Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2011.m99040

Vol. 4, No. 1, pp. 38-52
February 2011

Spectral Method for Nonlinear Stochastic Partial Differential Equations of Elliptic Type

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Received 30 October 2009; Accepted (in revised version) 26 April 2010

Available online 29 October 2010

Abstract. This paper is concerned with the numerical approximations of semi-linear stochastic partial differential equations of elliptic type in multi-dimensions. Convergence analysis and error estimates are presented for the numerical solutions based on the spectral method. Numerical results demonstrate the good performance of the spectral method.

AMS subject classifications: 65N35, 65N15, 65C30, 60H15

Key words: Stochastic differential equation, spectral method, error estimates, white noise.

1. Introduction

Many natural phenomena and engineering applications are described by stochastic partial differential equations (SPDEs). The study of numerical methods for approximating stochastic partial differential equations has been an active research area. Some fluid flow and other engineering related SPDEs were studied using polynomial chaos expansions in [12,19,20]. In [3,4,6,11,15], traditional finite element methods are applied to SPDEs with random coefficients. Numerical methods for SPDEs with white noise forcing terms have also been developed, analyzed, and tested by numerous authors [2,7–10,13,14,16,17].

The main purpose of this paper is to study the numerical approximations by a spectral method for nonlinear stochastic differential equations of elliptic type driven by an additive white noise:

$$\Delta u(x) - f(u(x)) = g(x) + \dot{W}(x), \quad \text{for } x \in D, \tag{1.1}$$

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with boundary condition $u(x) = 0$ for $x \in \partial D$. Here D is a bounded open set of $\mathbb{R}^d$, $g \in L^2(D)$, f is a continuous function satisfying certain regularity conditions given in Section 2, and

$$\dot{W}(x) = \frac{dW}{dx}(x)$$

is a white noise. Buckdahn and Pardoux have proved the existence and uniqueness of the weak solution for (1.1) in [5]. Besides, this solution is almost surely continuous on $\overline{D}$. In [9] Gyöngy and Martínez considered the finite difference approximation for (1.1). They converted (1.1) into an integral equation using the Green's function and obtained the convergence rates for the approximate solution under certain regularity assumptions on f.

In this paper, the white noise processes are approximated by piecewise constant random processes (as in [2] and [7]). First we introduce a rectangular partition of $D = [0,1]^d$. For each direction of $x = (x^1, x^2, \cdots, x^d) \in D$, there exists a partition $0 = x_1^j < x_2^j < \cdots < x_{N+1}^j = 1$ with $x_l^j = (l-1)h$, $\forall l \in \{1,\cdots,N+1\}$,$\forall j \in \{1,\cdots,d\}$, where $h = 1/N$. Then, $D = [0,1]^d$ is divided into disjointed cells

$$D_i = \Big\{x \in D \mid x_l^j \le x^j < x_{l+1}^j,\ \forall\, j \in \{1,\cdots,d\},\ \forall\, l \in \{1,\cdots,N\}\Big\}, \quad i = 1,\cdots,N^d,$$

and $|D_i| = 1/N^d$. A reasonable approximation to $\frac{dW}{dx}(x)$ is

$$\frac{d\widehat{W}}{dx}(x) = N^d \sum_{i=1}^{N^d} \eta_i \sqrt{|D_i|}\chi_i(x), \tag{1.2}$$

where

$$\sqrt{|D_i|}\eta_i = \int_{D_i} dW(x), \quad \text{for } i = 1,\cdots,N,$$

i.e., $\eta_i \in N(0,1)$ are independent identically distributed random variables, and

$$\chi_i(x) = \begin{cases} 1, & \text{if } x \in D_i, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\widehat{u}(x)$ be the approximation of $u(x)$ given by

$$\Delta\widehat{u}(x) - f(\widehat{u}(x)) = g(x) + \dot{\widehat{W}}(x), \quad \text{for } x \in D, \tag{1.3}$$

with boundary condition $\widehat{u} = 0$ for $x \in \partial D$.

The key to the error analysis of finite element and spectral methods is the regularity of the solution of the underlying SPDE. Unfortunately, As shown in [2] and other literatures, the required regularity conditions for the standard error estimates of the finite element method are not satisfied for the problem (1.1). With discretized process, we will show that the solution $\widehat{u}$ of the corresponding SPDE (1.3) has a certain regularity, which allows to obtain an error estimate.

The paper is organized as follows. In Section 2, we give several preliminary assumptions and lemmas that will be used in the proofs of convergence. In Section 3, we study the regularity of the solution $\widehat{u}$ for (1.3) and error estimates between $\widehat{u}$ and the solution u of (1.1). In Section 4, we apply a spectral method to (1.3) and present error estimates. In the last section, several numerical experiments are present.

2. Preliminary assumptions and lemmas

We consider $D=[0,1]^d$, $d=1,2,3$, and make the following assumptions on the SPDE (1.1).

Assumption 2.1. *The function f in (1.1) is of the form $f(x)=f_1(x)+f_2(x)$, where f_1 is a continuous non-decreasing and bounded function, $|f_1(x)|\le M$ and f_2 is a Lipschitz function with a Lipschitz constant L. Such function f satisfies the one-sided Lipschitz condition,*

$$(x-y)(f(x)-f(y))\ge -L(x-y)^2, \quad \forall\, x,\ y\in\mathbb{R}. \tag{2.1}$$

The numerical procedures are applied to the following weak and integral formulations of (1.1):

$$\int_D u(x)\Delta\phi(x)dx-\int_D f(u(x))\phi(x)dx=\int_D g(x)\phi(x)dx+\int_D \phi(x)dW(x), \tag{2.2}$$

for any $\phi\in C^2(D)$. The solution of (1.1) can be written as

$$u(x)=\int_D k(x,y)f(u(y))dy+\int_D k(x,y)g(y)dy+\int_D k(x,y)dW(y), \tag{2.3}$$

where $x,y\in D$. The last integral is understood in Ito's sense and $k(x,y)$ is the Green's function associated with the elliptic equation $\Delta v(x)=\psi(x)$ with zero boundary conditions. This means that the solution can be expressed as

$$v(x)=\int_D k(x,y)\psi(y)dy \quad \text{for any } \ \psi\in L^2(D).$$

Gyöngy and Martínez [9] expanded $k(x,y)$ in sine Fourier series on $[0,1]^d$,

$$k(x,y)=\sum_{\alpha\in\{1,2,\cdots\}^d}\frac{-2^d}{\pi^2|\alpha|^2}\varphi_\alpha(x)\varphi_\alpha(y), \qquad \varphi_\alpha(x)=\varphi_{\alpha_1}(x_1)\cdots\varphi_{\alpha_d}(x_d), \tag{2.4}$$

where $\varphi_{\alpha_i}(x_i)=\sin(\alpha_i\pi x_i)$, and series converges in $L^2(D,y)$ uniformly in x. They have proved that the kernel k has properties in the next lemma.

Lemma 2.1. *i) There exists a positive constant C_1 depending only on the dimension d such that*

$$\int_D |k(x,y)|^2 dy \le C_1. \tag{2.5}$$

ii) For every $\varepsilon > 0$, there exists a constant $C(d,\varepsilon)$, such that

$$\int_D |k(x,y) - k(z,y)|^2 dy \le C(d,\varepsilon)|x-z|^{4\gamma(d,\varepsilon)}, \tag{2.6}$$

where $\gamma(d,\varepsilon)$ is given by

$$\gamma(d,\varepsilon) = \begin{cases} \frac{1}{2}, & \text{if } d=1, \\ \frac{1}{2}-\varepsilon, & \text{if } d=2, \\ \frac{1}{4}-\varepsilon, & \text{if } d=3, \end{cases} \tag{2.7}$$

and

$$C(d,\varepsilon) = C(d)\left(\sum_{\alpha\in I^d} \frac{1}{|\alpha|^{d+\varepsilon}}\right)^2.$$

iii) There exists a positive constant a such that

$$\int_D \left(\int_D k(x,y)\phi(y)dy\right)\phi(x)dx \le -a\int_D \left(\int_D k(x,y)\phi(y)dy\right)^2 dx. \tag{2.8}$$

It is well known that for every $u \in W_0^{2,p}(D)$, there exists a constant $C_R > 0$ such that [1]

$$\|u\|_{2,p,D} \le C_R|\Delta u|_{0,p,D}, \qquad \|u\|_{1,p,D} \le C_R|\nabla u|_{0,p,D}. \tag{2.9}$$

In this paper we assume that the Lipschitz constant L has the following restriction.

Assumption 2.2. *The Lipschitz constant L in* (2.1) *satisfies*

$$L < \min\{a, C_R^{-2}\},$$

where a and C_R is the constant appearing in (2.8) *and* (2.9) *respectively.*

The condition $C_R L < 1$ and $C_R^2 L < 1$ are technical conditions used in the proofs of the main results (Theorem 3.1 and Theorem 4.1). By inequality (2.9), we know that $C_R \ge 1$, thus, $C_R^{-2} \le C_R^{-1}$.

3. The approximate problem

The approximate white noise processes are used to improve the regularity of the solution, so that standard analysis techniques in the spectral method can be applied. In this section, we study the regularity of the solution $\widehat{u}$ for the approximate problem (1.3) and error estimates between $\widehat{u}$ and the solution u of the original problem (1.1).

The weak formulation of (1.3) is given by

$$(\widehat{u},\Delta v)-(f(\widehat{u}),v)=(g,v)+\left(\frac{d\widehat{W}}{dx},v\right),\quad \forall\, v\in H_0^2(D). \tag{3.1}$$

And the integral form of solution for (1.3) can be written as

$$\widehat{u}(x)=\int_D k(x,y)f(\widehat{u}(y))dy+\int_D k(x,y)g(y)dy+\int_D k(x,y)d\widehat{W}(y). \tag{3.2}$$

In the following, we consider the regularity of the solution $\widehat{u}$ and give error estimates between the solution u of (1.1) and $\widehat{u}$. For this purpose, we collect some results in the next lemmas. Let E be the expectation and C be a positive constant whose value may be different from line to line.

Lemma 3.1. *There exists a positive constant C such that*

$$E\left\|\int_D k(x,y)dW(x)-\int_D k(x,y)d\widehat{W}(x)]\right\|_{L^2}^2\le C(d,\varepsilon)h^{4\gamma(d,\varepsilon)},$$

where $\gamma(d,\varepsilon)$ is as in (2.7).

Proof. By using (2.6) and the Hölder inequality, we have

$$\begin{aligned}
&E\left\|\int_D k(x,y)dW(x)-\int_D k(x,y)d\widehat{W}(x)]\right\|_{L^2}^2\\
&=E\int_D\left[\sum_{i=1}^{N^d}\int_{D_i}\left(k(x,y)-\frac{1}{|D_i|}\int_{D_i}k(z,y)dz\right)dW(x)\right]^2dy\\
&=\int_D\sum_{i=1}^{N^d}\int_{D_i}\left(\frac{1}{|D_i|}\int_{D_i}(k(x,y)-k(z,y))dz\right)^2dx\,dy\\
&\le\sum_{i=1}^{N^d}\int_{D_i}\frac{1}{|D_i|}\int_{D_i}\int_D(k(x,y)-k(z,y))^2dy\,dz\,dx\\
&\le N^d\sum_{i=1}^{N^d}\int_{D_i}\int_{D_i}C(d,\varepsilon)|x-z|^{4\gamma(d,\varepsilon)}dz\,dx\ \le\ C(d,\varepsilon)h^{4\gamma(d,\varepsilon)}.
\end{aligned}$$

The proof is complete. ◻

It is apparent that $\frac{d\widehat{W}}{dx} \in L^2(D)$ almost surely. However, the following lemma shows that $\|\frac{d\widehat{W}}{dx}\|_{L^2}$ is unbounded as $N \to \infty$.

Lemma 3.2. *White noise approximation processes defined in* (1.2) *satisfies*

$$E\left(\left\|\frac{d\widehat{W}}{dx}\right\|_{L^2}^2\right) = N^d.$$

Proof. As a piecewise constant function, $\frac{d\widehat{W}}{dx} \in L^2(D)$. Moreover,

$$E\left(\int_D (\frac{d\widehat{W}}{dx})^2 dx\right) = E\left(\sum_{i=1}^{N^d} \frac{1}{|D_i|^2}\int_{D_i} \eta_i^2 |D_i| dy\right) = \sum_{i=1}^{N^d} 1 = N^d.$$

The proof is complete. □

3.1. Regularity

In this subsection, we consider the regularity of the solution $\widehat{u}$ of the approximate problem (1.3).

Theorem 3.1. *Suppose that Assumptions* 2.1 *and* 2.2 *hold. The solution* $\widehat{u}$ *for* (1.3) *satisfies* $\widehat{u} \in H^2 \cap H_0^1$ *and*

$$E|\widehat{u}|_{H^2}^2 \le CN^d,$$

where C *depends only on* M, L, C_R *and* $f(0)$, $\|g\|_{L^2}$.

Proof. From Assumption 2.1, we have

$$\begin{aligned}\|f(x_1) - f(x_2)\|_{L^2} &\le \|f_1(x_1) - f_1(x_2)\|_{L^2} + \|f_2(x_1) - f_2(x_2)\|_{L^2} \\ &\le 2M + L\|x_1 - x_2\|_{L^2}.\end{aligned} \tag{3.3}$$

Based on (1.3), (3.3), Young inequality and (2.9), we have

$$\begin{aligned}\|\Delta\widehat{u}\|_{L^2}^2 &= (f(\widehat{u}) - f(0), \Delta\widehat{u}) + \left(f(0) + g(x) + \frac{d\widehat{W}}{dx}, \Delta\widehat{u}\right) \\ &\le (2M + L\|\widehat{u}\|_{L^2})\|\Delta\widehat{u}\|_{L^2} + C(\varepsilon)\|f(0) + g(x) + \frac{d\widehat{W}}{dx}\|_{L^2}^2 + \varepsilon\|\Delta\widehat{u}\|_{L^2}^2 \\ &\le C_R L\|\Delta\widehat{u}\|_{L^2}^2 + 2\varepsilon\|\Delta\widehat{u}\|_{L^2}^2 + C(\varepsilon)\left(M^2 + |f(0)|^2 + \|g(x)\|_{L^2}^2 + \|\frac{d\widehat{W}}{dx}\|_{L^2}^2\right).\end{aligned}$$

Letting $\varepsilon = \frac{1-C_R L}{4}$ and using Assumption 2.2, we obtain

$$\|\Delta\widehat{u}\|_{L^2}^2 \le C(\varepsilon)\left(M^2 + |f(0)|^2 + \|g(x)\|_{L^2}^2 + \|\frac{d\widehat{W}}{dx}\|_{L^2}^2\right).$$

Then it follows from Lemma 3.2 that

$$E|\widehat{u}|_{H^2}^2 = E\|\Delta\widehat{u}\|_{L^2}^2 \leq CN^d.$$

The proof is complete. □

3.2. Error estimates

In this subsection, we consider the error estimates between the solution $\widehat{u}$ of the approximate problem (1.3) and the solution u of the original problem (1.1).

Theorem 3.2. *Assume that Assumptions* 2.1 *and* 2.2 *hold. Let* u *and* $\widehat{u}$ *be the solution of* (1.1) *and* (1.3)*, respectively. Then, there exists a positive constant* h_0*, such that for* $h < h_0$*,*

$$E\|u-\widehat{u}\|_{L^2}^2 \leq Ch^{2\gamma(d,\varepsilon)},$$

where $\gamma(d,\varepsilon)$ *is given by* (2.7)*, and* C *is independent of* u *and* h*.*

Proof. From (2.3) and (3.2), we have

$$\begin{aligned} &u(x)-\widehat{u}(x)\\ &=\int_D k(x,y)[f(u(y))-f(\widehat{u}(y))]dy+\int_D k(x,y)[dW(y)-d\widehat{W}(y)]. \end{aligned} \tag{3.4}$$

Multiplying both sides by $f(u(x))-f(\widehat{u}(x))$, applying Assumption 2.1, (2.8), and integrating over D, we obtain

$$\begin{aligned} -L\|u-\widehat{u}\|_{L^2}^2 \leq& \int_D (u(x)-\widehat{u}(x))[f(u(x))-f(\widehat{u}(x))]dx\\ =& \int_D\int_D k(x,y)[f(u(y))-f(\widehat{u}(y))]dy[f(u(x))-f(\widehat{u}(x))]dx\\ &+\int_D\left(\int_D k(x,y)[dW(y)-d\widehat{W}(y)]\right)[f(u(x))-f(\widehat{u}(x))]dx\\ \leq& -a\int_D\left(\int_D k(x,y)[f(u(y))-f(\widehat{u}(y))]dy\right)^2dx\\ &+\left\|\int_D k(x,y)[dW(y)-d\widehat{W}(y)]\right\|_{L^2}\|f(u)-f(\widehat{u})\|_{L^2}. \end{aligned}$$

Applying (3.4) into the right hand of the above inequality, we have

$$\begin{aligned}-L\|u-\widehat{u}\|_{L^2}^2 \le -a\int_D\left(u(x)-\widehat{u}(x)-\int_D k(x,y)[dW(y)-d\widehat{W}(y)]\right)^2 dx\\ +\left\|\int_D k(x,y)[dW(y)-d\widehat{W}(y)]\right\|_{L^2}\|f(u)-f(\widehat{u})\|_{L^2}\\ \le -a\|u-\widehat{u}\|_{L^2}^2-a\|T(x)\|_{L^2}^2+2a\|u-\widehat{u}\|_{L^2}\|T(x)\|_{L^2}\\ +\|T(x)\|_{L^2}\Big(2M+L\|u-\widehat{u}\|_{L^2}\Big),\end{aligned}$$

where

$$T(x)=\int_D k(x,y)[dW(y)-d\widehat{W}(y)]. \tag{3.5}$$

This simplifies to

$$(a-L)\|u-\widehat{u}\|_{L^2}^2 \le (2a+L)\|u-\widehat{u}\|_{L^2}\|T(x)\|_{L^2}+2M\|T(x)\|_{L^2}.$$

We also have by Lemma 3.1

$$E\|T(x)\|_{L^2}^2 \le C(d,\varepsilon)h^{4\gamma(d,\varepsilon)}. \tag{3.6}$$

Thus, we obtain by using (3.6)

$$\begin{aligned}&(a-L)E\|u-\widehat{u}\|_{L^2}^2\\ \le\ &(2a+L)E(\|u-\widehat{u}\|_{L^2}^2)^{1/2}E(\|T(x)\|_{L^2}^2)^{1/2}+2ME(\|T(x)\|_{L^2}^2)^{1/2}\\ \le\ &(2a+L)C(d,\varepsilon)h^{2\gamma(d,\varepsilon)}E(\|u-\widehat{u}\|_{L^2}^2)^{1/2}+2MC(d,\varepsilon)h^{2\gamma(d,\varepsilon)}.\end{aligned} \tag{3.7}$$

By (3.7) and the facts that $L<a$,

$$h<h_0=\left(\frac{8M(a-L)}{(2a+L)^2}\right)^{\frac{1}{2\gamma(d,\varepsilon)}},$$

we obtain

$$E(\|u-\widehat{u}\|_{L^2}^2)\le Ch^{2\gamma(d,\varepsilon)}.$$

The proof is complete. □

In the particular case that f has only its Lipschitz part, i.e., f is a Lipschitz function, the convergence rate of the approximation $\widehat{u}$ to the solution u is slighter better.

Theorem 3.3. *Suppose $f_1=0$, i.e., f satisfies the Lipchitz condition*

$$|f(x_1)-f(x_2)|\le L|x_1-x_2|,$$

and Lipchitz constant L satisfies $C_1L<1$, where C_1 is a constant in (2.5). Then

$$E\|u-\widehat{u}\|_{L^2}^2\le Ch^{4\gamma(d,\varepsilon)},$$

where $\gamma(d,\varepsilon)$ is as in (2.7), and C is independent of u and h.

Proof. From (2.3), (3.2), Hölder inequality, and (2.5), we have

$$\begin{aligned}
&|u(x)-\widehat{u}(x)| \\
\le& \left|\int_D k(x,y)[f(u(y))-f(\widehat{u}(y))]dy\right| + \left|\int_D k(x,y)[dW(y)-d\widehat{W}(y)]\right| \\
\le& C_1^{1/2}\left(\int_D [f(u(y))-f(\widehat{u}(y))]^2 dy\right)^{1/2} + \left|\int_D k(x,y)[dW(y)-d\widehat{W}(y)]\right|. \qquad (3.8)
\end{aligned}$$

Squaring both sides of (3.8) and integrating over D, together with Hölder inequality and Lipchitz condition, we get that

$$\begin{aligned}
\|u(x)-\widehat{u}(x)\|_{L^2}^2 &\le C_1\|f(u)-f(\widehat{u})\|_{L^2}^2 + \|T(x)\|_{L^2}^2 + 2C_1^{1/2}\|f(u)-f(\widehat{u})\|_{L^2}\|T(x)\|_{L^2} \\
&\le C_1L(1+\varepsilon)\|u-\widehat{u}\|_{L^2}^2 + (1+C(\varepsilon))\|T(x)\|_{L^2}^2,
\end{aligned}$$

where the function T is given by (3.5). Set $\varepsilon = \frac{1-C_1L}{2}$. It follows from Lemma 3.1 and the fact that $C_1L<1$

$$E\|u-\widehat{u}\|_{L^2}^2 \le CE\|T(x)\|_{L^2}^2 \le Ch^{4\gamma(d,\varepsilon)}.$$

The proof is complete. □

4. Spectral method and error estimates

In this section, we consider the spectral approximation for (1.3) and the corresponding error estimates. Let $V=H^2(D)\cap H_0^1(D)$, and S_N be a N-dimensional subspace of V defined as

$$S_N = \mathrm{Span}\Big\{\phi_{\vec{k}}(x) = \phi_{k_1}(x^1)\cdots\phi_{k_d}(x^d),\ \vec{k}\in\{1,2,\cdots,N\}^d\Big\},$$

where $\phi_{k_j}(x^j)=\sin(\pi k_j x^j)$, $\vec{k}=\{k_1,\cdots,k_d\}$. So if $v\in S_N$, it can be expressed

$$v(x) = \sum_{\vec{k}\in\{1,2,\cdots,N\}^d} \hat{v}_{\vec{k}}\,\phi_{\vec{k}}(x),$$

where $\hat{v}_{\vec{k}}$ are constants. From the definition of S_N, it is easy to see that if $v\in S_N$, then $\Delta v\in S_N$.

Let $P_N: L^2(D)\to S_N$ be the orthogonal projection operator of V to S_N, defined by

$$(P_Nu,v)=(u,v), \quad \forall\, v\in S_N. \qquad (4.1)$$

We collect some properties of the orthogonal projection operator P_N in the next lemma.

Lemma 4.1. *The orthogonal projection operator $P_N : L^2 \to S_N$ has following properties,*

(i) $\|P_N\|_{L^2} = 1.$

(ii) $\Delta P_N u = P_N \Delta u, \quad \forall\, u \in V.$

(iii) *There exists a constant C, such that*

$$\|u - P_N u\|_{L^2} \le C N^{-2} |u|_{H^2}, \quad \forall\, u \in V.$$

Proof. (i) and (ii) are obvious. (iii) is proved in [18] for periodic functions. Using the odd periodic extension of V, the proof in [18] can be adopted to prove (iii). □

The spectral method for (1.3) is to find $\widehat{u}_N \in S_N \subset V$, such that

$$(\widehat{u}_N, \Delta v) - (f(\widehat{u}_N), v) = (g(x), v) + \left(\frac{d\widehat{W}}{dx}, v\right), \quad \forall\, v \in S_N. \tag{4.2}$$

In the following, we estimate the L^2 error between $\widehat{u}$ and $\widehat{u}_N$. We already have the orthogonal projection operator error $\widehat{u} - P_N \widehat{u}$, from Lemma 4.1. Thus, we only need to estimate the error $P_N \widehat{u} - \widehat{u}_N$.

Theorem 4.1. *If Assumptions* 2.1 *and* 2.2 *hold, then the approximation $\widehat{u}_N$ defined in* (4.2) *has the following approximation property,*

$$E\|P_N \widehat{u} - \widehat{u}_N\|_{H^1}^2 \le C h^{2-d/2},$$

where C is independent of $\widehat{u}$ and h.

Proof. By (3.1) and (4.1), the solution $\widehat{u}$ of (1.3) satisfies

$$(P_N \widehat{u}, \Delta v) - (f(\widehat{u}), v) = (g(x), v) + \left(\frac{d\widehat{W}}{dx}, v\right), \quad \forall\, v \in S_N. \tag{4.3}$$

Let $\theta = P_N \widehat{u} - \widehat{u}_N \in S_N$. Subtracting (4.3) from (4.2), we have

$$-(\nabla\theta, \nabla v) = (f(\widehat{u}) - f(\widehat{u}_N), v), \quad \forall\, v \in S_N. \tag{4.4}$$

Set $v = \theta \in S_N$ and we get by (2.1)

$$\begin{aligned}
-(\nabla\theta, \nabla\theta) &= (f(\widehat{u}) - f(P_N\widehat{u} - \theta), \theta) \\
&= (f(\widehat{u}) - f(P_N\widehat{u} - \theta), \widehat{u} - (P_N\widehat{u} - \theta)) - (f(\widehat{u}) - f(P_N\widehat{u} - \theta), \widehat{u} - P_N\widehat{u}) \\
&\ge -L\|\widehat{u} - P_N\widehat{u} + \theta\|_{L^2}^2 - (f(\widehat{u}) - f(P_N\widehat{u} - \theta), \widehat{u} - P_N\widehat{u}).
\end{aligned}$$

Then by Young and Hölder inequalities, (3.3) and (2.9), we have

$$\begin{aligned}
&\|\nabla\theta\|_{L^2}^2 \\
\le\ & L(1+\varepsilon)\|\theta\|_{L^2}^2 + L(1+C(\varepsilon))\|\widehat{u} - P_N\widehat{u}\|_{L^2}^2 + \|f(\widehat{u}) - f(P_N\widehat{u} - \theta)\|_{L^2}\|\widehat{u} - P_N\widehat{u}\|_{L^2} \\
\le\ & C_R^2 L(1+\varepsilon)\|\nabla\theta\|_{L^2}^2 + L(1+C(\varepsilon))\|\widehat{u} - P_N\widehat{u}\|_{L^2}^2 \\
&+ (2M + L\|\widehat{u} - P_N\widehat{u}\|_{L^2} + L\|\theta\|_{L^2})\|\widehat{u} - P_N\widehat{u}\|_{L^2} \\
\le\ & C_R^2 L(1+2\varepsilon)\|\nabla\theta\|_{L^2}^2 + 2L(1+C(\varepsilon))\|\widehat{u} - P_N\widehat{u}\|_{L^2}^2 + 2M\|\widehat{u} - P_N\widehat{u}\|_{L^2}.
\end{aligned}$$

It follows from Assumption 2.2 and Lemma 4.1

$$\begin{aligned}\|\nabla\theta\|_{L^2}^2 &\le 2CL(1+C(\varepsilon))\|\widehat{u}-P_N\widehat{u}\|_{L^2}^2+2CM\|\widehat{u}-P_N\widehat{u}\|_{L^2}\\ &\le Ch^4|\widehat{u}|_{H^2}^2+Ch^2|\widehat{u}|_{H^2}. \qquad (4.5)\end{aligned}$$

By (2.9) and Theorem 3.1, we obtain

$$\begin{aligned}E\|\theta\|_{H^1}^2 &\le E(C_R^2\|\nabla\theta\|_{L^2}^2)\le Ch^4E|\widehat{u}|_{H^2}^2+Ch^2(E|\widehat{u}|_{H^2}^2)^{1/2}\\ &\le Ch^{4-d}+Ch^{2-d/2}\le Ch^{2-d/2}.\end{aligned}$$

The proof is complete. □

As a direct consequence of Theorem 4.1 and Lemma 4.1 (iii), the following convergence result for the approximations $\widehat{u}_N$ by the spectral method is presented.

Theorem 4.2. *If Assumptions* 2.1 *and* 2.2 *hold, then the approximation* $\widehat{u}_N$ *defined in* (4.2) *converges to the solution* $\widehat{u}$ *of* (1.3) *in the mean of the* L^2*-norm*

$$E\|\widehat{u}-\widehat{u}_N\|_{L^2}^2\le C\left(h^{4-d}+h^{2-d/2}\right),$$

where C *is dependent only on* M, L, C_R, $f(0)$, $\|g\|_{L^2}$.

In the particular case that f has only its Lipschitz part, the convergence rate of the approximation $\widehat{u}_N$ to the solution $\widehat{u}$ is slighter better.

Theorem 4.3. *Suppose* f *satisfies the Lipchitz condition*

$$|f(x_1)-f(x_2)|\le L|x_1-x_2|,$$

and Lipchitz constant L *satisfies* $C_R^2L<1$, *where* C_R *is a constant in* (2.9). *The approximation* $\widehat{u}_N$ *defined in* (4.2) *converge to the solution* $\widehat{u}$ *of* (1.3) *in the mean of the* L^2*-norm*

$$E\|\widehat{u}-\widehat{u}_N\|_{L^2}^2\le Ch^{4-d},$$

where C *is dependent only on* M, L, C_R, $f(0)$, $\|g\|_{L^2}$.

Proof. It is similar to the frontal proof of the Theorem 4.1, we can obtain (4.4). Then set $v=\theta=P_N\widehat{u}-\widehat{u}_N\in S_N$, we have

$$\begin{aligned}(\nabla\theta,\nabla\theta) &= -(f(\widehat{u})-f(\widehat{u}_N),\theta)\\ &\le L\|\widehat{u}-\widehat{u}_N\|_{L^2}\|\theta\|_{L^2}\\ &\le L\|\theta\|_{L^2}^2+L\|\widehat{u}-P_N\widehat{u}\|_{L^2}\|\theta\|_{L^2}\\ &\le C_R^2L\|\nabla\theta\|_{L^2}^2+CC_RLh^2|\widehat{u}|_{H^2}\|\nabla\theta\|_{L^2},\end{aligned}$$

where the last inequality is obtained by (2.9) and Lemma 4.1 (iii). It follows from $C_R^2L<1$

$$|\theta|_{H^1}\le Ch^2|\widehat{u}|_{H^2}.$$

Then we get by (2.9) and Theorem 3.1,

$$E\|P_N\widehat{u}-\widehat{u}_N\|^2_{H^1}=E\|\theta\|^2_{H^1}\le C_R^2E|\theta|^2_{H^1}\le Ch^4E|\widehat{u}|^2_{H^2}\le Ch^{4-d}.$$

At last, we have that by Lemma 4.1 and Theorem 3.1,

$$\begin{aligned}E\|\widehat{u}-\widehat{u}_N\|^2_{L^2}&\le 2(E\|\widehat{u}-P_N\widehat{u}\|^2_{L^2}+E\|P_N\widehat{u}-\widehat{u}_N\|^2_{L^2})\\&\le Ch^4E|\widehat{u}|^2_{H^2}+Ch^{4-d}\\&\le Ch^{4-d}.\end{aligned}$$

The proof is complete. □

Finally we can obtain the error estimate of the approximate solution $\widehat{u}_N$ to the exact solution u of (1.1) in the next theorem.

Theorem 4.4. *i) Assume that Assumptions* 2.1 *and* 2.2 *hold. Let u be the solution of* (1.1) *and $\widehat{u}_N$ be the numerical solution defined by* (4.2)*. Then*

$$E\|u-\widehat{u}_N\|^2_{L^2}\le Ch^{2\gamma(d,\varepsilon)},$$

where $\gamma(d,\varepsilon)$ is given by (2.7)*, and C is independent of u and h.*

ii) Furthermore if $f_1=0$, i.e., f is a Lipchitz function such that

$$|f(x_1)-f(x_2)|\le L|x_1-x_2|,$$

and the Lipchitz constant L satisfies $L<\min\{C_R^{-2},C_1^{-1}\}$, then the approximation $\widehat{u}_N$ converge to the exact solution u in the mean of the L^2-norm,

$$E\|u-\widehat{u}_N\|^2_{L^2}\le Ch^{4\gamma(d,\varepsilon)},$$

where $\gamma(d,\varepsilon)$ is as in (2.7)*, and C is independent of u and h.*

5. Numerical experiments

In this section, we present numerical examples to demonstrate our theoretical results in the previous sections. We will consider both 1D and 2D linear problems.

The independent identically distributed random variables $\eta_i\in N(0,1)$ shall be simulated by using the random number generator of Matlab. Theoretically, the number of samples M should be chosen so that the error generated by the Monte Carlo method is in the same magnitude of the errors generated by spectral approximation. We shall use the Monte Carlo method to examine the following two types of errors

$$\begin{aligned}e1&=\|E(u)-E(\widehat{u}_N)\|_{L^2},\\e2&=|E(\|\widehat{u}_N\|^2_{L^2})-E(\|\widehat{u}_{2N}\|^2_{L^2})|,\end{aligned}$$

to check our theoretical error estimates. Obviously these errors can be controlled by the error $E(\|u-\widehat{u}_N\|^2_{L^2})$, but not equivalent to it. Nevertheless they could provide good indications about how the error $E(\|u-\widehat{u}_N\|^2_{L^2})$ behaves. Let $\bar{u}$ be the exact solution without white noise and $\bar{u}_N$ be the spectral approximate solution in the absence of the white noise. We also shall examine the error

$$e3=\|\bar{u}-\bar{u}_N\|_{L^2}.$$

Example 5.1. Let $D=[0,1]$ and exact solution be $\bar{u}(x)=\sin^2(\pi x)$ in the absence of the white noise. And consider $f(u)=0$, we have $E(u)=\bar{u}$. The 1D white noise is approximated by piecewise constant random processes as show in Fig. 1. We have that $h=1/N$ $=0.25$, 0.125, 0.0625, 0.03125 for $N=4$, 8, 16, 32. The computational results are displayed in Table 1.

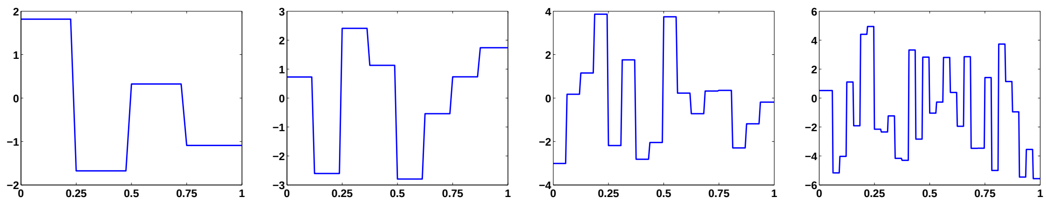

Figure 1: The approximations of white noise for partition $N=4,8,16,32$ in 1D domain $[0,1]$.

Table 1: Errors and convergence rates for 1D test.

M	N	e1	Rate	e2	Rate	e3	Rate
4000	4	1.5497e-2	-	2.2289e-3	-	1.5544e-2	-
16000	8	2.7758e-3	2.48	7.1688e-4	1.64	2.7023e-3	2.52
64000	16	5.5010e-4	2.34	1.6483e-4	2.12	4.7648e-4	2.50
256000	32	1.2111e-4	2.18	7.0149e-5	1.23	8.4188e-5	2.50

Example 5.2. Let $D=[0,1]\times[0,1]$ and exact solution be $\bar{u}(x,y)=\sin^2(\pi x)\sin^2(\pi y)$ in the absence of the white noise. And consider $f(u)=0$, we have $E(u)=\bar{u}$. The 2D white noise is approximated by piecewise constant random processes for $N=2,4,8,16$ as show in Fig. 2. The computational results are displayed in Table 2.

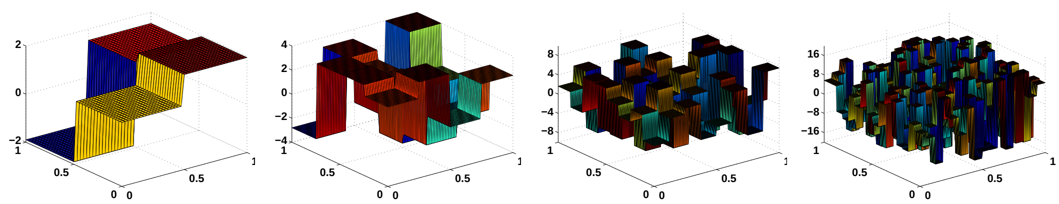

Figure 2: The approximations of white noise for partition $N=2,4,8,16$ in 2D domain $[0,1]\times[0,1]$.

Table 2: Errors and convergence rates for 2D test.

M	N	e1	Rate	e2	Rate	e3	Rate
2000	2	1.1016e-1	-	1.1299e-2	-	1.1025e-1	-
8000	4	1.3482e-2	3.03	9.0304e-4	3.65	1.3459e-2	3.03
32000	8	2.3406e-3	2.53	3.4261e-4	1.40	2.3402e-3	2.52
128000	16	4.3778e-4	2.42	1.0407e-4	1.72	4.1265e-4	2.50

In the two numerical examples, exact solutions without noise are very smooth and periodic functions, errors $e3$ are caused by the spectral method. The numbers of samples M are displayed in the first columns of the tables. We evaluate $E(\widehat{u}_N)$ by using the Monte Carlo method to examine errors $e1$ to see if we have used enough samples. The convergence rates of $e1$ are close to the convergence rates of $e3$ in the tables, which implies that our sample sizes are good enough to ensure the accuracy of the Monte Carlo method. From Theorem 4.4 ii), the error $(E\|u-\hat{u}_N\|^2_{L2})$ has second-order or less than second-order convergence rate for one or two dimensional case, respectively. Numerical results in the Tables 1 and 2 show that most of convergence rates for $e2$ are less than second-order. We believe that this is due to the sample errors rather than the spectral errors. Of cause that $e1$ and $e2$ are not equivalent to the error in Theorem 4.4. They just provide some indications.

6. Conclusions

Our work extended the work of [2, 7] from one-dimension to multi-dimensions and from linear problems to nonlinear problems. Future research on this subject includes numerical experiences for SPDEs with multiplicative noise forcing terms.

Acknowledgments This work is supported by the National Natural Science Foundation of China (10701014, 10871029), a grant from the Laboratory of Computational Physics, and the Foundation of China Academy of Engineering Physics. The authors thank the anonymous referees for helpful comments which improve the presentation of the paper.

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