Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2011.42s.4

Vol. **4**, No. 2, pp. 180-196
May 2011

Higher Order Triangular Mixed Finite Element Methods for Semilinear Quadratic Optimal Control Problems

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Received 6 September 2010; Accepted (in revised version) 16 November 2010

Available online 6 April 2011

Abstract. In this paper, we investigate a priori error estimates for the quadratic optimal control problems governed by semilinear elliptic partial differential equations using higher order triangular mixed finite element methods. The state and the co-state are approximated by the order k Raviart-Thomas mixed finite element spaces and the control is approximated by piecewise polynomials of order k ($k \geq 0$). A priori error estimates for the mixed finite element approximation of semilinear control problems are obtained. Finally, we present some numerical examples which confirm our theoretical results.

AMS subject classifications: 49J20, 65N30

Key words: a priori error estimates, semilinear optimal control problems, higher order triangular elements, mixed finite element methods.

1. Introduction

Optimal control problems governed by semilinear elliptic partial differential equations have been so widely met in all kinds engineering problems. Efficient numerical methods are critical for successful applications of optimal control problems in such cases. Recently, the finite element method of optimal control problems plays an important role in numerical methods for these problems, and the relevant literature is extensive, see, for example, [17, 22, 24, 27].

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Many contributions have been done to the priori error estimates of the standard finite element approximation, see, for example, Falk [11], Geveci [12]. But it is more difficult to obtain such error estimates for optimal control problems where the state equations are nonlinear or where there are inequality state constraints. While a priori error analysis for finite element discretization of optimal control problems governed by elliptic equations is discussed in many publications, see, for example, [16], there are only few published results on this topic for nonlinear optimal control problems, see, for example, Arada and Casas [1], Gunzburger and Hou [15].

In many control problems, the objective functional contains gradient of the state variables. Thus accuracy of gradient is important in numerical approximation of the state equations. In the finite element community, mixed finite element methods should be used for discretization of the state equations in such cases. In computational optimal control problems, mixed finite element methods are not widely used in engineering simulations. In particular there doesn't seem to exist much work on theoretical analysis of mixed finite element approximation of optimal control problems in the literature. More recently, we have done some preliminary work on sharp a posteriori error estimates, error estimates and superconvergence of mixed finite element methods for optimal control problems, see, for example, Chen *et al.* [6–9,23]. However, it doesn't seem to be straightforward to extend these existing techniques to the nonlinear optimal control problems.

For $1 \leq p < \infty$ and m any nonnegative integer let $W^{m,p}(\Omega) = \{v \in L^p(\Omega);\ D^\beta v \in L^p(\Omega) \text{ if } |\beta| \leq m\}$ denote the Sobolev spaces endowed with the norm

$$\| v\|_{m,p}^p = \sum_{|\beta|\leq m} \|D^\beta v\|_{L^p(\Omega)}^p,$$

and the semi-norm $| v |_{m,p}^p = \sum_{|\beta|=m} \|D^\beta v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : v |_{\partial\Omega} = 0\}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$. In addition C or c denotes a general positive constant independent of h.

In this paper we derive a priori error estimates of optimal order with respect to all discretization parameters for general semilinear convex quadratic optimal control problems using higher order triangular mixed finite element methods.

We consider the following semilinear quadratic optimal control problems:

$$\min_{u\in K\subset U} \left\{ \frac{1}{2}\|\vec{p} - \vec{p}_d\|^2 + \frac{1}{2}\|y - y_d\|^2 + \frac{\alpha}{2}\|u\|^2 \right\} \tag{1.1}$$

subject to the state equation

$$\begin{aligned} \mathrm{div}\vec{p} + \phi(y) &= f + Bu, && x \in \Omega, && (1.2)\\ \vec{p} &= -A\nabla y, && x \in \Omega, && (1.3)\\ y &= 0, && x \in \partial\Omega, && (1.4) \end{aligned}$$

where the bounded open set $\Omega \subset \mathbb{R}^2$, is a convex polygon or has the smooth boundary $\partial\Omega$. We shall assume that $f \in H^1(\Omega)$ and $\alpha > 0$ are given, and B is a continuous linear operator from $U = L^2(\Omega)$ to $H^1(\Omega)$. For any $R > 0$ the function $\phi(\cdot) \in W^{2,\infty}(-R,R)$, $\phi'(y) \in L^2(\Omega)$ for any $y \in H^1(\Omega)$, and $\phi'(y) \geq 0$. Furthermore, we assume the coefficient matrix $A(x) = (a_{i,j}(x))_{2\times 2} \in (W^{1,\infty}(\Omega))^{2\times 2}$ is a symmetric 2×2-matrix and there is a constant $c > 0$ satisfying for any vector $\mathrm{X} \in \mathbb{R}^2$, $\mathrm{X}'A\mathrm{X} \geq c\|\mathrm{X}\|^2_{\mathbb{R}^2}$. Here, K denotes the admissible set of the control variable, defined by

$$K = \left\{ u \in L^2(\Omega) : \int_\Omega u \geq 0 \right\}. \tag{1.5}$$

Now, we recall a result from Bonnans [5].

Lemma 1.1. *For every function $g \in L^p(\Omega)$, the solution y of*

$$-\mathrm{div}(A\nabla y) + \phi(y) = g \quad \text{in } \Omega, \quad y|_{\partial\Omega} = 0, \tag{1.6}$$

belongs to $H^1_0(\Omega) \cap W^{2,p}(\Omega)$. Moreover, there exists a positive constant C such that

$$\|y\|_{W^{2,p}(\Omega)} \leq C\|g\|_{L^p(\Omega)}. \tag{1.7}$$

Due to Lemma 1.1, the state equations (1.2)-(1.4) admit a unique solution in $H^1_0(\Omega) \cap H^2(\Omega)$.

Next, we introduce the co-state elliptic equation

$$-\mathrm{div}(A(\nabla z + \vec{p} - \vec{p}_d)) + \phi'(y)z = y - y_d, \quad x \in \Omega, \tag{1.8}$$

with the boundary condition

$$z = 0, \qquad x \in \partial\Omega. \tag{1.9}$$

The existence of a unique solution of (1.8) is justified by Lemma 1.1. Moreover, we make the following realistic assumption [6] on the regularity of the solution of the optimal control problems (1.1)-(1.4) and the co-state problems (1.8)-(1.9):

$$u \in W^{1,\infty}(\Omega), \quad y,\ z \in H^{2+s}(\Omega), \quad \text{for} \quad 0 < s \leq 1.$$

The outline of this paper is as follows. In Section 2, we construct the higher order triangular mixed finite element approximation for optimal control problems governed by semilinear elliptic equations. Furthermore, we briefly state the definitions and properties of some interpolation operators. In Section 3, we show that the properties of the control variable. Next, we derive a priori error estimates for the higher order Raviart-Thomas triangular mixed finite element solutions of the optimal control problems in Section 4. Numerical examples are presented in Section 5. Finally, we analyze the conclusion and the future work in Section 6.

2. Mixed methods of optimal control problems

We shall now describe the mixed finite element discretization of semilinear convex optimal control problems (1.1)-(1.4). Let

$$\vec{V} = H(\text{div};\Omega) = \left\{\vec{v} \in (L^2(\Omega))^2, \text{div}\vec{v} \in L^2(\Omega)\right\}, \quad W = L^2(\Omega).$$

The Hilbert space $\vec{V}$ is equipped with the following norm:

$$\|\vec{v}\|_{\text{div}} = \|\vec{v}\|_{H(\text{div};\Omega)} = (\|\vec{v}\|^2_{0,\Omega} + \|\text{div}\vec{v}\|^2_{0,\Omega})^{1/2}.$$

We recast (1.1)-(1.4) as the following weak form: find $(\vec{p}, y, u) \in \vec{V} \times W \times U$ such that

$$\min_{u\in K\subset U} \left\{\frac{1}{2}\|\vec{p} - \vec{p}_d\|^2 + \frac{1}{2}\|y - y_d\|^2 + \frac{\alpha}{2}\|u\|^2\right\}, \tag{2.1}$$

$$(A^{-1}\vec{p}, \vec{v}) - (y, \text{div}\vec{v}) = 0, \qquad \forall \vec{v} \in \vec{V}, \tag{2.2}$$

$$(\text{div}\vec{p}, w) + (\phi(y), w) = (f + Bu, w), \qquad \forall w \in W. \tag{2.3}$$

It is well known (see, e.g., [19]) that the optimal control problem (2.1)-(2.3) has a solution $(\vec{p}, y, u)$, and that a triplet $(\vec{p}, y, u)$ is the solution of (2.1)-(2.3) if and only if there is a co-state $(\vec{q}, z) \in \vec{V} \times W$ such that $(\vec{p}, y, \vec{q}, z, u)$ satisfies the following optimality conditions:

$$(A^{-1}\vec{p}, \vec{v}) - (y, \text{div}\vec{v}) = 0, \qquad \forall \vec{v} \in \vec{V}, \tag{2.4a}$$

$$(\text{div}\vec{p}, w) + (\phi(y), w) = (f + Bu, w), \qquad \forall w \in W, \tag{2.4b}$$

$$(A^{-1}\vec{q}, \vec{v}) - (z, \text{div}\vec{v}) = -(\vec{p} - \vec{p}_d, \vec{v}), \qquad \forall \vec{v} \in \vec{V}, \tag{2.4c}$$

$$(\text{div}\vec{q}, w) + (\phi'(y)z, w) = (y - y_d, w), \qquad \forall w \in W, \tag{2.4d}$$

$$(\alpha u + B^* z, \tilde{u} - u)_U \geq 0, \qquad \forall \tilde{u} \in K, \tag{2.4e}$$

where $(\cdot,\cdot)_U$ is the inner product of U, B^* is the adjoint operator of B. In the rest of the paper, we shall simply write the product as $(\cdot,\cdot)$ whenever no confusion should be caused.

For ease of exposition we will assume that Ω is polygon. Let $\mathcal{T}_h$ be regular triangulation of Ω. They are assumed to satisfy the angle condition which means that there is a positive constant C such that for all $T \in \mathcal{T}_h$, $C^{-1}h_T^2 \leq |T| \leq Ch_T^2$, where $|T|$ is the area of T, h_T is the diameter of T and $h = \max h_T$.

Let $\vec{V}_h \times W_h \subset \vec{V} \times W$ denote the Raviart-Thomas space [14] of the index k associated with the triangulation or rectangulation $\mathcal{T}_h$ of Ω, where $k \geq 0$. P_k denotes the space of polynomials of total degree at most k. Let $\vec{V}(T) = \{\vec{v} \in P_k^2(T) + x \cdot P_k(T)\}$, $W(T) = P_k(T)$. We define

$$\vec{V}_h := \{\vec{v}_h \in \vec{V} : \forall T \in \mathcal{T}_h, \vec{v}_h|_T \in \vec{V}(T)\},$$
$$W_h := \{w_h \in W : \forall T \in \mathcal{T}_h, w_h|_T \in W(T)\},$$
$$K_h := \{\tilde{u}_h \in K : \forall T \in \mathcal{T}_h, \tilde{u}_h|_T \in P_k(T)\}.$$

By the definition of finite element subspace, the mixed finite element discretization of (2.1)-(2.3) is as follows: compute $(\vec{p}_h, y_h, u_h) \in \vec{V}_h \times W_h \times K_h$ such that

$$\min_{u_h \in K_h} \left\{ \frac{1}{2}\|\vec{p}_h - \vec{p}_d\|^2 + \frac{1}{2}\|y_h - y_d\|^2 + \frac{\alpha}{2}\|u_h\|^2 \right\}, \tag{2.5}$$

$$(A^{-1}\vec{p}_h, \vec{v}_h) - (y_h, \mathrm{div}\vec{v}_h) = 0, \qquad \forall \vec{v}_h \in \vec{V}_h, \tag{2.6}$$

$$(\mathrm{div}\vec{p}_h, w_h) + (\phi(y_h), w_h) = (f + Bu_h, w_h), \qquad \forall w_h \in W_h. \tag{2.7}$$

It is well known that the optimal control problem (2.5)-(2.7) again has a solution $(\vec{p}_h, y_h, u_h)$, and that a triplet $(\vec{p}_h, y_h, u_h)$ is the solution of (2.5)-(2.7) if and only if there is a co-state $(\vec{q}_h, z_h) \in \vec{V}_h \times W_h$ such that $(\vec{p}_h, y_h, \vec{q}_h, z_h, u_h)$ satisfies the following optimality conditions:

$$(A^{-1}\vec{p}_h, \vec{v}_h) - (y_h, \mathrm{div}\vec{v}_h) = 0, \qquad \forall \vec{v}_h \in \vec{V}_h, \tag{2.8a}$$

$$(\mathrm{div}\vec{p}_h, w_h) + (\phi(y_h), w_h) = (f + Bu_h, w_h), \qquad \forall w_h \in W_h, \tag{2.8b}$$

$$(A^{-1}\vec{q}_h, \vec{v}_h) - (z_h, \mathrm{div}\vec{v}_h) = -(\vec{p}_h - \vec{p}_d, \vec{v}_h) \qquad \forall \vec{v}_h \in \vec{V}_h, \tag{2.8c}$$

$$(\mathrm{div}\vec{q}_h, w_h) + (\phi'(y_h)z_h, w_h) = (y_h - y_d, w_h), \qquad \forall w_h \in W_h, \tag{2.8d}$$

$$(\alpha u_h + B^* z_h, \tilde{u}_h - u_h) \geq 0, \qquad \forall \tilde{u}_h \in K_h. \tag{2.8e}$$

Let $P_h : W \to W_h$ be the orthogonal $L^2(\Omega)$-projection into W_h define by [2]:

$$(P_h w - w, \chi) = 0, \qquad w \in W, \quad \chi \in W_h, \tag{2.9}$$

which satisfies

$$\|P_h w - w\|_{0,q} \leq C\|w\|_{t,q} h^t, \qquad 0 \leq t \leq k+1, \text{ if } w \in W \cap W^{t,q}(\Omega), \tag{2.10a}$$

$$\|P_h w - w\|_{-r} \leq C\|w\|_t h^{r+t}, \qquad 0 \leq r, t \leq k+1, \text{ if } w \in H^t(\Omega), \tag{2.10b}$$

$$(\mathrm{div}\vec{v}, w - P_h w) = 0, \qquad w \in W, \vec{v} \in \vec{V}_h. \tag{2.10c}$$

Let $\pi_h : \vec{V} \to \vec{V}_h$ be the Raviart-Thomas projection [25], which satisfies

$$(\mathrm{div}(\pi_h \vec{v} - \vec{v}), w) = 0, \qquad \vec{v} \in \vec{V}, \ w \in W_h, \tag{2.11a}$$

$$\|\pi_h \vec{v} - \vec{v}\|_{0,q} \leq C\|\vec{v}\|_{t,q} h^t, \qquad 1/q < t \leq k+1, \text{ if } \vec{v} \in \vec{V} \cap W^{t,q}(\Omega)^2, \tag{2.11b}$$

$$\|\mathrm{div}(\pi_h \vec{v} - \vec{v})\|_0 \leq C\|\mathrm{div}\vec{v}\|_t h^t, \qquad 0 \leq t \leq k+1, \text{ if } \vec{v} \in \vec{V} \cap H^t(\mathrm{div};\Omega). \tag{2.11c}$$

We have the commuting diagram property

$$\mathrm{div} \circ \pi_h = P_h \circ \mathrm{div} : \ \vec{V} \to W_h \quad \text{and} \quad \mathrm{div}(I - \pi_h)\vec{V} \perp W_h, \tag{2.12}$$

where and after, I denote identity matrix.

Furthermore, we also define the standard L^2-orthogonal projection $Q_h : K \to K_h$, which satisfies: for any $\tilde{u} \in K$

$$(\tilde{u} - Q_h \tilde{u}, \tilde{u}_h)_U = 0, \qquad \forall \tilde{u}_h \in K_h, \tag{2.13a}$$

$$\|\tilde{u} - Q_h \tilde{u}\|_{-r,q,U} \leq C|\tilde{u}|_{t,q,U} h^{r+t}, \qquad 0 \leq t, r \leq k+1, \text{ for } \tilde{u} \in W^{t,q}(\Omega). \tag{2.13b}$$

For $\varphi \in W_h$, we shall write

$$\phi(\varphi)-\phi(\rho)=-\tilde{\phi}'(\varphi)(\rho-\varphi)=-\phi'(\rho)(\rho-\varphi)+\tilde{\phi}''(\varphi)(\rho-\varphi)^2, \tag{2.14}$$

where

$$\tilde{\phi}'(\varphi)=\int_0^1 \phi'(\varphi+s(\rho-\varphi))ds,$$
$$\tilde{\phi}''(\varphi)=\int_0^1 (1-s)\phi''(\rho+s(\varphi-\rho))ds$$

are bounded functions in $\bar{\Omega}$.

3. Properties of the control variable

In this section, we will show that the control variable of the optimal control problem (2.4a)-(2.4e) can be infinitely smooth if the special constraint set K defined as (1.5).

Lemma 3.1. *Let $(\vec{p},y,\vec{q},z,u)\in(\vec{V}\times W)^2\times K$ be the solution of (2.4a)-(2.4e). Then we have*

$$u=\max\left(0,\frac{\overline{B^*z}}{\alpha}\right)-\frac{B^*z}{\alpha}, \tag{3.1}$$

where $\overline{B^*z}=\int_\Omega B^*z/|\Omega|$ denotes the integral average on Ω of the function z.

Proof. If $\overline{B^*z}>0$, then $u=\frac{\overline{B^*z}-B^*z}{\alpha}$ and for any $v\in K$

$$\begin{aligned}(\alpha u+B^*z,v-u)&=\int_\Omega(\alpha u+B^*z)(v-u)\\&=\int_\Omega \overline{B^*z}\left(v-\frac{\overline{B^*z}-B^*z}{\alpha}\right)=\overline{B^*z}\int_\Omega v\geq 0.\end{aligned}$$

If $\overline{B^*z}\leq 0$, then $u=-\frac{B^*z}{\alpha}$ and $(\alpha u+B^*z,v-u)=0$. Now, for the costate solution z, since the solution of (2.4e) is unique, then we have proved the lemma. □

From Lemma 3.1, we can obtain the following regularity result for the control variable.

Lemma 3.2. *Let $(\vec{p},y,\vec{q},z,u)\in(\vec{V}\times W)^2\times K$ be the solution of (2.4a)-(2.4e). Assume that the data functions y_d, $\vec{p}_d$, and the domain Ω are infinitely smooth. Then the control function $u\in C^\infty(\bar{\Omega})$.*

Proof. By applying the regularity argument of elliptic problem (1.2)-(1.3), it is clear that $y\in H^2(\Omega)$, so that $\vec{p}\in H^1(\Omega)$. It follows from the costate equation (1.8) and the assumption of y_d, $\vec{p}_d$, we can obtain that $z\in H^2(\Omega)$. Using the relationship between the control and the costate $u=\max(0,\frac{\overline{B^*z}}{\alpha})-\frac{B^*z}{\alpha}$, then $u\in H^2(\Omega)$. Thus $y\in H^4(\Omega)$, $\vec{p}\in H^3(\Omega)$. By repeating the above process, we can conclude that $u\in C^\infty(\bar{\Omega})$. □

4. A priori error estimates

In the rest of the paper, we shall use some intermediate variables. For any control function $\tilde{u} \in K$, we first define the state solution $(\vec{p}(\tilde{u}), y(\tilde{u}), \vec{q}(\tilde{u}), z(\tilde{u}))$ associated with $\tilde{u}$ that satisfies

$$
\begin{aligned}
&(A^{-1}\vec{p}(\tilde{u}), \vec{v}) - (y(\tilde{u}), \mathrm{div}\vec{v}) = 0, && \forall \vec{v} \in \vec{V}, && (4.1a)\\
&(\mathrm{div}\vec{p}(\tilde{u}), w) + (\phi(y(\tilde{u})), w) = (f + B\tilde{u}, w), && \forall w \in W, && (4.1b)\\
&(A^{-1}\vec{q}(\tilde{u}), \vec{v}) - (z(\tilde{u}), \mathrm{div}\vec{v}) = -(\vec{p}(\tilde{u}) - \vec{p}_d, \vec{v}), && \forall \vec{v} \in \vec{V}, && (4.1c)\\
&(\mathrm{div}\vec{q}(\tilde{u}), w) + (\phi'(y(\tilde{u}))z(\tilde{u}), w) = (y(\tilde{u}) - y_d, w), && \forall w \in W. && (4.1d)
\end{aligned}
$$

Correspondingly, we define the discrete state solution $(\vec{p}_h(\tilde{u}), y_h(\tilde{u}), \vec{q}_h(\tilde{u}), z_h(\tilde{u}))$ associated with $\tilde{u} \in K$ that satisfies

$$
\begin{aligned}
&(A^{-1}\vec{p}_h(\tilde{u}), \vec{v}_h) - (y_h(\tilde{u}), \mathrm{div}\vec{v}_h) = 0, && \forall \vec{v}_h \in \vec{V}_h, && (4.2a)\\
&(\mathrm{div}\vec{p}_h(\tilde{u}), w_h) + (\phi(y_h(\tilde{u})), w_h) = (f + B\tilde{u}, w_h), && \forall w_h \in W_h, && (4.2b)\\
&(A^{-1}\vec{q}_h(\tilde{u}), \vec{v}_h) - (z_h(\tilde{u}), \mathrm{div}\vec{v}_h) = -(\vec{p}_h(\tilde{u}) - \vec{p}_d, \vec{v}_h), && \forall \vec{v}_h \in \vec{V}_h, && (4.2c)\\
&(\mathrm{div}\vec{q}_h(\tilde{u}), w_h) + (\phi'(y_h(\tilde{u}))z_h(\tilde{u}), w_h) = (y_h(\tilde{u}) - y_d, w_h), && \forall w_h \in W_h. && (4.2d)
\end{aligned}
$$

Thus, as we defined, the exact solution and its approximation can be written in the following way:

$$
\begin{aligned}
&(\vec{p}, y, \vec{q}, z) = (\vec{p}(u), y(u), \vec{q}(u), z(u)),\\
&(\vec{p}_h, y_h, \vec{q}_h, z_h) = (\vec{p}_h(u_h), y_h(u_h), \vec{q}_h(u_h), z_h(u_h)).
\end{aligned}
$$

By Lemma 2.1 in [25], we can obtain the following technical results:

Lemma 4.1. *Let $\omega \in \vec{V}$, $\varphi \in L^2(\Omega)^2$, and $\psi \in L^2(\Omega)$. If $\tau \in W_h$ satisfies*

$$
\begin{aligned}
&(A^{-1}\omega, \vec{v}_h) - (\tau, \mathrm{div}\vec{v}_h) = (\varphi, \vec{v}_h), && \forall \vec{v}_h \in \vec{V}_h,\\
&(\mathrm{div}\omega, w_h) + (\gamma\tau, w_h) = (\psi, w_h), && \forall w_h \in W_h,
\end{aligned}
$$

then, there exists a constant C such that

$$
\|\tau\|_0 \le C\big(h\|\omega\|_0 + h\|\mathrm{div}\omega\|_0 + \|\varphi\|_0 + \|\psi\|_0\big), \tag{4.3}
$$

for h sufficiently small.

Set some intermediate errors:

$$
\varepsilon_1 := \vec{p} - \vec{p}_h(u) \quad \text{and} \quad e_1 := y - y_h(u). \tag{4.4}
$$

To analyze the intermediate errors, let us first note the following error equations from (2.4a)-(2.4b) and (4.2a)-(4.2b) with the choice $\tilde{u} = u$:

$$
\begin{aligned}
&(A^{-1}\varepsilon_1, \vec{v}_h) - (e_1, \mathrm{div}\vec{v}_h) = 0, && \forall \vec{v}_h \in \vec{V}_h, && (4.5a)\\
&(\mathrm{div}\varepsilon_1, w_h) + (\tilde{\phi}'(y)e_1, w_h) = 0, && \forall w_h \in W_h. && (4.5b)
\end{aligned}
$$

By (2.9)-(2.11c) and Lemma 4.1, we can establish the following error estimates:

Theorem 4.1. *There is a positive constant C independent of h such that*

$$\|y - y_h(u)\|_0 \le Ch^{k+1}\|y\|_{k+1+\delta_{k0}}, \tag{4.6a}$$
$$\|\vec{p} - \vec{p}_h(u)\|_0 \le Ch^{k+1}\|y\|_{k+2}, \tag{4.6b}$$
$$\|\vec{p} - \vec{p}_h(u)\|_{\rm div} \le Ch^{k+1}\|y\|_{k+3}, \tag{4.6c}$$

where δ_{k0} is Dirac function.

Proof. Let $\tau = P_h y - y_h(u)$ and $\sigma = \pi_h\vec{p} - \vec{p}_h(u)$. Rewrite (4.5) in the form

$$(A^{-1}\varepsilon_1, \vec{v}_h) - (\tau, {\rm div}\vec{v}_h) = 0, \qquad \forall \vec{v}_h \in \vec{V}_h, \tag{4.7a}$$
$$({\rm div}\varepsilon_1, w_h) + (\tilde{\phi}'(y)\tau, w_h) = (\tilde{\phi}'(y)(P_h y - y), w_h), \qquad \forall w_h \in W_h. \tag{4.7b}$$

It follows from (2.10a) and Lemma 4.1 that

$$\|\tau\|_0 \le C\big(h\|\varepsilon_1\|_0 + h\|{\rm div}\varepsilon_1\|_0 + h^{k+1}\|y\|_{k+1+\delta_{k0}}\big). \tag{4.8}$$

Using again (2.10a) that

$$\begin{aligned} \|e_1\|_0 &= \|y - y_h(u)\|_0 \\ &= \|P_h y - y\|_0 + \|\tau\|_0 \\ &\le C\big(h\|\varepsilon_1\|_0 + h\|{\rm div}\varepsilon_1\|_0 + h^{k+1}\|y\|_{k+1+\delta_{k0}}\big). \end{aligned} \tag{4.9}$$

If we now again rewrite (4.5) as

$$(A^{-1}\sigma, \vec{v}_h) - (\tau, {\rm div}\vec{v}_h) = (A^{-1}(\pi_h\vec{p} - \vec{p}), \vec{v}_h), \qquad \forall \vec{v}_h \in \vec{V}_h, \tag{4.10a}$$
$$({\rm div}\sigma, w_h) = -(\tilde{\phi}'(y)e_1, w_h), \qquad \forall w_h \in W_h. \tag{4.10b}$$

Using the standard stability results of mixed finite element methods in [4], we can establish the following results:

$$\begin{aligned} \|\sigma\|_{\rm div} \le& C(\|\pi_h\vec{p} - \vec{p}\|_0 + \|e_1\|_0) \\ \le& C(h^{k+1}\|y\|_{k+2} + \|e_1\|_0). \end{aligned} \tag{4.11}$$

From (4.11), (2.11b), and the commuting diagram property (2.12) we now obtain the bounds

$$\begin{aligned} \|\varepsilon_1\|_0 &\le C(\|\pi_h\vec{p} - \vec{p}\|_0 + \|\sigma\|_0) \\ &\le C(h^{k+1}\|y\|_{k+2} + \|e_1\|_0), \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} \|{\rm div}\varepsilon_1\|_0 &\le C(\|{\rm div}(\pi_h\vec{p} - \vec{p})\|_0 + \|{\rm div}\sigma\|_0) \\ &= C(\|P_h \circ {\rm div}\vec{p} - {\rm div}\vec{p}\|_0 + \|{\rm div}\sigma\|_0) \\ &\le C(h^{k+1}\|y\|_{k+3} + \|e_1\|_0), \end{aligned} \tag{4.13}$$

which, when substituted into (4.9), yield the estimates

$$\|e_1\|_0 \le C(h\|e_1\|_0 + h^{k+1}\|y\|_{k+1+\delta_{k0}}). \tag{4.14}$$

Then (4.14) implies (4.6) holds if h is small enough. Applying (4.14) to (4.12) and (4.13) shows that (4.6) also hold. □

Now, we set some other intermediate errors:

$$\varepsilon_2 := \vec{q} - \vec{q}_h(u) \quad \text{and} \quad e_2 := z - z_h(u). \tag{4.15}$$

Let us note the following error equations from (2.4c)-(2.4d), (4.2c)-(4.2d), and (2.14):

$$(A^{-1}\varepsilon_2, \vec{v}_h) - (e_2, \mathrm{div}\vec{v}_h) = -(\varepsilon_1, \vec{v}_h), \qquad \forall \vec{v}_h \in \vec{V}_h, \tag{4.16a}$$

$$(\mathrm{div}\varepsilon_2, w_h) + (\phi'(y)e_2, w_h) = (e_1, w_h) - (z_h(u)\tilde{\phi}''(y)e_1, w_h), \qquad \forall w_h \in W_h. \tag{4.16b}$$

Using the argument similar to the proof of Theorem 4.1, we can also derive the following results:

Theorem 4.2. *There is a positive constant C independent of h such that*

$$\|z - z_h(u)\|_0 \le Ch^{k+1}\|y\|_{k+1+\delta_{k0}}, \tag{4.17a}$$

$$\|\vec{p} - \vec{p}_h(u)\|_0 \le Ch^{k+1}\|y\|_{k+2}, \tag{4.17b}$$

$$\|\vec{q} - \vec{q}_h(u)\|_{\mathrm{div}} \le Ch^{k+1}\|y\|_{k+3}, \tag{4.17c}$$

where δ_{k0} is Dirac function.

With the intermediate errors, we can decompose the errors as follows

$$\vec{p} - \vec{p}_h = \vec{p} - \vec{p}_h(u) + \vec{p}_h(u) - \vec{p}_h := \varepsilon_1 + \epsilon_1, \tag{4.18a}$$

$$y - y_h = y - y_h(u) + y_h(u) - y_h := e_1 + r_1, \tag{4.18b}$$

$$\vec{q} - \vec{q}_h = \vec{q} - \vec{q}_h(u) + \vec{q}_h(u) - \vec{q}_h := \varepsilon_2 + \epsilon_2, \tag{4.18c}$$

$$z - z_h = z - z_h(u) + z_h(u) - z_h := e_2 + r_2. \tag{4.18d}$$

From (2.8), (4.2), and (2.14), we have

$$(A^{-1}\epsilon_1, \vec{v}_h) - (r_1, \mathrm{div}\vec{v}_h) = 0, \qquad \forall \vec{v}_h \in \vec{V}_h,$$

$$(\mathrm{div}\epsilon_1, w_h) + (\tilde{\phi}'(y_h(u))r_1, w_h) = (B(u - u_h), w_h), \qquad \forall w_h \in W_h,$$

$$(A^{-1}\epsilon_2, \vec{v}_h) - (r_2, \mathrm{div}\vec{v}_h) = -(\vec{p}_h(u) - \vec{p}_h, \vec{v}_h), \qquad \forall \vec{v}_h \in \vec{V}_h,$$

$$(\mathrm{div}\epsilon_2, w_h) + (\phi'(y_h(u))r_2, w_h) = (y_h(u) - y_h, w_h) + (\tilde{\phi}''(y_h)z_h r_1, w_h), \qquad \forall w_h \in W_h.$$

The assumption that $A \in L^\infty(\Omega; \mathbb{R}^{2\times 2})$ implies that it is bounded that the inverse operator of the map $\{\epsilon_1, r_1\} : \mathbb{R}^3 \to \vec{V} \times W$ defined by the above saddle-point problem [4]:

$$\|\epsilon_1\|_{\mathrm{div}} + \|r_1\|_0 \le C\|u - u_h\|_0, \tag{4.19}$$

where the continuity of the linear operator B has been used. By applying (4.19), we have

$$\begin{aligned}\|\epsilon_2\|_{\text{div}} + \|r_2\|_0 &\le C\big(\|\vec{p}_h - \vec{p}_h(u)\|_0 + \|y_h - y_h(u)\|_0\big)\\ &\le C\|u-u_h\|_0. \end{aligned} \tag{4.20}$$

Let $(\vec{p}(u), y(u))$ and $(\vec{p}_h(u_h), y_h(u_h))$ be the solutions of (2.2)-(2.3) and (2.6)-(2.7), respectively. Let $J(\cdot) : K \to \mathbb{R}$ be a G-differential uniform convex functional near the solution u which satisfies the following form:

$$J(u) = \frac{1}{2}\|\vec{p} - \vec{p}_d\|^2 + \frac{1}{2}\|y - y_d\|^2 + \frac{\alpha}{2}\|u\|^2.$$

We assume that we have a sequence of uniform convex functional $J_h : K \to \mathbb{R}$:

$$\begin{aligned} J_h(u) &= \frac{1}{2}\|\vec{p}_h(u) - \vec{p}_d\|^2 + \frac{1}{2}\|y_h(u) - y_d\|^2 + \frac{\alpha}{2}\|u\|^2,\\ J_h(u_h) &= \frac{1}{2}\|\vec{p}_h - \vec{p}_d\|^2 + \frac{1}{2}\|y_h - y_d\|^2 + \frac{\alpha}{2}\|u_h\|^2. \end{aligned}$$

It can be shown that

$$\begin{aligned} (J'(u), v) &= (\alpha u + B^* z, v),\\ (J_h'(u), v) &= (\alpha u + B^* z_h(u), v),\\ (J_h'(u_h), v) &= (\alpha u_h + B^* z_h, v), \end{aligned}$$

where $z_h(u)$ is the solution of (4.1)-(4.2a) with $\tilde{u} = u_h$. An additional assumption is needed. We assume that the cost function J is strictly convex near the solution u, i.e., for the solution u there exists a neighborhood of u in L^2 such that J is convex in the sense that there is a constant $c > 0$ satisfying:

$$(J'(u) - J'(v), u - v) \ge c\|u - v\|_U^2, \tag{4.21}$$

for all v in this neighborhood of u. The convexity of $J(\cdot)$ is closely related to the second order sufficient optimality conditions of optimal control problems, which are assumed in many studies on numerical methods of the problem. For instance, in many references, the authors assume the following second order sufficiently optimality condition (see [1]): there is $c > 0$ such that $J''(u)v^2 \ge c\|v\|_0^2$.

From the assumption (4.21), by the proof contained in [1], there exists a constant $c > 0$ satisfying

$$(J_h'(v) - J_h'(u), v - u) \ge c\|v - u\|_U^2, \quad \forall v \in U_h. \tag{4.22}$$

In the following we estimate $\|u - u_h\|_0$ and then obtain the results:

Theorem 4.3. *Let $(\vec{p}, y, \vec{q}, z, u) \in (\vec{V} \times W)^2 \times K$ and $(\vec{p}_h, y_h, \vec{q}_h, z_h, u_h) \in (\vec{V}_h \times W_h)^2 \times K_h$ (2.4) and (2.8), respectively. We assume that*

$$u,\ \alpha u + B^* z \in H^{k+1}(\Omega).$$

Then, we have

$$\|u-u_h\|_0 \le Ch^{k+1}, \tag{4.23a}$$
$$\|\vec{p}-\vec{p}_h\|_{\rm div} + \|y-y_h\|_0 \le Ch^{k+1}, \tag{4.23b}$$
$$\|\vec{q}-\vec{q}_h\|_{\rm div} + \|z-z_h\|_0 \le Ch^{k+1}. \tag{4.23c}$$

Proof. We choose $\tilde{u}=u_h$ in (2.4e) and $\tilde{u}_h = Q_h u$ in (2.8e) to get that

$$(\alpha u + B^* z, u_h - u) \ge 0, \tag{4.24}$$

and

$$(\alpha u_h + B^* z_h, Q_h u - u_h) \ge 0. \tag{4.25}$$

Note that $Q_h u - u_h = Q_h u - u + u - u_h$ in (4.25) and add the two inequalities (4.24)-(4.25), we have

$$(\alpha u_h + B^* z_h - \alpha u - B^* z, u - u_h) + (\alpha u_h + B^* z_h, Q_h u - u) \ge 0. \tag{4.26}$$

By applying the uniform convexity of $J_h(\cdot)$ and (4.26), we obtain

$$\begin{aligned} c\|u-u_h\|_0^2 \le & (J_h'(u), u-u_h) - (J_h'(u_h), u-u_h) \\ = & (\alpha u + B^* z_h(u), u-u_h) - (\alpha u_h + B^* z_h, u-u_h) \\ \le & (B^* z_h(u) - B^* z, u-u_h) + (\alpha u_h - \alpha u, Q_h u - u) \\ & + (B^*(z_h - z_h(u)), Q_h u - u) + (B^*(z_h(u)-z), Q_h u - u) \\ & + (\alpha u + B^* z, Q_h u - u). \end{aligned} \tag{4.27}$$

Now, we estimate all terms at the right side of (4.27). From the continuity of the operator B and Lemma 4.2, we obtain

$$(B^* z_h(u) - B^* z, u-u_h) \le C\|z_h(u)-z\|_0 \cdot \|u-u_h\|_0 \le Ch^{k+1}\|u-u_h\|_0. \tag{4.28}$$

By δ-Caunchy inequality, and the approximation property (2.13b) of the projection Q_h, it is clear that

$$(\alpha u_h - \alpha u, Q_h u - u) \le Ch^{2(k+1)} + \delta\|u-u_h\|_0^2, \tag{4.29}$$

for any small $\delta > 0$. From (2.13b), (4.6a), and the continuity of the operator B, we have

$$(B^* z_h - B^* z_h(u), Q_h u - u) \le Ch^{2(k+1)} + \delta\|u-u_h\|_0^2, \tag{4.30a}$$
$$(B^* z_h(u) - B^* z, Q_h u - u) \le Ch^{2(k+1)}, \tag{4.30b}$$

where we use the estimates $\|Q_h u - u\|_0 \le Ch^{k+1}\|u\|_{k+1}$. As we can see,

$$\|\omega\|_{-k,\Omega} = \sup_{\varrho\in 0, \varrho\neq 0} \frac{(\omega,\varrho)}{\|\varrho\|_{k,\Omega}}, \tag{4.31}$$

and using the approximation property (2.13b), then we obtain

$$(\alpha u + B^* z, Q_h u - u) \leq C\|\alpha u + B^* z\|_{k+1,\Omega} \cdot \|Q_h u - u\|_{-k-1,\Omega} \leq Ch^{2(k+1)}. \tag{4.32}$$

From (4.27)-(4.32) we have

$$c\|u - u_h\|_0^2 \leq 2\delta\|u - u_h\|_0^2 + Ch^{2(k+1)}. \tag{4.33}$$

It is easy to obtain that

$$\|u - u_h\|_0 \leq Ch^{k+1}. \tag{4.34}$$

By using (4.18), Theorems 4.1–4.2, and (4.19)-(4.20), we derive

$$\begin{aligned} &\|\vec{p} - \vec{p}_h\|_{\text{div}} + \|y - y_h\|_0 \\ \leq &\|\vec{p} - \vec{p}_h(u)\|_{\text{div}} + \|y - y_h(u)\|_0 + \|\vec{p}_h(u) - \vec{p}_h\|_{\text{div}} + \|y_h(u) - y_h\|_0 \\ \leq &Ch^{k+1}, \end{aligned} \tag{4.35}$$

and

$$\begin{aligned} &\|\vec{q} - \vec{q}_h\|_{\text{div}} + \|z - z_h\|_0 \\ \leq &\|\vec{q} - \vec{q}_h(u)\|_{\text{div}} + \|z - z_h(u)\|_0 + \|\vec{q}_h(u) - \vec{q}_h\|_{\text{div}} + \|z_h(u) - z_h\|_0 \\ \leq &Ch^{k+1}. \end{aligned} \tag{4.36}$$

Then we derive the result (4.23b)-(4.23c). ☐

5. Numerical tests

In this section, we are going to validate the a priori error estimates for the error in the control, state, and co-state numerically. The optimization problems were dealt numerically with codes developed based on AFEPACK. The package is freely available and the details can be found at [18].

Our numerical example is the following optimal control problem:

$$\min_{u \in K} \left\{ \frac{1}{2}\|\vec{p} - \vec{p}_d\|^2 + \frac{1}{2}\|y - y_d\|^2 + \frac{1}{2}\|u\|^2 \right\}, \tag{5.1}$$

$$\text{div}\vec{p} + y^3 = Bu + f, \quad \vec{p} = -A\nabla y, \qquad x \in \Omega, \quad y|_{\partial\Omega} = 0, \tag{5.2}$$

$$\text{div}\vec{q} + 3y^2 z = y - y_d, \quad \vec{q} = -A(\nabla z + \vec{p} - \vec{p}_d), \qquad x \in \Omega, \quad z|_{\partial\Omega} = 0. \tag{5.3}$$

In our examples, we choose the domain $\Omega = [0,1] \times [0,1]$ and $A = B = I$. We present below two examples to illustrate the theoretical results of the optimal control problem.

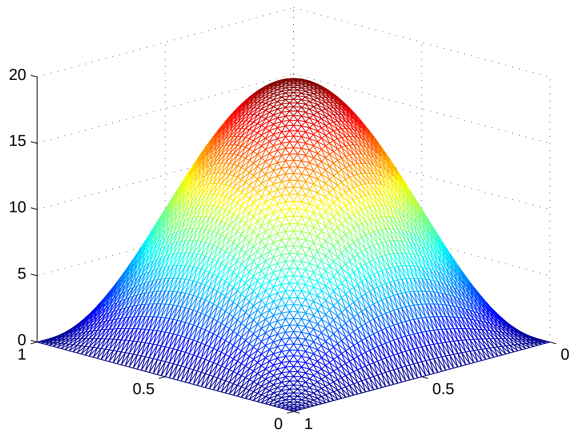

Figure 1: The profile of the control solution on the 64×64 mesh grids.

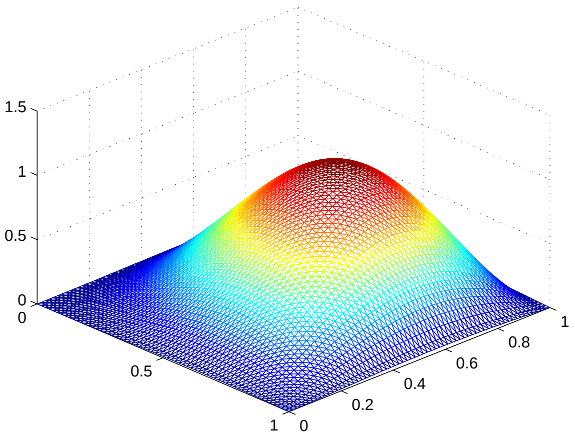

Figure 2: The profile of the control solution on the 64×64 mesh grids.

Example 5.1. We set the known functions as follows:

$$
\begin{aligned}
&y = \sin 2\pi x_1 \sin 2\pi x_2, \\
&z = -\sin 2\pi x_1 \sin 2\pi x_2, \\
&u_f = 2\pi^2 \sin \pi x_1 \sin \pi x_2, \\
&f = u_f + y^3 - u, \ u = u_f, \\
&y_d = (1 + 2\pi^4) y - 3y^2 z, \\
&\vec{p} = -\left(\pi \cos \pi x_1 \sin \pi x_2, \pi \sin \pi x_1 \cos \pi x_2\right), \\
&\vec{q} = -\left(\pi^3 \cos \pi x_1 \sin \pi x_2, \pi^3 \sin \pi x_1 \cos \pi x_2\right), \\
&\vec{p}_d = -\left(\pi(1+\pi^2) \cos \pi x_1 \sin \pi x_2, \pi(1+\pi^2) \sin \pi x_1 \cos \pi x_2\right).
\end{aligned}
$$

In this numerical implementation, the errors $\|u-u_h\|_0$, $\|\vec{p}-\vec{p}_h\|_{\text{div}}$, $\|y-y_h\|_0$, $\|\vec{q}-\vec{q}_h\|_{\text{div}}$, and $\|z-z_h\|_0$ obtained on RT0 mixed finite element approximation and RT1 mixed finite element approximation for state function are presented in Tables 1 and 2. The theoretical results can be observed clearly from the data. The profile of the numerical solution is plotted in Fig. 1.

Table 1: The numerical errors on RT0 mixed finite element for state function and piecewise constant functions for control function.

Resolution	Errors				
	$\|u-u_h\|$	$\|\vec{p}-\vec{p}_h\|$	$\|y-y_h\|$	$\|\vec{q}-\vec{q}_h\|$	$\|z-z_h\|$
16×16	3.56759e-02	1.62504e-01	3.56977e-02	1.62509e-01	3.56759e-02
32×32	1.78577e-02	8.13915e-02	1.78605e-02	8.13920e-02	1.78577e-02
64×64	8.93133e-03	4.07132e-02	8.93167e-03	4.07133e-02	8.93133e-03
128×128	4.46597e-03	2.03588e-02	4.46601e-03	2.03588e-02	4.46597e-03

Table 2: The numerical errors on RT1 mixed finite element for state function and piecewise linear functions for control function.

Resolution	Errors				
	$\|u-u_h\|$	$\|\vec{p}-\vec{p}_h\|$	$\|y-y_h\|$	$\|\vec{q}-\vec{q}_h\|$	$\|z-z_h\|$
16×16	2.45618e-02	3.51262e-03	2.48576e-02	3.46962e-02	2.45618e-02
32×32	6.14512e-03	8.80030e-04	6.18862e-03	8.68949e-03	6.14512e-03
64×64	1.53657e-03	2.20265e-04	1.55526e-03	2.17445e-03	1.53657e-03
128×128	3.84161e-04	5.51003e-05	3.88836e-04	5.43881e-04	3.84161e-04

Example 5.2. In this example we set the other known functions as follows:

$$
\begin{aligned}
&y=(x_1+x_2)\sin\pi x_1\sin\pi x_2,\\
&z=-(x_1+x_2)\sin\pi x_1\sin\pi x_2,\\
&u=\max(\bar{z},0)-z,\\
&f=2\pi^2(x_1+x_2)\sin\pi x_1\sin\pi x_2+\cos\pi x_1\sin\pi x_2+y^3-u,\\
&\vec{p}=-\big(\pi\cos\pi x_1\sin\pi x_2+\sin\pi x_1\sin\pi x_2,\pi\cos\pi x_2\sin\pi x_1+\sin\pi x_1\sin\pi x_2\big),\\
&\vec{p}_d=-\big(\pi\cos\pi x_1\sin\pi x_2+\sin\pi x_1\sin\pi x_2,\pi\cos\pi x_2\sin\pi x_1+\sin\pi x_1\sin\pi x_2\big),\\
&\vec{q}=\big(\pi\cos\pi x_1\sin\pi x_2+\sin\pi x_1\sin\pi x_2,\pi\cos\pi x_2\sin\pi x_1+\sin\pi x_1\sin\pi x_2\big),\\
&y_d=y+2\pi^2(x_1+x_2)\sin\pi x_1\cos\pi x_2-2\pi\cos\pi x_1\sin\pi x_2-2\pi\sin\pi x_1\cos\pi x_2-3y^2z.
\end{aligned}
$$

The profile of the numerical solution is presented in Fig. 2. From the error data on the uniform refined meshes, as listed in Tables 3 and 4, it can be seen that the priori error estimates results remains in our data.

The above examples obviously indicate that the resulting error estimates in Theorem 4.3 remains in output data. From the error data in our examples, it can be seen that the priori error estimates that we have mentioned is exact.

Table 3: The numerical errors on RT0 mixed finite element for state function and piecewise constant functions for control function.

Resolution	Errors				
	$\|\|u-u_h\|\|$	$\|\|\vec{p}-\vec{p}_h\|\|$	$\|\|y-y_h\|\|$	$\|\|\vec{q}-\vec{q}_h\|\|$	$\|\|z-z_h\|\|$
16×16	3.56759e-02	1.62504e-01	3.56977e-02	1.62509e-01	3.56758e-02
32×32	1.78578e-02	8.13915e-02	1.78605e-02	8.13920e-02	1.78577e-02
64×64	8.93133e-03	4.07132e-02	8.93167e-03	4.07133e-02	8.93133e-03
128×128	4.46597e-03	2.03588e-02	4.46601e-03	2.03588e-02	4.46597e-03

Table 4: The numerical errors on RT1 mixed finite element for state function and piecewise linear functions for control function.

Resolution	Errors				
	$\|\|u-u_h\|\|$	$\|\|\vec{p}-\vec{p}_h\|\|$	$\|\|y-y_h\|\|$	$\|\|\vec{q}-\vec{q}_h\|\|$	$\|\|z-z_h\|\|$
16×16	1.54735e-03	5.68057e-03	1.54740e-03	5.68596e-03	1.54735e-03
32×32	3.87330e-04	1.42815e-03	3.87334e-04	1.42866e-03	3.87330e-04
64×64	9.68635e-05	3.58018e-04	9.68637e-05	3.58072e-04	9.68635e-05
128×128	2.42181e-05	8.96263e-05	2.42178e-05	8.96330e-05	2.42181e-05

6. Conclusion and future works

We have presented the higher order triangular mixed finite element methods of the semilinear elliptic optimal control problems (1.1)-(1.4). By applying the priori error estimate results (see [26]) of the standard mixed finite element methods, we have established some error estimate results for both the state, the co-state and the control approximation with convergence order h^{k+1}. The priori error estimates for the general semilinear elliptic optimal control problems by mixed finite element methods seem to be new.

In our future work, we shall use the mixed finite element method to deal with the optimal control problems governed by nonlinear parabolic equations and convex boundary control problems. Furthermore, we shall consider a priori error estimates and superconvergence of optimal control problems governed by nonlinear parabolic equations or convex boundary control problems.

Acknowledgments The second author is supported by the Foundation for Talent Introduction of Guangdong Provincial Universities and Colleges, Pearl River Scholar Funded Scheme (2008), National Science Foundation of China (10971074).

The authors express their thanks to the referees for their helpful suggestions, which led to improvements of the presentation.

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