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Chebyshev Spectral Methods and the Lane-Emden Problem

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Abstract. The three-dimensional spherical polytropic Lane-Emden problem is $y_{rr} + (2/r)y_r + y^m = 0$, $y(0) = 1, y_r(0) = 0$ where $m \in [0,5]$ is a constant parameter. The domain is $r \in [0,\xi]$ where ξ is the first root of $y(r)$. We recast this as a nonlinear eigenproblem, with three boundary conditions and ξ as the eigenvalue allowing imposition of the extra boundary condition, by making the change of coordinate $x \equiv r/\xi$: $y_{xx} + (2/x)y_x + \xi^2 y^m = 0$, $y(0) = 1, y_x(0) = 0, y(1) = 0$. We find that a Newton-Kantorovich iteration always converges from an m-independent starting point $y^{(0)}(x) = \cos([\pi/2]x)$, $\xi^{(0)} = 3$. We apply a Chebyshev pseudospectral method to discretize x. The Lane-Emden equation has branch point singularities at the endpoint $x = 1$ whenever m is not an integer; we show that the Chebyshev coefficients are $a_n \sim \text{constant}/n^{2m+5}$ as $n \to \infty$. However, a Chebyshev truncation of $N = 100$ always gives at least ten decimal places of accuracy — much more accuracy when m is an integer. The numerical algorithm is so simple that the complete code (in Maple) is given as a one page table.

AMS subject classifications: 65L10, 65D05, 85A15

Key words: Lane-Emden, Chebyshev polynomial, pseudospectral.

1. Introduction

The Lane-Emden problem is

$$y_{rr} + (2/r)y_r + y^m = 0, \qquad y(0) = 1,\ y_r(0) = 0, \tag{1.1}$$

where $m \in [0,5]$ is a constant parameter. (This is the three-dimensional spherical polytropic case whose astrophysical context is given in the book by the Nobel Laureate "Black

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Hole" Chandrasekhar [7]; other variants are described in [10].) The goal of the astrophysical problem is to integrate this equation from the origin to its first zero, $r=\xi$. It is helpful to rescale the problem by defining

$$r=\xi x, \qquad x\equiv r/\xi. \tag{1.2}$$

The problem becomes

$$y_{xx}+(2/x)y_x+\xi^2 y^m=0, \qquad y(0)=1, y_x(0)=0, y(1)=0. \tag{1.3}$$

This is a nonlinear eigenvalue problem with the location of the first zero ξ as the eigenvalue. The eigenvalue is chosen so that the extra boundary condition is satisfied. Analytical solutions are known only for the exponent m equal to 0, 1 or 5 as catalogued in Table 1.

Table 1: Analytical exact solutions.

m	$y(r;m)$	ξ [first zero of $y(r;m)$]
0	$1-(1/6)r^2$	$\sqrt{6}$
1	$\sin(r)/r$	π
5	$1/\sqrt{1+r^2/3}$	∞

The Lane-Emden problem has a long history. Numerical tables for selected values of m were published as early as 1932. A small subset of the available literature is given in the bibliography table. This problem has become one of those benchmarks which are revisited repeatedly as every new numerical method is tested against it. In spite of this vast literature, recent applications of higher spectral methods have sloughed over important difficulties.

First, the Lane-Emden equation is *singular* at the right endpoint (where $y(1)=0$) whenever the order m is not equal to an integer. Singularities degrade the usual exponential rate of convergence of a spectral method to a finite order rate of convergence. That is to say, the error falls proportional to $1/N^k$ for some constant k, the so-called "algebraic order of convergence", where k depends on the type of singularity as will be explained in more detail below. Fortunately, it is possible to modify spectral methods so as to recover an exponential rate of convergence as will be explained later. Second, the Lane-Emden problem is a nonlinear eigenvalue problem. Although the differential equation is the second order, we need to satisfy three boundary conditions. This is possible because the problem also contains an eigenparameter space ξ which must be determined simultaneously with the solution to the differential equation.

2. Numerical strategies

One strategy is based upon the following theorem.

Theorem 2.1. *Suppose that $w(x,\xi)$ solves $w_{xx}+(2/x)w_x+\xi^2 w=0$ with $w(0)=1, w_x(0)=0, w(1)=0$. Then $v\equiv \omega w(x,\xi)$ solves $v_{xx}+(2/x)v_x+\Xi^2 v^m=0$ with $v(0)=\omega, v_x(0)=$*

$0, v(1) = 0$ *provided that*

$$\omega = \left(\frac{\xi^2}{\Xi^2}\right)^{1/(m-1)}. \tag{2.1}$$

Proof. Substitution of v into the equation is solved by v followed by cancellation of common factors. □

Table 2: Lane-Emden bibliography.

Airey [8]	Taylor series in x
Bender *et al* [2]	perturbation parameter is $m-1$
Chandrasekhar [7]	book including extensive tables
Chandrasekhar [6]	numerical table for $m = 3.25$
Davis [8]	review
He [9]	derives variational principle; no solutions
Horedt [10]	extensive tables; good discussion of other Lane-Emden cases (planar geometry, etc.)
Hunter [11]	Euler-accelerated power series; discussion of singularities in the complex plane
Liao [12]	homotopy perturbation method
Liu [13]	analytic approx. including $m \to \infty$
Mandelzweig & Tabakin [16]	review of quasi-linearization; LE, reduced to form without 1st deriv. is example with rate-of-convergence plot
Marzban, Tabrizidooz & Razzaghi [17]	Legendre domain decomposition
Mohan & Al-Bayaty [18]	power series about $x = 0$, $x = \xi$ and an arbitrary point. Expansions about $x = \xi$ for integer m only
Nouh [20]	power series in x^2, accelerated by Euler sum acceleration, then applied Padé
Parand *et al.* [21]	rational Legendre on $x \in [0, \infty]$; restricted to integer m
Pascual [22]	Padé approximants
Ramos [23]	homotopy perturbation applied to both differential eq. & Voltera integral formulation
Roxburgh & Stockman [24]	power series, Padé approximants and simple rational approximations
Sadler & Miller [25]	extensive tables, $m \in [1, 5]$
Shawagfeh [26]	Adomian decomposition
Yildirim & Öziş [28]	He's variational-iteration method

The theorem implies that one can solve the Lane-Emden differential equation as a boundary value problem with $\xi > 0$ as an arbitrary constant and with homogeneous Dirichlet boundary conditions. The Neumann condition at the origin, $w_x(0) = 1$, will not be satisfied. However,the condition can be satisfied by using the theorem to rescale the amplitude of the solution and simultaneously rescale the parameter ξ and thus determine the eigenvalue. We program this strategy and indeed were successful in solving the problem over most of parameter space. However, when $m = 1$, the problem becomes a linear eigenproblem which is insoluble unless ξ is equal to a discrete eigenvalue. Thus, the simple boundary value method fails for this value of the exponent m. Consequently, we developed a second numerical strategy which is more robust.

2.1. Nonlinear eigenvalue strategy

In this strategy, we discretized the spatial dependence by using collocation, also known as the "pseudo-spectral" method, on an N-point Chebyshev Lobatto grid. The numerical unknowns consisted of the N spectral coefficients a_j of the truncated Chebyshev polynomial series plus the eigenvalue ξ. We impose a total of $N+1$ constraints to implicitly determine the unknowns:

1. $(N-2)$ collocation conditions that the residual of the differential equation should be zero are imposed at the interior points of the Lobatto grid plus;
2. $y(0)=1$;
3. $y_x(0)=0$;
4. $y(1)=0$.

It is possible to impose *three* boundary conditions on a differential equation that is only of order *two* because the eigenvalue ξ is determined simultaneously with the solution so that the extra boundary condition is satisfied. This proved to be a very robust procedure. Because the differential equation is nonlinear, we employed a Newton-Kantorovich iteration. (This strategy is also known as "quasi-linearization".) We assume that the solution can be written

$$y(x) \approx y^{(n)}(x) + \delta^{(n)}(x), \qquad \xi \approx \xi^{(n)} + \epsilon^{(n)}, \tag{2.2}$$

where the correction $|\delta^{(n)}(x)|_\infty \ll |y^{(n)}(x)|_\infty$ and $|\epsilon^{(n)}| \ll \xi^{(n)}$. Substitution into the differential equation, neglect of terms quadratic in the correction space δ, and rearrangement gives a *linear* differential equation for the correction assuming that the previous iterate $y^{(n)}(x)$ is already known:

$$\begin{aligned} &\delta^{(n)}_{xx} + (2/x)\delta^{(n)}_x + m\left(\xi^{(n)}\right)^2\left(y^{(n)}\right)^{m-1}\delta^{(n)} \\ &= -\left\{y^{(n)}_{xx} + (2/x)y^{(n)}_x + \xi^2\left(y^{(n)}\right)^m\right\} \end{aligned} \tag{2.3}$$

with the boundary conditions

$$\delta^{(n)}_x(0) = 0, \qquad \delta^{(n)}(1) = 0. \tag{2.4}$$

The eigenvalue ξ is adjusted so that in addition

$$\delta^{(n)}(0) = 0. \tag{2.5}$$

A universally successful first guess for all $m \in [0,5)$ was

$$y^{(0)}(x) = \cos([\pi/2]\,x), \qquad \xi^{(0)} \equiv 3. \tag{2.6}$$

Thus, this problem is not representative of nonlinear problems in general where continuation in a parameter and other sophisticated techniques are needed to provide a satisfactory first guess. To discretize the Newton-Kantorovich iteration, begin by choosing N as the number of points in a Chebyshev-Lobatto grid:

$$x_j = (1/2)\left\{1 + \cos\left(t_j\right)\right\}, \qquad j = 1, 2, \cdots, \tag{2.7}$$

where

$$t_j \equiv \pi \frac{j-1}{N-1}, \qquad j = 1, 2, \cdots N. \tag{2.8}$$

The unknowns $y(x)$ and $\delta^{(n)}(x)$ are expanded as a truncated Chebyshev series:

$$y(x) = \sum_{j=0}^{N-1} a_j T_j(x), \qquad \delta^{(n)}(x) = \sum_{n=0}^{N-1} d_j^{(n)} T_j(x). \tag{2.9}$$

We will obtain N algebraic relations from (i) collocation of the differential equation at $(N-2)$ interior points plus (ii) imposition of the two homogeneous boundary conditions, as is standard in the discretization of any second order ODE. What makes this an eigenvalue problem is that ξ is added as the $(N+1)$-st unknown while the extra constraint is the third boundary condition, $y(0) = 1$. This is a normalization-of-amplitude condition is similar to the normalization of the eigenvectors performed by linear eigenvalue library routines. Define the differentiation matrices $\vec{\vec{D}}0$, $\vec{\vec{D}}1$, and $\vec{\vec{D}}2$ to have the elements

$$\begin{aligned}
D0_{ij} =& T_j(2x_i - 1) = \cos([j-1]t_i), \qquad\qquad j = 1, 2, \cdots N, \\
D1_{ij} =& T_{j,x}(2x_i - 1) = 2(j-1)\frac{\sin((j-1)t_i)}{\sin(t_i)}, \\
D2_{ij} =& T_{j,xx}(2x_i - 1) \\
=& 4\left\{-(j-1)^2 \frac{\cos([j-1]t_i)}{\sin^2(t_i)} + (j-1)\frac{\sin([j-1]t_i)\cos(t_i)}{\sin^3(t_i)}\right\},
\end{aligned} \tag{2.10}$$

where we have used the trigonometric connection between a Fourier cosine basis and the Chebyshev polynomials

$$T_n(\cos(t)) \equiv \cos(nt), \qquad \forall\, n, t. \tag{2.11}$$

The derivative formulas have numerators and denominators that vanish as $|x| \to 1$. Fortunately, the limits are known analytically:

$$D0_{1j} = 1, \tag{2.12}$$
$$D0_{Nj} = (-1)^{j+1}, \tag{2.13}$$
$$D1_{1j} = 2(j-1)^2, \tag{2.14}$$
$$D1_{Nj} = 2\,(-1)^{j+1}\,(j-1)^2, \tag{2.15}$$
$$D2_{1j} = 4(j-1)^2((j-1)^2-1)/3, \tag{2.16}$$
$$D2_{Nj} = 4(-1)^{j+1}\,(j-1)^2((j-1)^2-1)/3. \tag{2.17}$$

Let $\vec{\delta}$ be a vector of dimension $N+1$ whose first N elements are the Chebyshev coefficients of the function $\delta^{(n)}(x)$ and whose last element is $\xi^{(n)}$. The Newton iteration requires solving at each step the $(N+1)$-dimensional linear algebra problem

$$\vec{\vec{J}}\,\vec{\delta} = -\vec{r}. \tag{2.18}$$

The grid point values of $y^{(n)}(x)$ and its first two derivatives are organized into vectors of length N by

$$\vec{y} \equiv \vec{\vec{D0}}\vec{a}, \qquad \vec{yx} \equiv \vec{\vec{D1}}\vec{a}, \qquad \vec{yxx} \equiv \vec{\vec{D2}}\vec{a}. \tag{2.19}$$

The first $(N-2)$ elements of the residual vector come from the interior collocation conditions applied to the discretization of the Lane-Emden equation:

$$r_j = yxx_{j+1} + (2/x_{j+1})yx_{j+1} + \left(y_{j+1}\right)^m. \tag{2.20}$$

The next two elements come from the homogeneous boundary conditions imposed on $\delta^{(n)}(x)$:

$$r_{N-1} = r_N = 0. \tag{2.21}$$

The final element of the residual vector imposes the inhomogeneous boundary condition $y(0) = 1$:

$$r_{N+1} = y_1 - 1. \tag{2.22}$$

The elements of the Jacobian matrix are

$$\begin{aligned}
&J_{ij} = D2_{i+1,j} + (2/x_{i+1})D1_{i+1,j} + \xi^2 m y_{i+1}^{(m-1)} D0_{i+1,j}, && i = 1,\cdots,(N-2),\\
& && j = 1,\cdots,N,\\
&J_{N-1,j} = D0_{1,j}, && j = 1,\cdots,N,\\
&J_{N,j} = D1_{N,j}, && j = 1,\cdots,N,\\
&J_{N+1,j} = D0_{N,j}, && j = 1,\cdots,N,\\
&J_{i,N+1} = 2\xi^{(n)} y_{i+1}, && i = 1,\cdots,N,\\
&J_{N-1,N+1} = J_{N,N+1} = J_{N+1,N+1} = 0.
\end{aligned} \tag{2.23}$$

3. Singularities

When m is an *integer*, the Lane-Emden solution $y(x)$ is regular on the entire domain. When m is non-integral, then the term y^m has a *branch point singularity* at $x = 1$ where the boundary condition requires $y(1) = 0$. If $y(x)$ has a simple zero at $x = 1$, i. e.,

$$y(x) \approx Az + \text{higher order terms in z}, \tag{3.1}$$

where A is a constant (always *negative* for the Lane-Emden equation for $m \in [0,5]$ at least) and where

$$z \equiv x - 1, \tag{3.2}$$

then obviously

$$y^m \sim (-A)^m(-z)^m + \text{higher order terms in z}, \tag{3.3}$$

where we have chosen the branches to give the correct real-valued result when $A < 0$ and $z < 0$, as it is to the left of the boundary at $x = 1$. As noted by Hunter [11] — his variable ζ is our x^2, but the same principle applies — the singularity in $y(x)$ is a branch point of *order two greater* than that in y^m because it is the *second derivative* of $y(x)$ that must balance y^m

$$y(x) \sim Az - \frac{\xi^2(-A)^m}{(m+2)(m+1)}(-z)^{m+2} + \text{higher order terms in z}. \tag{3.4}$$

(Note that there is no inconsistency between the answer and the argument used to derive it because the leading branch point term is always decaying as fast as quadratically as $x \to 1$ while Az is linear.) In other words, when m is not an integer, $y(x)$ will have a branch point of order $m+2$ at the right endpoint, $x = 1$.

3.1. Implications for Chebyshev spectral methods

Elliott proved that if a function has a branch point singularity of the form $(1-x)^\phi$ where ϕ is not an integer, then the Chebyshev coefficients a_n would asymptotically be approximately

$$(1-x)^\phi \quad \Longrightarrow \quad a_n \sim \text{constant}/n^{2\phi+1}, \qquad n \gg 1. \tag{3.5}$$

When the Chebyshev coefficients decay asymptotically as $1/n^k$ for some constant $k<$ then k is the "algebraic order of convergence" in the language of [4]. In contrast, when a function $y(x)$ which is *singularity-free* on the interval is expanded in Chebyshev polynomials, the coefficients decrease *geometrically fast*, that is, fall like the terms of a geometric series with terms asymptotically proportional to p^n for some $|p| < 1$, or equivalently

$$a_n \sim \text{constant}\exp(-qn),$$

where $q = \log(p)$. For many functions with weak singularities (high algebraic convergence order k), it is common to see the Chebyshev coefficients display both an algebraic, power law rate of convergence (for large degree) and an exponential decay (for small and moderate n). The degree where there is a (usually rather sharp) transition is the "crossover degree". Examples of such transitions are given in [4]. It follows from the previous subsection and Elliott's theorem that when m is not an integer, the Chebyshev coefficients of the solution to the Lane-Embden equation will have an *algebraic* rate of convergence with convergence order

$$a_n \sim \text{constant}/n^{2m+5}. \tag{3.6}$$

However, the algebraic order is relatively high because the minimum order of convergence is always at least five. Does such a relatively high rate of algebraic convergence matter? The subtle answer: in a practical sense: No. However, the algebraic convergence is definitely observable as shown below.

3.2. Generic properties of Chebyshev series

To interpret graphs of Chebyshev coefficients, note that

$$|T_n(2x-1)| \leq 1, \qquad \forall n,\ x \in [0,1], \tag{3.7}$$

as follows trivially from the identity $T_n(cos(t)) = \cos(nt)$ and also that consequently the error in truncating a Chebyshev series by neglecting all terms of degree $N+1$ and higher is always bounded by the sum of the absolute values of the neglected terms.

$$\max_{x\in[0,1]} |f(x) - f_N(x)| \leq \sum_{n=N+1}^{\infty} |a_n|. \tag{3.8}$$

Unfortunately, the error of a Chebyshev series with algebraically-converging coefficients decreasing proportional to $1/n^k$ yields an error which falls only as $\mathcal{O}(1/N^{k-1})$ because of the following asymptotic inequality.

Theorem 3.1 (Bounds on Algebraically-Converging Sums). *For $k \geq 2$,*

$$\frac{1}{(k-1)(N+1)^{k-1}} < \sum_{n=N+1}^{\infty} n^{-k} < \frac{1}{(k-1)N^{k-1}}. \tag{3.9}$$

Proved in [1].

(Although this identity only gives an error *bound* and not the error itself, bounds based on summing the neglected coefficient are usually rather tight in practice as explained in the author's book [4].) Thus, a Chebyshev series with an algebraic index of convergence k will have an error falling only as $\mathcal{O}(1/N^{k-1})$.

3.3. Numerical results

The exact solution for $m = 0$ is a parabola which is given exactly as the sum of three Chebyshev coefficients:

$$u(x; m = 0) = (5/8) - (1/2)T_1(2x - 1) - (1/8)T_2(2x - 1) = 1 - x^2. \tag{3.10}$$

Fig. 1 shows the numerical solution for a tiny perturbation of this case, $m = 1/10,000$. The predicted fifth order convergence is observed with

$$a_n \sim 0.001/n^5$$

fitting the Chebyshev coefficients well for large n. Even so, because $a_{100} \sim 10^{-13}$, it is obvious that the accuracy of the 100-point Chebyshev pseudospectral approximation is extremely high. This is true even though the rate of convergence of the *error* is one less than the order of convergence of the *coefficients* as asserted by Theorem 3.1 and demonstrated for this value of m in Table 3.

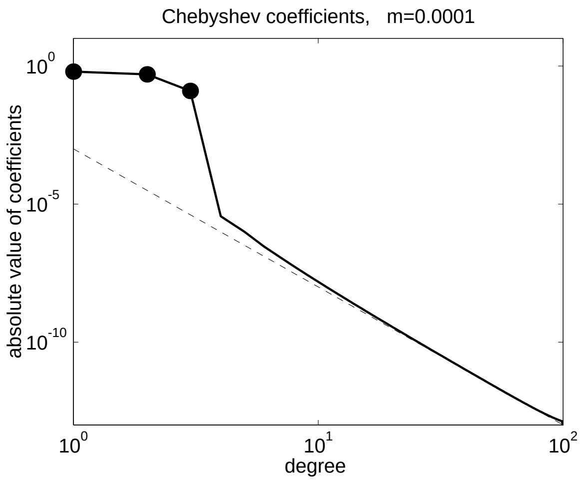

Figure 1: Chebyshev coefficients for $m = 1/10,000$ [thick solid]. These asymptote for large degree n to roughly $0.001/n^5$. The exact solution for $m = 0$ is a parabola whose three Chebyshev coefficients are marked by the black disks. The perturbation of changing m from zero to $1/10,000$ has modified the Chebyshev coefficients by rendering nonzero all coefficients of degree three and greater. However, because the sum of a series of Chebyshev coefficients is bounded by the absolute values of the included coefficients, it follows just from inspection of the tiny magnitude of $a_3, a_4, \cdots$ that the algebraically converging sum $\sum_{n=3}^{\infty}$ will modify $y(x)$ by less than 0^{-4}. Although the perturbation has drastically changed the quantitative appearance of a graph of Chebyshev coefficients, $u(x; m = 1/10,000)$ is still very well approximated by the parabola $u(x; m = 0)$.

When $m = 5/2$, the first 27 coefficients fall geometrically as shown in the left graph of Fig. 2. Beyond this crossover degree, the coefficients fall proportional to n^{-10}. These coefficients were computed using thirty digits of accuracy in Maple to show that the predicted theoretical behavior is truly there. It is difficult to see the power-law decay otherwise because the coefficient a_{26} has a magnitude smaller than 10^{-12}!. For all *practical* purposes, the convergence is geometric in spite of the branch point of order 4.5 of $y(x)$ at $y = 1$.

Table 3: L_∞ error for $m = 0.0001$.

truncation degree	max. pointwise error	$E_{N/2}/E_N$
10	1.836e-08	-
20	1.266e-09	14.51
40	8.2218e-11	15.40
80	5.009e-12	16.40

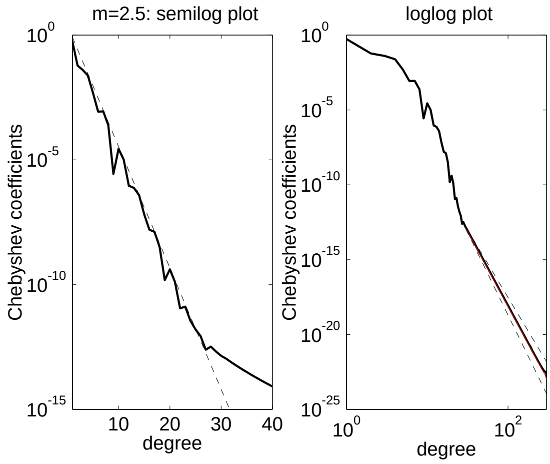

Figure 2: Chebyshev coefficients for $m = 5/2$. The left plot, which uses a linear scale in degree but a logarithmic scale for the coefficients, shows that the leading coefficients fall geometrically. The dashed line is the graph of $2.5\exp(-1.12n)$. The right graph is the same but with both scales logarithmic, and with the range of coefficients extended. The coefficients of degrees 27 to 400 are well-fit by the dashed line, indistinguishable from the graph of the coefficients, $75/n^{10}$. The other dashed guidelines, clearly not of the same slope as the thick curve of the coefficients, have slopes proportional to n^{-9} and n^{-11}.

Fig. 3 compares the Chebyshev coefficients for the exponent m from zero to 4.5 in increments of one-half. As m increases, the rate of geometric convergence steadily slows. However, when m is not an integer, there is an algebraically-converging tail that appears when the degree n is beyond the crossover degree for that function. This tail converges most slowly for smaller m; thus, only $m = 1/2$ has coefficients larger than 10^{-11} at $n = 100$, the right side of the graph. The crossover degree for high m is very large, and beyond the right edge of the graph, because for $m = 4.5$, for example, the geometrically-converging coefficients decrease more slowly than for any of the other nine exponents shown, but the algebraic tail is proportional to n^{-14}, and so only is visible at very large n.

Fig. 4 shows that a hundred Chebyshev polynomials suffice to give at ten decimal place accuracy for all m shown. Nevertheless, there are sharp down spikes in the error curve at $m = 0, 1, 2, 3$ because there is no singularity when m is an integer. The spike at $m = 4$ is muted because the singularity is so weak — the error falls at least as fast as $\mathcal{O}(N^{-12})$ for $m > 4$ — that it hardly matters except at considerably higher truncation, computed in more than the 2×10^{-16} precision of Matlab.

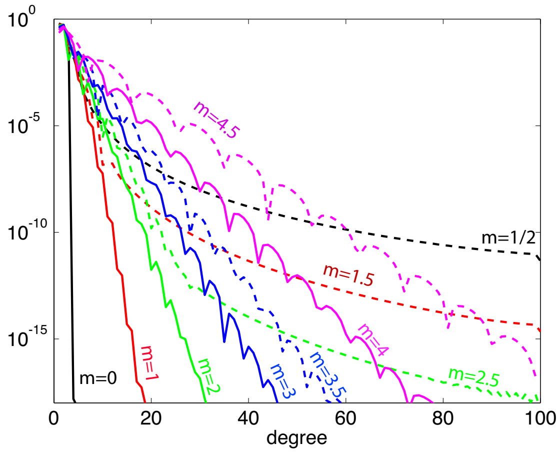

Figure 3: Absolute values of Chebyshev coefficients of $y(x; m)$ for ten different values of the exponent m. The coefficients for $m = k$ and $m = k+1/2$ (same color, dashed) are similar for small degree, but then differ dramatically for large degree. When m is an integer, there is no singularity and the exponential decay continues to all degrees. When m is not an integer, the coefficients have a "long tail" of decay as an inverse power law proportional to $n^{-(2m+5)}$.

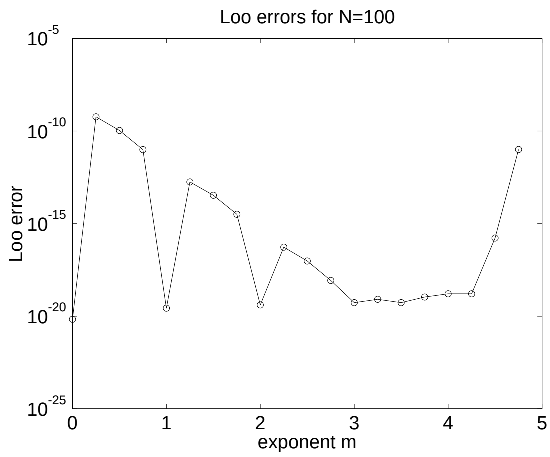

Figure 4: Maximum pointwise error (L_∞ norm error) in truncating the Chebyshev series for the Lane-Emden function after $N = 100$ terms, computed by comparison with numerical solutions computed at much higher resolution.

4. Old table/new table/very new table

Chandrasekhar [6] published tables of the Lane-Emden function for $m = 3.25$. He gives the first root as $\xi = 8.01894$, and then tabulates $y(r; 3.25)$ and five derivative quantities at intervals of one-tenth in r up to $r = \xi$, a total of 486 six digit numbers. This is old style of table-making: list a lot of values in x so that low order interpolation can fill the gaps in the tabulated values. A much more efficient way is to simply list the Chebyshev coefficients of $y(x; 3.25)$. Derived quantities such as the first and second derivatives can be evaluated at an arbitrary point by using the trigonometric functions given in (2.10) with t_i replaced by $t = \arccos(x)$. Only thirty numbers suffice to give an accuracy to less than 5×10^{-11}, as determined by bounding the neglected coefficients and rounding of the numbers in the table. Summation of the Chebyshev series is not a low order interpolation, but has the full specified accuracy at any point $x \in [0,1]$. The new style — Chebyshev style — of table-making is much more efficient [14, 15, 19]. This can be extended to bivariate Chebyshev series as in [3,27]. However, the Chebyshev pseudospectral/Newton-Kantorovich algorithm is so simple that it is preferable here to apply a third strategy, which is to give the complete computer code to solve the Lane-Embden equation as in Table 6.

Table 4: Eigenvalues.

m	ξ	$dy/dx(1)$
0	$\sqrt{6} = 2.44948974278318$	-2
0.5	2.752698054065	-1.3763410
1	π	-1
1.5	3.65375373621912 608	-7.428128210
2	4.352874595946124676973570 0	-0.553897420877303672
2.5	5.355275459010779	-0.408419619543
3	6.896848619376960375454528	-0.292631615154770425343767 5
3.25	8.018937527	-0.24314991318450
3.5	9.535805344244850444	-0.1982587757607830
4	14.97154634883809509761106 6	-0.120043038478702539
4.5	31.836463244694285264	-0.0545851734317770
5	∞	0

Table 5: Chebyshev coefficients for $y(x;3.25)$ to within an L_∞ error of 5×10^{-11}.

n	a_n	n	a_n
0	0.3974001738440	15	2.8371063e-06
1	-0.5222925358655	16	2.930727e-07
2	0.1363576404586	17	-7.512164e-07
3	4.53209425128e-03	18	4.383967e-07
4	-0.0286527474641	19	-1.416943e-07
5	0.0178489951998	20	7.2662e-09
6	-6.1516194336e-03	21	2.31240e-08
7	5.808182397e-04	22	-1.67198e-08
8	8.106935493e-04	23	6.4938e-09
9	-6.449182037e-04	24	-9.950e-10
10	2.658761754e-04	25	-6.239e-10
11	-4.90112935e-05	26	6.092e-10
12	-1.97167283e-05	27	-2.799e-10
13	2.25847399e-05	28	6.48e-11
14	-1.10220727e-05	29	1.24e-11

Table 6: Maple code for solving the Lane-Emden equation.

```
Digits:=30; # variable precision;  N:=20; # no. of interpolation pts;
m:= 0.5;  # Lane-Emden exponent; itermax:= 20; # no. of Newton iters.;
with(LinearAlgebra);  D0Mat:= Matrix(N,N): D1Mat:= Matrix(N,N):
D2Mat:= Matrix(N,N): xCheb:= Vector(N):
fa:= Vector(N+1,orientation=column): ya0:= Vector(N,orientation=column):
for ii from 1 by 1 to N do # compute grid points xCheb;
ta[ii]:=evalf( Pi*(ii-1)/(N-1)); xCheb[ii]:=evalf(0.5*(1+cos(ta[ii]))); od:
for ii from 2  to  (N-1)  do  # differentiation matrices:
t:=ta[ii];  ss:=evalf( sin(t)) ;     cc:=evalf( cos(t));
for j from 1 by 1 to N do
D0Matrix[ii,j]:=evalf(cos((j-1)*t));   pt:=evalf(-(j-1)*sin((j-1)*t) );
ptt:=evalf(- (j-1)*(j-1)*D0Mat[ii,j]); D1Mat[ii,j]:=evalf(-2*pt/ss);
D2Mat[ii,j]:=evalf(4*(ptt/(ss*ss)-cc*pt/(ss*ss*ss))); od: od:
# apply non-trig formulas at the endpoints;
for j from 1 to N  do D0Mat[1,j]  := 1;
D0Mat[N,j]:= (-1)^(j+1);  D1Mat[1,j]:=evalf(2*(j-1)^2);
D1Mat[N,j]:=evalf(2*(-1)^j *(j-1)^2);
D2Mat[1,j]:=evalf((j-1)^2 *((j-1)^2 -1 )*(4/3));
D2Mat[N,j]:=evalf(-(-1)^j *(j-1)^2 *( (j-1)^2 - 1)*4/3) ; od:
for ii from 1 to N do ya0[ii]:=evalf(cos((Pi/2)*xCheb[ii]));od: xi0:=3.0;
# ya0,  xi0 are first guess for Newton iteration;
a0:=LinearSolve(<D0Matrix|ya0 >); # Cheb coeffs of ya0(x);
a:=a0: xi:=xi0; Jacobian:=Matrix(N+1,N+1): for iter from 1 to itermax do
# begin Newton-Kantorovich iter.; ya:=MatrixVectorMultiply(D0Mat,a);
for ii from 1 to (N-2) do fa[ii]:= - xi*xi*ya[ii+1]**m;
for j from 1 to N do fa[ii]:= fa[ii] - D2Mat[ii+1,j]*a[j]
   - (2 / xCheb[ii+1] ) * D1Mat[ii+1,j]*a[j]; od:  od:  yatzero:=0;
for j from 1 to N do yatzero:=evalf(yatzero + D0Mat[N,j]*a[j]): od;
fa[N+1]:=-( yatzero-1); fa[N-1]:=0; fa[N]:=0; Jacobian[N+1,N+1]:= 0;
for ii from 1 to (N-2) do for j from 1 by 1 to N do
Jacobian[ii,j]:= D2Matrix[ii+1,j]+(2 /xCheb[ii+1])*D1Mat[ii+1,j]
+ xi*xi * m * ya[ii+1]**(m-1) * D0Mat[ii+1,j] ; od:  od:
for j from 1 to N do Jacobian[N-1,j]:= D0Mat[1,j];
Jacobian[N,j] := D1Mat[N,j];   Jacobian[N+1,j]:= D0Mat[N,j]; od:
for ii from 1 to (N-2) do Jacobian[ii,N+1] := 2*xi* ya[ii+1]**m;  od:
Jacobian[N,N+1]:= 0;      Jacobian[N-1,N+1]:= 0;
delta_a_and_xi:=LinearSolve(  <Jacobian|fa>);
for j from 1 to N do a[j]:= evalf( a[j] + delta_a_and_xi[j] ); od:
xidelta:=delta_a_and_xi[N+1]; xi:=xi+xidelta; print(iter,xidelta); od:
print(xi);  for j to N do printf("%18.17e ...\n",a[j]); od:
```

5. Mappings and difficulties

The endpoint singularity can be weakened by applying a change-of-coordinate ("mapping") $x = f(t)$ where t is the new computational coordinate, and where the mapping function $f(t)$ is chosen so that the standard Chebyshev grid in t gives a grid in x such that points are much denser near the endpoints than in the Chebyshev grid with the same number of points. Such mappings are thoroughly discussed in the Chapter 16 of the author's book [4]. However, the unmapped Chebyshev method does so well that the Lane-Emden problem is a poor exemplar of the uses of mapping. In the limit the power of the nonlinearity m tends to five, the first root ξ in radius tends to infinity. It follows that

$$\lim_{m\to 5} y(x;m) = \delta(x),$$

a Dirac delta function. Again, a mapping can be used to mitigate the singularity. One possible strategy is to solve the problem on a semi-infinite domain in the original coordinate r as done by [21]. However, apparently to duck the singularity at $r=\xi$ where $y(r)$ is zero, they restricted their computations to integer m only. Interior singularities are more damaging than endpoint singularities with a branchpoint of order ϕ giving Chebyshev coefficients proportional $1/n^{\phi+1}$ and an error falling as $1/N^{\phi}$. For m near five, however, $\phi \approx 7$ meaning that seventh order convergence would be still be achieved. The root can be computed directly from the (mapped) Chebyshev series as explained in the review [5]. We have not actually implemented this strategy or any form of mapping because brute force application of Chebyshev polynomials is very effective except when m is *very* close to five, or when accuracy beyond ten decimal places is required.

6. Summary

The Lane-Emden problem is an interesting illustration of the effect of weak endpoint singularities on the rate of convergence of Chebyshev series. Although the Lane-Emden function is singular at the right endpoint whenever m is not an integer, there is no difficulty in calculating the function to ten decimal places using fewer than a hundred Chebyshev polynomials. And yet when we plot accuracy versus N, the effects of the branch points is clearly evident. For small m, the coefficients decay *exponentially* for small and moderate N, followed by a rather abrupt transition to a "tail" for larger N that falls proportional n^k for some k. For the Lane-Emden function with exponent $m \in [0,5]$, singularity analysis shows that $k = 2m+5$, so the coefficients always display at least a fifth order rate of convergence in the coefficients and fourth order in the error. The Lane-Emden problem is useless as a test of nonlinear iterations. In most problems, devising a first guess within the radius of convergence of Newton's iteration is difficult; perturbation theory about a limiting (often linear case) followed by continuation in the parameter is the usual remedy, accompanied by underrelaxation, line search and trust region modifications to Newton's method. None of this is needed here. The choice

$$y^{(0)}(x) \equiv \cos([\pi/2]x), \qquad \xi^{(0)} = 3$$

works for all $m \in [0,5]$.

The Lane-Emden problem is not a good test of differential equation solving methods either because it is so smooth that even a poor algorithm will yield accurate results.

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References

[1] G. BASZENSKI AND F. DELVOS, *Error-estimates for sine series expansions*, Math. Nach., 139 (1988), pp. 155–166.

[2] C. M. BENDER, K. A. MILTON, S. S. PINSKY, AND L. M. SIMMONS, JR., *A new perturbative approach to nonlinear problems*, J. Math. Phys., 30 (1989), pp. 1447–1455.

[3] J. P. BOYD, *A Chebyshev polynomial method for computing analytic solutions to eigenvalue problems with application to the anharmonic oscillator*, J. Math. Phys., 19 (1978), pp. 1445–1456.

[4] ———, *Chebyshev and Fourier Spectral Methods*, Dover, Mineola, New York, 2d ed., 2001. 665 pp. Heavily revised and updated second edition of Boyd(1989).

[5] ———, *Computing the zeros, maxima and inflection points of Chebyshev, Legendre and Fourier series: Solving transcendental equations by spectral interpolation and polynomial rootfinding*, J. Eng. Math., 56 (2006), pp. 203–219. Errata: $(-1)a_{j-1}/a_N$ in the last line of (2) should be $(-1)a_{j-1}/(2a_N)$.

[6] S. CHANDRASEKHAR, *The Lane-Emden function* $\theta_{3.25}$, Astrophysical Journal, 89 (1939), pp. 116–118.

[7] ———, *An Introduction to the Study of Stellar Structure*, Dover, New York, 1967.

[8] H. T. DAVIS, *Introduction to Nonlinear Differential and Integral Equations*, Dover, Mineola, New York, 1962.

[9] J.-H. HE, *Variational approach to the Lane-Emden equation*, Appl. Math. Comput., 143 (2003), pp. 539–541.

[10] G. P. HOREDT, *7-digit tables of Lane-Emden functions*, Astrophys. Space Sci., 126 (1986), pp. 357–408.

[11] C. HUNTER, *Series solutions for polytropes and the isothermal sphere*, Month. Notic. R. Astro. Soc., 328 (2001), pp. 839–847.

[12] S. LIAO, *A new analytic algorithm of Lane-Emden type equations*, Appl. Math. Comput., 142 (2003), pp. 1–16.

[13] F. K. LIU, *Polytropic gas spheres: An approximate analytic solution of the Lane-Emden equation*, Month. Notic. R. Astro. Soc., 281 (1996), pp. 1197–1205.

[14] Y. L. LUKE, *The Special Functions and Their Approximations*, vol. I & II, Academic Press, New York, 1969.

[15] ———, *Mathematical Functions and Their Approximations*, Academic Press, New York, 1975. 566 pp.

[16] V. B. MANDELZWEIG AND F. TABAKIN, *Quasilinearization approach to nonlinear problems in physics with application to nonlinear ODEs*, Comput. Phys. Commun., 141 (2001), pp. 268–281.

[17] H. R. MARZBAN, H. R. TABRIZIDOOZ, AND M. RAZZAGHI, *Hybrid functions for nonlinear initial-value problems with applications to Lane-Emden type equations*, Phys. Lett. A, 372 (2008), pp. 5883–5886.

[18] C. MOHAN AND A. R. AL-BAYATY, *Power-series solutions of the Lane-Emden equation*, Astrophys. Space Sci., 73 (1980), pp. 227–239.

[19] G. NÉMETH, *Mathematical Approximation of Special Functions: Ten Papers on Chebyshev Expansions*, Nova Science Publishers, New York, 1992. 200 pp.

[20] M. I. NOUH, *Accelerated power series solution of polytropic and isothermal gas sphe*, New Astron., 9 (2004), pp. 467–473.
[21] K. PARAND, M. SHAHINI, AND M. DEHGAHN, *Rational Legendre pseudospectral approach for solving nonlinear differential equations of Lane-Emden type*, J. Comput. Phys., 228 (2009), pp. 8830–8840.
[22] P. PASCUAL, *Lane-Emden equation and Padé approximants*, Astron. Astrophys., 60 (1977), pp. 161–163.
[23] J. I. RAMOS, *Series approach to the lane-emden equation and comparison with the homotopy perturbation method*, Chaos, Solitons & Fractals, 38 (2008), pp. 400–408.
[24] I. W. ROXBURGH AND L. M. STOCKMAN, *Power series solutions of the polytrope equations*, Mon. Not. R. Astron. Soc., 4303 (1999), pp. 466–470.
[25] D. H. SADLER AND J. C. P. MILLER, *Tables of the Lane-Emden function*, in Mathematical Tables, vol. 2, Office of the British Association, London, 1932.
[26] N. T. SHAWAGFEH, *Nonperturbative approximate solution for Lane-Emden equation*, J. Math. Phys., 34 (1993), pp. 3867–4372.
[27] V. B. SHEOREY, *Double Chebyshev expansions for wave functions*, Comput. Phys. Commun., 12 (1976), pp. 125–134.
[28] A. YILDIRIM AND T. ÖZIŞ, *Solutions of singular IVPs of Lane-Emden type by the variational iteration method*, Nonlinear Analysis, 70 (2009), pp. 2480–2484.