Numer. Math. Theor. Meth. Appl. Vol. **5**, No. 3, pp. 403-422
doi: 10.4208/nmtma.2012.m11006 August 2012

The Simultaneous Approximation Average Errors for Bernstein Operators on the r-Fold Integrated Wiener Space

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Received 7 March 2011; Accepted (in revised version) 25 October 2011

Available online 3 July 2012

Abstract. For weighted approximation in L_p-norm, we determine strongly asymptotic orders for the average errors of both function approximation and derivative approximation by the Bernstein operators sequence on the r-fold integrated Wiener space.

AMS subject classifications: 41A28, 65D05, 41A25

Key words: Bernstein operators, weighted L_p-norm, r-fold integrated Wiener space, average error.

1. Introduction

Let F be a real separable Banach space equipped with a probability measure μ on the Borel field of F. Let H be another normed space with norm $\|\cdot\|$ such that F is continuously embedded in H. Any $A : F \to H$ such that $f \mapsto \|f - A(f)\|$ is a measurable mapping is called an approximation operator (or just approximation). The p-average error of A is defined as

$$e_p(A, \|\cdot\|, F, \mu) = \left(\int_F \|f - A(f)\|^p \mu(df) \right)^{\frac{1}{p}}.$$

Since in practice the underlying function is usually given via its values at finitely many points, the approximation operator $A(f)$ is often considered depending on some function values about f only. Many papers such as [1-4] studied the complexity of computing an ε-approximation in the average case setting. We observe that all results used before rely on spline function approximation (see [5]). It is well known that the optimal spline function approximation operators depend on the smoothness property of covariance kernels, and it is not known if they are good approximation operators for derivative approximation. The simultaneous approximation problem for smooth functions is an important research topic in approximation theory and application. We want to consider the simultaneous approximation problem in the average case setting. Since the Bernstein operator approximation

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is the most important approximation method which depends on its values at equidistant interpolation points only, and it is widely used in practice, we will discuss its simultaneous approximation average errors on the r-fold integrated Wiener space.

Denote

$$F_0 = \{f \in C[0,1] : f(0) = 0\},$$

and for every $f \in F_0$ set

$$\|f\|_C := \max_{0\le t\le 1} |f(t)|.$$

Then $(F_0, \|\cdot\|_C)$ becomes a separable Banach space. Denote by $\mathfrak{B}(F_0)$ the Borel field of $(F_0, \|\cdot\|_C)$, and by ω_0 the Wiener measure on $\mathfrak{B}(F_0)$ (see [5]). Now we introduce integral operator $T_r, r \ge 1$, on F_0 as follows.

Let $r \ge 0$ be an integer. Define for $r = 0$

$$(T_0 g)(t) = g(t), \ \ \forall g \in F_0,$$

and for $r \ge 1$

$$(T_r g)(t) = \int_0^t g(u) \cdot \frac{(t-u)^{r-1}}{(r-1)!} du.$$

Obviously we have

$$(T_r g)^{(s)}(t) = (T_{r-s} g)(t), \ \ \forall 0 \le s \le r, \tag{1.1}$$

and for an arbitrary $g \in F_0$,

$$T_r g \in F_r = \left\{ f \in C^{(r)}[0,1] : f^{(k)}(0) = 0 \text{ for } k = 0,1,\cdots,r \right\}.$$

It is well known that T_r is a bijective mapping from F_0 to F_r. The r-fold integrated Wiener measure ω_r on F_r is defined by induced measure $\omega_r = T_r \omega_0$, i.e., for $A \subset F_r$,

$$\omega_r(A) = \omega_0(\{g \in F_0 | T_r g \in A\}).$$

From [5, p. 70] and [5, p. 71] we know

$$\int_{F_0} f(s) f(t) \omega_0(df) = \min\{s,t\}, \ \ \forall s,t \in [0,1], \tag{1.2}$$

$$\int_{F_r} f(s) f(t) \omega_r(df) = \int_0^1 \frac{(s-u)_+^r (t-u)_+^r \, du}{(r!)^2}. \tag{1.3}$$

Where $z_+ = z$ if $z > 0$ and $z_+ = 0$ otherwise.

For $\varrho \in L_1[0,1]$, $\varrho \ge 0$, the weighted L_p-norm of $f \in C[0,1]$ is defined by

$$\|f\|_{p,\varrho} = \left(\int_0^1 |f(t)|^p \cdot \varrho(t) dt \right)^{\frac{1}{p}}$$

and we simply write $\|\cdot\|_p$ if $\varrho(t)=1$.

For $f\in C[0,1]$, the nth Bernstein polynomial of f is

$$B_n(f,x)=\sum_{k=0}^{n} f(t_k)p_{n,k}(x), \tag{1.4}$$

where

$$t_k=\frac{k}{n},\ \ p_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k},\ \ k=0,\cdots,n. \tag{1.5}$$

On the one hand, we will consider the average errors of function approximation by the Bernstein operators (1.4). We obtain

Theorem 1.1. *Let $B_n(f,x)$ be defined as* (1.4)*. Then for an arbitrary $\varrho\in L_1[0,1]$, $\varrho>0$, $\varrho(x)$ is continuous on $(-1,1)$ and $1\le p<\infty$, we have that*

(1). *For $r=0$,*

$$e_p\left(B_n,\|\cdot\|_{p,\varrho},F_0,\omega_0\right)=C_{p,\varrho,0}n^{-\frac{1}{4}}+o\left(n^{-\frac{1}{4}}\right),$$

where

$$C_{p,\varrho,0}=\left(\frac{2\sqrt{2}-1}{2\sqrt{\pi}}\right)^{\frac{1}{2}}\left(v_p\cdot\int_0^1(x(1-x))^{\frac{p}{4}}\cdot\varrho(x)dx\right)^{\frac{1}{p}},$$

and v_p is the p-th absolute moment of the standard normal distribution, i.e.,

$$v_p=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|x|^pe^{-\frac{x^2}{2}}dx.$$

Meanwhile, here and in the following the notation $a_n=o(b_n)$ for sequences $\{a_n\}$ and $\{b_n\}$ means that $\lim_{n\to\infty}a_n/b_n=0$.

(2). *For $r=1$,*

$$e_p\left(B_n,\|\cdot\|_{p,\varrho},F_1,\omega_1\right)=C_{p,\varrho,1}n^{-\frac{3}{4}}+o\left(n^{-\frac{3}{4}}\right),$$

where

$$C_{p,\varrho,1}=\left(\frac{2-\sqrt{2}}{3\sqrt{\pi}}\right)^{\frac{1}{2}}\left(v_p\cdot\int_0^1(x(1-x))^{\frac{3p}{4}}\cdot\varrho(x)dx\right)^{\frac{1}{p}}.$$

(3). *For $r\ge 2$,*

$$e_p\left(B_n,\|\cdot\|_{p,\varrho},F_r,\omega_r\right)=C_{p,\varrho,r}n^{-1}+o\left(n^{-1}\right),$$

where

$$C_{p,\varrho,r}=\frac{1}{2\sqrt{2r-3}(r-2)!}\left(v_p\cdot\int_0^1x^{rp-\frac{p}{2}}(1-x)^p\cdot\varrho(x)dx\right)^{\frac{1}{p}}.$$

For $1 \le p, q < \infty$, by Hölder's inequality we know that there exists C_1, C_2 such that

$$\begin{aligned} & C_1 e_{\min\{p,q\}} \left(B_n, \|\cdot\|_{\min\{p,q\},\varrho}, F_r, \omega_r\right) \\ & \le e_p \left(B_n, \|\cdot\|_{q,\varrho}, F_r, \omega_r\right) \le C_2 e_{\max\{p,q\}} \left(B_n, \|\cdot\|_{\max\{p,q\},\varrho}, F_r, \omega_r\right). \end{aligned}$$

Hence from Theorem 1.1 it follows that

Corollary 1.1. *Let $B_n(f,x)$ be defined as (1.4). Then for an arbitrary $\varrho \in L_1[0,1]$, $\varrho > 0$, $\varrho(x)$ is continuous on $(0,1)$ and $1 \le p, q < \infty$, we have*

$$\begin{aligned} & e_p \left(B_n, \|\cdot\|_{q,\varrho}, F_0, \omega_0\right) \asymp n^{-\frac{1}{4}}, \\ & e_p \left(B_n, \|\cdot\|_{q,\varrho}, F_1, \omega_1\right) \asymp n^{-\frac{3}{4}}, \\ & e_p \left(B_n, \|\cdot\|_{q,\varrho}, F_r, \omega_r\right) \asymp n^{-1}, \quad r \ge 2. \end{aligned}$$

Here and in the following the notation $a_n \asymp b_n$ for sequences $\{a_n\}$ and $\{b_n\}$ of positive numbers means the existence of a constant C independent of n such that $a_n/C \le b_n \le C a_n$, and C may be different in different expressions.

Remark. For $r = 0$, the result of Corollary 1.1 can be found in [6].

On the other hand, we will consider the average errors of derivative approximation by the Bernstein operators (1.4). For $f \in C^{(s)}[0,1]$, $\varrho \in L_1[0,1]$, $\varrho \ge 0$ and $1 \le p < \infty$, we define

$$\|f\|_{p,s,\varrho} = \left(\int_0^1 \left| f^{(s)}(x) \right|^p \varrho(x) dx \right)^{\frac{1}{p}}$$

and we simply write $\|f\|_{p,\varrho}$ if $s = 0$.

Theorem 1.2. *Let $B_n(f,x)$ be defined as (1.4). Then for an arbitrary $\varrho \in L_1[0,1]$, $\varrho > 0$, $\varrho(x)$ is continuous on $(-1,1)$ and $1 \le p < \infty$, we have that*

(1). *For $s = r$,*

$$e_p \left(B_n, \|\cdot\|_{p,r,\varrho}, F_r, \omega_r\right) = C_{p,\varrho,0} n^{-\frac{1}{4}} + o\left(n^{-\frac{1}{4}}\right).$$

(2). *For $s = r - 1$,*

$$e_p \left(B_n, \|\cdot\|_{p,r-1,\varrho}, F_r, \omega_r\right) = C_{p,\varrho,1} n^{-\frac{3}{4}} + o\left(n^{-\frac{3}{4}}\right).$$

(3). *For $s \le r - 2$,*

$$e_p \left(B_n, \|\cdot\|_{p,s,\varrho}, F_r, \omega_r\right) = C_{p,\varrho,r,s} n^{-1} + o\left(n^{-1}\right),$$

where

$$C_{p,\varrho,r,s} = \frac{1}{2(r-s)!} \left(\nu_p \cdot \int_0^1 |D_{r,s}(x)|^{\frac{p}{2}} \cdot \varrho(x) dx \right)^{\frac{1}{p}}$$

and

$$\begin{aligned} &D_{r,s}(x)\\ &=\frac{x^{2r-2s-1}}{(2n(r-s)!)^2}\left[\frac{s^2(s-1)^2x^2}{2r-2s+1}+\frac{(r-s)^2s^2(1-2x)^2}{(2r-2s-1)}+\frac{(r-s)^2(r-s-1)^2(1-x)^2}{2r-2s-3}\right]\\ &\quad-\frac{sx^{2r-2s}}{(2n(r-s-1)!)^2}\left[\frac{2(s-1)(r-s-1)(1-x)}{(2r-2s-1)(r-s)}-\frac{s(s-1)(1-2x)}{(r-s)^2}+(1-x)(1-2x)\right]. \end{aligned}$$

Similar to Corollary 1.1, by Theorem 1.2 we have

Corollary 1.2. *Let $B_n(f,x)$ be defined as* (1.4). *Then for an arbitrary $\varrho\in L_1[0,1]$, $\varrho>0$, $\varrho(x)$ is continuous on $(0,1)$ and $1\le p,q<\infty$, we have*

$$\begin{aligned} &e_p\left(B_n,\|\cdot\|_{q,r,\varrho},F_r,\omega_r\right)\asymp n^{-\frac{1}{4}},\\ &e_p\left(B_n,\|\cdot\|_{q,r-1,\varrho},F_r,\omega_r\right)\asymp n^{-\frac{3}{4}},\\ &e_p\left(B_n,\|\cdot\|_{q,s,\varrho},F_r,\omega_r\right)\asymp n^{-1},\quad s\le r-2. \end{aligned}$$

2. Some Lemmas

We will list and prove some useful lemmas in this section.

Lemma 2.1. *Let $s\ge t$. Then*

$$\int_{F_1} f(s)f(t)\omega_1(df)=\frac{t^3}{3}+\frac{(s-t)t^2}{2},\tag{2.1}$$

$$\int_{F_1} f'(t)f(s)\omega_1(df)=\frac{t^2}{2}+t(s-t),\tag{2.2}$$

$$\int_{F_1} f(t)f'(s)\omega_1(df)=\frac{t^2}{2}.\tag{2.3}$$

Proof. From (1.3) we obtain (2.1). We only prove (2.2). The proof of (2.3) is similar. Let $f=T_1g$, then (1.1) gives

$$\begin{aligned} &\int_{F_1} f'(t)f(s)\omega_1(df)=\int_{F_0}(T_1g)'(t)(T_1g)(s)\omega_0(dg)\\ &=\int_{F_0} g(t)\int_0^s g(u)du\omega_0(dg)=\int_{F_0}\int_0^s g(t)g(u)du\omega_0(dg). \end{aligned}\tag{2.4}$$

Using Fubini's theorem and (1.2), we have

$$\begin{aligned} &\int_{F_0}\int_0^s g(t)g(u)du\omega_0(dg)=\int_0^s du\int_{F_0} g(t)g(u)\omega_0(dg)\\ &=\int_0^s \min\{t,u\}du=\int_0^t udu+\int_t^s tdu=\frac{t^2}{2}+t(s-t). \end{aligned}\tag{2.5}$$

This completes the proof of Lemma 2.1. □

From the proof of Theorem 1.5.2 in [7, p. 17] we obtain

Lemma 2.2. *For fixed* $0 < \delta < \frac{1}{2}$, $\alpha > \frac{1}{3}$, *the asymptotic relation*

$$\begin{aligned} p_{n,k}(x) &= \binom{n}{k} x^k(1-x)^{n-k} \\ &\cong \frac{1}{\sqrt{2\pi x(1-x)n}} exp\left[-\frac{n}{2x(1-x)}\left(\frac{k}{n}-x\right)^2\right] = P_{n,k}(x), \end{aligned} \tag{2.6}$$

holds uniformly for all $x \in [\delta, 1-\delta]$ *and for all values of* k *satisfying the inequality*

$$\left|\frac{k}{n}-x\right| \le n^{-\alpha}. \tag{2.7}$$

In other words,

$$\lim_{n\to\infty} \frac{p_{n,k}(x)}{P_{n,k}(x)} = 1 \tag{2.8}$$

uniformly for all k *satisfying* (2.7).

Let $X = x(1-x)$ and let

$$T_{n,s}(x) := \sum_{\nu=0}^{n} (\nu - nx)^s p_{n,\nu}(x), \quad n = 1, 2, \cdots, s = 0, 1, \cdots. \tag{2.9}$$

Lemma 2.3. ([8]) *For a fixed* $s = 0, 1, \cdots$, $T_{n,s}(x)$ *is a polynomial in* x *of degree* $\le s$, *and*

$$T_{n,2s}(x) = \sum_{j=0}^{s} a_{j,s}(X) n^j X^j, \quad T_{n,2s+1}(x) = (1-2x)\sum_{j=1}^{s} b_{j,s}(X) n^j X^j, \tag{2.10}$$

where $a_{j,s}$, $b_{j,s}$ *are polynomials of degree* $\le s-j$, *with coefficients independent of* n. *Specifically*

$$\begin{aligned} &T_{n,0}(x) = 1, \quad T_{n,1}(x) = 0, \\ &T_{n,2}(x) = nX, \ T_{n,3}(x) = n(1-2x)X, \ \cdots. \end{aligned} \tag{2.11}$$

3. Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1.

Proof. From [5, p. 107, (28)] we obtain

$$e_p^p\left(B_n, \|\cdot\|_{p,\varrho}, F_r, \omega_r\right) = \nu_p \cdot \int_0^1 \left(\int_{F_r} \left|f(x) - B_n(f,x)\right|^2 \omega_r(df)\right)^{\frac{p}{2}} \cdot \varrho(x)dx. \tag{3.1}$$

From (1.4) it follows that

$$\begin{aligned}\int_{F_r}\left|f(x)-B_n(f,x)\right|^2\omega_r(df)=&\int_{F_r}f^2(x)\omega_r(df)-2\sum_{k=0}^{n}p_{n,k}(x)\int_{F_r}f(x)f(t_k)\omega_r(df)\\&+\sum_{k=0}^{n}\sum_{m=0}^{n}p_{n,k}(x)p_{n,m}(x)\int_{F_r}f(t_k)f(t_m)\omega_r(df)\\=&A_{r,1}(x)-2A_{r,2}(x)+A_{r,3}(x).\end{aligned}\tag{3.2}$$

For $r=0$, from (1.2) it follows that

$$A_{0,1}(x)=x.\tag{3.3}$$

From [8] we obtain

$$\sum_{k=0}^{n}p_{n,k}(x)=1,\tag{3.4}$$

$$\sum_{k=0}^{n}t_kp_{n,k}(x)=x,\tag{3.5}$$

$$\sum_{k=0}^{n}t_k^2p_{n,k}(x)=x^2+\frac{x(1-x)}{n}.\tag{3.6}$$

From (2.11), (3.4)-(3.6) and a simple computation, we get

$$\sum_{k=0}^{n}t_k^3p_{n,k}(x)=x^3+\frac{3x^2(1-x)}{n}+\frac{x(1-x)(1-2x)}{n^2}.\tag{3.7}$$

From (1.2), (3.4), (3.5), the identity $\min\{a,b\}=\frac{a+b-|a-b|}{2}$ and a simple computation we obtain

$$A_{0,2}(x)=\frac{1}{2}\sum_{k=0}^{n}\Big(x+t_k-|x-t_k|\Big)p_{n,k}(x)=x-\frac{1}{2}\sum_{k=0}^{n}|x-t_k|p_{n,k}(x).\tag{3.8}$$

Similarly, we have

$$A_{0,3}(x)=x-\frac{1}{2}\sum_{k=0}^{n}\sum_{m=0}^{n}|t_k-t_m|p_{n,k}(x)p_{n,m}(x).\tag{3.9}$$

From (3.2), (3.3), (3.8) and (3.9) we obtain

$$\begin{aligned}&\int_{F_0}\left|f(x)-B_n(f,x)\right|^2\omega_0(df)\\=&\sum_{k=0}^{n}|x-t_k|p_{n,k}(x)-\frac{1}{2}\sum_{k=0}^{n}\sum_{m=0}^{n}|t_k-t_m|p_{n,k}(x)p_{n,m}(x)\\=&I_1(x)-\frac{I_2(x)}{2}.\end{aligned}\tag{3.10}$$

For a fixed $0<\delta<\frac{1}{2}$ and $n\geq\delta^{-\frac{12}{5}}$, if $x\in\left[t_s,t_{s+1}\right]\cap[\delta,1-\delta]$, then $1\leq s\pm n^{\frac{7}{12}}\leq n$ and

$$I_1(x)=\sum_{|k-s|>\left[n^{\frac{7}{12}}\right]}|t_k-x|p_{n,k}(x)+\sum_{|k-s|\leq\left[n^{\frac{7}{12}}\right]}|t_k-x|p_{n,k}(x), \tag{3.11}$$

where $[x]$ denote the integer part of x. From [7, p. 15, (8)] we know

$$\sum_{|k-s|>\left[n^{\frac{7}{12}}\right]}p_{n,k}(x)\leq Cn^{-2}. \tag{3.12}$$

Here and in the following the constant C is independent of n and may be different in different expressions. Combining $0\leq t_k,x\leq 1$ and (3.12) we get

$$\sum_{|k-s|>\left[n^{\frac{7}{12}}\right]}|t_k-x|p_{n,k}(x)\leq Cn^{-2}. \tag{3.13}$$

For $s-\left[n^{\frac{7}{12}}\right]\leq k\leq s+\left[n^{\frac{7}{12}}\right]$, from (2.8) we know

$$|t_k-x|p_{n,k}(x)=\frac{1+o(1)}{\sqrt{2\pi x(1-x)n}}\left|\frac{k}{n}-x\right|\exp\left[-\frac{n}{2x(1-x)}\left(\frac{k}{n}-x\right)^2\right]. \tag{3.14}$$

By (3.14) we obtain

$$\begin{aligned}&\sum_{|k-s|\leq\left[n^{\frac{7}{12}}\right]}|t_k-x|p_{n,k}(x)\\=&\frac{1+o(1)}{\sqrt{2\pi x(1-x)n}}\sum_{|k-s|\leq\left[n^{\frac{7}{12}}\right]}\left|x-\frac{k}{n}\right|\exp\left[-\frac{n}{2x(1-x)}\left(x-\frac{k}{n}\right)^2\right].\end{aligned} \tag{3.15}$$

Denote

$$F(s)=s\cdot\exp\left[-\frac{n}{2x(1-x)}s^2\right].$$

For $\frac{k-1}{n}\leq t\leq\frac{k+1}{n}$, by the Lagrange intermediate value theorem we know

$$\begin{aligned}&\left(x-\frac{k}{n}\right)\exp\left[-\frac{n}{2x(1-x)}\left(x-\frac{k}{n}\right)^2\right]-(x-t)\exp\left[-\frac{n}{2x(1-x)}(x-t)^2\right]\\=&F'(\xi)\left(t-\frac{k}{n}\right),\end{aligned} \tag{3.16}$$

where $\xi\in\left(x-\frac{k+1}{n},x-\frac{k-1}{n}\right)\subset\left(-n^{-\frac{5}{12}}-\frac{1}{n},n^{-\frac{5}{12}}+\frac{1}{n}\right)$. By a simple computation we yield that

$$|F'(\xi)|\leq C. \tag{3.17}$$

From (3.16) and (3.17) it follows that

$$\begin{aligned}&\left|x-\frac{k}{n}\right|\exp\left[-\frac{n}{2x(1-x)}\left(x-\frac{k}{n}\right)^2\right]\\ &=|x-t|\exp\left[-\frac{n}{2x(1-x)}(x-t)^2\right]+o\left(n^{-\frac{7}{12}}\right).\end{aligned} \tag{3.18}$$

Integrating two side of (3.18) about t in $\left[x-\frac{s-k+1}{n},x-\frac{s-k}{n}\right]$, we obtain

$$\begin{aligned}&\left|x-\frac{k}{n}\right|\exp\left[-\frac{n}{2x(1-x)}\left(x-\frac{k}{n}\right)^2\right]\\ &=n\int_{x-\frac{s+1-k}{n}}^{x-\frac{s-k}{n}}|x-t|\exp\left[-\frac{n}{2x(1-x)}(x-t)^2\right]dt+o\left(n^{-\frac{7}{12}}\right).\end{aligned} \tag{3.19}$$

By (3.19) and a simple computation we obtain

$$\begin{aligned}&\sum_{|k-s|\le\left[n^{\frac{7}{12}}\right]}\left|x-\frac{k}{n}\right|\exp\left[-\frac{n}{2x(1-x)}\left(x-\frac{k}{n}\right)^2\right]\\ &=n\int_{\frac{nx-1-\left[n^{\frac{7}{12}}\right]}{n}}^{x+\frac{\left[n^{\frac{7}{12}}\right]}{n}}|x-t|\exp\left[-\frac{n}{2x(1-x)}(x-t)^2\right]dt+o(1)\\ &=n\int_{x-n^{-\frac{5}{12}}}^{x+n^{-\frac{5}{12}}}|x-t|\exp\left[-\frac{n}{2x(1-x)}(x-t)^2\right]dt\\ &\quad-n\int_{x+\frac{\left[n^{\frac{7}{12}}\right]}{n}}^{x+n^{-\frac{5}{12}}}|x-t|\exp\left[-\frac{n}{2x(1-x)}(x-t)^2\right]dt\\ &\quad+n\int_{\frac{nx-1-\left[n^{\frac{7}{12}}\right]}{n}}^{x-n^{-\frac{5}{12}}}|x-t|\exp\left[-\frac{n}{2x(1-x)}(x-t)^2\right]dt+o(1)\\ &=n\int_{x-n^{-\frac{5}{12}}}^{x+n^{-\frac{5}{12}}}|x-t|\exp\left[-\frac{n}{2x(1-x)}(x-t)^2\right]dt+o(1).\end{aligned} \tag{3.20}$$

Let $s=\sqrt{\frac{n}{2x(1-x)}}(x-t)$. Then

$$\begin{aligned}&\int_{x-n^{-\frac{5}{12}}}^{x+n^{-\frac{5}{12}}}|x-t|\exp\left[-\frac{n}{2x(1-x)}(x-t)^2\right]dt\\ &=\frac{2x(1-x)}{n}\int_{-\frac{\sqrt[12]{n}}{\sqrt{2x(1-x)}}}^{\frac{\sqrt[12]{n}}{\sqrt{2x(1-x)}}}|s|e^{-s^2}ds=\frac{4x(1-x)}{n}\int_0^{\frac{\sqrt[12]{n}}{\sqrt{2x(1-x)}}}se^{-s^2}ds\end{aligned}$$

$$=\frac{4x(1-x)}{n}\int_0^{+\infty} se^{-s^2}ds-\frac{4x(1-x)}{n}\int_{\frac{12\sqrt[12]{n}}{\sqrt{2x(1-x)}}}^{+\infty} se^{-s^2}ds. \tag{3.21}$$

By a direct computation we obtain

$$\int_0^{+\infty} se^{-s^2}ds=\frac{1}{2}. \tag{3.22}$$

By the convergence of the improper integral $\int_0^{+\infty} se^{-s^2}ds$ we obtain

$$\lim_{n\to\infty}\int_{\frac{\sqrt[12]{n}}{\sqrt{2x(1-x)}}}^{+\infty} se^{-s^2}ds=0. \tag{3.23}$$

From (3.11), (3.13), (3.15) and (3.20)-(3.23) it follows that

$$I_1(x)=\frac{\sqrt{2x(1-x)}}{\sqrt{\pi n}}+o\left(n^{-\frac{1}{2}}\right). \tag{3.24}$$

By the same discussion and a direct computation we obtain

$$\begin{aligned}I_2(x)=&\frac{\sqrt{2x(1-x)}}{\pi\sqrt{n}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|s-t|e^{-s^2-t^2}dsdt+o\left(n^{-\frac{1}{2}}\right)\\=&\frac{\sqrt{x(1-x)}}{\sqrt{\pi n}}+o\left(n^{-\frac{1}{2}}\right).\end{aligned} \tag{3.25}$$

From (3.10), (3.24) and (3.25) we get

$$\int_{F_0}\left|f(x)-B_n(f,x)\right|^2\omega_0(df)=\frac{\left(2\sqrt{2}-1\right)\sqrt{x(1-x)}}{2\sqrt{\pi n}}+o\left(n^{-\frac{1}{2}}\right). \tag{3.26}$$

For $r=1$, from (2.1) we obtain

$$A_{1,1}(x)=\frac{x^3}{3}. \tag{3.27}$$

By (2.1), (3.2), (3.4) and (3.5) we verify that

$$\begin{aligned}A_{1,2}(x)=&\sum_{k=0}^{s}\left[\frac{t_k^3}{3}+\frac{(x-t_k)t_k^2}{2}\right]p_{n,k}(x)+\sum_{k=s+1}^{n}\left[\frac{x^3}{3}+\frac{(t_k-x)x^2}{2}\right]p_{n,k}(x)\\=&\sum_{k=0}^{s}\left[\frac{t_k^3}{3}+\frac{(x-t_k)t_k^2}{2}-\frac{x^3}{3}-\frac{(t_k-x)x^2}{2}\right]p_{n,k}(x)+\sum_{k=0}^{n}\left[\frac{x^3}{3}+\frac{(t_k-x)x^2}{2}\right]p_{n,k}(x)\\=&\frac{1}{6}\sum_{k=0}^{s}(x-t_k)^3p_{n,k}(x)+\sum_{k=0}^{n}\left[\frac{x^3}{3}+\frac{(t_k-x)x^2}{2}\right]p_{n,k}(x)\\=&\frac{x^3}{3}+\frac{1}{6}\sum_{k=0}^{s}(x-t_k)^3p_{n,k}(x).\end{aligned} \tag{3.28}$$

From (2.1), the identity $\min\{a,b\}=\frac{a+b-|a-b|}{2}$ and a direct computation we obtain

$$\begin{aligned}
A_{1,3}(x)=&\sum_{k=0}^{n}\sum_{m=0}^{n}\left(\frac{\left(\min\{t_k,t_m\}\right)^3}{3}+\frac{|t_k-t_m|\left(\min\{t_k,t_m\}\right)^2}{2}\right)p_{n,k}(x)p_{n,m}(x)\\
=&\sum_{k=0}^{n}\sum_{m=0}^{n}\left(\frac{\left(t_k+t_m-|t_k-t_m|\right)^3}{24}+\frac{|t_k-t_m|\left(t_k+t_m-|t_k-t_m|\right)^2}{8}\right)p_{n,k}(x)p_{n,m}(x)\\
=&\sum_{k=0}^{n}\sum_{m=0}^{n}\frac{-t_k^3-t_m^3+3t_kt_m^2+3t_k^2t_m}{12}p_{n,k}(x)p_{n,m}(x)\\
&+\frac{1}{12}\sum_{k=0}^{n}\sum_{m=0}^{n}|t_k-t_m|^3p_{n,k}(x)p_{n,m}(x).
\end{aligned}\tag{3.29}$$

From (3.4)-(3.7) we obtain

$$\begin{aligned}
&\sum_{k=0}^{n}\sum_{m=0}^{n}\frac{-t_k^3-t_m^3+3t_kt_m^2+3t_k^2t_m}{12}p_{n,k}(x)p_{n,m}(x)\\
=&\frac{1}{2}\sum_{k=0}^{n}t_kp_{n,k}(x)\sum_{k=0}^{n}t_k^2p_{n,k}(x)-\frac{1}{6}\sum_{k=0}^{n}p_{n,k}(x)\sum_{k=0}^{n}t_k^3p_{n,k}(x)\\
=&\frac{x^3}{3}-\frac{x(1-x)(1-2x)}{6n^2}.
\end{aligned}\tag{3.30}$$

From (3.2), (3.27)-(3.30) it follows that

$$\begin{aligned}
&\int_{F_1}\left|f(x)-B_n(f,x)\right|^2\omega_1(df)\\
=&-\frac{1}{3}\sum_{k=0}^{s}(x-t_k)^3p_{n,k}(x)+\frac{1}{12}\sum_{k=0}^{n}\sum_{m=0}^{n}|t_k-t_m|^3p_{n,k}(x)p_{n,m}(x)-\frac{x(1-x)(1-2x)}{6n^2}\\
=&\frac{-J_1(x)}{3}+\frac{J_2(x)}{12}+J_3(x).
\end{aligned}\tag{3.31}$$

Similar to the proof of (3.24), we have

$$\begin{aligned}
J_1(x)=&\frac{2\sqrt{2}(x(1-x))^{\frac{3}{2}}}{\sqrt{\pi}n^{\frac{3}{2}}}\int_0^{+\infty}s^3e^{-s^2}ds+o\left(n^{-\frac{3}{2}}\right)\\
=&\frac{\sqrt{2}(x(1-x))^{\frac{3}{2}}}{\sqrt{\pi}n^{\frac{3}{2}}}+o\left(n^{-\frac{3}{2}}\right).
\end{aligned}\tag{3.32}$$

By the same discussion and a direct computation we obtain

$$\begin{aligned}
J_2(x)=&\frac{2\sqrt{2}(x(1-x))^{\frac{3}{2}}}{\pi n^{\frac{3}{2}}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|s-t|^3e^{-s^2-t^2}dsdt+o\left(n^{-\frac{3}{2}}\right)\\
=&\frac{8(x(1-x))^{\frac{3}{2}}}{\sqrt{\pi}n^{\frac{3}{2}}}+o\left(n^{-\frac{3}{2}}\right).
\end{aligned}\tag{3.33}$$

From (3.31)– (3.33) we get

$$\int_{F_1} \left|f(x) - B_n(f,x)\right|^2 \omega_1(df) = \frac{\left(2-\sqrt{2}\right)(x(1-x))^{\frac{3}{2}}}{3\sqrt{\pi}n^{\frac{3}{2}}} + o\left(n^{-\frac{3}{2}}\right). \tag{3.34}$$

For $r \geq 2$, by Taylor formula we have

$$\begin{aligned} f\left(\frac{\nu}{n}\right) =& f(x) + \left(\frac{\nu}{n} - x\right) f'(x) + \left(\frac{\nu}{n} - x\right)^2 \cdot \frac{f''(\xi_\nu)}{2} \\ =& f(x) + \left(\frac{\nu}{n} - x\right) f'(x) + \left(\frac{\nu}{n} - x\right)^2 \cdot \frac{f''(x)}{2} + \left(\frac{\nu}{n} - x\right)^2 \cdot \left(\frac{f''(\xi_\nu) - f''(x)}{2}\right), \end{aligned} \tag{3.35}$$

where ξ_ν is between in x and $\frac{\nu}{n}$. Hence

$$\left|f''(\xi_\nu) - f''(x)\right| \leq \omega\left(f'', \left|\frac{\nu}{n} - x\right|\right), \tag{3.36}$$

where $\omega(f,t)$ is the modulus of continuity of f in the uniform norm. Hence, by (3.35) and a simple computation we obtain

$$\begin{aligned} B_n(f,x) - f(x) =& \frac{x(1-x)f''(x)}{2n} + \sum_{\nu=0}^{n} \left(\frac{\nu}{n} - x\right)^2 \cdot \left(\frac{f''(\xi_\nu) - f''(x)}{2}\right) p_{n,\nu}(x) \\ =& K_1(f,x) + K_2(f,x). \end{aligned} \tag{3.37}$$

Let $f = T_r g$. By (3.37), (1.1) and (1.3) we obtain

$$\begin{aligned} &\int_{F_r} |K_1(f,x)|^2 \omega_r(df) = \frac{(x(1-x))^2}{4n^2} \int_{F_r} |f''(x)|^2 \omega_r(df) \\ =& \frac{(x(1-x))^2}{4n^2} \int_{F_0} |(T_r g)''(x)|^2 \omega_0(dg) = \frac{(x(1-x))^2}{4n^2} \int_{F_0} |(T_{r-2}g)(x)|^2 \omega_0(dg) \\ =& \frac{(x(1-x))^2}{4n^2} \int_{F_{r-2}} |f(x)|^2 \omega_{r-2}(df) \\ =& \frac{(x(1-x))^2}{4((r-2)!)^2 n^2} \int_0^x (x-u)^{2r-4} du = \frac{x^{2r-1}(1-x)^2}{4(2r-3)((r-2)!)^2 n^2}. \end{aligned} \tag{3.38}$$

From (3.36), (3.37), (2.11) and (3.12) it follows that

$$\begin{aligned} &|K_2(f,x)| \\ \leq& \omega\left(f'', \frac{1}{\sqrt[12]{n^5}}\right) \sum_{\left|\frac{\nu}{n} - x\right| \leq \frac{1}{\sqrt[12]{n^5}}} \left(\frac{\nu}{n} - x\right)^2 p_{n,\nu}(x) + \omega(f'',1) \sum_{\left|\frac{\nu}{n} - x\right| > \frac{1}{\sqrt[12]{n^5}}} \left(\frac{\nu}{n} - x\right)^2 p_{n,\nu}(x) \\ \leq& \omega\left(f'', \frac{1}{\sqrt[12]{n^5}}\right) \sum_{\nu=0}^{n} \left(\frac{\nu}{n} - x\right)^2 p_{n,\nu}(x) + \frac{C\omega(f'',1)}{n^2} \\ \leq& \frac{C\omega\left(f'', \frac{1}{\sqrt[12]{n^5}}\right)}{n}. \end{aligned} \tag{3.39}$$

From [9, p. 86] we obtain

$$\int_{F_0}\left|\omega\left(g,\frac{1}{n}\right)\right|^2\omega_0(dg)\le\frac{C\ln n}{n}. \tag{3.40}$$

For $r=2$, let $f=T_2g$, then by (1.1), (3.39) and (3.40) we obtain

$$\begin{aligned}&\int_{F_2}|K_2(f,x)|^2\omega_2(df)\le\frac{C}{n^2}\int_{F_2}\omega\left(f'',\frac{1}{\sqrt[12]{n^5}}\right)^2\omega_2(df)\\&=\frac{C}{n^2}\int_{F_0}\omega\left((T_2g)'',\frac{1}{\sqrt[12]{n^5}}\right)^2\omega_0(dg)=\frac{C}{n^2}\int_{F_0}\omega\left(g,\frac{1}{\sqrt[12]{n^5}}\right)^2\omega_0(dg)\\&\le\frac{C\ln n}{n^{\frac{29}{12}}}.\end{aligned} \tag{3.41}$$

For $r>2$, let $f=T_rg$, then by $f''(0)=0$ and the Lagrange Intermediate Value Theorem it follows that

$$\omega\left(f'',\frac{1}{\sqrt[12]{n^5}}\right)\le\frac{\|f^{(3)}\|_C}{\sqrt[12]{n^5}}=\frac{\|(T_rg)^{(3)}\|_C}{\sqrt[12]{n^5}}. \tag{3.42}$$

Let c satisfy that $\|(T_rg)^{(3)}\|_C=|(T_rg)^{(3)}(c)|$. Noticed that $(T_rg)^{(3)}(0)=0$, we have

$$\begin{aligned}\|(T_rg)^{(3)}\|_C&=|(T_rg)^{(3)}(c)-(T_rg)^{(3)}(0)|\le\omega((T_rg)^{(3)},c)\le\omega((T_rg)^{(3)},1)\\&\le\left(\sqrt[12]{n^5}+1\right)\omega\left((T_rg)^{(3)},\frac{1}{\sqrt[12]{n^5}}\right)\\&=\left(\sqrt[12]{n^5}+1\right)\omega\left(f^{(3)},\frac{1}{\sqrt[12]{n^5}}\right).\end{aligned} \tag{3.43}$$

Combing (3.42) and (3.43) it follows that

$$\omega\left(f'',\frac{1}{\sqrt[12]{n^5}}\right)\le 2\omega\left(f^{(3)},\frac{1}{\sqrt[12]{n^5}}\right). \tag{3.44}$$

Using the same method we conclude that

$$\omega\left(f'',\frac{1}{\sqrt[12]{n^5}}\right)\le 2^{r-2}\omega\left(f^{(r)},\frac{1}{\sqrt[12]{n^5}}\right). \tag{3.45}$$

Similar to (3.41), from (3.39), (3.40) and (3.45) we get

$$\int_{F_r}|K_2(f,x)|^2\omega_r(df)\le\frac{C\ln n}{n^{\frac{29}{12}}}. \tag{3.46}$$

From (3.37), (3.38), (3.41) and (3.46) it follows that

$$\int_{F_r} |B_n(f,x) - f(x)|^2 \omega_r(df) = \frac{x^{2r-1}(1-x)^2}{4(2r-3)((r-2)!)^2 n^2} + o\left(\frac{1}{n^2}\right). \tag{3.47}$$

From (3.1), (3.26), (3.34) and (3.47) we obtain the desired results of Theorem 1.1. □

4. Proof of Theorem 1.2

In this section, we give the proof of the Theorem 1.2.

Proof. From [5, p. 107, (28)] we obtain

$$e_p^p\left(B_n, \|\cdot\|_{p,s,\varrho}, F_r, \omega_r\right) = v_p \cdot \int_0^1 \left(\int_{F_r} \left|f^{(s)}(x) - B_n^{(s)}(f,x)\right|^2 \omega_r(df)\right)^{\frac{p}{2}} \cdot \varrho(x)dx. \tag{4.1}$$

From [8, p. 306] it follows that

$$B_n^{(s)}(f,x) = n(n-1)\cdots(n-s+1)\sum_{v=0}^{n-s} \Delta_{1/n}^s f\left(\frac{v}{n}\right) p_{n-s,v}(x), \tag{4.2}$$

where $\Delta_h^k f(x)$ is the k^{th} difference of the function f at the point x with step h. For $f \in C^{(k)}[0,1]$, it is well known that

$$\Delta_h^k f(x) = \int_0^h dy_1 \cdots \int_0^h f^{(k)}(x + y_1 + \cdots + y_k)dy_k. \tag{4.3}$$

For $s = r$, from (4.3) and the integral intermediate theorem we obtain

$$\begin{aligned} \Delta_{1/n}^s f\left(\frac{v}{n}\right) &= \int_0^{1/n} dy_1 \cdots \int_0^{1/n} f^{(s)}\left(\frac{v}{n} + y_1 + \cdots + y_s\right) dy_s \\ &= \frac{1}{n^s} f^{(s)}\left(\frac{v}{n} + \xi_v\right), \end{aligned} \tag{4.4}$$

where $0 \le \xi_v \le \frac{s}{n}$. It is easy to verify that $t \in \left[\frac{v}{n}, \frac{v+s}{n}\right]$ implies $\left|t - \frac{v}{n-s}\right| \le \frac{s}{n}$. Hence

$$\begin{aligned} &\left|\Delta_{1/n}^s f\left(\frac{v}{n}\right) - \frac{1}{n^s} f^{(s)}\left(\frac{v}{n-s}\right)\right| \\ =&\frac{1}{n^s}\left|f^{(s)}\left(\frac{v}{n} + \xi_v\right) - f^{(s)}\left(\frac{v}{n-s}\right)\right| \le \frac{1}{n^s}\omega\left(f^{(s)}, \frac{s}{n}\right). \end{aligned} \tag{4.5}$$

From (4.2) and (4.5) it follows that

$$\begin{aligned}B_n^{(s)}(f,x)=&\frac{(n-1)\cdots(n-s+1)}{n^{s-1}}\sum_{v=0}^{n-s}f^{(s)}\left(\frac{v}{n-s}\right)p_{n-s,v}(x)\\&+\frac{(n-1)\cdots(n-s+1)}{n^{s-1}}\sum_{v=0}^{n-s}\left(f^{(s)}\left(\frac{v}{n}+\xi_v\right)-f^{(s)}\left(\frac{v}{n-s}\right)\right)p_{n-s,v}(x)\\=&\frac{(n-1)\cdots(n-s+1)}{n^{s-1}}\sum_{v=0}^{n-s}f^{(s)}\left(\frac{v}{n-s}\right)p_{n-s,v}(x)+\omega\left(f^{(s)},\frac{s}{n}\right)B(f,x)\\=&\frac{(n-1)\cdots(n-s+1)}{n^{s-1}}B_{n-s}(f^{(s)},x)+\omega\left(f^{(s)},\frac{s}{n}\right)B(f,x),\end{aligned}\tag{4.6}$$

where $|B(f,x)|\le 1$. By (4.6) and a simple computation we obtain

$$\begin{aligned}B_n^{(s)}(f,x)-f^{(s)}(x)=&\frac{(n-1)\cdots(n-s+1)}{n^{s-1}}\left(B_{n-s}\left(f^{(s)},x\right)-f^{(s)}(x)\right)\\&+\left(\frac{(n-1)\cdots(n-s+1)}{n^{s-1}}-1\right)f^{(s)}(x)+\omega\left(f^{(s)},\frac{s}{n}\right)B(f,x)\\=&(1+o(1))\left(B_{n-s}\left(f^{(s)},x\right)-f^{(s)}(x)\right)\\&-\frac{(1+o(1))s(s-1)f^{(s)}(x)}{2n}+\omega\left(f^{(s)},\frac{s}{n}\right)B(f,x).\end{aligned}\tag{4.7}$$

Similar to the proof of (3.46), we have that for $0\le s\le r$

$$\begin{aligned}\int_{F_r}\left(\omega\left(f^{(s)},\frac{s}{n}\right)\right)^2\omega_r(df)=&\int_{F_0}\left(\omega\left((T_rg)^{(s)},\frac{s}{n}\right)\right)^2\omega_0(dg)\\\le&\,2^{2(r-s)}\int_{F_0}\left(\omega\left(g,\frac{s}{n}\right)\right)^2\omega_0(dg)\le\frac{C\ln n}{n}.\end{aligned}\tag{4.8}$$

By (3.26) we obtain

$$\begin{aligned}&\int_{F_r}\left|B_{n-r}\left(f^{(r)},x\right)-f^{(r)}(x)\right|^2\omega_r(df)\\=&\int_{F_0}\left|B_{n-r}\left((T_rg)^{(r)},x\right)-(T_rg)^{(r)}(x)\right|^2\omega_0(dg)\\=&\int_{F_0}\left|B_{n-r}\left(g,x\right)-g(x)\right|^2\omega_0(dg)\\=&\frac{\left(2\sqrt{2}-1\right)\sqrt{x(1-x)}}{2\sqrt{\pi n}}+o\left(\frac{1}{\sqrt{n}}\right).\end{aligned}\tag{4.9}$$

From (1.1) and (1.2) it follows that

$$\int_{F_r}\left(f^{(r)}(x)\right)^2\omega_r(df)=\int_{F_0}\left((T_rg)^{(r)}(x)\right)^2\omega_0(dg)=\int_{F_0}\left(g(x)\right)^2\omega_0(dg)=x.\tag{4.10}$$

From (4.7)-(4.10) we obtain that

$$\int_{F_r} \left| f^{(r)}(x) - B_n^{(r)}(f,x) \right|^2 \omega_r(df) = \frac{\left(2\sqrt{2}-1\right)\sqrt{x(1-x)}}{2\sqrt{\pi n}} + o\left(\frac{1}{\sqrt{n}}\right). \tag{4.11}$$

For $s < r$, by (4.3) and the Newton-Leibniz formula we conclude that

$$\begin{aligned}
\Delta_{1/n}^{s} f\left(\frac{\nu}{n}\right) =& \int_0^{1/n} dy_1 \cdots \int_0^{1/n} f^{(s)}\left(\frac{\nu}{n-s}\right) dy_s \\
&+ \int_0^{1/n} dy_1 \cdots \int_0^{1/n} \left(f^{(s)}\left(\frac{\nu}{n} + y_1 + \cdots + y_s\right) - f^{(s)}\left(\frac{\nu}{n-s}\right)\right) dy_s \\
=& \frac{1}{n^s} f^{(s)}\left(\frac{\nu}{n-s}\right) + \int_0^{1/n} dy_1 \cdots \int_0^{1/n} dy_s \int_{\frac{\nu}{n-s}}^{\frac{\nu}{n}+y_1+\cdots+y_s} f^{(s+1)}(t) dt \\
=& \frac{1}{n^s} f^{(s)}\left(\frac{\nu}{n-s}\right) + \int_0^{1/n} dy_1 \cdots \int_0^{1/n} dy_s \int_{\frac{\nu}{n-s}}^{\frac{\nu}{n}+y_1+\cdots+y_s} f^{(s+1)}\left(\frac{\nu}{n-s}\right) dt \\
&+ \int_0^{1/n} dy_1 \cdots \int_0^{1/n} dy_s \int_{\frac{\nu}{n-s}}^{\frac{\nu}{n}+y_1+\cdots+y_s} \left(f^{(s+1)}(t) - f^{(s+1)}\left(\frac{\nu}{n-s}\right)\right) dt.
\end{aligned} \tag{4.12}$$

By a direct computation we get

$$\begin{aligned}
&\int_0^{1/n} dy_1 \cdots \int_0^{1/n} dy_s \int_{\frac{\nu}{n-s}}^{\frac{\nu}{n}+y_1+\cdots+y_s} f^{(s+1)}\left(\frac{\nu}{n-s}\right) dt \\
=& \left(\frac{s}{2n^{s+1}} - \frac{\nu s}{n^{s+1}(n-s)}\right) f^{(s+1)}\left(\frac{\nu}{n-s}\right).
\end{aligned} \tag{4.13}$$

Similar to the proof of (4.5), we have that for an arbitrary t in (4.12), we have

$$\left| f^{(s+1)}(t) - f^{(s+1)}\left(\frac{\nu}{n-s}\right)\right| \le \omega\left(f^{(s+1)}, \frac{s}{n}\right). \tag{4.14}$$

By (4.14) we obtain

$$\begin{aligned}
&\left| \int_0^{1/n} dy_1 \cdots \int_0^{1/n} dy_s \int_{\frac{\nu}{n-s}}^{\frac{\nu}{n}+y_1+\cdots+y_s} \left(f^{(s+1)}(t) - f^{(s+1)}\left(\frac{\nu}{n-s}\right)\right) dt \right| \\
\le& \frac{\left|ns - s^2 - 2\nu s\right|}{2n^{s+1}(n-s)} \omega\left(f^{(s+1)}, \frac{s}{n}\right) \le \frac{|s|}{2n^{s+1}} \omega\left(f^{(s+1)}, \frac{s}{n}\right).
\end{aligned} \tag{4.15}$$

Similar to (4.6), from (4.2) and (4.12)-(4.15) it follows that

$$\begin{aligned} B_n^{(s)}(f,x) =& \frac{(n-1)\cdots(n-s+1)}{n^{s-1}} B_{n-s}(f^{(s)},x) + \frac{s(n-1)\cdots(n-s+1)}{2n^s} B_{n-s}(f^{(s+1)},x) \\ &- \frac{s(n-1)\cdots(n-s+1)}{n^s} \sum_{v=0}^{n-s} \frac{v}{n-s} f^{(s+1)}\left(\frac{v}{n-s}\right) p_{n-s,v}(x) \\ &+ \frac{1}{n}\omega\left(f^{(s+1)}, \frac{s}{n}\right) A(f,x), \end{aligned} \tag{4.16}$$

where $|A(f,x)| \leq s$. It is easy to verify that

$$\begin{aligned} &\sum_{v=0}^{n-s} \frac{v}{n-s} f^{(s+1)}\left(\frac{v}{n-s}\right) p_{n-s,v}(x) \\ =& x \sum_{w=0}^{n-s-1} f^{(s+1)}\left(\frac{w+1}{n-s}\right) p_{n-s-1,w}(x) \\ =& x \sum_{w=0}^{n-s-1} f^{(s+1)}\left(\frac{w}{n-s-1}\right) p_{n-s-1,w}(x) \\ &+ x \sum_{w=0}^{n-s-1} \left(f^{(s+1)}\left(\frac{w+1}{n-s}\right) - f^{(s+1)}\left(\frac{w}{n-s-1}\right)\right) p_{n-s-1,w}(x) \\ =& x B_{n-s-1}\left(f^{(s+1)},x\right) + x \sum_{w=0}^{n-s-1} \left(f^{(s+1)}\left(\frac{w+1}{n-s}\right) - f^{(s+1)}\left(\frac{w}{n-s-1}\right)\right) p_{n-s-1,w}(x). \end{aligned} \tag{4.17}$$

From

$$\left|\frac{w+1}{n-s} - \frac{w}{n-s-1}\right| \leq \frac{1}{n-s}, \quad \forall 0 \leq w \leq n-s-1, \tag{4.18}$$

we get

$$\left| x \sum_{w=0}^{n-s-1} \left(f^{(s+1)}\left(\frac{w+1}{n-s}\right) - f^{(s+1)}\left(\frac{w}{n-s-1}\right)\right) p_{n-s-1,w}(x)\right| \leq \omega\left(f^{(s+1)}, \frac{1}{n-s}\right). \tag{4.19}$$

Combining (4.16), (4.17) and (4.19), we obtain

$$\begin{aligned} B_n^{(s)}(f,x) =& \frac{(n-1)\cdots(n-s+1)}{n^{s-1}} B_{n-s}(f^{(s)},x) + \frac{s(n-1)\cdots(n-s+1)}{2n^s} B_{n-s}(f^{(s+1)},x) \\ &- \frac{s(n-1)\cdots(n-s+1)}{n^s} x B_{n-s-1}\left(f^{(s+1)},x\right) + \frac{1}{n}\omega\left(f^{(s+1)}, \frac{s}{n-s}\right) C(f,x), \end{aligned}$$

where $|C(f,x)| \le s+1$. Similar to (4.7), by above relation we conclude that

$$\begin{aligned}
&B_n^{(s)}(f,x) - f^{(s)}(x)\\
=&(1+o(1))\left(B_{n-s}\left(f^{(s)},x\right) - f^{(s)}(x)\right) - \frac{(1+o(1))s(s-1)f^{(s)}(x)}{2n}\\
&+\frac{s(n-1)\cdots(n-s+1)}{2n^s}\left(B_{n-s}(f^{(s+1)},x) - f^{(s+1)}(x)\right) + \frac{s(n-1)\cdots(n-s+1)}{2n^s}f^{(s+1)}(x)\\
&-\frac{s(n-1)\cdots(n-s+1)}{n^s}x\left(B_{n-s-1}\left(f^{(s+1)},x\right) - f^{(s+1)}(x)\right)\\
&-\frac{s(n-1)\cdots(n-s+1)}{n^s}xf^{(s+1)}(x) + \frac{1}{n}\omega\left(f^{(s+1)},\frac{s}{n-s}\right)C(f,x)\\
=&(1+o(1))\left(B_{n-s}\left(f^{(s)},x\right) - f^{(s)}(x)\right) - \frac{(1+o(1))s(s-1)f^{(s)}(x)}{2n}\\
&+\frac{(1+o(1))}{2n}\left(B_{n-s}(f^{(s+1)},x) - f^{(s+1)}(x)\right) + \frac{(1+o(1))}{2n}(1-2x)f^{(s+1)}(x)\\
&-\frac{(1+o(1))}{n}x\left(B_{n-s-1}\left(f^{(s+1)},x\right) - f^{(s+1)}(x)\right) + \frac{1}{n}\omega\left(f^{(s+1)},\frac{s}{n-s}\right)C(f,x).
\end{aligned} \tag{4.20}$$

For an arbitrary $1 \le m \le r$, we have

$$\begin{aligned}
&\int_{F_r}\left|B_{n-m}\left(f^{(m)},x\right) - f^{(m)}(x)\right|^2\omega_r(df)\\
=&\int_{F_0}\left|B_{n-m}\left((T_rg)^{(m)},x\right) - (T_rg)^{(m)}(x)\right|^2\omega_0(dg)\\
=&\int_{F_0}\left|B_{n-m}\left(T_{r-m}g,x\right) - (T_{r-m}g)(x)\right|^2\omega_0(dg)\\
=&\int_{F_{r-m}}\left|B_{n-m}\left(f,x\right) - f(x)\right|^2\omega_{r-m}(df).
\end{aligned} \tag{4.21}$$

By (1.3) we obtain

$$\begin{aligned}
&\int_{F_r}\left|f^{(m)}(x)\right|^2\omega_r(df)\\
=&\int_{F_0}\left|(T_rg)^{(m)}(x)\right|^2\omega_0(dg) = \int_{F_0}\left|(T_{r-m}g)(x)\right|^2\omega_0(dg)\\
=&\int_{F_{r-m}}\left|f(x)\right|^2\omega_{r-m}(df) = \frac{1}{((r-m)!)^2}\int_0^x (x-u)^{2(r-m)}du\\
=&\frac{x^{2r-2m+1}}{((r-m)!)^2(2r-2m+1)}.
\end{aligned} \tag{4.22}$$

For $s = r-1$, from (4.8), (4.20)– (4.22) and (3.34) we obtain

$$\int_{F_r}\left|f^{(r-1)}(x) - B_n^{(r-1)}(f,x)\right|^2\omega_r(df) = \frac{\left(2-\sqrt{2}\right)(x(1-x))^{\frac{3}{2}}}{3\sqrt{\pi}n^{\frac{3}{2}}} + o\left(n^{-\frac{3}{2}}\right). \tag{4.23}$$

For $s \le r-2$, from (3.37), (4.8), (4.20)– (4.22) and a simple discussion it follows that

$$\int_{F_r} \left| f^{(s)}(x) - B_n^{(s)}(f,x) \right|^2 \omega_r(df) = D_{r,s}(x) + o\left(\frac{1}{n^2}\right), \tag{4.24}$$

where

$$\begin{aligned} D_{r,s}(x) =& \frac{1}{4n^2} \int_{F_r} \left| s(s-1)f^{(s)}(x) - x(1-x)f^{(s+2)}(x) + s(1-2x)f^{(s+1)} \right|^2 \omega_r(df) \\ =& \frac{s^2(s-1)^2}{4n^2} \int_{F_r} \left| f^{(s)}(x) \right|^2 \omega_r(df) + \frac{x^2(1-x)^2}{4n^2} \int_{F_r} \left| f^{(s+2)}(x) \right|^2 \omega_r(df) \\ &+ \frac{s^2(1-2x)^2}{4n^2} \int_{F_r} \left| f^{(s+1)}(x) \right|^2 \omega_r(df) - \frac{s(s-1)x(1-x)}{2n^2} \int_{F_r} f^{(s)}(x) f^{(s+2)}(x) \omega_r(df) \\ &+ \frac{s^2(s-1)(1-2x)}{2n^2} \int_{F_r} f^{(s)}(x) f^{(s+1)}(x) \omega_r(df) \\ &- \frac{xs(1-x)(1-2x)}{2n^2} \int_{F_r} f^{(s+1)}(x) f^{(s+2)}(x) \omega_r(df). \end{aligned} \tag{4.25}$$

From $f \in F_r$ we know $f^{(s)}(0) = f^{(s+1)}(0) = 0$. Hence the Newton-Leibniz formula gives

$$f^{(s)}(x) = \int_0^x f^{(s+1)}(t)dt = \int_0^x dt \int_0^t f^{(s+2)}(u)du. \tag{4.26}$$

From Fubini's Theorem, (4.26), (1.1) and (1.3) it follows that

$$\begin{aligned} &\int_{F_r} f^{(s)}(x) f^{(s+2)}(x) \omega_r(df) \\ =& \int_0^x dt \int_0^t du \int_{F_r} f^{(s+2)}(u) f^{(s+2)}(x) \omega_r(df) \\ =& \int_0^x dt \int_0^t du \int_{F_0} (T_r g)^{(s+2)}(u) (T_r g)^{(s+2)}(x) \omega_0(dg) \\ =& \int_0^x dt \int_0^t du \int_{F_0} (T_{r-s-2} g)(u) (T_{r-s-2} g)(x) \omega_0(dg) \\ =& \int_0^x dt \int_0^t du \int_{F_{r-s-2}} f(u) f(x) \omega_{r-s-2}(df) \\ =& \frac{1}{((r-s-2)!)^2} \int_0^x dt \int_0^t du \int_0^u (x-v)^{r-s-2} (u-v)^{r-s-2} dv \\ =& \frac{1}{((r-s-2)!)^2} \int_0^x dv \int_v^x dt \int_v^t (x-v)^{r-s-2} (u-v)^{r-s-2} du \\ =& \frac{x^{2r-2s-1}}{(r-s)!(r-s-2)!(2r-2s-1)}. \end{aligned} \tag{4.27}$$

Similarly, we have

$$\int_{F_r} f^{(s)}(x)f^{(s+1)}(x)\omega_r(df)=\frac{x^{2r-2s}}{2((r-s)!)^2}, \tag{4.28}$$

$$\int_{F_r} f^{(s+1)}(x)f^{(s+2)}(x)\omega_r(df)=\frac{x^{2r-2s-2}}{2((r-s-1)!)^2}. \tag{4.29}$$

From (4.25), (4.22) and (4.27)-(4.29) it follows that

$$\begin{aligned}&D_{r,s}(x)\\ =&\frac{x^{2r-2s-1}}{(2n(r-s)!)^2}\left[\frac{s^2(s-1)^2x^2}{2r-2s+1}+\frac{(r-s)^2s^2(1-2x)^2}{(2r-2s-1)}+\frac{(r-s)^2(r-s-1)^2(1-x)^2}{2r-2s-3}\right]\\ &-\frac{sx^{2r-2s}}{(2n(r-s-1)!)^2}\left[\frac{2(s-1)(r-s-1)(1-x)}{(2r-2s-1)(r-s)}-\frac{s(s-1)(1-2x)}{(r-s)^2}+(1-x)(1-2x)\right].\end{aligned} \tag{4.30}$$

By using (4.1), (4.11), (4.23), (4.24) and (4.30), we obtain the desired estimate of Theorem 1.2. □

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