations under multiple-precision arithmetic. Nevertheless, no theoretical results are provided to justify the high accuracy numerically obtained. Some efforts are made to implement the spectral methods to solve the second-kind Volterra integral equations. In [14], a spectral method is suggested, but spectral accuracy is not observed for most of the computations. Tang and Xu in [12] develop a novel spectral Legendre-collocation method. Actually this is the first spectral approach for which the spectral accuracy can be justified both theoretically and numerically. Inspired by the work in [12], Chen and Tang [4] implement the spectral Chebyshev-collocation method to solve the second kind Volterra integral equation with weakly singular kernel $(t-s)^{-\frac{1}{2}}k(t,s)$, where $k(t,s)$ is a smooth function. Then they [5] extend the approach in [4] to the second kind Volterra integral equation with more general weakly singular kernel $(t-s)^{\alpha}k(t,s)$, where $-1<\alpha\leq 0$ and $k(t,s)$ is a smooth function. The spectral accuracy of these approaches is verified both theoretically and numerically in [4] and [5]. Xie and Tang [7] develop spectral and pseudo-spectral Galerkin methods based on the general Jacobi weight to solve the second-kind Volterra integral equation. They give a rigorous proof of the spectral convergence in $L^2_{\omega^{\alpha,\beta}}$ and L^∞ norms. Actually, the success of the spectral method for the second-kind Volterra integral equations is the main motivation for our work in the second-kind integro-differential equations.

Unlike the standard spectral and pseudo-spectral Galerkin methods, the spectral and pseudo-spectral Petrov-Galerkin methods allow the trial and test function spaces to be different. Lin et.al, in [8] introduce the Petrov-Galerkin finite element (PGFE) method for Volterra integro-differential equations. It is proved that the PGFE solution u_h and its derivative u_h' have optimal convergence rates $\mathcal{O}(h^{m+1})$ and $\mathcal{O}(h^m)$ in L^∞ norm, respectively. After using some postprocessing techniques, the convergence rate of u_h reaches $\mathcal{O}(h^{2m})$ at the nodes of the mesh. Tang [13] discusses the collocation method to solve the first-order Volterra integro-differential equation with a singular kernel function $(t-s)^{-\alpha}k(t,s,u(s))(0<\alpha<1)$. For grading exponents $r>\frac{m}{2-\alpha}$ of the graded mesh, the collocation solution has the convergence rate $\mathcal{O}(N^{-m})$ in L^∞ norm. Besides, Brunner, *et.al*, in [2] present the $hp-$discontinuous Galerkin method for Volterra integro-differential equations with singular kernels. It is proved both theoretically and numerically that the DG solution based on geometrically graded meshes has the exponential convergence rate in L^2 and L^∞ norms. Inspired by these works, we will show that both spectral and pseudo-spectral Petrov-Galerkin methods for Eq. (1.1) could produce numerical solutions with exponential convergence accuracy.

The purpose of this work is to provide numerical methods for the second-kind Volterra integro-differential equations with initial condition based on spectral and pseudo-spectral Petrov-Galerkin approaches. For some spectral and pseudo-spectral Jacobi-Petrov-Galerkin approaches, a rigorous error analysis which theoretically justifies the spectral rate of convergence of our approaches is provided. This paper is organized as follows. In Section 2, we demonstrate the implementation of the spectral and pseudo-spectral Petrov-Galerkin approaches for the underlying equation. Some lemmas useful for the convergence analysis will be provided in Section 3. The convergence analysis for both spectral and pseudo-spectral Jacobi-Petrov-Galerkin methods in $L^2_{\omega^{\alpha,\beta}}$ and L^∞ norms, with some assumptions on the weight function $\omega^{\alpha,\beta}(x)$, will be given in Section 4 and Section 5, respectively. Numerical experiments are carried out in Section 6, which will be used to validate the theoretical results in Section 4 and Section 5.

2. The implementation of the spectral and pseudo-spectral Petrov-Galerkin methods

By introducing the integral operator K defined as

$$Ku(x) = \int_{-1}^{x} k(x,s)u(s)ds,$$

Eq. (1.1) can be reformulated as

$$\begin{cases} u'(x) + Ku(x) = g(x), & x \in [-1,1], \\ u(-1) = 0. \end{cases} \tag{2.1}$$

We will adopt the spectral and pseudo-spectral Jacobi-Petrov-Galerkin methods to solve this underlying problem.

Let us demonstrate the numerical implementation of the spectral Jacobi-Petrov-Galerkin approach first. Denote P_N a space consisting of polynomials defined on $[-1,1]$ with degree at most N, $\phi_j(x)$ is the $j-$th Jacobi polynomial corresponding to the weight function $\omega^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$, with $\alpha, \beta > -1, j = 0,1,\cdots,N$. As a result,

$$P_N = \text{span}\{\phi_0(x), \phi_1(x), \cdots, \phi_N(x)\}.$$

Define the polynomial space V_N as follows,

$$V_N = \{u : u \in P_N, u(-1) = 0\}.$$

Our aim is to find $u_N \in V_N$ such that

$$(u_N', v_N)_{\omega^{\alpha,\beta}} + (Ku_N, v_N)_{\omega^{\alpha,\beta}} = (g, v_N)_{\omega^{\alpha,\beta}}, \qquad \forall v_N \in P_{N-1}, \tag{2.2}$$

where

$$(u,v)_{\omega^{\alpha,\beta}} = \int_{-1}^{1} \omega^{\alpha,\beta}(x)u(x)v(x)dx$$

is the continuous inner product. Set

$$u_N(x) = \sum_{j=0}^{N-1} \hat{u}_j(\phi_j(x) + s_j\phi_{j+1}(x)),$$

where s_j is a constant chosen by the condition

$$\phi_j(-1) + s_j\phi_{j+1}(-1) = 0.$$

It is worthwhile to point out, when $\phi_i(x)$ is the Legendre or Chebyshev polynomial, obviously $s_j = 1, j = 0,1,\cdots,N-1$. Substituting it into (2.2) and taking $v_N = \phi_i(x), i = 0,1,\cdots,N-1$, we obtain

$$\sum_{j=0}^{N-1}(\phi_i,\phi_j' + s_j\phi_{j+1}')_{\omega^{\alpha,\beta}}\hat{u}_j + \sum_{j=0}^{N-1}(\phi_i,K(\phi_j + s_j\phi_{j+1}))_{\omega^{\alpha,\beta}}\hat{u}_j = (\phi_i,g)_{\omega^{\alpha,\beta}}, \tag{2.3}$$

which leads to an equation of the matrix form

$$(A+B)\hat{U}_{N-1} = g_{N-1}, \tag{2.4}$$

where

$$\hat{U}_{N-1} = [\hat{u}_0,\hat{u}_1,\cdots,\hat{u}_{N-1}]^T, \qquad A(i,j) = (\phi_i,\phi_j' + s_j\phi_{j+1}')_{\omega^{\alpha,\beta}},$$
$$B(i,j) = (\phi_i,K(\phi_j + s_j\phi_{j+1}))_{\omega^{\alpha,\beta}}, \qquad g_{N-1}(i) = (\phi_i,g)_{\omega^{\alpha,\beta}}.$$

Now we turn to describe the pseudo-spectral Jacobi-Petrov-Galerkin method. For this purpose, set $s = s(x,\theta) = \frac{x-1}{2} + \frac{x+1}{2}\theta, \theta \in [-1,1]$. It is clear that

$$Ku(x) = \int_{-1}^{x} k(x,s)u(s)ds = \int_{-1}^{1} \widetilde{k}(x,s(x,\theta))u(s(x,\theta))d\theta \tag{2.5}$$

with $\widetilde{k}(x,s(x,\theta)) = \frac{x+1}{2}k(x,s(x,\theta))$. Using $N-$point Gauss-Legendre quadrature formula to approximate (2.5) yields

$$Ku(x) = \int_{-1}^{1} \widetilde{k}(x,s(x,\theta))u(s(x,\theta))d\theta \approx \sum_{n=0}^{N-1} \widetilde{k}(x,s(x,\theta_n))u(s(x,\theta_n))v_n, \tag{2.6}$$

where $\{\theta_n\}_{n=0}^{N-1}$ are the N-degree Legendre-Gauss points, and $\{v_n\}_{n=0}^{N-1}$ are the corresponding Legendre weights. On the other hand, instead of the continuous inner product, the discrete inner product will be implemented in (2.2) and (2.3), i.e.,

$$(u,v)_{\omega^{\alpha,\beta}} \approx (u,v)_{\omega^{\alpha,\beta},N-1} = \sum_{m=0}^{N-1} u(x_m)v(x_m)\omega_m^{\alpha,\beta}, \tag{2.7}$$

where $\{x_m\}_{m=0}^{N-1}$ and $\{\omega_m^{\alpha,\beta}\}_{m=0}^{N-1}$ are the $N-$degree Jacobi-Gauss points and their corresponding Jacobi weights, respectively. As a result,

$$(u,v)_{\omega^{\alpha,\beta}}=(u,v)_{\omega^{\alpha,\beta},N-1},\qquad \text{if}\quad uv\in P_{2N-1}.$$

If we substitute (2.6) and (2.7) into (2.2) and (2.3), respectively, then the pseudo-spectral Jacobi-Petrov-Galerkin method is to find

$$u_N^{(1)}(x)=\sum_{j=0}^{N-1}\hat{u}_j^{(1)}(\phi_j(x)+s_j\phi_{j+1}(x))\in V_N, \tag{2.8}$$

such that

$$\begin{aligned}&(u_N^{(1)\prime},v_N)_{\omega^{\alpha,\beta},N-1}+\left(\sum_{n=0}^{N-1}\widetilde{k}(x,s(x,\theta_n))u_N^{(1)}(s(x,\theta_n))v_n,v_N\right)_{\omega^{\alpha,\beta},N-1}\\ =&(g,v_N)_{\omega^{\alpha,\beta},N-1},\qquad \forall v_N\in P_{N-1},\end{aligned} \tag{2.9}$$

where $\{\hat{u}_j^{(1)}\}_{j=0}^{N-1}$ are determined by

$$\begin{aligned}&\sum_{j=0}^{N-1}(\phi_i,\phi_j'+s_j\phi_{j+1}')_{\omega^{\alpha,\beta},N-1}\hat{u}_j^{(1)}\\ &\quad+\sum_{j=0}^{N-1}\left(\phi_i,\sum_{n=0}^{N-1}\widetilde{k}(x,s(x,\theta_n))(\phi_j(s(x,\theta_n))+s_j\phi_{j+1}(s(x,\theta_n)))v_n\right)_{\omega^{\alpha,\beta},N-1}\hat{u}_j^{(1)}\\ =&(\phi_i,g)_{\omega^{\alpha,\beta},N-1},\qquad i=0,1,\cdots,N-1.\end{aligned} \tag{2.10}$$

Denoting $\hat{U}_{N-1}^{(1)}=[\hat{u}_0^{(1)},\hat{u}_1^{(1)},\cdots,\hat{u}_{N-1}^{(1)}]^T$, (2.10) yields an equation of the matrix form

$$(A^{(1)}+B^{(1)})\hat{U}_{N-1}^{(1)}=g_{N-1}^{(1)}, \tag{2.11}$$

where

$$\begin{aligned}&A^{(1)}(i,j)=(\phi_i,\phi_j'+s_j\phi_{j+1}')_{\omega^{\alpha,\beta},N-1},\\ &B^{(1)}(i,j)=\left(\phi_i,\sum_{n=0}^{N-1}\widetilde{k}(x,s(x,\theta_n))(\phi_j(s(x,\theta_n))+s_j\phi_{j+1}(s(x,\theta_n)))v_n\right)_{\omega^{\alpha,\beta},N-1},\\ &g_{N-1}^{(1)}(i)=(\phi_i,g)_{\omega^{\alpha,\beta},N-1}.\end{aligned}$$

It is worthwhile to point out that the known recurrence formula for Jacobi polynomials can be used to calculate $\phi_i(x)$ in the two approaches mentioned above.

3. Some useful lemmas

Define a weighted space as

$$L^2_{\omega^{\alpha,\beta}}(I) = \Big\{ v : v \quad \text{is measureable and} \quad \|v\|_{\omega^{\alpha,\beta}} < \infty \Big\},$$

where

$$\|v\|_{\omega^{\alpha,\beta}} = \left(\int_I \omega^{\alpha,\beta}(x) v^2(x) dx \right)^{\frac{1}{2}}.$$

Further, define

$$H^m_{\omega^{\alpha,\beta}}(I) = \Big\{ v : D^k v \in L^2_{\omega^{\alpha,\beta}}(I), \quad 0 \le k \le m \Big\},$$

equipped with the norm

$$\|v\|_{m,\omega^{\alpha,\beta}} = \left(\sum_{k=0}^{m} \|D^k v\|^2_{\omega^{\alpha,\beta}} \right)^{\frac{1}{2}},$$

with $D^k v = \frac{d^k v}{dx^k}$. When $\omega^{\alpha,\beta}(x) = 1, L^2_{\omega^{\alpha,\beta}}(I), H^m_{\omega^{\alpha,\beta}}(I)$ and $\|\cdot\|_{\omega^{\alpha,\beta}}$ are denoted simply by $L^2(I), H^m(I)$ and $\|\cdot\|$, respectively.

First we define the orthogonal projection $\pi_N^{\alpha,\beta} : L^2_{\omega^{\alpha,\beta}}(I) \to P_N$ such that, for any $u \in L^2_{\omega^{\alpha,\beta}}(I)$,

$$(\pi_N^{\alpha,\beta} u, v_N)_{\omega^{\alpha,\beta}} = (u, v_N)_{\omega^{\alpha,\beta}}, \qquad \forall v_N \in P_N. \tag{3.1}$$

Secondly, $I_N^{\alpha,\beta}$ denotes the interpolation operator of u based on $N+1-$degree Jacobi-Gauss points corresponding to the weight function $\omega^{\alpha,\beta}(x)$ with $\alpha, \beta > -1$.

In the following, we will give some useful lemmas which play a significant role in the convergence analysis later.

According to [3], we have the following lemmas.

Lemma 3.1. *Suppose that* $v \in H^m_{\omega^{\alpha,\beta}}(I)$.
(i) *If* $\alpha, \beta > -1$, *then for any* l *such that* $1 \le l \le m$:

$$\|v - \pi_N^{\alpha,\beta} v\|_{\omega^{\alpha,\beta}} \le C N^{-m} \|D^m v\|_{\omega^{\alpha,\beta}}, \tag{3.2}$$

$$\|v - \pi_N^{\alpha,\beta} v\|_{H^l_{\omega^{\alpha,\beta}}(I)} \le C N^{2l - \frac{1}{2} - m} \|D^m v\|_{\omega^{\alpha,\beta}}. \tag{3.3}$$

(ii) *If* $-1 < \alpha, \beta \le 0$, *then*

$$\|v - \pi_N^{\alpha,\beta} v\|_{L^\infty(I)} \le C N^{\frac{3}{4} - m} \|D^m v\|_{\omega^{\alpha,\beta}}. \tag{3.4}$$

Proof. The conclusion in (i) is a classical one, so we only prove (ii). It is straightforward that

$$\|w\| \le C \|w\|_{\omega^{\alpha,\beta}}, \quad \|w\|_{H^1(I)} \le C \|w\|_{H^1_{\omega^{\alpha,\beta}}}, \tag{3.5}$$

when $-1<\alpha,\beta\leq 0$. Consequently, the implementation of the Sobolev inequality ([3], p. 490)

$$\|w\|_{L^\infty(I)}\leq C\|w\|^{1/2}\|w\|^{1/2}_{H^1(I)} \tag{3.6}$$

and (3.5) leads to

$$\|w\|_{L^\infty(I)}\leq C\|w\|^{1/2}_{\omega^{\alpha,\beta}}\|w\|^{1/2}_{H^1_{\omega^{\alpha,\beta}}(I)} \tag{3.7}$$

for $-1<\alpha,\beta\leq 0$. In terms of the estimates in (i), we have, for $-1<\alpha,\beta\leq 0$,

$$\|v-\pi_N^{\alpha,\beta}v\|_{L^\infty(I)}\leq CN^{\frac{3}{4}-m}\|D^m v\|_{\omega^{\alpha,\beta}}. \tag{3.8}$$

This completes the proof. □

Lemma 3.2. *Suppose that $v\in H^m_{\omega^{\alpha,\beta}}(I)$.*
(i) *If $\alpha,\beta>-1$, then*

$$\|v-I_N^{\alpha,\beta}v\|_{\omega^{\alpha,\beta}}\leq CN^{-m}\|D^m v\|_{\omega^{\alpha,\beta}}. \tag{3.9}$$

(ii) *If $\omega^{\alpha,\beta}$ is the Legendre weight, i.e., $\alpha=\beta=0$, then for any l such that $1\leq l\leq m$:*

$$\|v-I_N^{0,0}v\|_{H^l(I)}\leq CN^{2l-\frac{1}{2}-m}\|D^m v\|, \tag{3.10}$$

$$\|v-I_N^{0,0}v\|_{L^\infty(I)}\leq CN^{\frac{3}{4}-m}\|D^m v\|. \tag{3.11}$$

(iii) *If $\omega^{\alpha,\beta}$ is the Chebyshev weight, i.e., $\alpha=\beta=-\frac{1}{2}$, then*

$$\|v-I_N^{-\frac{1}{2},-\frac{1}{2}}v\|_{L^\infty(I)}\leq CN^{\frac{1}{2}-m}\|D^m v\|_{\omega^{-\frac{1}{2},-\frac{1}{2}}}. \tag{3.12}$$

Proof. The conclusion in (i) is also a classical one. The first estimate in (ii) can be found in [3] (p. 289) and leads to the second estimate in (ii), by using (i) and the Sobolev inequality (3.6). The estimate in (iii) can be seen in [3] (p. 297). □

Lemma 3.3. *If $v\in H^m_\omega(I)$ with $\alpha,\beta>-1, m\geq 1$ and $\phi\in P^N$, then we have*

$$|(v,\phi)_{\omega^{\alpha,\beta}}-(v,\phi)_{\omega^{\alpha,\beta},N}|\leq CN^{-m}\|D^m v\|_{\omega^{\alpha,\beta}}\|\phi\|_{\omega^{\alpha,\beta}}.$$

Proof. As the discrete inner product is based on the $N+1$-degree Jacobi-Gauss points corresponding to the weight function $\omega^{\alpha,\beta}(x)$, we have

$$(v,\phi)_{\omega^{\alpha,\beta},N}=(I_N^{\alpha,\beta}v,\phi)_{\omega^{\alpha,\beta},N}=(I_N^{\alpha,\beta}v,\phi)_{\omega^{\alpha,\beta}}.$$

Therefore,

$$|(v,\phi)_{\omega^{\alpha,\beta}}-(v,\phi)_{\omega^{\alpha,\beta},N}|=|(v-I_N^{\alpha,\beta}v,\phi)_{\omega^{\alpha,\beta}}|\leq\|v-I_N^{\alpha,\beta}v\|_{\omega^{\alpha,\beta}}\|\phi\|_{\omega^{\alpha,\beta}},$$

which, combined with Lemma 3.2, implies the conclusion. □

Lemma 3.4. *For each bounded function $v(x)$, there exists a constant C, independent of v, such that*

$$\sup_N \|I_N^{\alpha,\beta} v\|_{\omega^{\alpha,\beta}} \le C\|v\|_\infty,$$

where $I_N^{\alpha,\beta} v = \sum_{n=0}^N v(x_n)h_n(x)$ is the interpolation of v, with $h_n(x)$, the Lagrange interpolation basis functions based on $(N+1)$-degree Jacobi-Gauss points corresponding to the weight function $\omega^{\alpha,\beta}(x)$ with $\alpha,\beta > -1$.

Proof. As the $(N+1)$-points Jacobi-Gauss quadrature formulas are accurate for the polynomials with degree no more than $2N+1$, direct calculation shows that

$$\begin{aligned}\|I_N^{\alpha,\beta} v\|_{\omega^{\alpha,\beta}} &= \int_{-1}^1 \omega^{\alpha,\beta}(x)\Big(I_N^{\alpha,\beta} v\Big)^2 dx = \sum_{n=0}^N v^2(x_n)\omega_n^{\alpha,\beta} \\ &\le \|v\|_\infty^2 \sum_{n=0}^N \omega_n^{\alpha,\beta} = \gamma_0 \|v\|_\infty^2,\end{aligned}$$

where $\gamma_0 = (\phi_0,\phi_0)_{\omega^{\alpha,\beta}}$. As a consequence,

$$\sup_N \|I_N^{\alpha,\beta} v\|_{\omega^{\alpha,\beta}} \le C\|v\|_\infty,$$

with $C = \sqrt{\gamma_0}$. □

Lemma 3.5. (Gronwall inequality) *If a non-negative integrable function $E(x)$ satisfies*

$$E(x) \le C_1 \int_{-1}^x E(s)ds + G(x), \qquad -1 \le x \le 1, \tag{3.13}$$

where $G(x)$ is an integrable function, then

$$\|E\|_{L^p_{\omega^{\alpha,\beta}}(I)} \le C\|G\|_{L^p_{\omega^{\alpha,\beta}}(I)}, \qquad p \ge 1, \tag{3.14}$$

where $\|E\|_{L^p_{\omega^{\alpha,\beta}}(I)} = \big(\int_{-1}^1 \omega^{\alpha,\beta}|E|^p dx\big)^{1/p}$ with $\alpha,\beta > -1$. In particular,

$$\|E\|_{L^\infty(I)} \le C\|G\|_{L^\infty(I)}. \tag{3.15}$$

Remark 3.1. The proof of Lemmas 3.1-3.4 has been shown in [7]. For completeness, we just list it above.

4. Convergence analysis for spectral Jacobi-Petrov-Galerkin method

According to (2.2) and the definition of the projection operator $\pi_N^{\alpha,\beta}$, the spectral Jacobi-Petrov-Galerkin solution u_N satisfies

$$u_N' + \pi_{N-1}^{\alpha,\beta} K u_N = \pi_{N-1}^{\alpha,\beta} g. \tag{4.1}$$

Theorem 4.1. *Suppose that u_N is the spectral Jacobi-Petrov-Galerkin solution determined by* (2.2) *with α and β satisfying one of the following conditions, i.e.,* (i) $-1 < \alpha, \beta < 1$; (ii) $\alpha = 0, \beta > -1$; (iii) $\alpha > -1, \beta = 0$; (iv) $\alpha > -1, -1 < \beta \leq 0$, *then we have the following error estimate*

$$\|u - u_N\|_{\omega^{\alpha,\beta}} \leq CN^{-m}\|D^{m+1}u\|_{\omega^{\alpha,\beta}}.$$

Proof. First we prove the existence and uniqueness of the spectral Petrov-Galerkin solution. When $g = 0$, (4.1) can be written as

$$u_N' + \pi_{N-1}^{\alpha,\beta} K u_N = 0.$$

In terms of the fact that

$$u_N' + \pi_{N-1}^{\alpha,\beta} K u_N = u_N' + K u_N - (K u_N - \pi_{N-1}^{\alpha,\beta} K u_N),$$

it is clear that, by integrating on $[-1, x], x \in (-1, 1)$, we have

$$u_N = -\int_{-1}^{x} K u_N ds + \int_{-1}^{x} (K u_N - \pi_{N-1}^{\alpha,\beta} K u_N) ds, \tag{4.2}$$

where $u_N(-1) = 0$ is used. On the other hand, as the kernel function $k(x, s)$ is smooth, we obtain

$$\begin{aligned}\left|\int_{-1}^{x} K u ds\right| \leq & \int_{-1}^{x}\int_{-1}^{s} |k(s,t)u(t)| dt ds \\ \leq & C\int_{-1}^{x}\int_{-1}^{s} |u(t)| dt ds \leq C\int_{-1}^{x} |u(s)| ds.\end{aligned} \tag{4.3}$$

The combination of (4.2) and (4.3) leads to

$$|u_N| \leq C\int_{-1}^{x} |u_N(s)| ds + \left|\int_{-1}^{x} J_1 ds\right|,$$

with $J_1 = K u_N - \pi_{N-1}^{\alpha,\beta} K u_N$. This, combined with Lemma 3.5, leads to

$$\|u_N\|_{\omega^{\alpha,\beta}} \leq C\left\|\int_{-1}^{x} J_1 ds\right\|_{\omega^{\alpha,\beta}}. \tag{4.4}$$

We will use the claim that

$$\left\|\int_{-1}^{x} u(s) ds\right\|_{\omega^{\alpha,\beta}}^2 \leq C\|u\|_{\omega^{\alpha,\beta}}^2, \tag{4.5}$$

when α and β satisfy one of (i)-(iv). Actually, when α and β satisfy one of the assumptions (i)-(iii), (4.5) holds according to [6] (p. 239). On the other hand, if α and β satisfy (iv), then

$$\begin{aligned}\left\|\int_{-1}^{x} u(s)ds\right\|^2_{\omega^{\alpha,\beta}} &= \int_{-1}^{1} \omega^{\alpha,\beta}(x)\left(\int_{-1}^{x} u(s)ds\right)^2 dx \\ &\le C\int_{-1}^{1} \omega^{\alpha,\beta}(x)\int_{-1}^{x} u^2(s)dsdx \\ &= C\int_{-1}^{1} u^2(s)\int_{s}^{1}(1-x)^{\alpha}(1+x)^{\beta}dxds \\ &\le C\int_{-1}^{1} u^2(s)(1+s)^{\beta}\int_{s}^{1}(1-x)^{\alpha}dxds \\ &\le C\int_{-1}^{1}(1-s)^{\alpha}(1+s)^{\beta}u^2(s)ds = C\|u\|^2_{\omega^{\alpha,\beta}}. \end{aligned} \tag{4.6}$$

In virtue of Lemma 3.1 and (4.5),

$$\begin{aligned}\left\|\int_{-1}^{x} J_1 ds\right\|_{\omega^{\alpha,\beta}} &\le C\|J_1\|_{\omega^{\alpha,\beta}} \le CN^{-1}\left\|k(x,x)u_N(x)+\int_{-1}^{x}\partial_x k(x,s)u_N(s)ds\right\|_{\omega^{\alpha,\beta}} \\ &\le CN^{-1}\left(\|u_N\|_{\omega^{\alpha,\beta}}+\left\|\int_{-1}^{x}u_N(s)ds\right\|_{\omega^{\alpha,\beta}}\right) \le CN^{-1}\|u_N\|_{\omega^{\alpha,\beta}}. \end{aligned} \tag{4.7}$$

The combination of (4.4) and (4.7) leads to

$$\|u_N\|_{\omega^{\alpha,\beta}} \le CN^{-1}\|u_N\|_{\omega^{\alpha,\beta}},$$

which implies, when N is large enough such that $C/N < 1$, $u_N = 0$. Hence, the spectral Petrov-Galerkin solution u_N is existent and unique as V_N is finite-dimensional.

Subtracting (4.1) from (2.1) yields

$$u' - u'_N + Ku - \pi^{\alpha,\beta}_{N-1}Ku_N = g - \pi^{\alpha,\beta}_{N-1}g. \tag{4.8}$$

Set $e = u - u_N$. Direct computation shows that

$$\begin{aligned} &Ku - \pi^{\alpha,\beta}_{N-1}Ku_N \\ =&Ku - \pi^{\alpha,\beta}_{N-1}Ku + \pi^{\alpha,\beta}_{N-1}K(u-u_N) \\ =&Ku - \pi^{\alpha,\beta}_{N-1}Ku + K(u-u_N) - \left(K(u-u_N) - \pi^{\alpha,\beta}_{N-1}K(u-u_N)\right) \\ =&(g-u') - \pi^{\alpha,\beta}_{N-1}(g-u') + Ke - (Ke - \pi^{\alpha,\beta}_{N-1}Ke) \\ =&g - \pi^{\alpha,\beta}_{N-1}g - (u' - \pi^{\alpha,\beta}_{N-1}u') + Ke - (Ke - \pi^{\alpha,\beta}_{N-1}Ke). \end{aligned} \tag{4.9}$$

Inserting (4.9) into (4.8) yields

$$e' + Ke - (u' - \pi_{N-1}^{\alpha,\beta} u') - (Ke - \pi_{N-1}^{\alpha,\beta} Ke) = 0. \tag{4.10}$$

By integrating on $[-1, x], x \in (-1, 1)$, we obtain

$$e(x) = -\int_{-1}^{x} Keds + \int_{-1}^{x} (u' - \pi_{N-1}^{\alpha,\beta} u')ds + \int_{-1}^{x} (Ke - \pi_{N-1}^{\alpha,\beta} Ke)ds, \tag{4.11}$$

where $e(-1) = 0$ is used. Therefore,

$$|e| \leq \left|\int_{-1}^{x} Keds\right| + \left|\int_{-1}^{x} (u' - \pi_{N-1}^{\alpha,\beta} u')ds\right| + \left|\int_{-1}^{x} (Ke - \pi_{N-1}^{\alpha,\beta} Ke)ds\right|, \tag{4.12}$$

which, together with (4.3), leads to

$$|e| \leq C \int_{-1}^{x} |e|ds + \left|\int_{-1}^{x} J_2 ds\right| + \left|\int_{-1}^{x} J_3 ds\right|, \tag{4.13}$$

with $J_2 = u' - \pi_{N-1}^{\alpha,\beta} u', J_3 = Ke - \pi_{N-1}^{\alpha,\beta} Ke$.

By (4.13), (4.6) and Lemma 3.5, we have

$$\|e\|_{\omega^{\alpha,\beta}} \leq C \left(\left\| \int_{-1}^{x} J_2 ds \right\|_{\omega^{\alpha,\beta}} + \left\| \int_{-1}^{x} J_3 ds \right\|_{\omega^{\alpha,\beta}} \right) \leq C \Big(\|J_2\|_{\omega^{\alpha,\beta}} + \|J_3\|_{\omega^{\alpha,\beta}} \Big). \tag{4.14}$$

By Lemma 3.1 and (4.5),

$$\|J_2\|_{\omega^{\alpha,\beta}} = \|u' - \pi_{N-1}^{\alpha,\beta} u'\|_{\omega^{\alpha,\beta}} \leq CN^{-m} \|D^{m+1} u\|_{\omega^{\alpha,\beta}}, \tag{4.15}$$

$$\begin{aligned} \|J_3\|_{\omega^{\alpha,\beta}} &\leq CN^{-1} \left\| k(x,x)e(x) + \int_{-1}^{x} \partial_x k(x,s)e(s)ds \right\|_{\omega^{\alpha,\beta}} \\ &\leq CN^{-1} \left(\|e\|_{\omega^{\alpha,\beta}} + \| \int_{-1}^{x} |e|ds \|_{\omega^{\alpha,\beta}} \right) \\ &\leq CN^{-1} \|e\|_{\omega^{\alpha,\beta}}, \end{aligned} \tag{4.16}$$

under the assumptions on α and β above. Combining (4.14)–(4.16), we obtain, when N is large enough such that $C/N < 1$,

$$\|e\|_{\omega^{\alpha,\beta}} = \|u - u_N\|_{\omega^{\alpha,\beta}} \leq CN^{-m} \|D^{m+1} u\|_{\omega^{\alpha,\beta}}.$$

This completes the proof. □

Now we investigate the L^{∞}-error estimate.

Theorem 4.2. *Suppose that u_N is the spectral Jacobi-Petrov-Galerkin solution determined by* (2.2) *with* $-1<\alpha,\beta\le 0$. *Then we have the following error estimate*

$$\|u-u_N\|_{L^\infty}\le CN^{\frac{3}{4}-m}\|D^{m+1}u\|_{\omega^{\alpha,\beta}}. \tag{4.17}$$

Proof. In terms of Lemma 3.5 and (4.13),

$$\|e\|_{L^\infty}\le C\left(\left\|\int_{-1}^{x}J_2ds\right\|_{L^\infty}+\left\|\int_{-1}^{x}J_3ds\right\|_{L^\infty}\right)\le C\Big(\|J_2\|_{L^\infty}+\|J_3\|_{L^\infty}\Big). \tag{4.18}$$

By Lemma 3.1,

$$\begin{aligned}\|J_2\|_{L^\infty(I)}\le& CN^{\frac{3}{4}-m}\|D^{m+1}u\|_{\omega^{\alpha,\beta}}, &(4.19)\\ \|J_3\|_{L^\infty(I)}\le& CN^{-\frac{1}{4}}\left\|k(x,x)e(x)+\int_{-1}^{x}\partial_x k(x,s)e(s)ds\right\|_{\omega^{\alpha,\beta}}\\ \le& CN^{-\frac{1}{4}}\|e\|_{L^\infty(I)}. &(4.20)\end{aligned}$$

Combining (4.18)-(4.20), we obtain, when N is large enough such that $CN^{-1/4}<1$, the desired result (4.17). □

5. Convergence analysis for pseudo-spectral Jacobi-Petrov-Galerkin method

As $I_{N-1}^{\alpha,\beta}$ is the interpolation operator which is based on the N-degree Jacobi-Gauss points, in terms of (2.9), the pseudo-spectral Petrov-Galerkin solution $u_N^{(1)}\in V_N$ satisfies

$$\begin{aligned}&\left(I_{N-1}^{\alpha,\beta}\sum_{n=0}^{N-1}\widetilde{k}(x,s(x,\theta_n))u_N^{(1)}(s(x,\theta_n))v_n,v_N\right)_{\omega^{\alpha,\beta}}+({u_N^{(1)}}',v_N)_{\omega^{\alpha,\beta}}\\ =&(I_{N-1}^{\alpha,\beta}g,v_N)_{\omega^{\alpha,\beta}},\qquad \forall v_N\in P_{N-1},\end{aligned} \tag{5.1}$$

where

$$\begin{aligned}&\sum_{n=0}^{N-1}\widetilde{k}(x,s(x,\theta_n))u_N^{(1)}(s(x,\theta_n))v_n\\ =&\int_{-1}^{1}\widetilde{k}(x,s(x,\theta))u_N^{(1)}(s(x,\theta))d\theta\\ &-\left(\int_{-1}^{1}\widetilde{k}(x,s(x,\theta))u_N^{(1)}(s(x,\theta))d\theta-\sum_{n=0}^{N-1}\widetilde{k}(x,s(x,\theta_n))u_N^{(1)}(s(x,\theta_n))v_n\right)\\ =&\int_{-1}^{x}k(x,s)u_N^{(1)}(s)ds-Q(x)=Ku_N^{(1)}-Q(x),\end{aligned} \tag{5.2}$$

with

$$
\begin{aligned}
Q(x)=&\int_{-1}^{1}\widetilde{k}(x,s(x,\theta))u_N^{(1)}(s(x,\theta))d\theta-\sum_{n=0}^{N-1}\widetilde{k}(x,s(x,\theta_n))u_N^{(1)}(s(x,\theta_n))v_n\\
=&\Big(\widetilde{k}(x,s(x,\cdot)),u_N^{(1)}(s(x,\cdot))\Big)-\Big(\widetilde{k}(x,s(x,\cdot)),u_N^{(1)}(s(x,\cdot))\Big)_{N-1},
\end{aligned}
\tag{5.3}
$$

in which $(\cdot,\cdot)$ represents the continuous inner product with respect to θ and $(\cdot,\cdot)_{N-1}$ is the corresponding discrete inner product defined by the N-degree Gauss-Legendre quadrature formula. The combination of (5.1) and (5.2), yields

$$
\begin{aligned}
&(u_N^{(1)'},v_N)_{\omega^{\alpha,\beta}}+(I_{N-1}^{\alpha,\beta}Ku_N^{(1)}-I_{N-1}^{\alpha,\beta}Q,v_N)_{\omega^{\alpha,\beta}}\\
=&(I_{N-1}^{\alpha,\beta}g,v_N)_{\omega^{\alpha,\beta}},\qquad \forall v_N\in P_{N-1},
\end{aligned}
\tag{5.4}
$$

which gives rise to

$$
u_N^{(1)'}+I_{N-1}^{\alpha,\beta}Ku_N^{(1)}-I_{N-1}^{\alpha,\beta}Q=I_{N-1}^{\alpha,\beta}g. \tag{5.5}
$$

By the discussion above, (2.9), (5.1) and (5.5) are equivalent.

We first consider an auxiliary problem, i.e., we want to find $R_Nu\in V_N$ such that

$$
((R_Nu)',v_N)_{\omega^{\alpha,\beta},N-1}+(KR_Nu,v_N)_{\omega^{\alpha,\beta},N-1}=(g,v_N)_{\omega^{\alpha,\beta},N-1},\quad \forall v_N\in P_{N-1}. \tag{5.6}
$$

In terms of the definition of $I_{N-1}^{\alpha,\beta}$, (5.6) can be written as

$$
((R_Nu)',v_N)_{\omega^{\alpha,\beta}}+(I_{N-1}^{\alpha,\beta}KR_Nu,v_N)_{\omega^{\alpha,\beta}}=(I_{N-1}^{\alpha,\beta}g,v_N)_{\omega^{\alpha,\beta}},\quad \forall v_N\in P_{N-1}, \tag{5.7}
$$

which is equivalent to

$$
(R_Nu)'+I_{N-1}^{\alpha,\beta}KR_Nu=I_{N-1}^{\alpha,\beta}g. \tag{5.8}
$$

Lemma 5.1. *Suppose R_Nu is determined by* (5.6).
(i) *If α and β satisfy one the following assumptions, i.e., $-1<\alpha,\beta<1$, or $\alpha=0,\beta>-1$, or $\alpha>-1,\beta=0$, or $-1<\beta\le 0,\alpha>-1$, we have*

$$
\|u-R_Nu\|_{\omega^{\alpha,\beta}}\le CN^{-m}\|D^{m+1}u\|_{\omega^{\alpha,\beta}}.
$$

(ii) *If $\omega^{\alpha,\beta}(x)$ is the Legendre weight, i.e., $\alpha=\beta=0$, then we have*

$$
\|u-R_Nu\|_{L^\infty}\le CN^{\frac{3}{4}-m}\|D^{m+1}u\|.
$$

If $\omega^{\alpha,\beta}(x)$ is the Chebyshev weight, i.e., $\alpha=\beta=-\frac{1}{2}$, then we have

$$
\|u-R_Nu\|_{L^\infty}\le CN^{\frac{1}{2}-m}\|D^{m+1}u\|_{\omega^{-\frac{1}{2},-\frac{1}{2}}}.
$$

Proof. (i) The existence and uniqueness of $R_N u$ and the $L^2_{\omega^{\alpha,\beta}}$ error estimate of $u - R_N u$ can be verified in a similar way as those for the spectral Jacobi-Petrov-Galerkin solution u_N in the proof of Theorem 4.1, with $\pi^{\alpha,\beta}_{N-1}$ replaced by $I^{\alpha,\beta}_{N-1}$. For simplicity, we omit it here.

(ii) Subtracting (5.8) from (2.1), yields

$$u' - (R_N u)' + Ku - I^{\alpha,\beta}_{N-1} K R_N u = g - I^{\alpha,\beta}_{N-1} g. \tag{5.9}$$

Set $\epsilon = u - R_N u$. Direct computation shows that

$$\begin{aligned}
&Ku - I^{\alpha,\beta}_{N-1} K R_N u \\
=&Ku - I^{\alpha,\beta}_{N-1} Ku + I^{\alpha,\beta}_{N-1} K(u - R_N u) \\
=&Ku - I^{\alpha,\beta}_{N-1} Ku + K(u - R_N u) - \Big(K(u - R_N u) - I^{\alpha,\beta}_{N-1} K(u - R_N u) \Big) \\
=&(g - u') - I^{\alpha,\beta}_{N-1}(g - u') + K\epsilon - (K\epsilon - I^{\alpha,\beta}_{N-1} K\epsilon) \\
=&g - I^{\alpha,\beta}_{N-1} g - (u' - I^{\alpha,\beta}_{N-1} u') + K\epsilon - (K\epsilon - I^{\alpha,\beta}_{N-1} K\epsilon).
\end{aligned} \tag{5.10}$$

Inserting (5.10) into (5.9) yields

$$\epsilon' - (u' - I^{\alpha,\beta}_{N-1} u') + K\epsilon - (K\epsilon - I^{\alpha,\beta}_{N-1} K\epsilon) = 0. \tag{5.11}$$

By integrating on $[-1, x], x \in (-1, 1)$, we obtain

$$\epsilon = -\int_{-1}^{x} K\epsilon ds + \int_{-1}^{x} J_4 ds + \int_{-1}^{x} J_5 ds,$$

which implies

$$\begin{aligned}
|\epsilon| \le& \Big| \int_{-1}^{x} K\epsilon ds \Big| + \Big| \int_{-1}^{x} J_4 ds \Big| + \Big| \int_{-1}^{x} J_5 ds \Big| \\
\le& \int_{-1}^{x} |\epsilon| ds + \Big| \int_{-1}^{x} J_4 ds \Big| + \Big| \int_{-1}^{x} J_5 ds \Big|,
\end{aligned} \tag{5.12}$$

where $J_4 = u' - I^{\alpha,\beta}_{N-1} u', J_5 = K\epsilon - I^{\alpha,\beta}_{N-1} K\epsilon$. Here (4.3) and $\epsilon(-1) = 0$ is used. By Lemma 3.5,

$$\|\epsilon\|_{L^\infty} \le C \Big(\Big\| \int_{-1}^{x} J_4 ds \Big\|_{L^\infty} + \Big\| \int_{-1}^{x} J_5 ds \Big\|_{L^\infty} \Big) \le C \Big(\|J_4\|_{L^\infty} + \|J_5\|_{L^\infty} \Big). \tag{5.13}$$

Actually, by Lemma 3.2,

$$\|J_4\|_{L^\infty} \le C N^{\theta - m} \|D^{m+1} u\|_{\omega^{\alpha,\beta}}, \tag{5.14}$$

where $\theta = \frac{3}{4}$ when $\omega^{\alpha,\beta}$ is the Legendre weight, and $\theta = \frac{1}{2}$ when $\omega^{\alpha,\beta}$ is the Chebyshev weight. Besides,

$$\|J_5\|_{L^\infty} \le C N^{-\gamma} \Big\| k(x,x)\epsilon(x) + \int_{-1}^{x} \partial_x k(x,s)\epsilon(s) ds \Big\|_{\omega^{\alpha,\beta}} \le C N^{-\gamma} \|\epsilon\|_{L^\infty}, \tag{5.15}$$

where $\gamma = \frac{1}{4}$ when $\omega^{\alpha,\beta}$ is the Legendre weight, and $\gamma = \frac{1}{2}$ when $\omega^{\alpha,\beta}$ is the Chebyshev weight. Combining (5.13), (5.14) and (5.15), when N is big enough such that $CN^{-\gamma} < 1$, we obtain

$$\|\epsilon\|_{L^\infty} \le CN^{\theta-m}\|D^{m+1}u\|_{\omega^{\alpha,\beta}}.$$

□

Subtracting (5.5) from (5.8), leads to

$$(R_N u)' - u_N^{(1)'} + I_{N-1}^{\alpha,\beta}K(R_N u - u_N^{(1)}) + I_{N-1}^{\alpha,\beta}Q = 0,$$

which can be simplified as, by setting $E = R_N u - u_N^{(1)}$,

$$E' + I_{N-1}^{\alpha,\beta}KE + I_{N-1}^{\alpha,\beta}Q = 0. \tag{5.16}$$

Theorem 5.1. *Suppose that the solution of* (2.1) *is sufficiently smooth. For the pseudo-spectral Jacobi-Petrov-Galerkin solution* $u_N^{(1)}$ *satisfying* (2.9)*, the following results hold:*
(i) *if* $-1 < \alpha, \beta \le 0$*, we have*

$$\|u - u_N^{(1)}\|_{\omega^{\alpha,\beta}} \le CN^{-m}\|D^{m+1}u\|_{\omega^{\alpha,\beta}} + CM_m N^{-m}\|u\|_{\omega^{\alpha,\beta}}.$$

(ii) *if* $0 < \alpha = \beta < 1$*, we have*

$$\|u - u_N^{(1)}\|_{\omega^{\alpha,\beta}} \le CN^{-m}\|D^{m+1}u\|_{\omega^{\alpha,\beta}} + CM_m N^{-m+\alpha}\|u\|_{\omega^{\alpha,\beta}},$$

where $M_m = \max_{x\in I}(\frac{x+1}{2})^m(\int_{-1}^x |\partial_s^m k(x,s)|^2 ds)^{\frac{1}{2}}$.

Proof. We first prove the existence and uniqueness of the pseudo-spectral Jacobi-Petrov-Galerkin solution. As the dimension of V_N is finite and (2.9) and (5.5) are equivalent, we only need to prove that the solution of (5.5) is $u_N^{(1)} = 0$ when $g = 0$. For this purpose, we consider the equation

$$u_N^{(1)'} + I_{N-1}^{\alpha,\beta}Ku_N^{(1)} - I_{N-1}^{\alpha,\beta}Q = 0,$$

which can be written as

$$u_N^{(1)'} + Ku_N^{(1)} - I_{N-1}^{\alpha,\beta}Q - (Ku_N^{(1)} - I_{N-1}^{\alpha,\beta}Ku_N^{(1)}) = 0.$$

Set

$$J_6 = I_{N-1}^{\alpha,\beta}Q, \qquad J_7 = Ku_N^{(1)} - I_{N-1}^{\alpha,\beta}Ku_N^{(1)}.$$

Using the same technique in the proof of Theorem 4.1, leads to

$$|u_N^{(1)}(x)| \le C\int_{-1}^x |u_N^{(1)}|ds + \Big|\int_{-1}^x J_6 ds\Big| + \Big|\int_{-1}^x J_7 ds\Big|.$$

By Lemma 3.5 and (4.5), under the assumptions on α and β above, we have

$$\begin{aligned}\|u_N^{(1)}\|_{\omega^{\alpha,\beta}} \le& C\Big(\Big\|\int_{-1}^{x} J_6 ds\Big\|_{\omega^{\alpha,\beta}} + \Big\|\int_{-1}^{x} J_7 ds\Big\|_{\omega^{\alpha,\beta}}\Big)\\ \le& C\Big(\|J_6\|_{\omega^{\alpha,\beta}} + \|J_7\|_{\omega^{\alpha,\beta}}\Big). \end{aligned} \tag{5.17}$$

On the other hand, according to Lemma 3.4,

$$\|J_6\|_{\omega^{\alpha,\beta}} = \|I_{N-1}^{\alpha,\beta} Q(x)\|_{\omega^{\alpha,\beta}} \le C\|Q(x)\|_{L^\infty(I)}. \tag{5.18}$$

By the expression of $Q(x)$ in (5.3) and Lemma 3.3, we have

$$\begin{aligned}|Q(x)| &\le CN^{-m}\frac{x+1}{2}\|\partial_\theta^m k(x,s(x,\cdot))\|\|u_N^{(1)}(s(x,\cdot))\|\\ &\le CN^{-m}\Big(\frac{x+1}{2}\Big)^m\Big(\int_{-1}^{x}|\partial_s^m k(x,s)|^2 ds\Big)^{\frac{1}{2}}\Big(\int_{-1}^{x}|u_N^{(1)}(s)|^2 ds\Big)^{\frac{1}{2}}.\end{aligned}$$

Set $M_m = \max_{x\in I}(\frac{x+1}{2})^m(\int_{-1}^{x}|\partial_s^m k(x,s)|^2 ds)^{\frac{1}{2}}$. Therefore,

$$\|Q(x)\|_{L^\infty(I)} \le CM_m N^{-m}\|u_N^{(1)}\|, \tag{5.19}$$

which, together with (5.18), yields

$$\|J_6\|_{\omega^{\alpha,\beta}} \le CM_m N^{-m}\|u_N^{(1)}\|. \tag{5.20}$$

If $-1 < \alpha, \beta \le 0$, obviously, we have

$$\|u_N^{(1)}\| \le C\|u_N^{(1)}\|_{\omega^{\alpha,\beta}}.$$

Therefore,

$$\|J_6\|_{\omega^{\alpha,\beta}} \le CM_m N^{-m}\|u_N^{(1)}\|_{\omega^{\alpha,\beta}}. \tag{5.21}$$

On the other hand, according to ([3], p. 282),

$$\|\phi\| \le CN^{\alpha}\|\phi\|_{\omega^{\alpha,\alpha}}, \quad \forall \phi \in P_N,$$

where $\omega^{\alpha,\alpha}(x) = (1-x^2)^\alpha$ with $\alpha \ge 0$ and C is a positive constant independent of N. Hence, when $0 < \alpha = \beta < 1$,

$$\|J_6\|_{\omega^{\alpha,\beta}} \le CM_m N^{-m+\alpha}\|u_N^{(1)}\|_{\omega^{\alpha,\beta}}. \tag{5.22}$$

The implementation of Lemma 3.2 implies

$$\begin{aligned}\|J_7\|_{\omega^{\alpha,\beta}} \le& CN^{-1}\left\|k(x,x)u_N^{(1)}(x) + \int_{-1}^{x}\partial_x k(x,s)u_N^{(1)}(s)ds\right\|_{\omega^{\alpha,\beta}}\\ \le& CN^{-1}\left(\|u_N^{(1)}\|_{\omega^{\alpha,\beta}} + \left\|\int_{-1}^{x} u_N^{(1)}(s)ds\right\|_{\omega^{\alpha,\beta}}\right)\\ \le& CN^{-1}\|u_N^{(1)}\|_{\omega^{\alpha,\beta}}, \end{aligned} \tag{5.23}$$

where (4.5) is used again under the assumptions on α and β. Based on (5.17), (5.21) and (5.23), when $-1<\alpha,\beta\le 0$ and $C(N^{-1}+M_mN^{-m})<1$, $u_N^{(1)}=0$. On the other hand, by (5.17), (5.22) and (5.23), when $0<\alpha=\beta<1$ and $C(N^{-1}+M_mN^{-m+\alpha})<1$, $u_N^{(1)}=0$. As a result, the existence and uniqueness of the pseudo-spectral Jacobi-Petrov-Galerkin solution $u_N^{(1)}$ is proved.

Now we turn to the $L^2_{\omega^{\alpha,\beta}}$ error estimate. Actually (5.16) can be transformed into

$$E'+KE+I_{N-1}^{\alpha,\beta}Q-(KE-I_{N-1}^{\alpha,\beta}KE)=0,$$

which yields

$$|E|\le C\int_{-1}^{x}|E(s)|ds+\left|\int_{-1}^{x}J_6ds\right|+\left|\int_{-1}^{x}J_8ds\right|, \tag{5.24}$$

with $J_8=KE-I_{N-1}^{\alpha,\beta}KE$. By Lemma 3.5 and (4.5), under the assumptions on α and β, we have

$$\begin{aligned}\|E\|_{\omega^{\alpha,\beta}}\le& C\left(\left\|\int_{-1}^{x}J_6ds\right\|_{\omega^{\alpha,\beta}}+\left\|\int_{-1}^{x}J_8ds\right\|_{\omega^{\alpha,\beta}}\right)\\ \le& C\left(\|J_6\|_{\omega^{\alpha,\beta}}+\|J_8\|_{\omega^{\alpha,\beta}}\right).\end{aligned} \tag{5.25}$$

By Lemma 3.2 and (4.5),

$$\begin{aligned}\|J_8\|_{\omega^{\alpha,\beta}}\le& CN^{-1}\left\|k(x,x)E(x)+\int_{-1}^{x}\partial_x k(x,s)E(s)ds\right\|_{\omega^{\alpha,\beta}}\\ \le& CN^{-1}\|E\|_{\omega^{\alpha,\beta}}.\end{aligned} \tag{5.26}$$

In terms of (5.21), (5.25) and (5.26), when $-1<\alpha,\beta\le 0$, we have

$$\|E\|_{\omega^{\alpha,\beta}}\le CM_mN^{-m}\|u_N^{(1)}\|_{\omega^{\alpha,\beta}}\le CM_mN^{-m}\left(\|e^{(1)}\|_{\omega^{\alpha,\beta}}+\|u\|_{\omega^{\alpha,\beta}}\right), \tag{5.27}$$

when $C/N<1$. By (5.22), (5.25) and (5.26), when $0<\alpha=\beta<1$, we have

$$\|E\|_{\omega^{\alpha,\beta}}\le CM_mN^{-m+\alpha}\|u_N^{(1)}\|_{\omega^{\alpha,\beta}}\le CM_mN^{-m+\alpha}\left(\|e^{(1)}\|_{\omega^{\alpha,\beta}}+\|u\|_{\omega^{\alpha,\beta}}\right). \tag{5.28}$$

when $C/N<1$. By the triangular inequality,

$$\|u-u_N^{(1)}\|_{\omega^{\alpha,\beta}}\le\|u-R_Nu\|_{\omega^{\alpha,\beta}}+\|R_Nu-u_N^{(1)}\|_{\omega^{\alpha,\beta}}, \tag{5.29}$$

together with Lemma 5.1, (5.27) and (5.28), the desired conclusions can be obtained when $CM_mN^{-m}<1$ or $CM_mN^{-m+\alpha}<1$ for $-1<\alpha,\beta\le 0$ or $0<\alpha=\beta<1$, respectively. □

Theorem 5.2. *Suppose that the solution of* (2.1) *is sufficiently smooth. For the pseudo-spectral Jacobi-Petrov-Galerkin solution* $u_N^{(1)}$, *such that* (2.9) *holds, we have the following estimates:*
(i) *when* $\omega^{\alpha,\beta}$ *is the Legendre weight,*

$$\|u-u_N^{(1)}\|_{L^\infty} \le CN^{\frac{3}{4}-m}\|D^{m+1}u\|_{\omega^{\alpha,\beta}} + CM_mN^{\frac{1}{2}-m}\|u\|;$$

(ii) *when* $\omega^{\alpha,\beta}$ *is the Chebyshev weight,*

$$\|u-u_N^{(1)}\|_{L^\infty} \le CN^{\frac{1}{2}-m}\|D^{m+1}u\|_{\omega^{\alpha,\beta}} + CM_mN^{-m}\log N\|u\|,$$

where $M_m = \max\limits_{x\in I}(\frac{x+1}{2})^m(\int_{-1}^x |\partial_s^m k(x,s)|^2 ds)^{\frac{1}{2}}$.

Proof. Implementing Lemma 3.5 and (5.24), we have

$$\|E\|_{L^\infty} \le C\left(\left\|\int_{-1}^x J_6 ds\right\|_{L^\infty} + \left\|\int_{-1}^x J_8 ds\right\|_{L^\infty}\right) \le C\Big(\|J_6\|_{L^\infty} + \|J_8\|_{L^\infty}\Big). \tag{5.30}$$

On the other hand,

$$\begin{aligned}\|J_6\|_{L^\infty(I)} &= \|I_N^{\alpha,\beta}Q(x)\|_{L^\infty(I)} \\ &\le \max_{0\le n\le N}|Q(x_n)|\max_I\sum_{n=0}^N|h_n(x)| \\ &\le CM_mN^{-m}L_N(\alpha,\beta)\|u_N^{(1)}\|,\end{aligned} \tag{5.31}$$

with $L_N(\alpha,\beta) = L_N(0,0) = \mathscr{O}(N^{\frac{1}{2}})$ for the Legendre case and $L_N(\alpha,\beta) = L_N(-\frac{1}{2},-\frac{1}{2}) = \mathscr{O}(\log N)$ for the Chebyshev case. According to Lemma 3.2,

$$\|J_8\|_{L^\infty} \le CN^{-\eta}\left\|k(x,x)E(x) + \int_{-1}^x \partial_x k(x,s)E(s)ds\right\|_{\omega^{\alpha,\beta}} \le CN^{-\eta}\|E\|_{L^\infty}, \tag{5.32}$$

with $\eta = \frac{1}{4}$ when $\omega^{\alpha,\beta}(x)$ is the Legendre weight and $\eta = \frac{1}{2}$ when $\omega^{\alpha,\beta}(x)$ is the Chebyshev weight. Combining (5.30)–(5.32) yields

$$\|E\|_{L^\infty} \le CM_mN^{-m}L_N(\alpha,\beta)\Big(\|u\| + \|e^{(1)}\|_{L^\infty}\Big), \tag{5.33}$$

provided that $CN^{-\eta} < 1$. By the triangular inequality,

$$\|u-u_N^{(1)}\|_{L^\infty} \le \|u-R_Nu\|_{L^\infty} + \|R_Nu-u_N^{(1)}\|_{L^\infty}. \tag{5.34}$$

Implementing Lemma 5.1, (5.33) and (5.34), we have

$$\|u-u_N^{(1)}\|_{L^\infty} \le CN^{\gamma-m}\|D^{m+1}u\|_{\omega^{\alpha,\beta}} + CM_mN^{-m}L_N(\alpha,\beta)\Big(\|u\| + \|e^{(1)}\|_{L^\infty}\Big),$$

where

$$\gamma = \frac{3}{4}, \qquad L_N(\alpha,\beta) = L_N(0,0) = \mathcal{O}(N^{\frac{1}{2}})$$

for the Legendre case and

$$\gamma = \frac{1}{2}, \qquad L_N(\alpha,\beta) = L_N\Big(-\frac{1}{2},-\frac{1}{2}\Big) = \mathcal{O}(\log N)$$

for the Chebyshev case. As a result, we obtain

$$\|u-u_N^{(1)}\|_{L^\infty} \le CN^{\frac{3}{4}-m}\|D^{m+1}u\|_{\omega^{\alpha,\beta}} + CM_mN^{\frac{1}{2}-m}\|u\|,$$

when $\omega^{\alpha,\beta}(x)$ is the Legendre weight and $CM_mN^{\frac{1}{2}-m} < 1$, and

$$\|u-u_N^{(1)}\|_{L^\infty} \le CN^{\frac{1}{2}-m}\|D^{m+1}u\|_{\omega^{\alpha,\beta}} + CM_mN^{-m}\log N\|u\|,$$

when $\omega^{\alpha,\beta}(x)$ is the Chebyshev weight and $CM_mN^{-m}\log N < 1$. □

6. Numerical experiments

The efficiency of spectral or pseudo-spectral Legendre-Petrov-Galerkin methods and Chebyshev-Petrov-Galerkin methods will be demonstrated in the following as two special cases of the spectral or pseudo-spectral Jacobi-Petrov-Galerkin approaches.

Example 6.1. Consider the Volterra integro-differential equation (1.1) with $k(x,s) = \frac{9}{4}\exp(\frac{9(1+x)(1+s)}{4})$. The corresponding exact solution is given by $u(x) = e^{\frac{3}{2}(1+x)} - 1$.

First we implement the numerical scheme (2.3) based on the spectral Legendre-Petrov-Galerkin and Chebyshev-Petrov-Galerkin methods to solve this example. Table 1 illustrates the L^∞ and L^2 errors of the spectral Legendre-Petrov-Galerkin method which are also shown in Fig. 1. Next the L^∞ and $L^2_{\omega^{\alpha,\beta}}$ errors of the spectral Chebyshev-Petrov-Galerkin method are demonstrated in Table 2 and Fig. 2. Clearly the desired spectral accuracy is obtained in these approaches.

Table 1: The errors of spectral Legendre-Petrov-Galerkin method.

N	4	6	8	10	12	14
L^∞-error	4.80e-2	8.48e-4	8.47e-6	5.28e-8	2.21e-10	6.86e-13
L^2-error	2.67e-2	3.41e-4	2.61e-6	1.32e-8	4.74e-11	2.86e-13

Table 2: The errors of spectral Chebyshev-Petrov-Galerkin method.

N	4	6	8	10	12	14
L^∞-error	4.20e-2	5.56e-4	5.20e-6	2.93e-8	1.11e-10	1.97e-12
L^2_ω-error	4.40e-2	5.61e-4	4.12e-6	2.10e-8	7.37e-11	1.85e-12

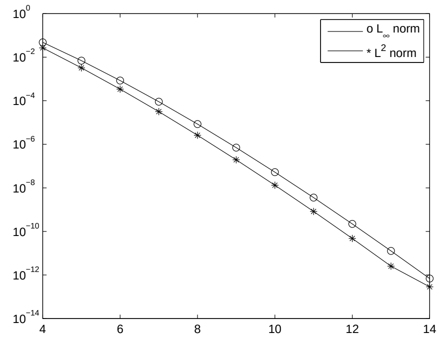

Figure 1: L^∞ and L^2 errors of spectral Legendre-Petrov-Galerkin method versus N.

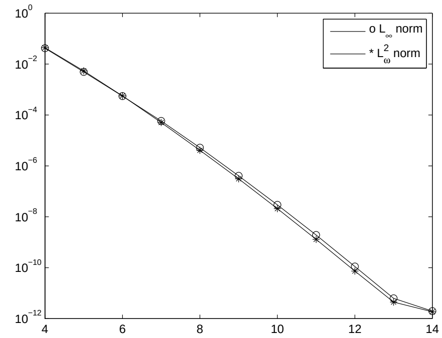

Figure 2: L^∞ and $L^2_{\omega^{\alpha,\beta}}$ errors of spectral Chebyshev-Petrov-Galerkin method versus N.

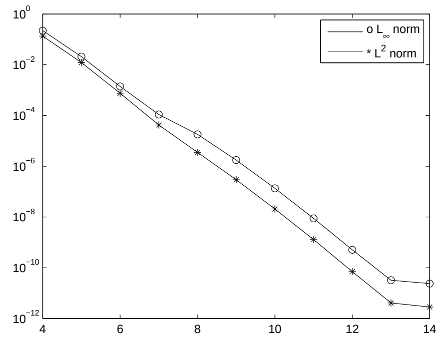

Figure 3: L^∞ and L^2 errors of pseudo-spectral Legendre-Petrov-Galerkin method versus N.

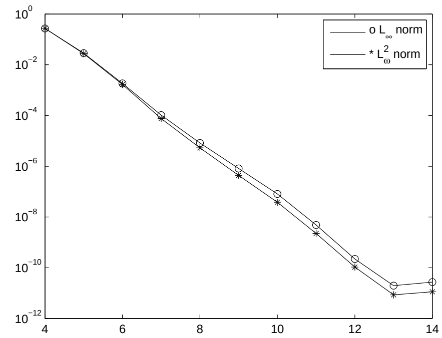

Figure 4: L^∞ and $L^2_{\omega^{\alpha,\beta}}$ errors of pseudo-spectral Chebyshev-Petrov-Galerkin method versus N.

Table 3: The errors of pseudo-spectral Legendre-Petrov-Galerkin method.

N	4	6	8	10	12	14
L^∞-error	2.20e-1	1.39e-3	1.80e-5	1.35e-7	5.12e-10	2.37e-11
L^2-error	1.35e-1	7.50e-4	3.52e-6	2.08e-8	7.06e-11	2.85e-12

Table 4: The errors of pseudo-spectral Chebyshev-Petrov-Galerkin method.

N	4	6	8	10	12	14
L^∞-error	2.68e-1	1.84e-3	8.28e-6	8.07e-8	2.21e-10	2.73e-11
L^2_ω-error	2.73e-1	1.66e-3	5.33e-6	3.82e-8	1.07e-10	1.13e-11

Next we turn to the numerical scheme (2.10) based on the pseudo-spectral Legendre-Petrov-Galerkin and Chebyshev-Petrov-Galerkin methods to solve the example above. Table 3 illustrates the L^∞ and L^2 errors of the pseudo-spectral Legendre-Petrov-Galerkin method which are also shown in Fig. 3. Next the L^∞ and $L^2_{\omega^{\alpha,\beta}}$ errors of the pseudo-spectral Chebyshev-Petrov-Galerkin method are demonstrated in Table 4 and Fig 4. Once again the desired spectral accuracy is obtained.

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